Lectures on Celestial Holography

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Abstract

These notes consist of 5 lectures on celestial holography given at the Higgs summer school 2022. After an introduction to soft theorems, we discuss the infrared problem in QED and show that asymptotic symmetries provide a new perspective and a potential resolution to the problem. We then review how semiclassically, the subleading soft graviton theorem implies an enhancement of the Lorentz symmetry of scattering in four-dimensional asymptotically flat spacetimes to Virasoro. This leads to the construction of celestial amplitudes as $\mathcal{S}$-matrices computed in a basis of boost eigenstates. Both massless and massive asymptotic states are recast as insertions on the celestial sphere transforming as global conformal primaries under the Lorentz $\mathrm{SL}(2,\mathbb{C})$. We finally review celestial symmetries and the constraints they impose on celestial scattering and show how the celestial perspective provides new insights into gravity in asymptotically flat spacetimes.

Contents

1 Introduction

AdS/CFT [1–3] provided a concrete realization of the holographic principle [4,5]: a theory of gravity in an arbitrary number of dimensions should be dual to a quantum theory in one dimension less. A concrete realization of this duality in any but asymptotically negatively curved backgrounds remains an important open problem. The goal of these lectures is to review some of the recent developments addressing this problem in asymptotically flat spacetimes (AFS).

In the past decade we learned that gravity and gauge theory in AFS are governed in the infrared by a triangular equivalence: soft theorems can be recast as conservation laws associated with large gauge symmetries [6,7], while memory effects are an observable signature thereof [8, 9]. (See [10] for a detailed review.) These developments lead to the proposal that gravity in four-dimensional (4D) AFS may be dual to a theory living on the "celestial sphere" at infinity [11]. Among others, this proposal is backed up by evidence that the Lorentz symmetry of scattering in 4D AFS is enhanced to Virasoro [12], as well as the existence of a stress tensor constructed from a particular subleading soft graviton mode in the bulk [13]. In the first lecture we will see how this follows from the subleading soft graviton theorem. In the second lecture we formulate scattering in AFS in terms of a new observable: the celestial amplitude. We show how celestial amplitudes re-express the $\mathcal{S}$-matrix in a basis of boost eigenstates [14, 15]. (In contrast, conventional scattering amplitudes are computed in a plane wave basis.) Such a construction exists for scattering of both massive and massless particles which is illustrated with a calculation of the tree-level celestial amplitude of two massless and one massive scalars [16]. In the final lecture we describe some recent developments centered around the theme of celestial symmetries. We show that both bulk translation symmetry as well as the soft theorems imply the existence of celestial currents which constrain the celestial amplitudes. We will see for example that Poincaré symmetry can be used to completely fix celestial three-point functions and constrain four-point functions [17, 18], while subleading and subsubleading soft theorems can be used to completely fix the leading OPE coefficients in a collinear expansion of gluons and gravitons [19]. We finally show that celestial theories contain an infinity of soft currents and compute their algebra in some examples [20, 21].

We have tried to give a self-contained overview of this rapidly growing field by choosing a particular path through the subject. Many fascinating recent developments have been left out. Explicit constructions of tree-level celestial amplitudes have appeared in [11, 14, 22–27]. Loop corrections were addressed in [28–30] while properties of bulk scattering such as the double copy and connections to ambitwistor strings have been worked out in [31–34]. Celestial symmetries in both gravity and gauge theories, as well as their constraints on celestial amplitudes have been discussed in [17, 18, 20, 21, 35–47]. Analytic properties of celestial four-point functions in the complex boost-weight plane have been worked out in [48, 49] and conformal block expansions were computed in [50–53]. Infrared divergences and related aspects were discussed in [54–62]. We hope these lectures provide a bridge between the earlier developments reviewed in [10] and more recent results.

Lecture 1 outline:

- Basics of QFT: fields and quantization (basis of solutions to KG, plane wave expansions, comments on curved spacetimes?);
- S-matrix definition and assumptions;
- Symmetries and Ward identities $\rightarrow$ soft theorems!

2 QFT basics

2.1 Free particles

Consider the equations of motion for a spin$-s$ free field $\Phi(x)$

$$\mathcal{D}\cdot\Phi(x)=0, \tag{2.1}$$

where $\mathcal{D}$ is a differential operator and $\Phi(x):\mathcal{M}\rightarrow\mathcal{F}$ is a map from spacetime $\mathcal{M}$ to field space $\mathcal{F}$. In these lectures we will restrict to 4-dimensional spacetimes. In quantum field theory, $\mathcal{M}=\mathbb{R}^{1,3}$. In (quantum) gravity, $\mathcal{M}$ is a manifold (locally $\mathbb{R}^{1,3}$). Examples (in flat space)

$$\begin{aligned} s=0: &\quad \mathcal{D}=\Box+m^2, \quad \Phi(x)=\phi(x),\\ s=1/2: &\quad \mathcal{D}=\gamma^\mu\partial_\mu+mI, \quad \Phi(x)=\psi(x). \end{aligned} \tag{2.2}$$

Comment: $\mathcal{D}$ can always be traded for a (in general matrix-valued) second order differential operator, while $\Phi(x)$ can also be an s-index symmetric traceless tensor field. We work in mostly + metric signature in which case (in flat space) $\Box = -\eta^{\mu\nu}\partial_\mu\partial_\nu$.

In Minkowski space the space of solutions to (2.1) splits as

$$\mathcal{S} = \mathcal{S}_p \oplus \bar{\mathcal{S}}_p, \tag{2.3}$$

where $\mathcal{S}_p$ and $\bar{\mathcal{S}}_p$ are positive and negative frequency subspaces. To see this, note that Minkowski space can be foliated with constant time (Cauchy) slices Σ_t. The solutions to (2.1) are completely specified by data $(\Phi, \partial_t\Phi)$ on such a Cauchy slice (say Σ_0). With any Cauchy slice we then associate an inner product (Klein-Gordon)

$$(\alpha, \beta) = \langle \alpha, \beta \rangle_{KG} = \int d^3x n^a j_a(\alpha, \beta), \tag{2.4}$$

where $\nabla^a j_a = 0$ and n^a is the normal to Σ. This condition (+ Stokes' theorem) ensures that (2.4) is independent on the slice,

$$\int_{\Sigma_i} d^3x n^a j_a(\alpha, \beta) - \int_{\Sigma_f} d^3x n^a j_a(\alpha, \beta) = \int_{\mathcal{M}_*} d^4x n^a \nabla_a j^a = 0. \tag{2.5}$$

figure. j is defined by

$$j = -i(\alpha^* d\beta - \beta^* d\alpha) \tag{2.6}$$

and for complex scalar fields reduces to the familiar expression

$$j_a = -i(\phi^* \partial_a \phi - \phi \partial_a \phi^*). \tag{2.7}$$

Note that

$$(\alpha, \beta) = -(\beta^*, \alpha^*) \tag{2.8}$$

therefore, the inner product is not positive definite on the whole solution space. The positive frequency subspace is the subspace of $\mathcal{S}$ on which (2.4) is positive definite.

For the scalar KG equation, positive frequency modes are

$$\phi_p(x) = e^{ip\cdot x}, \quad p^0 = \sqrt{\vec{p}^2 + m^2} \tag{2.9}$$

and form a *basis* for $\mathcal{S}_p$. Similarly, ϕ_p^* form a basis for $\bar{\mathcal{S}}_p$. A generic solution ϕ_p can then be expanded as

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3 2p^0} \left(a_p^\dagger \phi_p + a_p \phi_p^* \right). \tag{2.10}$$

What singles out the basis (2.9) is that the $\phi(x)$ diagonalize the momentum generators P. (In fact, the expansion (2.10) could have been derived by imposing Poincaré symmetries alone!) We will see later that a different choice exists that diagonalizes the Lorentz generators instead.

2.2 S-matrix

The S-matrix is a fundamental observable in quantum field theory (and we think quantum gravity too). This matrix computes the amplitude for a collection of particles in the far past to evolve into a collection of particles in the far future. In QFT one assumes that:

- Particles are freely moving in the far past and the far future; a consequence of the fact that:
- Interactions are localized in space and time;

Formally, an S-matrix element is the overlap between in and out states $|\Psi^-_\alpha\rangle, |\Psi^+_\beta\rangle$ defined at $t \to \pm\infty$ respectively, namely

$$S_{\alpha\beta} = \langle \Psi^+_\beta | \Psi^-_\alpha \rangle. \tag{2.11}$$

Here Ψ are taken to be non-interacting and hence transform under the homogeneous Lorentz group as a product of one-particle states

$$U(\Lambda, a)\Psi_{p_1,\sigma_1;\cdots} = e^{-ia_\mu((\Lambda p_1)^\mu + \cdots)} \sum_{\sigma'_1\cdots} D^{(j_1)}_{\sigma'_1\sigma_1} \cdots \Psi_{\Lambda p_1, \sigma'_1;\cdots} \tag{2.12}$$

where p_i, σ_i are momentum and helicity and we have suppressed the dependence on any other quantum numbers. Λ, a are Lorentz elements and translation parameters respectively. For massive particles, D are $SO(3)$ little group elements. For massless particles, they are simply phases.

Exercise 2.1. *i) Let $P_\mu, L_{\mu\nu}$ be the translation and Lorentz generators of the Poincaré algebra. Using the standard commutation relations of this algebra, show that*

$$P^2 \equiv P_\mu P^\mu, \quad W^2 \equiv W_\mu W^\mu, \quad W_\mu = -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma} L^{\nu\rho} P^\sigma \tag{2.13}$$

are Casimirs of the Poincaré algebra (ie. they commute with all generators).
ii) Show that when acting on a rest frame eigenstate of P_μ, $\bar{p}^\mu = (m, 0, 0, 0)$ W_μ reduces to w_μ where

$$w_0 = 0, \quad w_i = \frac{1}{2} m \varepsilon_{ijk} L_{jk} \equiv m J_i, \quad w^2 \to m^2 J^2. \tag{2.14}$$

iii) Find an additional operator that commutes with P and deduce that massive states in Poincaré representations are labelled by $|m, s; \vec{p}, \lambda\rangle$ where λ is an eigenvalue of $J \cdot \vec{p}$.
iv) Show that massive Poincaré states transform as in (2.12).

Exercise 2.2. *i) Show that $\{J_3, L_1 \equiv J_1 + K_2, L_2 \equiv J_2 - K_1\}$ leave massless momenta $q = \omega(1, 0, 0, 1)$ invariant.*
ii) Show that $L_1^2 + L_2^2$ is a quadratic Casimir and argue that massless states ought to be annihilated by L_1, L_2 and are hence specified by their eigenvalue under J_3 only. (Hint: Note that $[J_3, L_\pm] = \pm L_\pm$, where $L_\pm = L_1 \pm i L_2$.)
iii) Conclude that massless states transform as in (2.12) *where the Wigner rotation D is simply a phase.*

Ψ^+, Ψ^- live in the *same* Hilbert space: out states can be expanded in terms of in states and vice-versa. $S_{\alpha\beta}$ are the coefficients in this expansion. Completeness of in/out states implies that the S-matrix is unitary,

$$\int d\beta S^*_{\beta\gamma} S_{\beta\alpha} = \langle \Psi^+_\gamma | \Psi^+_\alpha \rangle = \delta(\gamma - \alpha). \tag{2.15}$$

Note that (2.11) is written in the Heisenberg picture and hence Ψ do not depend on time: all time dependence is in the operators.

It will be convenient to rewrite (2.11) in terms of matrix elements of free particle states, Φ[1]

$$\Psi(\pm\infty) = \Omega(\pm\infty)\Phi \tag{2.16}$$

and define an S-operator such that

$$\langle \Phi^{\text{out}} | S | \Phi^{\text{in}} \rangle \equiv \langle \Psi^+_\beta | \Psi^-_\alpha \rangle \implies S = \Omega^\dagger(+\infty)\Omega(-\infty) = U(\infty, -\infty). \tag{2.17}$$

Moreover

$$U(\tau, \tau_0) = e^{iH_0\tau} e^{-iH(\tau-\tau_0)} e^{-iH_0\tau_0}. \tag{2.18}$$

We will see later that:

- In theories with massless particles, asymptotic states are never really free;
- It is important to distinguish between timelike and null infinities: symmetries manifest at null infinity, much larger than Poincaré;
- It seems like a good idea to look for asymptotic states that make these symmetries manifest;
- Gains: a better understanding of IR divergences, observables that live on the sphere - trade 4D bulk for CFT_2 like theory, use CFT tools to better understand the bulk.

3 Soft theorems

Soft theorems arise as conservation laws associated with asymptotic/large gauge symmetries [6, 7, 63, 64]. In this section we illustrate this connection by studying the (tree-level) subleading soft graviton theorem and the implied Virasoro symmetry of the $\mathcal{S}$-matrix. This section is a review of [12] and [13].

We start by introducing a universal[2] relation obeyed by scattering amplitudes in any theory with massless particles. For simplicity, we focus on tree-level scattering. In gravity and gauge theory, the scattering of high-energy charged particles is accompanied by radiation. The radiation can be described as a collection of quanta (photons, gravitons, ...) of different energies. When the energy carried away by one such quantum is small, the scattering amplitude factorizes [65, 66]

$$\lim_{\omega\to 0} \mathcal{A}^\pm_{n+1}(q) = \left[S^{(0)\pm}_n + S^{(1)\pm}_n + \mathcal{O}(\omega) \right] \mathcal{A}_n. \tag{3.1}$$

[1] Free in that they always evolve with the free Hamiltonian.

[2] Universal here means independent of the nature of other particles involved in the scattering process.

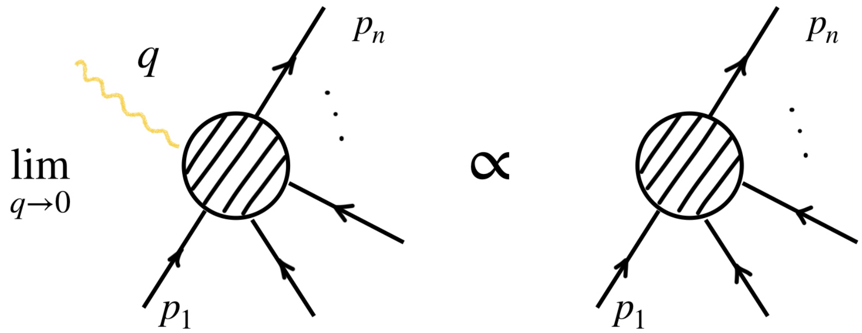

Figure 1: The soft limit relates an amplitude with a low-energy massless particle to the same amplitude without the massless particle.

Here $\mathcal{A}_{n+1}$ is a scattering amplitude of n generic particles of four-momenta $p_1, ..., p_n$ and one massless particle of four-momentum $q = (\omega, \vec{q})$ and positive or negative helicity. $\mathcal{A}_n$ is the same scattering amplitude in the absence of the massless particle. This limit is illustrated in figure 1 and will be referred to as the soft limit.

$S_n^{(0)\pm}$ and $S_n^{(1)\pm}$ are the leading and subleading soft factors respectively, which take the form [65–67]

$$S_n^{(0)\pm} = \frac{\kappa}{2}\sum_{k=1}^{n}\frac{(p_k\cdot\varepsilon^{\pm}(q))^2}{p_k\cdot q}\ , \quad S_n^{(1)\pm} = -\frac{i\kappa}{2}\sum_{k=1}^{n}\frac{\varepsilon^{\pm}(q)\cdot p_k}{p_k\cdot q}q\cdot\mathcal{J}_k\cdot\varepsilon^{\pm}(q)\ , \quad \kappa=\sqrt{32\pi G} \tag{3.2}$$

in gravity[3] and

$$S_n^{(0)\pm} = \sum_{k=1}^{n} Q_k\frac{p_k\cdot\varepsilon^{\pm}(q)}{p_k\cdot q}\ , \quad S_n^{(1)\pm} = -i\sum_{k=1}^{n} Q_k\frac{q\cdot\mathcal{J}_k\cdot\varepsilon^{\pm}(q)}{p_k\cdot q} \tag{3.3}$$

in quantum electrodynamics. G and Q_k are Newton's constant and the charges of the n particles respectively. We expressed the polarization tensor $\varepsilon^{\pm}_{\mu\nu}$ of the graviton as [4]

$$\varepsilon^{\pm}_{\mu\nu}(q) = \varepsilon^{\pm}_{\mu}(q)\varepsilon^{\pm}_{\nu}(q), \tag{3.4}$$

where $\varepsilon^{\pm}_{\mu}(q)$ is the polarization of a helicity-1 particle obeying

$$\varepsilon^{\pm}(q)\cdot q = 0, \qquad \varepsilon^{\pm}(q)\cdot\varepsilon^{\pm}(q) = 0, \qquad \varepsilon^{\pm}(q)\cdot\varepsilon^{\mp}(q) = 1. \tag{3.5}$$

$\mathcal{J}_k$ is the total angular momentum of particle k. For simplicity, we will work in units where

$$8\pi G = 1, \quad \kappa = \sqrt{32\pi G} = 2. \tag{3.6}$$

[3] In gravity there is also a sub-subleading soft theorem [67] with $S_n^{(2)\pm} = -\frac{\kappa}{4}\sum_{k=1}^{n}\frac{(q\cdot J_k\cdot\varepsilon^{\pm})^2}{q\cdot p_k}$.

[4] We pick a gauge in which the graviton is transverse and traceless, $q^{\mu}\varepsilon_{\mu\nu} = q^{\nu}\varepsilon_{\mu\nu} = \varepsilon^{\mu}{}_{\mu} = 0$.

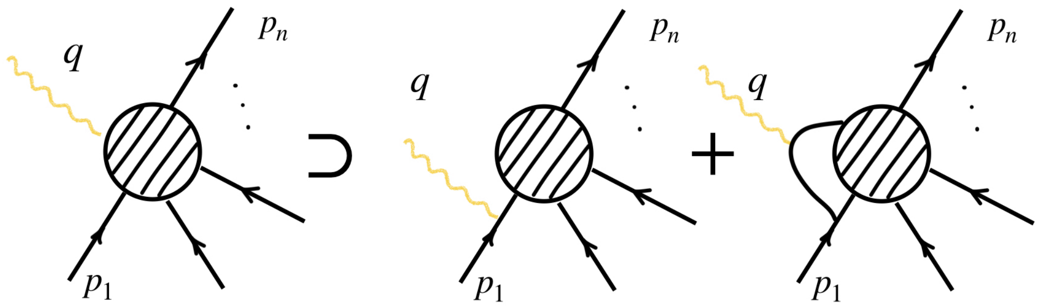

Figure 2: In the soft limit, the amplitude will include contributions from Feynman diagrams where the soft particle attaches to external and internal lines. Diagrams where the soft particle attaches to an internal line are subleading in the soft limit.

Exercise 3.1. *Restricting to massless scalars $k = 1, \cdots n$ and using the parameterizations (6.3) for massless momenta and $\varepsilon^+ \propto \partial_z q(z, \bar{z})$, $\varepsilon^- \propto \partial_{\bar{z}} q(z, \bar{z})$ (with normalization of $\varepsilon^\pm$ to be determined) show that the soft factors in QED are*

$$S_n^{(0)+} = \sum_{k=1}^{n} \frac{Q_k}{\omega(z - z_k)}, \quad S_n^{(1)+} = \sum_{k=1}^{n} \frac{Q_k \eta_k}{\omega_k(z - z_k)} \left(\omega_k \partial_{\omega_k} + (\bar{z} - \bar{z}_k)\partial_{\bar{z}_k}\right). \tag{3.7}$$

Deduce the expressions for the leading and subleading soft graviton factors in this parameterization.

Notice that the soft theorem (3.1) captures the behavior of the scattering amplitude in an expansion around $\omega = 0$. The leading term in (3.1) has a simple pole at $\omega = 0$ which can be understood by considering the Feynman diagrams contributing to the scattering of $n + 1$ particles, as shown in figure 2. In particular, as $\omega \to 0$, the leading order contribution comes from diagrams where the massless particle attaches to an external line. In this limit, an internal propagator goes on-shell and the amplitude develops a pole in q

$$\lim_{\omega \to 0} \mathcal{A}_{n+1}(q) = \left[\sum_{k=1}^{n} -i\frac{V_k(\varepsilon, p_k)}{2p_k \cdot q} + \mathcal{O}(\omega^0)\right] \mathcal{A}_n, \tag{3.8}$$

where $V_k(\varepsilon, p_k)$ is the leading term as $\omega \to 0$ in the (momentum-space) coupling at vertex k. The remaining diagrams, where the massless particle attaches to an internal line remain finite as $\omega \to 0$.

The analysis of subleading terms in the (tree-level) soft expansion was carried out explicitly in gauge theory [65] and, more recently in gravity using on-shell amplitudes techniques [67] . The "brute-force" computation is lengthy and subtle,[5] yet a number of

[5]This is not only because of many sources of subleading corrections coming from both classes of diagrams in figure 2, but also because at subleading order in the soft expansion, momenta of other external particles have to be deformed to obey overall momentum conservation.

apparently miraculous cancellations yield the final result (3.1) universal in gravity [68,69],[6] with the subleading soft graviton factor taking the simple form in (3.2).

It is often the case in physics that simple answers found as a result of complicated calculations point towards an underlying symmetry of the theory. Indeed, as we will review in section 5.5, the subleading soft graviton theorem is nothing but a consequence of an infinite-dimensional enhancement of the Lorentz symmetry of the $\mathcal{S}$-matrix [12]. Moreover, a certain mode of the soft graviton will be identitfied with the generator of this symmetry in section 5.6 by recasting the subleading soft graviton theorem as the Virasoro-Ward identities of an insertion of the stress-tensor in a 2D conformal correlation function [13].

Exercise 3.2. *Derive the leading soft graviton theorem.*

4 Infrared issues

Outline Lecture 1 (second half) and Lecture 2:

- The IR problem; the standard solution; FK dressed states;
- Revisiting the assumptions: motivate the infinity of charges and vacua!
- A new resolution, comments on general gauge theories
- Next: What are these charges? Transition to gravity.

4.1 The problem

In gauge theory and gravity low energy photons, gluons, gravitons running in loops give rise to infrared divergences. Let us specialize to QED. Note that amplitudes with and without N virtual exchanges are related by the addition of N pairs of soft factors (3.3) "glued" together by N factors of the photon propagator

$$\Pi_{\mu\nu} = \frac{-i\eta_{\mu\nu}}{q^2 - i\epsilon}. \tag{4.1}$$

We hence find that an $N-$ loop diagram is related to the tree level diagram by a factor

$$\frac{1}{N!}\left(\frac{1}{2}\int \frac{d^4q}{(2\pi)^4} A\right)^N, \tag{4.2}$$

where

$$A = \sum_{n,m} Q_n Q_m \frac{-i\eta_{\mu\nu}}{q^2 - i\epsilon} \frac{p_m^\mu p_n^\nu}{(p_m \cdot q + i\epsilon)(p_n \cdot q - i\epsilon)} \tag{4.3}$$

and the signs distinguishing between incoming and outgoing hard momenta have been absorbed in the definition of p_n. The sign of q is left manifest and leads to the difference

[6] The subleading soft photon theorem may receive non-universal corrections from a short list of operators [69, 70].

in the the sign of the $i\epsilon$ prescription. Note that since one sums over all permutations of how internal lines can be attached, division by the appropriate symmetry factor

$$2^N N! \tag{4.4}$$

is necessary. Power counting shows that the integral in (4.3) is soft infrared divergent and diverges as

$$\int_\lambda^\Lambda \omega^3 d\omega \frac{1}{\omega^4} \propto \log \frac{\Lambda}{\lambda}, \tag{4.5}$$

where Λ, λ are UV and IR cutoffs respectively. We see that summing over all N leads to exponentiation of these divergences. Because the remaining angular integral in (4.3) has a real part coming from when the propagator is put on shell

$$\frac{-i\eta_{\mu\nu}}{q^2 + i\epsilon} \to \pi\delta(q^2), \tag{4.6}$$

the exponentiation sets all scattering amplitudes to zero []. This is illustrated in figure **??** Figure. (A proper justification of (7.10) requires one to evaluate the integral in (4.3) using contours. Careful consideration of which sides the contour can be closed is necessary - see [].) For completeness we give here the result of the real part of the exponent (which can be checked to be manifestly negative and divergent)

$$-\alpha \ln \frac{\Lambda}{\lambda} \equiv Re\left[\int \frac{d^4q}{(2\pi)^4} A\right] = -\frac{1}{8\pi^2} \sum_{n,m} Q_m Q_n \eta_n \eta_m \beta_{mn}^{-1} \log\left(\frac{1+\beta_{mn}}{1-\beta_{mn}}\right) \ln \frac{\Lambda}{\lambda}, \tag{4.7}$$

with

$$\beta_{mn} = \sqrt{1 - \frac{m_n^2 m_m^2}{p_m \cdot p_n}} \tag{4.8}$$

the relative velocity of a particle in the rest frame of the other. α and its generalization in non-abelian gauge theories is called the cusp anomalous dimension as it can be reproduced by a cuspy Wilson line computation [].

A similar analysis applies to gravity. A much more complicated (and yet unsolved) story can be told for QCD (or more generally non-abelian gauge theories). See [].

Exercise 4.1. *Analyze the pole structure of the integrand A given in* (4.3). *In which cases do we get real and imaginary contributions?*

4.2 The standard solution

The textbook resolution to the IR problem in QED is to accept that S-matrices are IR divergent and eliminate the IR divergences at the level of *observable* cross-sections. By observable we mean the following: in the real world, cross-sections or the probability for a certain number of electrons and positrons to scatter are not measurable since any such scattering process is accompanied by the emission of infinite numbers of photons with energies below the sensitivity of detectors. Observables are instead inclusive cross-sections that sum over all processes involving arbitrary soft photons emissions [Figure.]

Order by order in perturbation theory, it can be shown that such inclusive cross-sections are infrared finite. The soft theorem (3.2) is instrumental in this analysis.

The lack of an $\mathcal{S}$-matrix is bothersome for a theorist for many reasons. To list a few

- In practical applications, causality, unitarity and crossing imply constraints on low-energy effective field theories. These analysis rely on the existence of the S-matrix, eg. no meaningful bounds on low-energy couplings can be obtained otherwise [].

- In quantum gravity, we believe the S-matrix is a unitary operator. IR divergences preclude the definition of such an operator. Non-perturbative scattering amplitudes should diagnose the black hole information problem.

4.3 The Faddeev-Kulish (FK) proposal

An IR-finite definition of the S-matrix in QED was proposed in []. It relies on the observation that photons, gravitons, etc. mediate long range forces. As such, the standard assumption in QFT that asymptotic states are non-interacting fails. To see this consider the standard QED interaction describing the coupling of photons to electrons

$$V = \int j_\mu(x) A^\mu(x) = -e \int : \bar{\psi}(x)\gamma_\mu \psi(x) : A^\mu(x)\ d^3x \tag{4.9}$$

in the interaction representation. We can rewrite this in terms of the quantized fields

$$\psi(x) = \frac{1}{(2\pi)^4} \int \sum_n \left(\frac{m}{p_0}\right) \sum_n \left(b_n(\vec{p}) w_n(\vec{p}) e^{-ip\cdot x} + d_n^\dagger(\vec{p}) v_n(\vec{p}) e^{ip\cdot x}\right) d^3p \tag{4.10}$$

$$\bar{\psi}(x) = \frac{1}{(2\pi)^4} \int \sum_n \left(\frac{m}{p_0}\right) \sum_n \left(b_n^\dagger(\vec{p}) \bar{w}_n(\vec{p}) e^{ip\cdot x} + d_n(\vec{p}) \bar{v}_n(\vec{p}) e^{-ip\cdot x}\right) d^3p \tag{4.11}$$

$$A_\mu(x) = \frac{1}{(2\pi)^4} \int \left(a_\mu^\dagger(\vec{k}) e^{ik\cdot x} + a_\mu(\vec{k}) e^{-ik\cdot x}\right) \frac{d^3k}{2k_0} \tag{4.12}$$

where ψ is a fermion field and A_μ is a photon field. The resulting expression for $V(t)$ is an integral over the momenta $\vec{p}, \vec{q}$ and $\vec{k}$ of the fermions and photons, which are related by the equation $\vec{p} + \vec{k} = \vec{q}$.

FK then study the asymptotic behavior of this expression for $|t| \to \infty$. In this limit, all the terms in (4.9) can be split into two groups. The terms of the first group contain two creation operators or two annihilation operators of charged particles. The argument of the exponential function characterizing the time dependence of these terms is proportional to

$$(\vec{p}^2 + m^2)^{1/2} + ((\vec{p} + \vec{k})^2 + m^2)^{1/2} \pm k_0 \tag{4.13}$$

which is non-zero for all $\vec{p}$ and $\vec{k}$. Such terms therefore decrease sufficiently rapidly as $|t| \to \infty$. The terms of the second group have an argument of the exponential function proportional to

$$(\vec{p}^2 + m^2)^{1/2} - ((\vec{p} + \vec{k})^2 + m^2)^{1/2} \pm k_0 \tag{4.14}$$

which vanishes for $\vec{k} = 0$ for all $\vec{p}$. These terms are the ones that determine the desired asymptotic behavior of the operator $V(t)$. Note that $\vec{k} \to 0$ corresponds to the soft regime, so it is the soft photons that are responsible for the long-range interactions. We can therefore set $\vec{k} = 0$ in all the slowly varying functions, i.e., in the creation and annihilation operators b_n and d_n and the spinors. In addition, the expressions for the latter simplify considerably because of the orthogonality conditions for solutions of the Dirac equation. As a result, one arrives at a simple expression for the interaction potential (4.9) as $|t| \to \infty$

$$V_{as}(t) = \frac{1}{(2\pi)^4} \int J^{\mu}_{as}(\vec{k}, t) \left[a^{\dagger}_{\mu}(-\vec{k}) + a_{\mu}(\vec{k}) \right] \frac{d^3k}{(2k_0)} \tag{4.15}$$

where

$$J^{\mu}_{as}(\vec{k}, t) = -e \int p^{\mu} e^{i\frac{p\cdot k}{p_0}t} \rho(\vec{p}) \frac{d^3p}{p_0} \tag{4.16}$$

and

$$\rho(\vec{p}) = \sum_n [b^{\dagger}_n(\vec{p}) b_n(\vec{p}) - d^{\dagger}_n(\vec{p}) d_n(\vec{p})] = \rho_-(\vec{p}) - \rho_+(\vec{p}). \tag{4.17}$$

Exercise 4.2. *Work out the details leading to* (4.15)*.*

FK note that the expression for the asymptotic current only depends on the charge distribution of the particles and is therefore universal: a similar formula can be obtained in the case of charged particles with arbitrary spin, the corresponding density $\rho(\vec{p})$ is replaced by a sum over the charged particles in the system.

The operator of the asymptotic current $J^{\mu}_{as}(\vec{k}, t)$ has a simple physical meaning. A state of charged particles with given momenta

$$\Psi(p_1, s_1, \ldots p_n, s_n | q_1, l_1, \ldots q_m, l_m) = b^{\dagger}_{s_1}(\vec{p_1}) \ldots b^{\dagger}_{s_n}(\vec{p_n}) d^{\dagger}_{l_1}(\vec{q_1}) \ldots d^{\dagger}_{l_m}(\vec{q_m}) |0\rangle \tag{4.18}$$

is an eigenstate for this operator; the corresponding eigenvalue

$$j_{\mu}(\vec{k}, t | \vec{p}_1, \ldots, \vec{p}_n; \vec{q}_1, \ldots, \vec{q}_m) = \sum_{j=1}^{m} j_{\mu}(\vec{k}, t | q_j) - \sum_{j=1}^{n} j_{\mu}(\vec{k}, t | p_j) \tag{4.19}$$

where

$$j_{\mu}(\vec{k}, t | p) = e \frac{p_{\mu}}{p_0} \exp\left[i \frac{\vec{k}\vec{p}}{p_0} t \right] \tag{4.20}$$

is the classical current of point charges moving along straight lines with momenta $p_i, q_j,\ i = 1, \ldots, n;\ j = 1, \ldots, m$. In this sense, the asymptotic interaction operator is a relativistic generalization of the nonrelativistic asymptotic Coulomb potential []. The asymptotic dynamics of the system is hence effectively governed by the operator

$$H_{as}(t) = H_0 + V_{as}(t) \tag{4.21}$$

where H_0 is the free Hamiltonian of photons and fermions. FK proceed by solving the Schrodinger equations for this asymptotic interacting Hamiltonian. The upshot is that solutions turn out to be coherent states of soft photons. They argue that IR finite scattering amplitudes are obtained by evaluating the S-matrix in this basis.

4.3.1 Asymptotic Dynamics

To find the asymptotic dynamics, FK proceed to solve the Schrodinger equation governing the evolution of the asymptotic states

$$i\frac{d}{dt}|\Psi_{as}(t)\rangle = H_{as}(t)|\Psi(t)\rangle, \quad |\Psi(t)\rangle = U(t)|\Psi_0\rangle. \tag{4.22}$$

If $H_{\text{as}} = H_0$ (like in the previous lecture), then $|\Psi_0\rangle$ would be the same as $|\Phi\rangle$ (defined in eq. (2.16)). Instead, $|\Psi_0\rangle$ is a Heisenberg picture state that evolves according to the asymptotic Hamiltonian, $U = U_{\text{as}}$ in (4.34). We can determine U by noting that (4.22) translates into a differential equation for the asymptotic evolution operator

$$i\frac{d}{dt}U(t) = H_{as}(t)U(t). \tag{4.23}$$

Switching to the interaction picture, we let

$$U_{as}(t) = e^{-iH_0t}Z(t) \tag{4.24}$$

in which case the equation for $Z(t)$ reduces to

$$i\frac{d}{dt}Z(t) = V_{as}^I(t)Z(t) \tag{4.25}$$

where

$$V_{as}^I(t) = e^{iH_0t}V_{as}(t)e^{-iH_0t}. \tag{4.26}$$

This expression for the operator $V_{as}^I(t)$ differs from formula (4.15) only by the presence of the factors e^{ik_0t} and e^{-ik_0t} multiplying $a_\mu^\dagger(k)$ and $a_\mu(k)$.

Since $V_{as}^I(t)$ is linear in photon operators, the commutator

$$[V_{as}^I(t_1), V_{as}^I(t_2)] = Q(t_1, t_2) \tag{4.27}$$

is a c-number and trivially commutes with $V_{as}^I(t)$ for all t, t_1, t_2. This property enables one to disentangle the T-product explicitly and to find the general solution of Eq. (4.31)

$$Z(t) = T\exp\left[-i\int^t V_{as}^I(\tau)d\tau\right], \tag{4.28}$$

namely

$$Z(t) = \exp\left[-i\int^t V_{as}^I(\tau)d\tau - \frac{1}{2}\int^t d\tau\int^\tau dsQ(\tau, s)\right]. \tag{4.29}$$

FK choose initial conditions such that

$$\int^t e^{is\tau}d\tau = \frac{1}{is}e^{ist}, \tag{4.30}$$

which amounts to discarding terms that do not commute asymptotically with the momentum.

The solution to (4.23) is therefore

$$Z(t) = \exp[R(t)] \exp[i\Phi(t)] \tag{4.31}$$

where

$$R(t) = \frac{e}{(2\pi)^{3/2}} \int \frac{p_\mu}{p \cdot k} \left[a^\dagger_\mu(\vec{k}) e^{i\frac{k\cdot p}{p_0}t} - a_\mu(\vec{k}) e^{-i\frac{k\cdot p}{p_0}t} \right] \rho(\vec{p}) d\vec{p} \frac{d\vec{k}}{(2k_0)^{1/2}} \tag{4.32}$$

and

$$\Phi(t) = \frac{e^2}{8\pi} \int : \rho(\vec{p})\rho(\vec{q}) : \frac{p \cdot q}{((p\cdot q)^2 - m^4)^{1/2}} \text{sign}[t] \ln \frac{|t|}{t_0} d\vec{p} d\vec{q}. \tag{4.33}$$

The evaluation of the integrals leading to Eq. (4.33) can be found in the appendix of []. It is natural to call Φ the phase operator. This phase is related to the imaginary part of Weinberg's exponentiated IR divergence (4.3). As for the first contribution to (4.31), it can be shown that it is responsible for the cancellation of infrared divergences arising from photon loops!

Thus, the final expression for the operator of the asymptotic dynamics has the form

$$U_{as}(t) = \exp[-iH_0 t] \exp[i\Phi(t)] \exp[R(t)] \tag{4.34}$$

where the operators $R(t)$ and $\Phi(t)$ are defined by formulas (4.32) and (4.33), respectively. We note that these operators commute. Following the steps leading to the definition of the S-matrix in section 2.2, in the presence of long range interactions, the $\mathcal{S}$-matrix must be defined as the limit of the operator

$$\mathcal{S}(t_1, t_2) = U^\dagger_{as}(t_1) \exp[-iH(t_1 - t_2)] U_{as}(t_2) \tag{4.35}$$

as $t_1 \to \infty$ and $t_2 \to -\infty$. The expression on the right hand side differs from the Dyson $\mathcal{S}$-matrix for finite times (cf. (2.18))

$$\mathcal{S}_D(t_1, t_2) = \exp[iH_0 t_1] \exp[-iH(t_1 - t_2)] \exp[-iH_0 t_2] \tag{4.36}$$

by the outer factors of the type $\exp[R(t) + i\Phi(t)]$. Equivalently, scattering amplitudes can be interpreted as evaluating the standard Dyson S-matrix in a basis of coherent states of soft photons. Order by order in perturbation theory, the soft photon clouds can be shown to cancel the contributions from virtual photons discussed before.

Questions remain:

- Tedious argument, some of the steps sketchy (eg. the stationary phase/large times argument);
- The new S-matrix takes us out of the Fock space (or equivalently the coherent states don't live in the Fock space because of IR divergences). [Revisit this argument]
- Massless particles don't live at $t \to \pm\infty$.
- In gravity and QCD there are non-linearities and collinear divergences - argument needs refinement, not known if refinement exists [].
- Comments on Wald's new paper;

4.4 An alternate symmetry-inspired derivation

In this section we present a new take on infrared divergences and the construction of infrared finite S-matrices. So far we learned that theories with massless particles typically have long-range interactions. This implies the textbook (quantum mechanics/QFT) assumption in scattering theory, namely that interactions only happen over finite regions and time intervals is invalidated. Asymptotic states don't just consist of free particles, but include also coherent states of soft photons.

There is another simple observation which together with the soft theorem can be used to construct a simple, more intuitive argument for why we should scatter coherent states instead of free particle states: in theories with massless particles, the vacuum is not unique. To see this consider a state consisting of a superposition of soft photons with different polarizations

$$|\Omega\rangle = \lim_{\omega\to 0}\int d^2z\Omega(z,\bar{z})a^\dagger(\omega\hat{q}(z,\bar{z}))|0\rangle \ \text{ is such that } \begin{cases} H|\Omega(z,\bar{z})\rangle = H|0\rangle = 0 \\ \langle\Omega(z,\bar{z})|0\rangle = 0. \end{cases} \tag{4.37}$$

We hence see that $|\Omega\rangle$ has both zero energy and is orthogonal to the vacuum! There is a functions on the sphere worth of states degenerate with the vacuum. We will show in the next lecture that the soft theorem

$$\lim_{\omega\to 0}\langle\text{out}|a_\pm(\omega\hat{q})\mathcal{S}|\text{in}\rangle = S_\pm^{(0)}\langle\text{out}|\mathcal{S}|\text{in}\rangle + \mathcal{O}(\omega^0), \tag{4.38}$$

where the leading soft factor was given in (3.3) can be re-expressed as the Ward identity for the conservation of large gauge charge, namely

$$\langle\text{out}|Q^+\mathcal{S} - \mathcal{S}Q^-|\text{in}\rangle = 0. \tag{4.39}$$

Here $|\text{in}\rangle, |\text{out}\rangle$ consist of finite energy particles only and the charges can be split into two components

$$Q^\pm = Q_S^\pm + Q_H^\pm, \tag{4.40}$$

where

$$\begin{aligned} &Q_S \leftrightarrow \text{soft photon operator,} \\ &Q_H \leftrightarrow \text{operator whose action on asymptotic states } \propto \text{soft factor} \end{aligned} \tag{4.41}$$

and the superscripts are to distinguish action on incoming and outgoing states (not to be confused with helicities).

By a similar argument to the one above

$$Q_S|\Omega_0\rangle \tag{4.42}$$

can be shown to be degenerate to the vacuum $|\Omega_0\rangle$ (physically, Q_S creates a soft photon and states differing by the number of zero energy photons are orthogonal), unless $|\Omega_0\rangle$ happens to be an eigenstate of Q_S. But recall that in textbook QFT, the vacuum is assumed to be unique. The only option then is that $|\Omega_0\rangle$ is an eigenstate with eigenvalue q_S

$$Q_S|\Omega_0\rangle = q_S|\Omega_0\rangle. \tag{4.43}$$

Now rewriting (5.54) using (4.40)

$$\langle \text{out}|Q_S^+\mathcal{S} - \mathcal{S}Q_S^-|\text{in}\rangle = S^{(0)}\langle \text{out}|\mathcal{S}|\text{in}\rangle \tag{4.44}$$

and applying (4.43) on the LHS, we find

$$\left(q_S^{\text{out}} - q_S^{\text{in}}\right)\langle \text{out}|\mathcal{S}|\text{in}\rangle = S^{(0)}\langle \text{out}|\mathcal{S}|\text{in}\rangle. \tag{4.45}$$

But since the vacuum is typically assumed unique, $q_S^{\text{out}} = q_S^{\text{in}}$ and so the only way for (4.45) to be obeyed is if the scattering amplitude vanishes! But this is the same as the statement that IR divergences exponentiate in QED and set scattering amplitudes to zero!

On the other hand, if we allow for vacuum degeneracy, and moreover, vacuum transitions (ie. distinct incoming and outgoing vacua)

$$q_S^{\text{out}} \neq q_S^{\text{in}} \tag{4.46}$$

we can have a non-vanishing S-matrix together with a selection rule

$$q_S^{\text{out}} - q_S^{\text{in}} = S^{(0)}. \tag{4.47}$$

But this condition implies the asymptotic states are coherent states of soft photons as constructed by Faddeev and Kulish. To see this, as we will see later

$$Q_S \propto \lim_{\omega\to 0}\omega(a^\dagger(\omega\hat{q}) + a(\omega\hat{q})) \tag{4.48}$$

and the eigenvalue equation

$$Q_S|\Omega_0\rangle = q_s|\Omega_0\rangle \tag{4.49}$$

implies $|\Omega_0\rangle$ has to be coherent, of FK type. In the discussion session we will study the properties of these coherent states and the implications for observables in QED.

Exercise 4.3. *Show explicitly that FK dressings allow one to construct eigenstates of the soft charge with different eigenvalues.*

Lecture 3

- Boundaries and gauge symmetries
- AFS, null boundaries and asymptotic symmetries
- Stationary phase argument, symmetry/conservation law $\implies$ soft theorem.

5 Asymptotic symmetries

5.1 Gauge symmetries and boundaries

In gauge theories physical states are subject to constraints

$$G_0[\xi]|\Psi_{\text{phys}}\rangle = 0. \tag{5.1}$$

For example, Gauss's law in free E&M requires that all physical states obey

$$\int_\Sigma \xi d \star F|\Psi_{\rm phys}\rangle = 0 \iff \int_\Sigma \xi n^\mu \nabla^\nu F_{\mu\nu}|\Psi_{\rm phys}\rangle = 0, \tag{5.2}$$

where Σ is a Cauchy slice in spacetime and n^ν is normal to Σ. The constraints should generate gauge transformations

$$\{G_0[\xi], A\} = d\xi. \tag{5.3}$$

In the presence of boundaries however, the generator of gauge transformations may nor be G_0 but instead

$$G = G_0 - B \tag{5.4}$$

where B is a boundary contribution. While G_0 was such that

$$\{G_0[\xi], G_0[\rho]\} = G_0[[\xi,\rho]] \tag{5.5}$$

and hence imposing that G_0 vanishes on physical states is consistent with this algebra,

$$\{G[\xi], G[\rho]\} = G[[\xi,\rho]] + K. \tag{5.6}$$

This implies that G is non-vanishing and is hence promoted to a symmetry generator.

In E&M (5.2) can be rewritten as

$$G_0[\xi]|\Psi_{\rm phys}\rangle = \left(\int_{\partial\Sigma} \xi \star F - \int_\Sigma d\xi \star F\right)|\Psi_{\rm phys}\rangle = 0 \tag{5.7}$$

or in coordinates

$$G_0[\xi]|\Psi_{\rm phys}\rangle = \left(\int_{\partial\Sigma} dx^\nu \xi n^\mu F_{\mu\nu} - \int_\Sigma \nabla^\nu \xi n^\mu F_{\mu\nu}\right)|\Psi_{\rm phys}\rangle = 0 \tag{5.8}$$

and one can indeed show that it is $G = G_0 - B$ that generate gauge transfrormations. Since G acts non-trivially on physical states as $G|\Psi_{\rm phys}\rangle = -B|\Psi_{\rm phys}\rangle$, they should be thought of as true charges.

The take-home message is that in the presence of boundaries, gauge transformations get promoted to true symmetries, aka large gauge transformations. In the next lecture we will study an example of this phenomenon in gravity in 4D asymptotically flat spacetimes. We will see how asymptotic diffeomorphisms can be associated with charges whose action on phase space is non trivial and whose conservation laws implies Ward identities which are equivalent to the soft theorems discussed before.

5.2 Penrose diagram of Minkowski space

Penrose diagrams are a convenient tool for studying physics at "infinity" as they preserve the causal structure of spacetime while mapping "infinity" to the boundary of a finite region. In this section we review how this works for Minkowski spacetime.

The Minkowski metric takes the form

$$ds^2 = -dt^2 + d\vec{x}^2 = -dt^2 + dr^2 + r^2 d\Omega_2^2, \tag{5.9}$$

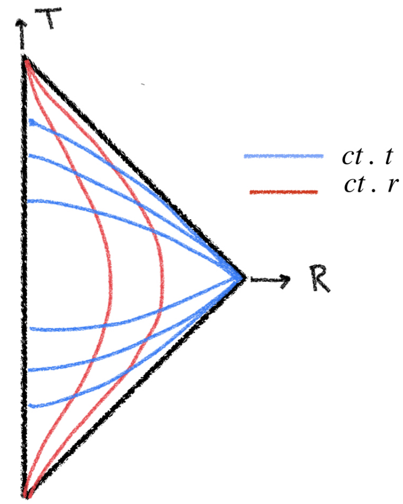

Figure 3: Penrose diagram of Minkowski space.

where

$$d\Omega_2^2 = d\theta^2 + (\sin\theta)^2 d\varphi^2 \tag{5.10}$$

is the metric on the unit two-sphere. It will be convenient to introduce retarded and advanced coordinates u, v

$$u = t - r, \qquad v = t + r, \tag{5.11}$$

and coordinates $(z, \bar{z})$ related to the angular coordinates (θ, ϕ) by a stereographic projection

$$z = \cot\frac{\theta}{2} e^{i\varphi}, \qquad \bar{z} = \cot\frac{\theta}{2} e^{-i\varphi}. \tag{5.12}$$

In retarded coordinates $(u, r, z, \bar{z})$ the metric (5.9) becomes

$$ds^2 = -du^2 - 2du dr + 2r^2\gamma_{z\bar{z}} dz d\bar{z}, \qquad \gamma_{z\bar{z}} = \frac{2}{(1+z\bar{z})^2} \tag{5.13}$$

and similarly, in advanced coordinates $(v, r, z, \bar{z})$

$$ds^2 = -dv^2 + 2dv dr + 2r^2\gamma_{z\bar{z}} dz d\bar{z}. \tag{5.14}$$

The asymptotic structure of (5.9) can be understood by introducing coordinates (T, R) related to (t, r) by

$$u = \tan U, \quad v = \tan V, \qquad T = U + V, \quad R = V - U, \tag{5.15}$$

in which case (5.9) reduces to

$$\begin{aligned} ds^2 &= \Omega^2(T,R)\left(-dT^2 + dR^2 + 2r^2(R,T)\sin^2 R\,\gamma_{z\bar{z}} dz d\bar{z}\right), \\ \Omega^{-2}(T,R) &= 4\cos^2\frac{1}{2}(T-R)\cos^2\frac{1}{2}(T+R). \end{aligned} \tag{5.16}$$

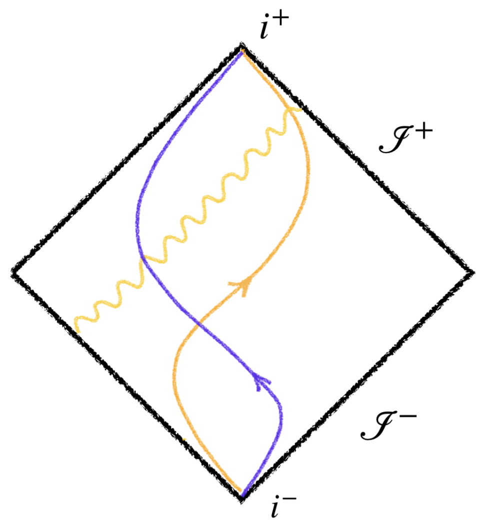

Figure 4: Penrose diagram of Minkowski space where each pair of points represents a two-sphere. Massive particles come in from i^- and go out at i^+, while massless particles enter and exit spacetime at $\mathcal{I}^\pm$.

In the original coordinates, Minkowski space is covered by $r > 0$, $-\infty < u < v < \infty$, therefore the ranges of the new coordinates are $-\frac{\pi}{2} < U < V < \frac{\pi}{2}$ and $0 < R < \pi$. This is illustrated in figure 3.

Exercise 5.1. *a) Plot the lines of constant r and t in the (R, T) plane.*

b) Plot the lines of $\frac{r-r_0}{t}$ for different values of r_0 in the (R, T) plane.

It will be convenient to unfold this diagram to represent antipodal points on the spheres. Future null infinity ($\mathcal{I}^+$) is defined by taking $r \to \infty$ for fixed u, while past null infinity ($\mathcal{I}^-$) is reached by taking $r \to \infty$ for fixed v. In a free theory, massless particles follow lines of unit slope and cross points on the spheres at $\mathcal{I}^\mp$ at retarded times v, u. This is illustrated in figure 4. Massive particles never reach $\mathcal{I}^\pm$, but only past and future timelike infinities $i^\mp$ ($t \to \mp\infty$).

5.3 Asymptotically flat spacetimes

Asymptotically flat spacetimes have the same causal structure as Minkowski space at infinity. An asymptotically flat spacetime admits an expansion in powers of r^{-1} around the Minkowski metric (5.13) near $\mathcal{I}^+$[7]

$$
\begin{aligned}
ds^2 = & - du^2 - 2dudr + 2r^2\gamma_{z\bar{z}}dzd\bar{z}, \\
& + \frac{2m_B}{r}du^2 + rC_{zz}dz^2 + rC_{\bar{z}\bar{z}}d\bar{z}^2 + 2g_{uz}dudz + 2g_{u\bar{z}}dud\bar{z} + ...
\end{aligned}
\tag{5.17}
$$

Solving the Einstein equations[8]

$$
R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = T^M_{\mu\nu} \tag{5.18}
$$

order by order in a large-r expansion[9] one finds [71]

$$
g_{uz} = \frac{1}{2}D^zC_{zz} + \frac{1}{6r}C_{zz}D_zC^{zz} + \frac{2}{3r}N_z + \mathcal{O}(r^{-2}), \tag{5.19}
$$

where D_z is the covariant derivative associated with $\gamma_{z\bar{z}}$. Here m_B and N_z are the Bondi mass aspect and angular momentum aspect respectively, while

$$
N_{zz} = \partial_u C_{zz} \tag{5.20}
$$

is the outgoing news tensor. They are all functions of $(u, z, \bar{z})$.

m_B, C_{zz}, N_z are not all independent. They are related by constraint equations.[10] The uu constraint gives [71]

$$
\partial_u m_B = \frac{1}{4}D_z^2N^{zz} + \frac{1}{4}D_{\bar{z}}^2N^{\bar{z}\bar{z}} - \frac{1}{2}T^{M(2)}_{uu} - \frac{1}{4}N_{zz}N^{zz}, \tag{5.21}
$$

while the uz constraint reduces to

$$
\begin{aligned}
\partial_u N_z = & \frac{1}{4}D_z\left(D_z^2C^{zz} - D_{\bar{z}}^2C^{\bar{z}\bar{z}}\right) - T^M_{uz} + \partial_z m_B + \frac{1}{16}D_z\partial_u\left(C_{zz}C^{zz}\right) \\
& - \frac{1}{4}\left(N^{zz}D_zC_{zz} + N_{zz}D_zC^{zz}\right) - \frac{1}{4}D_z\left(C^{zz}N_{zz} - N^{zz}C_{zz}\right).
\end{aligned}
\tag{5.22}
$$

We defined

$$
T^{M(2)}_{uu} = \lim_{r\to\infty} r^2 T^M_{uu}. \tag{5.23}
$$

Exercise 5.2. ***Optional** Verify* (5.21) *and* (5.22)*.*

The square of N_{zz} measures the energy carried by gravitational radiation. We learn from (5.21) that the Bondi mass (the integrated Bondi mass aspect over the sphere) is roughly speaking a measure of the net energy contained in spacetime excluding the parts carried off to infinity by null matter and gravitational waves: as energy is radiated away, the Bondi mass decreases. Similarly, the change in N_z with retarded time (integrated over the sphere) measures the amount of angular momentum carried away by null matter and gravitational radiation.

The analogous equations near $\mathcal{I}^-$ can be found in [12].

[7] We are working in Bondi gauge defined by $\partial_r \det\left(\frac{g_{AB}}{r^2}\right) = 0$ and $g_{rr} = g_{rA} = 0$ where A, B run over the transverse indices $z, \bar{z}$. In these coordinates gravitational waves propagate radially outwards (equivalently, lines of constant $u, z, \bar{z}$ are null) and the wavefronts are spherical.

[8] We set $\kappa = 2$.

[9] The leading terms in the uu and uz components of the matter stress tensor are taken to be $\mathcal{O}(r^{-2})$.

[10] These are the components of (5.18) along the tangent to $\mathcal{I}^+$.

5.4 Asymptotic symmetries

The asymptotic symmetry group of (5.17) has beeen proposed to be the extended BMS$^+$ group in [71]. This is generated by vector fields ξ^+ that preserve the asymptotic fall-off of (5.17) with r namely

$$\mathcal{L}_{\xi^+} g_{ur} = \mathcal{O}(r^{-2}), \quad \mathcal{L}_{\xi^+} g_{uz} = \mathcal{O}(1), \quad \mathcal{L}_{\xi^+} g_{zz} = \mathcal{O}(r), \quad \mathcal{L}_{\xi^+} g_{uu} = \mathcal{O}(r^{-1}). \tag{5.24}$$

Solving (5.24) order by order in a large-r expansion, such vector fields are found to be of the form [12, 71]

$$\begin{aligned} \xi^+ = &\left(1 + \frac{u}{2r}\right) Y^{+z}\partial_z - \frac{u}{2r} D^{\bar{z}} D_z Y^{+z} \partial_{\bar{z}} - \frac{1}{2}(u+r) D_z Y^{+z} \partial_r + \frac{u}{2} D_z Y^{+z} \partial_u + c.c. \\ &+ f^+ \partial_u - \frac{1}{r}\left(D^z f^+ \partial_z + D^{\bar{z}} f^+ \partial_{\bar{z}}\right) + D^z D_z f^+ \partial_r + ..., \end{aligned} \tag{5.25}$$

where $f^+(z, \bar{z})$ is an arbitrary function on $\mathcal{S}^2$ and $Y^+(z, \bar{z})$ is a conformal Killing vector on $\mathcal{S}^2$

$$\partial_{\bar{z}} Y^{+z} = 0. \tag{5.26}$$

Exercise 5.3. *Use* (5.24) *and* (5.17) *to derive* (5.25)*.*

One easy way to see that (5.26) ought to hold is to notice that under Lorentz transformations,

$$\mathcal{L}_{Y^+} g_{\bar{z}\bar{z}} = 2r^2 \gamma_{z\bar{z}} \partial_{\bar{z}} Y^{+z} + \mathcal{O}(r). \tag{5.27}$$

Imposing that (5.24) are obeyed immediately leads to (5.26).

Globally, (5.26) admits six solutions

$$\begin{aligned} &Y^{+z}_{12} = -iz, \quad Y^{+z}_{13} = \frac{1}{2}(1+z^2), \quad Y^{+z}_{23} = \frac{i}{2}(1-z^2), \\ &Y^{+z}_{03} = -z, \quad Y^{+z}_{01} = \frac{1}{2}(1-z^2), \quad Y^{+z}_{02} = \frac{i}{2}(1+z^2) \end{aligned} \tag{5.28}$$

corresponding to the three Lorentz rotations and three boosts (see appendix B). Locally, there are infinitely many solutions $Y^z \propto z^n$.

For the remainder of this section, we restrict to the subgroup of asymptotic symmetries generated by (5.25) with $f = 0$ which are known as superrotations [72].[11] In this case, the vector fields (5.25) that map $\mathcal{I}^+$ to itself are

$$\xi^+\Big|_{\mathcal{I}^+} = Y^{+z}\partial_z + \frac{u}{2} D_z Y^{+z} \partial_u + c.c.. \tag{5.29}$$

The infinitesimal BMS$^+$ transformations (5.29) act on the metric components as follows

$$\begin{aligned} \delta_{Y^+} C_{zz} &= \frac{1}{2}\left(D_z Y^{+z} + D_{\bar{z}} Y^{+\bar{z}}\right)(u\partial_u - 1) C_{zz} + \mathcal{L}_{Y^+} C_{zz} - u D_z^3 Y^{+z}, \\ \delta_{Y^+} N_{zz} &= \partial_u \delta_{Y^+} C_{zz} = \frac{u}{2}\left(D_z Y^{+z} + D_{\bar{z}} Y^{+\bar{z}}\right)\partial_u N_{zz} + \mathcal{L}_{Y^+} N_{zz} - D_z^3 Y^{+z}. \end{aligned} \tag{5.30}$$

[11] Conversely, supertranslations are the subset of symmetries (5.25) with $Y^+ = 0$ and $f \neq 0$.

Upon quantization, (5.30) imply the existence of "charges"[12] under which an outgoing Fock state[13] transforms as

$$Q^+(Y^+)|\mathrm{out}\rangle = i\delta_{Y^+}|\mathrm{out}\rangle, \tag{5.31}$$

where

$$Q^+(Y^+) = Q^+_H + Q^+_S \tag{5.32}$$

and [72]

$$\begin{aligned} Q^+_H &= \frac{1}{4}\int_{\mathcal{I}^+} dud^2z\gamma_{z\bar{z}}\left(uD_zY^zN_{zz}N^{zz} - Y^zD_z(C_{zz}N^{zz}) - 2Y^zC_{zz}D_zN^{zz} + \text{matter}\right) + c.c., \\ Q^+_S &= -\frac{1}{2}\int_{\mathcal{I}^+} dud^2zD^3_zY^{+z}uN^z_{\ \bar{z}} + c.c.. \end{aligned} \tag{5.33}$$

We show in appendix A that (5.33) reproduce the symmetry action (5.30). Using the canonical commutation relations [73]

$$[N_{\bar{z}\bar{z}}(u,z,\bar{z}), C_{ww}(u',w,\bar{w})] = 2i\gamma_{z\bar{z}}\delta^{(2)}(z-w)\delta(u-u'), \tag{5.34}$$

it then follows that for transformations parameterized by $Y^+ = (Y^z, 0)$ [12]

$$Q^+_H|\mathrm{out}\rangle = i\sum_{k\in\mathrm{out}}\left(\mathcal{L}_{Y^{+z_k}} - \frac{\omega_k}{2}D_{z_k}Y^{+z_k}\partial_{\omega_k}\right)|\mathrm{out}\rangle. \tag{5.35}$$

Similar formulas hold near $\mathcal{I}^-$.

5.5 Recovering the Virasoro symmetry from the soft theorem

An independent action of BMS$^+$ and BMS$^-$ on $\mathcal{I}^+$ and $\mathcal{I}^-$ leads to an ambiguity in defining scattering in AFS. In particular, upon specifying data at $\mathcal{I}^+$, the S-matrix provides a map between in and out states *up to* a BMS transformation.

A solution to this problem was proposed in [63] where it was shown that the gravitational scattering problem in AFS becomes well defined upon imposing the antipodal matching conditions

$$f(z,\bar{z})\Big|_{\mathcal{I}^+_-} = f(z,\bar{z})\Big|_{\mathcal{I}^-_+}, \quad Y^+(z,\bar{z})\Big|_{\mathcal{I}^+_-} = Y^-(z,\bar{z})\Big|_{\mathcal{I}^-_+}. \tag{5.36}$$

Here, points on the sphere at $\mathcal{I}^+_-$ are antipodally related to points at $\mathcal{I}^-_+$, $(z,\bar{z})\Big|_{\mathcal{I}^+_-} = (-\frac{1}{\bar{z}}, -\frac{1}{z})\Big|_{\mathcal{I}^-_+}$. Moreover, upon imposing the boundary condition[14]

$$N_z(z,\bar{z})\Big|_{\mathcal{I}^+_-} = N_z(z,\bar{z})\Big|_{\mathcal{I}^-_+}, \tag{5.37}$$

[12]We haven't shown these are conserved yet; conservation will be implied by the subleading soft graviton theorem.

[13]The radiative data consists of the modes of N_{zz} hence the action of Q^+ on the radiative part of $|\mathrm{out}\rangle$ is related to the commutator $[Q^+, N_{zz}]$ [6].

[14]Using the constraint (5.22) the superrotation charges can be put into the simpler form $Q^+(Y^+) = \int d^2z\,(Y_{\bar{z}}N_z + Y_zN_{\bar{z}})$ [10].

the charges at $\mathcal{I}^-_+$ and $\mathcal{I}^+_-$ obey

$$Q^+ = Q^-. \tag{5.38}$$

It then makes sense to study the constraint imposed by conservation of Q on the $\mathcal{S}$-matrix, namely

$$\langle \text{out}|Q^+\mathcal{S} - \mathcal{S}Q^-|\text{in}\rangle = 0. \tag{5.39}$$

Using the split (5.32) into soft and hard parts, (5.39) is equivalent to

$$\langle \text{out}|Q^+_S\mathcal{S} - \mathcal{S}Q^-_S|\text{in}\rangle = -\langle \text{out}|Q^+_H\mathcal{S} - \mathcal{S}Q^-_H|\text{in}\rangle. \tag{5.40}$$

5.5.1 Stationary phase interlude

How do we go from the classical expressions for the charges (5.33) to computing their action on asymptotic particle states? The key is that metric perturbations such as C_{zz} are (asympotically) related to gravitons upon quantization. Consider the Fourier expansion

$$h^{\text{out}}_{\mu\nu} = \sum_{\alpha=\pm}\int \frac{d^3\vec{q}}{(2q^0)(2\pi)^3}\left[\varepsilon^{*\alpha}_{\mu\nu}a^{\text{out}}_\alpha(\vec{q})e^{iq\cdot x} + \varepsilon^{\alpha}_{\mu\nu}a^{\text{out}\dagger}_\alpha(\vec{q})e^{-iq\cdot x}\right] \tag{5.41}$$

of the the linearized perturbation $h_{\mu\nu}$ about a Minkowski background

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}. \tag{5.42}$$

Here the creation and annihilation operators obey the standard commutation relations

$$[a^{\text{out}}_\alpha(\vec{q}), a^{\text{out}\dagger}_\beta(\vec{q}\,')] = \delta_{\alpha\beta}2q^0(2\pi)^3\delta^3(\vec{q}-\vec{q}\,'). \tag{5.43}$$

Now using the parameterization

$$q(\omega, z, \bar{z}) = \omega\hat{q}(z,\bar{z}) = \frac{\omega}{1+z\bar{z}}\left(1+z\bar{z}, z+\bar{z}, -i(z-\bar{z}), 1-z\bar{z}\right) \tag{5.44}$$

for massless momenta together with

$$x = (u+r, r\hat{x})\,, \quad \hat{x} = \frac{1}{1+w\bar{w}}\left(w+\bar{w}, -i(w-\bar{w}), 1-w\bar{w}\right), \tag{5.45}$$

we find

$$q\cdot x = -(u+r)\omega + \omega\vec{q}(z,\bar{z})\cdot\hat{x}(w,\bar{w}) = -u\omega - \omega r(1-\cos\theta). \tag{5.46}$$

Then

$$h^{\text{out}}_{\mu\nu} = \sum_{\alpha=\pm}\frac{1}{8\pi^2}\int_0^\infty d\omega\omega\int_0^\pi d\theta\sin\theta\left[\varepsilon^{*\alpha}_{\mu\nu}a^{\text{out}}_\alpha(\vec{q})e^{-i\omega u - i\omega r(1-\cos\theta)} + \varepsilon^{\alpha}_{\mu\nu}a^{\text{out}\dagger}_\alpha(\vec{q})e^{i\omega u + i\omega r(1-\cos\theta)}\right] \tag{5.47}$$

and at large $r\to\infty$ one can apply the stationary phase approximation to deduce that the exponents are dominated by $\cos\theta = 1$ and so

$$\begin{aligned} h^{\text{out}}_{\mu\nu}(u,r,\hat{x}) &= \sum_{\alpha=\pm}\frac{1}{8\pi^2}\int_0^\infty d\omega\omega\int_0^\pi d\theta\theta\left[\varepsilon^{*\alpha}_{\mu\nu}(\hat{x})a^{\text{out}}_\alpha(\hat{x})e^{-i\omega u + i\omega r\frac{\theta^2}{2}} + \varepsilon^{\alpha}_{\mu\nu}(\hat{x})a^{\text{out}\dagger}_\alpha(\hat{x})e^{i\omega u - i\omega r\frac{\theta^2}{2}}\right] \\ &= \frac{1}{8\pi^2 ir}\sum_{\alpha=\pm}\int_0^\infty d\omega\left[\varepsilon^{*\alpha}_{\mu\nu}(\hat{x})a^{\text{out}}_\alpha(\hat{x})e^{-i\omega u} - \varepsilon^{\alpha}_{\mu\nu}(\hat{x})a^{\text{out}\dagger}_\alpha(\hat{x})e^{i\omega u}\right]. \end{aligned} \tag{5.48}$$

Note that the coefficients are evaluated at $\theta = 0$ or equivalently $\vec{q} = \hat{x}$ since $\vec{q} \cdot \hat{x} = 1$. Moreover, the contribution from the $\theta = \pi$ limit of integration vanishes in the large r limit upon introducing a regulator.

Exercise 5.4. *Work out the details of the stationary phase approximation.*

Finally

$$h_{ww} = \frac{\partial x^\mu}{\partial w}\frac{\partial x^\mu}{\partial w} h^{\text{out}}_{\mu\nu} = -\frac{ir}{8\pi^2}\hat{\epsilon}_{ww}\int_0^\infty d\omega \left[a_+^{\text{out}}(\hat{x})e^{-i\omega u} - a_-^{\text{out}\dagger}(\hat{x})e^{i\omega u}\right] \tag{5.49}$$

where we used that

$$\partial_w x^\mu \varepsilon^-_\mu = \frac{\sqrt{2}r}{1+w\bar{w}}, \quad \partial_w x^\mu \varepsilon^+_\mu = 0, \quad \hat{\epsilon}_{ww} = \partial_w x^\mu \varepsilon^-_\mu \partial_w x^\nu \varepsilon^-_\nu \equiv \frac{2}{(1+w\bar{w})^2} \tag{5.50}$$

and hence

$$C_{zz} = \lim_{r\to\infty}\frac{\kappa}{r}h_{zz} = -\frac{i\kappa}{8\pi^2}\hat{\epsilon}_{zz}\int_0^\infty d\omega \left[a_+^{\text{out}}(\hat{x}(z,\bar{z}))e^{-i\omega u} - a_-^{\text{out}\dagger}(\hat{x}(z,\bar{z}))e^{i\omega u}\right]. \tag{5.51}$$

Comment on arguments z vs. w.

Returning to the definition of the soft charge, we note that it involves the integral

$$\int du u N_{\bar{z}\bar{z}} = \int du u \partial_u C_{\bar{z}\bar{z}} \tag{5.52}$$

and using the mode expansion (5.51) we find [12]

$$N^{(1)}_{\bar{z}\bar{z}} = \frac{i}{4\pi}\hat{\epsilon}_{\bar{z}\bar{z}}\lim_{\omega\to 0}(1+\omega\partial_\omega)\left[a_-^{\text{out}}(\omega\hat{x}) - a_+^{\text{out}\dagger}(\omega\hat{x})\right]. \tag{5.53}$$

Therefore Q_S picks out a particular subleading[15] soft graviton mode!

For simplicity we now restrict to the case when all asymptotic particles but the soft insertion are scalars. Using the subleading soft relation (3.2), the LHS of (5.40) reduces to [12]

$$\begin{aligned}\langle\text{out}|Q_S^+\mathcal{S} - \mathcal{S}Q_S^-|\text{in}\rangle &= -\frac{i}{4\pi}\int d^2z D_z^3 Y^z \hat{\epsilon}_{\bar{z}\bar{z}} S^{(1)-}\langle\text{out}|\mathcal{S}|\text{in}\rangle \\ &= -i\sum_{k\in\text{in,out}}\left(Y^{z_k}\partial_{z_k} - \frac{\omega_k}{2}D_{z_k}Y^{z_k}\partial_{\omega_k}\right)\langle\text{out}|\mathcal{S}|\text{in}\rangle.\end{aligned} \tag{5.54}$$

In the second line, we integrated by parts and used the parameterizations (5.44) of momenta for which the subleading soft factor of a negative helicity graviton becomes

$$S^{(1)-} = \sum_k \left(\frac{(z-z_k)(1+z\bar{z}_k)}{(\bar{z}_k-\bar{z})(1+z_k\bar{z}_k)}\omega_k\partial_{\omega_k} + \frac{(z-z_k)^2}{\bar{z}_k-\bar{z}}\partial_{z_k}\right). \tag{5.55}$$

But the RHS of (5.54) is nothing but the action of the hard charge Q_H on scalar asymptotic states. We conclude the subleading soft theorem implies conservation of the charges (5.32), hence the enhancement of Lorentz symmetry to Virasoro.

[15] $1+\omega\partial_\omega$ projects out the leading soft pole.

5.6 A 2D stress tensor for 4D gravity

(5.54) also implies that gravity in AFS has a remarkable feature: the existence of a subleading soft graviton mode whose insertions in the quantum gravity $\mathcal{S}$-matrix behaves like the stress tensor of a 2D CFT. To see this, we simply set

$$Y^{z_k} = \frac{1}{z - z_k} \tag{5.56}$$

in (5.54). Upon defining[16]

$$T_{zz} \equiv i \int d^2w \frac{1}{z-w} D_w^2 D^{\bar{w}} N_{\bar{w}\bar{w}}^{(1)}, \tag{5.57}$$

(5.54) reduces to

$$\langle T_{zz} \mathcal{O}_1 ... \mathcal{O}_n \rangle = \sum_{k=1}^{n} \left[\frac{\hat{h}_k}{(z-z_k)^2} + \frac{\Gamma^{z_k}_{z_k z_k}}{z-z_k} \hat{h}_k + \frac{1}{z-z_k} \partial_{z_k} \right] \langle \mathcal{O}_1 ... \mathcal{O}_n \rangle. \tag{5.58}$$

This is the Ward identity of a stress tensor in a conformal field theory on a curved background [74]. Note however that the weight[17] $\hat{h}_k = -\frac{1}{2}\omega_k \partial_{\omega_k}$ is a differential operator which acts non-diagonally on S-matrix elements in a basis of momentum eigenstates. In the next section we will introduce a new basis of asymptotic states which diagonalize the action of $\hat{h}_k$. The scattering problem in AFS will be then reformulated in terms of an observable living on the celestial sphere: the *celestial amplitude*.

6 Celestial amplitudes

Lecture 4

- Conformal primary basis in relation to mode expansion (in first section)
- Massive, massless (already done, but revisit logic)
- Amplitudes structures and celestial symmetries; 4-point massless, higher points, practical uses? Relation to twistors?

Cautionary note: In this section we will work in flat retarded coordinates in which the Minkowski metric takes the form

$$ds^2 = -dudr + 2r^2 dz d\bar{z} \tag{6.1}$$

while null momenta

$$q(\omega, z, \bar{z}) = \frac{1}{\sqrt{2}} \omega \hat{q}(z, \bar{z}) \tag{6.2}$$

[16]Note that this operator is directly related to the soft charges $Q_S^{\pm}$ evaluated at $Y^z = \frac{1}{z-w}$.

[17]For external states of spin s_k, the weights generalize to $\hat{h}_k = \frac{s_k - \omega_k \partial_{\omega_k}}{2}$ and (5.58) gets corrected by a spin connection term, see [13] for the general formula.

are parameterized in terms of the null vector

$$\hat{q}(z,\bar{z}) = (1+z\bar{z}, z+\bar{z}, -i(z-\bar{z}), 1-z\bar{z})\,. \tag{6.3}$$

We start by introducing a basis for scattering in AFS which diagonalizes asymptotic boosts as opposed to momentum generators. We show how to formulate scattering in this basis, with celestial amplitudes defining observables living on the sphere at infinity. We illustrate this construction by computing the celestial amplitude of two massless and one massive scalars. This section is based on [14] and [15].

6.1 Conformal primary wavefunctions

Scalar conformal primary wavefunctions are solutions to the wave equation

$$\left(\nabla^2 - m^2\right)\Psi = 0, \tag{6.4}$$

which are "highest weight" with respect to the Lorentz $SO(1,3)$. We start by identifying the associated highest weight states, then impose they are solutions to (6.4). A representation of the Lorentz generators is

$$J^{\mu\nu} = L^{\mu\nu} + S^{\mu\nu}, \tag{6.5}$$

where

$$L^{\mu\nu} = -\left(x^\mu\partial^\nu - x^\nu\partial^\mu\right) \tag{6.6}$$

is the orbital angular momentum generator and $S^{\mu\nu}$ is the spin generator. For simplicity, we focus on scalars in which case $S^{\mu\nu}=0$.

(6.6) consist of rotations

$$J_1 = -(x^2\partial_{x^3} - x^3\partial_{x^2}), \quad J_2 = x^1\partial_{x^3} - x^3\partial_{x^1}, \quad J_3 = -(x^1\partial_{x^2} - x^2\partial_{x^1}) \tag{6.7}$$

and boosts

$$K_1 = -(x^0\partial_{x^1} + x^1\partial_{x^0}), \quad K_2 = -(x^0\partial_{x^2} + x^2\partial_{x^0}), \quad K_3 = -(x^0\partial_{x^3} + x^3\partial_{x^0}). \tag{6.8}$$

These generators obey the standard Lorentz algebra

$$\begin{aligned}
[J_i, J_j] &= \epsilon_{ijk}J_k, \\
[K_i, K_j] &= -\epsilon_{ijk}J_k, \\
[J_i, K_j] &= \epsilon_{ijk}K_k,
\end{aligned} \tag{6.9}$$

while the linear combinations (B.1) of (6.7), (6.8) in appendix B obey the $\mathrm{SL}(2,\mathbb{C})$ commutation relations

$$[L_m, L_n] = (m-n)L_{m+n}, \quad [\bar{L}_m, \bar{L}_n] = (m-n)\bar{L}_{m+n}. \tag{6.10}$$

We now notice that

$$\Psi_\Delta \propto \frac{1}{(x^0+x^3)^\Delta} \tag{6.11}$$

obeys

$$(L_0 + \bar{L}_0)\Psi_\Delta = \Delta\Psi_\Delta, \quad (L_0 - \bar{L}_0)\Psi_\Delta = 0 \tag{6.12}$$

as well as

$$L_1\Psi_\Delta = \bar{L}_1\Psi_\Delta = 0. \tag{6.13}$$

In other words, (6.11) diagonalizes boosts in the x^3 direction and obeys the highest weight condition (6.13).

One could have done the same analysis starting with a set of rotated bulk Lorentz generators,

$$J'_i = R_{ij}J_j, \quad K'_i = R_{ij}K_j, \tag{6.14}$$

where

$$R = \begin{pmatrix} \cos\hat{\varphi}\cos\hat{\theta} & \sin\hat{\varphi}\cos\hat{\theta} & -\sin\hat{\theta} \\ -\sin\hat{\varphi} & \cos\hat{\varphi} & 0 \\ \cos\hat{\varphi}\sin\hat{\theta} & \sin\hat{\varphi}\sin\hat{\theta} & \cos\hat{\theta} \end{pmatrix} \equiv \begin{pmatrix} \hat{n}_1 \\ \hat{n}_2 \\ \hat{n}_3 \end{pmatrix}.$$

Multiplication by an arbitrary function f of the Lorentz invariant x^2 will preserve both the eigenvalue and highest weight conditions (6.12), (6.13), hence in general, a highest weight solution diagonal with respect to K'_3 will be[18]

$$\Psi_\Delta(\hat{q};x) = \frac{f(x^2)}{(\hat{q}\cdot x)^\Delta}, \quad \hat{q} \propto (1, \hat{n}_3) = \hat{q}(z,\bar{z}), \tag{6.15}$$

with $\hat{q}(z,\bar{z})$ in (5.44).

Finally, we require that (6.15) obeys the wave equation. Plugging (6.15) into (6.4), we find the following differential equation for f [15]

$$4x^2 f''(x^2) - 4(\Delta-2)f'(x^2) - m^2 f(x^2) = 0. \tag{6.16}$$

The solutions to (6.16) are linear combination of Bessel functions (of first kind)

$$f(x^2) = \left(\sqrt{-x^2}\right)^{\Delta-1}\left[c_1 I_{\Delta-1}(m\sqrt{x^2}) + c_2 I_{-\Delta+1}(m\sqrt{x^2})\right], \tag{6.17}$$

where c_1, c_2 are (Δ-dependent) constants. Imposing that (6.17) decays to 0 as $x^2 \to \infty$ picks out the linear combination proportional to the Bessel function of second kind

$$f(x^2) \propto \left(\sqrt{-x^2}\right)^{\Delta-1} K_{\Delta-1}(m\sqrt{x^2}). \tag{6.18}$$

We conclude that up to normalization, the massive conformal primary wavefunctions take the form[19]

$$\Psi_\Delta(\hat{q};x) \propto \frac{\left(\sqrt{-x^2}\right)^{\Delta-1}}{(\hat{q}\cdot x)^\Delta} K_{\Delta-1}(m\sqrt{x^2}). \tag{6.19}$$

[18] $(\hat{\theta},\hat{\varphi})$ are related to $z, \bar{z}$ via the stereographic projection.

[19] An $i\epsilon$ prescription distinguishes between in and out states [15].

Under Lorentz transformations, both $\hat{q}$ and x transform linearly

$$\begin{aligned} \hat{q}^\mu(z', \bar{z}') &= \left|\frac{\partial \vec{z}'}{\partial \vec{z}}\right|^{1/2} \Lambda^\mu{}_\nu \hat{q}^\nu(z, \bar{z}), \\ x^{\mu'} &= \Lambda^\mu{}_\nu x^\nu \end{aligned} \tag{6.20}$$

hence (6.19) obeys

$$\Psi_\Delta(\Lambda^\mu{}_\nu x; \vec{z}'(\vec{z})) = \left|\frac{\partial \vec{z}'}{\partial \vec{z}}\right|^{-\Delta/2} \Psi_\Delta(x; \vec{z}). \tag{6.21}$$

Exercise 6.1. *Show that the conformal primary wavefunctions obey* (6.21).

In the next section we give an alternate derivation of (6.19) which will lead to a representation of (6.19) as Fourier transforms of AdS_3 bulk-to-boundary propagators.

6.2 Milne slicing

The conformal compactification of Minkowski space in section 5.2 obscures one aspect of scattering in AFS: all massive particles enter (exit) spacetime at a point, $i^-(i^+)$, so how are we supposed to distingush between different asymptotics? The key is to resolve past and future timelike infinities by introducing the new coordinates [75, 76]

$$\begin{aligned} t^2 - r^2 &= \tau^2, \\ \rho\tau &= r. \end{aligned} \tag{6.22}$$

In $(\tau, \rho, z, \bar{z})$ coordinates, (5.9) becomes

$$ds^2 = -d\tau^2 + \tau^2 \left(\frac{d\rho^2}{1+\rho^2} + 2\rho^2 \gamma_{z\bar{z}} dz d\bar{z}\right) = -d\tau^2 + \tau^2 ds_{\mathbb{H}_3}. \tag{6.23}$$

We learn that slices of constant τ correspond to hyperboloids of radius τ, while $\rho = \frac{r}{t}\left(1 - \frac{r^2}{t^2}\right)^{-1/2}$ is constant whenever r/t is constant. Since $t = \tau\sqrt{1+\rho^2}$, as $\tau \to \infty$, $t \to \infty$ for fixed $(\rho, z, \bar{z})$. We illustrate the foliations of the past and future light-cones (also known as Milne wedges) with $\mathbb{H}_3$ slices in figure 5.

Parameterizing the trajectory of massive particle of constant momentum $\vec{p}$ and energy E by

$$\vec{r} = \vec{r}_0 + \frac{t}{E}\vec{p}, \tag{6.24}$$

we find that as $t \to \infty$,

$$\rho \to \frac{|\vec{p}|}{m}, \quad \frac{\vec{r}}{r} \to \frac{\vec{p}}{p}. \tag{6.25}$$

Hence at late times, massive particles asymptote to fixed $(\rho, z, \bar{z})$, or equivalently points on the unit hyperboloid at i^+. Similarily, the Rindler wedges can be foliated with dS_3 slices. This slicing is easily obtained by letting

$$\tau = i\widetilde{\tau}, \quad \rho = -i\widetilde{\rho} \tag{6.26}$$

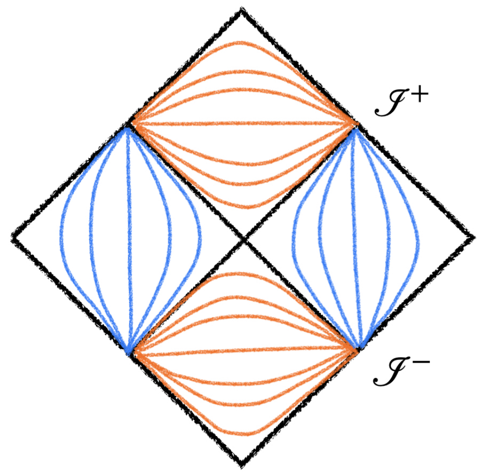

Figure 5: Minkowski space split into four regions: the past and future lightcones are covered by $\mathbb{H}_3$ slices while the causally disconnected Rindler regions are covered by dS_3 slices.

in (6.23) in which case

$$ds^2 = d\widetilde{\tau}^2 + \widetilde{\tau}^2\left(\frac{d\widetilde{\rho}^2}{1-\widetilde{\rho}^2} + 2\widetilde{\rho}^2\gamma_{z\bar{z}}dzd\bar{z}\right). \tag{6.27}$$

This dS_3 slicing won't be discussed further herein, but see [77] for an analysis of associated conformal primary solutions.

The proper, orthocronous Lorentz group acts as the group of isometries on the $\mathbb{H}_3$ slices for $t > 0$. To find solutions to (6.4) that preserve slices of constant τ it is convenient to express ∇^2 with respect to the coordinates in (6.23) .

Exercise 6.2. *Using*

$$\nabla^2\Psi = \frac{1}{\sqrt{g}}\partial_\mu\left(g^{\mu\nu}\sqrt{g}\partial_\nu\Psi\right) \tag{6.28}$$

show that in the coordinates (6.23), (6.4) *becomes*

$$\left[\frac{1}{\rho\tau^2}\left(\left(3\rho^2+2\right)\partial_\rho + \rho\left(\rho^2+1\right)\partial_\rho^2 - \rho\tau\left(3\partial_\tau + \tau\partial_\tau^2\right)\right) + \frac{\Box_{S^2}}{\rho^2\tau^2}\right]\Psi = m^2\Psi. \tag{6.29}$$

Setting

$$\rho = \sinh\eta, \tag{6.30}$$

(6.29) becomes

$$\left[\frac{1}{\tau^2}\left(\partial_\eta^2 + 2\coth\eta\partial_\eta\right) - 3\frac{\partial_\tau}{\tau} - \partial_\tau^2 + \frac{\Box_{S^2}}{\sinh^2\eta\tau^2}\right]\Psi = m^2\Psi. \tag{6.31}$$

This equation can be solved by separation of variables

$$\Psi = \phi_{p,l}(\eta)\varphi_p(\tau)Y_{lm}(z,\bar{z}) \tag{6.32}$$

where

$$\begin{gathered}\left(\partial_\eta^2 + 2\coth\eta\partial_\eta + \frac{-l(l+1)}{\sinh^2\eta} - p^2\right)\phi_{p,l}(\eta) = 0,\\ \left(-3\frac{\partial_\tau}{\tau} - \partial_\tau^2 + \frac{p^2}{\tau^2} - m^2\right)\varphi_p(\tau) = 0\end{gathered} \tag{6.33}$$

and

$$\Box_{S^2}Y_{lm} = -l(l+1)Y_{lm}. \tag{6.34}$$

Note that p is a free parameter which cancels in (6.31). We recognize the first equation in (6.33) as the massive wave equation in AdS_3 while the second equation has two linearly independent solutions which can be taken to be

$$\varphi_p(\tau) = \frac{I_{\sqrt{1+p^2}}(m\tau)}{\tau}, \quad \frac{K_{\sqrt{1+p^2}}(m\tau)}{\tau}. \tag{6.35}$$

Choosing the second solution as it decays at $\tau \to \infty$, we recover the τ dependence in (6.19) upon identifying

$$\Delta - 1 = \sqrt{1+p^2}. \tag{6.36}$$

Using (6.36) it is a standard exercise in AdS_3 to show that the first equation in (6.33) can be written in terms of the $\mathrm{SL}(2,\mathbb{C})$ generators (B.1) (upon an appropriate coordinate transformation)

$$\left(4L_0^2 - 2L_{-1}L_1 - 2L_1L_{-1}\right)\phi_{p,l} = \left(4\bar{L}_0^2 - 2\bar{L}_{-1}\bar{L}_1 - 2\bar{L}_1\bar{L}_{-1}\right)\phi_{p,l} = \Delta(\Delta-2)\phi_{p,l}. \tag{6.37}$$

Using the $\mathrm{SL}(2,\mathbb{C})$ commutation relations (6.10) and imposing that $L_1\phi_{p,l} = 0$, (6.37) reduces to (6.12), (6.13) and we recover the solutions (6.19).

Note that (6.33) admit more general solutions which don't obey the highest weight condition (6.13). These can be used to construct the unitary principal series representations of $\mathrm{SL}(2,\mathbb{C})$. This complementary calculation is detailed for the dS slicing of Minkowski space (6.27) in [77]. A discussion of conformal primary solutions of (6.4) in $(2,2)$ signature can be found in [78].

6.3 Integral representation

The conformal primary wavefunctions (6.19) admit the Fourier representation

$$\Psi_\Delta(x;\vec{z}) = \int_{\mathbb{H}_3} d^3\hat{p}\, G_\Delta(\hat{p};\vec{z})e^{im\hat{p}\cdot X}, \tag{6.38}$$

where the momenta

$$\hat{p}(y,w,\bar{w}) = \frac{1}{2y}\left(1+y^2+w\bar{w}, w+\bar{w}, -i(w-\bar{w}), 1-y^2-w\bar{w}\right) \tag{6.39}$$

are in one-to-one correspondence with points on the unit hyperboloid at i^+ and

$$G_\Delta(y, w, \bar{w}; z, \bar{z}) = \left(\frac{y}{y^2 + |z - w|^2}\right)^\Delta \tag{6.40}$$

is the bulk-to-boundary propagator in AdS_3 [79]. As they are weighted integrals of plane waves, they automatically solve the wave equation. That they transform as (6.21) under $\mathrm{SL}(2, \mathbb{C})$ follows from the transformation property

$$G_\Delta(\hat{p}'; \hat{q}') = \left|\frac{\partial \vec{z}'}{\partial \vec{z}}\right|^{-\Delta/2} G_\Delta(\hat{p}; \hat{q}) \tag{6.41}$$

of (6.40). The Fourier transform (6.38) can be evaluated to recover the conformal primary wavefunctions (6.19).

6.4 Massless particles

The massless conformal primary wavefunctions can be obtained from (6.19) by taking the $m \to 0$ limit (assuming $\mathrm{Re}(\Delta) > 1$). Using the expansion

$$\begin{aligned} K_{\Delta-1}(x) = {} & x^{-\Delta}\left(2^{\Delta-2}\Gamma(\Delta-1)x + \mathcal{O}(x^2)\right) \\ & + x^\Delta\left(2^{-\Delta}\Gamma(1-\Delta)x^{-1} + 2^{-2-\Delta}\frac{\Gamma(1-\Delta)}{\Delta}x + \mathcal{O}(x^2)\right), \end{aligned} \tag{6.42}$$

we find

$$\varphi_\Delta(\hat{q}; x) = \lim_{m\to 0} \Psi_\Delta(\hat{q}; x) \propto \frac{1}{(\hat{q}\cdot x)^\Delta}. \tag{6.43}$$

The integral representation of the massless conformal primary wavefunctions can also be derived from (6.38) by taking the limit $m \to 0$ for fixed $\omega \equiv m/(2y)$. In this limit (6.39) becomes null, (6.40) becomes proportional to [80]

$$\lim_{y\to 0} G_\Delta(y, z, \bar{z}; w, \bar{w}) = \frac{\pi}{\Delta - 1} y^{2-\Delta}\delta^{(2)}(z - w) + \frac{y^\Delta}{(w - z)^{2\Delta}} + \mathcal{O}(y^{4-\Delta}) \tag{6.44}$$

and upon evaluating the integral in (6.38) we recover (6.43).[20]

Exercise 6.3. *Evaluate* (6.38) *in the limit* (6.44) *and show that the massless conformal wavefunction indeed reduces to* (6.43)*.*

6.5 Conformal primary wavefunctions: summary

In this section we summarize what we have learned so far. We have shown that there exist solutions to the scalar wave equations that diagonalize Lorentz boosts in and rotations

[20] In fact (6.44) gives two contributions in the massless limit: the conformal primary (6.43) and its shadow. Since these solutions are not linearly independent, [15] argued it is sufficient to restrict to (6.43).

around a particular null direction or towards a point on the celestial sphere. In the massive case these take the form

$$\Psi_\Delta(\hat{q};x) \propto \frac{\left(\sqrt{-x^2}\right)^{\Delta-1}}{(\hat{q}\cdot x)^\Delta} K_{\Delta-1}(m\sqrt{x^2}). \tag{6.45}$$

while in the massless case they reduce to

$$\varphi_\Delta(\hat{q};x) = \lim_{m\to 0} \Psi_\Delta(\hat{q};x) \propto \frac{1}{(\hat{q}\cdot x)^\Delta}. \tag{6.46}$$

For massless particles, the bulk energy effectively gets traded for a scaling dimension Δ. The scaling dimension can be seen as a "Rindler energy" as it diagonalizes the Rindler (or rather Milne) time evolution operator which is nothing but a boost.

In the same way as plane waves form a basis of on-shell one-particle states in Minkowski in that any bulk (scalar) field can be decomposed as

$$\Phi(x) = \int \frac{d^3\vec{q}}{(2\pi)^3} \left[a^{\text{out}}(\vec{q})e^{iq\cdot x} + a^{\text{out}\dagger}(\vec{q})e^{-iq\cdot x}\right] \tag{6.47}$$

massless scalar conformal primary wavefunctions also form a basis (at least) provided that $\Delta = 1 + i\lambda$, for $\lambda \in \mathbb{R}$, namely

$$\Phi(x) = \int_{-\infty}^{\infty} d\lambda \int d^2z \left[\widetilde{a}^{\text{out}}_\lambda(\hat{q})\varphi_{1+i\lambda}(\hat{q}(z,\bar{z});x) + \widetilde{a}^{\text{out}\dagger}_\lambda(\hat{q})\varphi^*_{1-i\lambda}(\hat{q}(z,\bar{z});x)\right]. \tag{6.48}$$

Here $\widetilde{a}, \widetilde{a}^\dagger$ should be thought of boundary (CCFT) operators. For example $a^{\text{out}}_\lambda(\hat{q})$ creates an outgoing state in the CCFT while $a^{\text{in}\dagger}$ creates and incoming state in the CCFT and can be extracted from the bulk field $\Phi(x)$ via the Klein Gordon inner product

$$\widetilde{a}^{\text{out}}_\lambda(\hat{q}) = \langle \Phi(x), \varphi_{1+i\lambda}(\hat{q};x)\rangle_\Sigma \tag{6.49}$$

where the Cauchy slice Σ is taken to be $\mathcal{I}^+$. This can be shown using the orthogonality condition for massless outgoing conformal primary wavefunctions []

$$\langle \varphi_{1+i\nu_1}, \varphi_{1+i\nu_2}\rangle = 8\pi^4\delta(\nu_1-\nu_2)\delta^2(z_1-z_2) \tag{6.50}$$

Note that on the other hand

$$\langle \varphi_{1+i\nu_1}, \varphi^*_{1+i\nu_2}\rangle = \int_0^\infty d\omega_1 \int_0^\infty d\omega_2 \omega_1^{\Delta_1-1}\omega_2^{\Delta_2^*-1}\langle e^{i\omega_1\hat{q}_1\cdot X}, e^{-i\omega_2\hat{q}_2\cdot X}\rangle = 0. \tag{6.51}$$

This construction can be generalized to both spinning and massive particles. For massless spinning particles the story is simple: the CPW are simply obtained by dressing the conformal primary wavefunctions by appropriate frame fields

$$m^\mu = \epsilon^\mu_+ + \frac{\epsilon_+\cdot X}{(-q\cdot X)}q^\mu\,, \quad \bar{m}^\mu = \epsilon^\mu_- + \frac{\epsilon_-\cdot X}{(-q\cdot X)}q^\mu\,. \tag{6.52}$$

which satisfy $m\cdot\bar{m} = 1$ and transform with $\Delta = 0$ and, respectively, $J = +1$ and $J = -1$.

We define spin-one and spin-two conformal primary wavefunctions by

$$\begin{aligned} A_{\Delta,J=+1} &= m\varphi_\Delta\,, \quad A_{\Delta,J=-1} = \bar{m}\varphi_\Delta\,, \\ h_{\Delta,J=+2} &= mm\varphi_\Delta\,, \quad h_{\Delta,J=-2} = \bar{m}\bar{m}\varphi_\Delta\,. \end{aligned} \tag{6.53}$$

An analogous construction for conformal primary wavefunctions with half-integer spin using a decomposition of the null tetrad into a spin frame can be found in [84]. The case of spinning massive particles is more subtle and less examples have been worked out so far.

6.6 Celestial diamonds

An important role in the following will be played by conformal primary wavefunctions at particular (half-)integer dimensions Δ. We first exploit tools from conformal field theory to understand the possible representations arising in CCFT. An ubiquitous feature of CCFT is the presence of primary descendant operators. These are constructed from wavefunctions that (as the name suggests) are both primaries and descendants with respect to the global conformal group. This is only possible at particular weights $(h, \bar{h})$ and we can find these values by imposing that

$$L_1 L_{-1}^k |h, \bar{h}\rangle = 0. \tag{6.54}$$

We can now use an implication of the Lorentz algebra, namely that

$$[L_1, L_{-1}^k] = k(L_{-1})^{k-1}(2L_0 + k - 1) \tag{6.55}$$

to deduce that (6.54) implies

$$2h + k - 1 = 0 \implies h = \frac{1-k}{2}, \quad k \in \mathbb{N}. \tag{6.56}$$

The primary descendant has $h = \frac{k+1}{2}$ which correponds to $h \to 1 - h$. One can run the same argument for $\bar{L}$ as well as for $L, \bar{L}$ simultaneously. Note that the latter class is typically not encountered in unitary CFT such primary descendats would be associated with primary operators of

$$h = \frac{1-k}{2}, \quad \bar{h} = \frac{1-\bar{k}}{2}, \quad k, \bar{k} \in \mathbb{N} \tag{6.57}$$

which are ruled out by unitarity. However, as we will see later, they play an important role in CCFT.

Conformal primary wavefunctions for different dimensions give rise to symmetry generators in the celestial CFT. Let's look at the $s = 0$ diamond associated with the leading soft graviton theorem or supertranslation symmetry. At the left corner is the radiative (leading) soft graviton, while at the right corner is its shadow transform. The shadow is an intertwiner between $\mathrm{SL}(2, \mathbb{C})$ representations which maps primary operators to primary operators

$$\widetilde{\mathcal{O}}_{1-h,1-\bar{h}} = \int d^2w \frac{1}{(z-w)^{2-2h}(\bar{z}-\bar{w})^{2-2\bar{h}}} \mathcal{O}_{h,\bar{h}}. \tag{6.58}$$

at the bottom lies the supertranslation charge. A similar diamond exists for the canonically conjugate operators (or wavefunctions) which give rise to the Faddeev Kulish dressings discussed before. For $s = 1$ we get a negative helicity subleading soft graviton at the left, its shadow - stress tensor - at the right and the superrotation charge at the bottom. For $s = 2$ the diamond degenerates to a line, while for $s \geq 3$ we get the tower of soft symmetries that will be discussed later in the course.

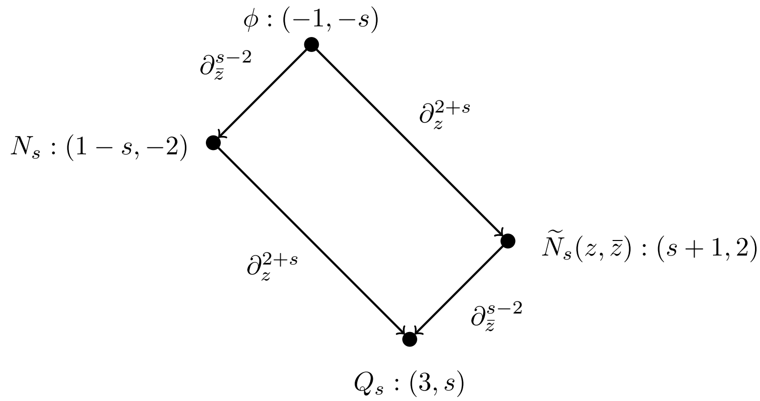

Figure 6: Diamond associated with a negative helicity soft graviton of dimension $\Delta = 1 - s$ for $s \leq 3$. Operators connected by long edges have weights related by $(h, \bar{h}) \leftrightarrow (1 - h, \bar{h})$. Operators connected by short edges have $(h, \bar{h}) \leftrightarrow (h, 1 - \bar{h})$. Diagonally opposite corners are related by $(h, \bar{h}) \leftrightarrow (1 - h, 1 - \bar{h})$.

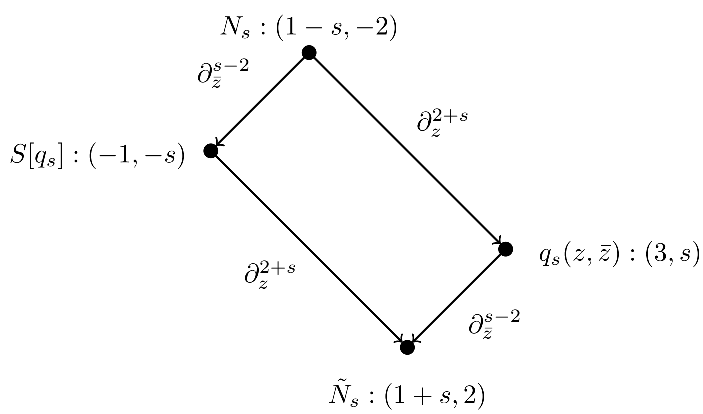

Figure 7: Diamond associated with a negative helicity soft graviton of dimension $\Delta = 1 - s$ for $s \geq 3$. Operators connected by long edges have weights related by $(h, \bar{h}) \leftrightarrow (1 - h, \bar{h})$. Operators connected by short edges have $(h, \bar{h}) \leftrightarrow (h, 1 - \bar{h})$. Diagonally opposite corners are related by $(h, \bar{h}) \leftrightarrow (1 - h, 1 - \bar{h})$.

6.7 A conformal primary basis for scattering in AFS

Using the map (6.38) from plane wave solutions to conformal primary solutions of the scalar wave equation, one can relate momentum space scattering amplitudes $\mathcal{A}$ to scattering amplitudes $\widetilde{\mathcal{A}}$ in a conformal primary basis

$$\widetilde{\mathcal{A}}(\Delta_i, z_i, \bar{z}_i) = \prod_{i=1}^{n} \int_{H_3} \frac{d^3\hat{p}_i}{p_i^0} G_{\Delta_i}(\hat{p}_i; z_i, \bar{z}_i)\mathcal{A}(\epsilon_i m_i \hat{p}_i), \tag{6.59}$$

where $\epsilon_i = \pm 1$ depending on whether the i^{th} particle is incoming or outgoing. The transformation of (6.21) under $\text{SL}(2,\mathbb{C})$ implies that (6.59) transform as correlators of 2D (global) conformal primary operators

$$\widetilde{\mathcal{A}}(\Delta_i, \vec{z}_i'(\vec{z}_i)) = \prod_{i=1}^{n} \left| \frac{\partial \vec{z}_i'}{\partial \vec{z}_i} \right|^{-\Delta_i/2} \widetilde{\mathcal{A}}(\Delta_i, \vec{z}_i). \tag{6.60}$$

(6.59) is the defining relation of a *celestial amplitude.*

It can be shown that [15]

- The massive conformal primary wavefunctions (6.38) form a basis of solutions to the wave equation provided $\Delta = 1 + i\lambda,\ \lambda \geq 0$. For such Δ, these solutions are complete, linearly independent and orthogonal.[21]
- The massless conformal primary wavefunctions form a basis of solutions to the wave equation provided $\Delta = 1 + i\lambda,\ \lambda \in \mathbb{R}$.

The construction of conformal primary wavefunctions and celestial amplitudes generalizes for spinning particles. Photons and gravitons are discussed in [15, 81], fermions were analyzed in [82, 83] while arbitrary spins are addressed in [41, 84].

6.8 Example: 2 massless and 1 massive scalars at tree–level

In this section we illustrate the construction of celestial amplitudes with a computation of the tree-level celestial amplitude for two massless and one massive scalars [16]. We start with the momentum space 3-point interaction

$$\mathcal{A}(\hat{p}_i) = g\delta^{(4)}(\omega_1 \hat{q}_1 + \omega_2 \hat{q}_2 - m\hat{p}). \tag{6.61}$$

The associated celestial amplitude is then

$$\begin{aligned} \widetilde{\mathcal{A}}(\Delta_i, z_i, \bar{z}_i) = g \prod_{i=1}^{2} \left(\int_0^\infty d\omega_i \omega_i^{\Delta_i - 1} \right) \int_0^\infty \frac{dy}{y^3} \int d^2 w \left(\frac{y}{y^2 + |z_3 - w|^2} \right)^{\Delta_3} \\ \times \delta^{(4)}(\omega_1 \hat{q}_1 + \omega_2 \hat{q}_2 - m\hat{p}). \end{aligned} \tag{6.62}$$

[21] Notice that in this case, p^2 (the effective "mass" on the AdS_3 slices in (6.33)) will be complex

Using the parameterizations of momenta

$$\begin{aligned}\hat{q} &= (1+z\bar{z}, z+\bar{z}, -i(z-\bar{z}), 1-z\bar{z}) \,,\\ \hat{p} &= \frac{1}{2y}\left(1+y^2+w\bar{w}, w+\bar{w}, -i(w-\bar{w}), 1-y^2-w\bar{w}\right)\end{aligned} \tag{6.63}$$

and evaluating the integrals over $y, \vec{w}$ and ω_2, (6.62) reduces to (see appendix C for details)

$$\widetilde{\mathcal{A}}(\Delta_i, z_i, \bar{z}_i) = \frac{g m^{2\Delta_2+\Delta_3-4}}{2^{2\Delta_2-\Delta_3-2}|z_{12}|^{2\Delta_2-2\Delta_3}} \int_0^\infty d\omega \frac{\omega^{\Delta_1-\Delta_2+\Delta_3-1}}{(m^2|z_{23}|^2+4|z_{12}|^2|z_{13}|^2\omega^2)^{\Delta_3}}. \tag{6.64}$$

Upon a change of variables, the remaining integral becomes proportional to the standard integral

$$\int_0^1 dt t^{\alpha-1}(1-t)^{\beta-1} = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \equiv B(\alpha,\beta) \tag{6.65}$$

and we conclude

$$\begin{aligned}\widetilde{\mathcal{A}}(\Delta_i, z_i, \bar{z}_i) &= \frac{C(\Delta_1,\Delta_2,\Delta_3)}{|z_{12}|^{\Delta_1+\Delta_2-\Delta_3}|z_{13}|^{\Delta_1+\Delta_3-\Delta_2}|z_{23}|^{\Delta_2+\Delta_3-\Delta_1}},\\ C(\Delta_1,\Delta_2,\Delta_3) &= g\frac{m^{\Delta_1+\Delta_2-4}}{2^{\Delta_1+\Delta_2-1}} B\left(\frac{\Delta_{12}+\Delta_3}{2}, \frac{-\Delta_{12}+\Delta_3}{2}\right).\end{aligned} \tag{6.66}$$

7 Celestial symmetries

This section contains a review of celestial symmetries and the constraints they impose on celestial amplitudes. The ideas and calculations summarized in this section are detailed in [17–21, 41, 85].

Lecture 4, 5

- Symmetry, Ward identities, MHV and beyond.
- OPEs: soft symmetry algebras.

7.1 Poincaré action on the celestial sphere

We begin by discussing the Poincaré symmetry of celestial amplitudes. As shown in [17] the Lorentz generators act on operators $\mathcal{O}_{h,\bar{h}}(z,\bar{z})$ as

$$\begin{aligned} L_0 &= 2(z\partial_z + h), \quad L_- = \partial_z, \quad L_+ = z^2\partial_z + 2zh,\\ \bar{L}_0 &= 2(\bar{z}\partial_{\bar{z}} + \bar{h}), \quad \bar{L}_- = \partial_{\bar{z}}, \quad \bar{L}_+ = \bar{z}^2\partial_{\bar{z}} + 2\bar{z}\bar{h}.\end{aligned} \tag{7.1}$$

Lorentz symmetry of scattering in AFS is equivalent to global conformal symmetry of celestial amplitudes

$$\mathcal{L}_I \widetilde{\mathcal{A}}_n = \bar{\mathcal{L}}_I \widetilde{\mathcal{A}}_n = 0, \tag{7.2}$$

where $\mathcal{A}_n$ is an n-point celestial amplitude

$$\mathcal{L}_I = \sum_{k=1}^{n} L_{I,k}, \quad \bar{\mathcal{L}}_I \equiv \sum_{k=1}^{n} \bar{L}_{I,k} \tag{7.3}$$

and I runs over $-1, 0, 1$. (7.2) is a familiar property of correlation functions in 2D CFT.

Additionally, bulk translation invariance implies that

$$\mathcal{P}_\mu \widetilde{\mathcal{A}}_n = 0, \quad \mathcal{P}_\mu \equiv \sum_{k=1}^{n} P_{\mu,k}. \tag{7.4}$$

Celestial translation generators act on massless particles as weight-shifting operators

$$P_{\mu,k} = \epsilon_k \hat{q}_\mu(z_k, \bar{z}_k) e^{\partial_{\Delta_k}}, \tag{7.5}$$

where $\epsilon_k = \pm 1$ distinguishes between incoming and outgoing particles. To see this, we can start with the momentum space action

$$\hat{P}_k A(q_1, ..., q_n) = \epsilon_k \omega_k \hat{q}_k A(q_1, ..., q_n) \tag{7.6}$$

and rewrite it in a conformal primary basis by taking a Mellin transform

$$\begin{aligned} P_k \widetilde{\mathcal{A}}(\Delta_1, ...\Delta_n) &= \prod_{j=1}^{n} \left(\int_0^\infty d\omega_j \omega_j^{\Delta_j - 1} \right) \epsilon_k \omega_k \hat{q}_k A(q_1, ..., q_n) \\ &= \prod_{\substack{j=1 \\ j \neq k}}^{n} \left(\int_0^\infty d\omega_j \omega_j^{\Delta_j - 1} \right) \int_0^\infty d\omega_k \omega_k^{\Delta_k + 1 - 1} \epsilon_k \hat{q}_k A(q_1, ..., q_n) \\ &= \epsilon_k \hat{q}_k \widetilde{\mathcal{A}}(\Delta_1, ..., \Delta_k + 1, ..., \Delta_n). \end{aligned} \tag{7.7}$$

We conclude that for massless scattering, (7.4) relates celestial amplitudes involving operators of shifted weights

$$\widetilde{\mathcal{A}}_n(\Delta_1 + 1, \Delta_2, ..., \Delta_n) + \widetilde{\mathcal{A}}_n(\Delta_1, \Delta_2 + 1, ..., \Delta_n) + \cdots + \widetilde{\mathcal{A}}_n(\Delta_1, ..., \Delta_n + 1) = 0. \tag{7.8}$$

For massive scalars, (7.5) are replaced by [18]

$$P^\mu = \frac{m}{2} \left[\left(\partial_z \partial_{\bar{z}} q^\mu + \frac{\partial_{\bar{z}} q^\mu \partial_z + \partial_z q^\mu \partial_{\bar{z}}}{\Delta - 1} + \frac{q^\mu \partial_z \partial_{\bar{z}}}{(\Delta - 1)^2} \right) e^{-\partial_\Delta} + \frac{\Delta q^\mu}{\Delta - 1} e^{\partial_\Delta} \right]. \tag{7.9}$$

(7.9) is determined by imposing the on-shell condition

$$P_\mu P^\mu = -m^2, \tag{7.10}$$

as well as the Poincaré algebra

$$[P_\mu, P_\nu] = 0, \quad [M_{\mu\nu}, P_\rho] = \eta_{\mu\rho} P_\nu - \eta_{\nu\rho} P_\mu. \tag{7.11}$$

Exercise 7.1. *Verify* (7.9) *satisfy* (7.10) *and* (7.11).

The momentum generators for spinning particles can be found in [41]. It is interesting to notice that in addition to off-diagonal terms in dimension $\Delta = h + \bar{h}$, they also contain off-diagonal terms in spin $J = h - \bar{h}$.

7.2 Constraints from Poincaré symmetry

(7.1) and (7.5) imply that any celestial 4-point function can be put into the form

$$\widetilde{\mathcal{A}}_4 = K_{h_i,\bar{h}_i}(z_i,\bar{z}_i)\delta(z-\bar{z})f^{h_i,\bar{h}_i}(z,\bar{z}), \tag{7.12}$$

where

$$K_{h_i,\bar{h}_i}(z_i,\bar{z}_i) = \prod_{i<j=1}^{4} z_{ij}^{h/3-h_i-h_j}\bar{z}_{ij}^{\bar{h}/3-\bar{h}_i-\bar{h}_j}, \quad h = \sum_{i=1}^{4} h_i \tag{7.13}$$

and

$$z = \frac{z_{13}z_{24}}{z_{12}z_{34}}, \quad \bar{z} = \frac{\bar{z}_{13}\bar{z}_{24}}{\bar{z}_{12}\bar{z}_{34}} \tag{7.14}$$

are the 2D conformally invariant cross-ratios. For massless scattering, (7.14) are related to the bulk variables (Mandelstam invariants) by

$$z = -\frac{t}{s}, \quad s = -(p_1+p_2)^2, \quad t = -(p_1+p_3)^2. \tag{7.15}$$

Additionally, in this case momentum conservation (7.4) implies that

$$\sum_{j=1}^{4} K_{h_j+\frac{1}{2},\bar{h}_j+\frac{1}{2}}(z_i,\bar{z}_i)f^{h_j+\frac{1}{2},\bar{h}_j+\frac{1}{2}}(z,\bar{z}) = 0. \tag{7.16}$$

Exercise 7.2. *Show that*

$$\sum_{j=1}^{4} K_{h_j+\frac{1}{2},\bar{h}_j+\frac{1}{2}}(z_i,\bar{z}_i) = 0. \tag{7.17}$$

Since the conformally covariant factor (7.13) is translationally invariant by itself, (7.16) can be non-trivially obeyed if

$$f^{h_i+\frac{1}{2},\bar{h}_i+\frac{1}{2}}(z,\bar{z}) = f^{h_j+\frac{1}{2},\bar{h}_j+\frac{1}{2}}(z,\bar{z}), \quad \forall i,j. \tag{7.18}$$

By induction it can be shown that (7.18) implies that

$$f^{h_i,\bar{h}_i}(z,\bar{z}) = f^{\beta,J_i}(z,\bar{z}), \quad \beta = \sum_{i=1}^{4}(h_i+\bar{h}_i) = \sum_{i=1}^{4}\Delta_i. \tag{7.19}$$

More generally, (7.4) will be obeyed by celestial amplitudes with both massive and massless particles, provided $P_{\mu,k}$ are chosen appropriately. For example, consider the three-point amplitude of two massless and one massive scalars computed in section 6.8. The same result can be perhaps more easily recovered by considering the constraint

$$\left(P_1+P_2+P_3^{(m)}\right)\widetilde{\mathcal{A}}_3(1,2,3^{(m)}) = 0, \tag{7.20}$$

where P_1, P_2 are the massless momentum generators in (7.5) while $P_3^{(m)}$ is the massive momentum (7.9). Global conformal invariance fixes

$$\widetilde{\mathcal{A}}(1,2,3^{(m)}) = \frac{C(\Delta_1,\Delta_2,\Delta_3)}{|z_{12}|^{\Delta_1+\Delta_2-\Delta_3}|z_{23}|^{\Delta_2+\Delta_3-\Delta_1}|z_{13}|^{\Delta_1+\Delta_3-\Delta_2}}, \tag{7.21}$$

hence (7.20) leads to the following recursion relations on the 3-point coefficients [18]

$$\begin{aligned}
\left(\frac{\Delta_{12}^2}{4} - \frac{(\Delta_3-1)^2}{4}\right) C_{\Delta_1,\Delta_2,\Delta_3-1} + \Delta_3(\Delta_3-1)C_{\Delta_1,\Delta_2,\Delta_3+1} &= 0, \\
4\epsilon_2(\Delta_3-1)C_{\Delta_1,\Delta_2+1,\Delta_3} + m\epsilon_3(\Delta_3-1-\Delta_{12})C_{\Delta_1,\Delta_2,\Delta_3-1} &= 0, \\
4\epsilon_1(\Delta_3-1)C_{\Delta_1+1,\Delta_2,\Delta_3} + m\epsilon_3(\Delta_3-1+\Delta_{12})C_{\Delta_1,\Delta_2,\Delta_3-1} &= 0,
\end{aligned} \tag{7.22}$$

where $\Delta_{12} \equiv \Delta_1 - \Delta_2$.

Exercise 7.3. *Show that* (7.22) *are solved by*

$$C(\Delta_1,\Delta_2,\Delta_3) = B\left(\frac{\Delta_1-\Delta_2+\Delta_3}{2}, \frac{\Delta_2-\Delta_1+\Delta_3}{2}\right) c_{\Delta_1,\Delta_2,\Delta_3}, \tag{7.23}$$

where

$$\begin{aligned}
c_{\Delta_1,\Delta_2,\Delta_3-1} &= c_{\Delta_1,\Delta_2,\Delta_3+1}, \\
c_{\Delta_1+1,\Delta_2,\Delta_3} &= c_{\Delta_1,\Delta_2+1,\Delta_3}, \\
c_{\Delta_1+1,\Delta_2,\Delta_3} &= c_{\Delta_1,\Delta_2,\Delta_3-1}.
\end{aligned} \tag{7.24}$$

The first constraint in (7.24) implies c is perodic in Δ_3 of period 2, the second implies periodicity of period 1 in Δ_1 and Δ_2 up to dependence of $\Delta_1+\Delta_2$ while additionally, the last constraint implies c is periodic of period 1 in Δ_1, Δ_3 up to dependence of $\sum_{i=1}^{3}\Delta_i$ in which case the period becomes 2. The periodic function is set to a constant by requiring the inverse Mellin transform to be well defined.

7.3 Conformally soft symmetries

We saw in section 5.5 that the subleading soft theorem implies a conservation law associated with an infinite-dimensional Virasoro symmetry. This is one example of a more general equivalence between soft theorems and conservation laws associated with large gauge symmetries. For example, it was shown in [7] that the leading soft photon theorem implies that soft photons behave as $U(1)$ currents. As such, their insertions into $\mathcal{S}$-matrices obey Ward identities of the form

$$\begin{aligned}
\langle J_z \mathcal{O}_1(\omega_1,z_1,\bar{z}_1)...\mathcal{O}_n(\omega_n,z_n,\bar{z}_n)\rangle &\equiv \lim_{\omega\to 0}\omega\langle \mathcal{O}^+(\omega,z,\bar{z})\mathcal{O}_1(\omega_1,z_1,\bar{z}_1)...\mathcal{O}_n(\omega_n,z_n,\bar{z}_n)\rangle \\
&= \sum_{k=1}^{n}\frac{Q_k}{z-z_k}\langle \mathcal{O}_1(\omega_1,z_1,\bar{z}_1)...\mathcal{O}_n(\omega_n,z_n,\bar{z}_n)\rangle.
\end{aligned} \tag{7.25}$$

Similarly, the leading soft gluon theorem can be recast as a Kac-Moody symmetry generated by non-abelian currents J_z^a obeying the Ward identities [64]

$$\begin{aligned}
\langle J_z^a \mathcal{O}_1(\omega_1,z_1,\bar{z}_1)...\mathcal{O}_n(\omega_n,z_n,\bar{z}_n)\rangle &\equiv \lim_{\omega\to 0}\omega\langle \mathcal{O}^{+,a}(\omega,z,\bar{z})\mathcal{O}_1(\omega_1,z_1,\bar{z}_1)...\mathcal{O}_n(\omega_n,z_n,\bar{z}_n)\rangle \\
&= \sum_{k=1}^{n}\frac{1}{z-z_k}\langle \mathcal{O}_1(\omega_1,z_1,\bar{z}_1)...T_k^a\mathcal{O}_k...\mathcal{O}_n(\omega_n,z_n,\bar{z}_n)\rangle.
\end{aligned} \tag{7.26}$$

We would like to reexpress (7.25) and (7.26) in a conformal primary basis and identify the celestial representations of these symmetry generators. Since the currents were constructed from low-energy limits of bulk photons and gluons, while celestial operators involve integrals over photons and gluons of all energies, it is a-priori not immediately obvious how to construct the celestial currents. One hint is that in conventional CFT_d, currents saturate unitarity bounds[22] and hence the dimension of a spin-j current is constrained to be [86]

$$\Delta = d + j - 2. \tag{7.27}$$

In particular, positive helicity conformally soft photons and gluons should correspond to operators of weights[23] $(h, \bar{h}) = (1, 0)$, while negative helicity ones should have $(h, \bar{h}) = (0, 1)$. They should be associated with abelian and non-abelian symmetries on the celestial sphere.

The simplest way to show that this guess is indeed correct is to start with the Mellin representation

$$\mathcal{O}^+_\Delta(z, \bar{z}) = \int_0^\infty d\omega \omega^{\Delta-1} \mathcal{O}^+(\omega, z, \bar{z}) \tag{7.28}$$

and notice that [85]

$$\begin{aligned} \lim_{\Delta \to 1} (\Delta - 1)\mathcal{O}^+_\Delta(z, \bar{z}) &= \lim_{\Delta \to 1} \int_0^\infty d\omega (\Delta - 1)\omega^{\Delta-1} \mathcal{O}^+(\omega, z, \bar{z}) \\ &= 2 \int_0^\infty d\omega \delta(\omega) \omega \mathcal{O}^+(\omega, z, \bar{z}) = \lim_{\omega \to 0} \omega \mathcal{O}^+(\omega, z, \bar{z}). \end{aligned} \tag{7.29}$$

In the last line we have used the identity[24]

$$\lim_{\epsilon \to 0} \frac{\epsilon}{2} |x|^{\epsilon - 1} = \delta(x). \tag{7.30}$$

More generally

$$\begin{aligned} \lim_{\Delta \to -n} (\Delta + n)\mathcal{O}^+_\Delta(z, \bar{z}) &= \lim_{\Delta \to -n} (\Delta + n) \int_0^{\omega_*} d\omega \omega^{\Delta-1} \mathcal{O}^+(\omega, z, \bar{z}) \\ &= \lim_{\Delta \to -n} (\Delta + n) \sum_k \int_0^{\omega_*} d\omega \omega^{\Delta+k-1} O^+_k(z, \bar{z}) \\ &= O^+_n(z, \bar{z}), \end{aligned} \tag{7.31}$$

where we expanded

$$\mathcal{O}^+(\omega, z, \bar{z}) = \sum_k \omega^k O^+_k(z, \bar{z}) \tag{7.32}$$

for $\omega \ll \omega_*$ and assumed that insertions of $\mathcal{O}^+(\omega, z, \bar{z})$ into $\mathcal{S}$-matrices have fast enough falls offs with energy[25] in which case the high-energy part of the Mellin integral will be

[22] No analog bounds are known to exist in CCFT. Moreover, as we will see there is an infinite tower of negative dimension operators arising from soft limits in the bulk.

[23] The conformal weights are related to the conformal dimensions Δ and the spin J by $h = \frac{\Delta+J}{2}$, $\bar{h} = \frac{\Delta-J}{2}$.

[24] This holds provided that x has compact support.

[25] An exponential fall-off $\lim_{\omega\to\infty}\langle \mathcal{O}(\omega, z, \bar{z}) \cdots \rangle \sim e^{-\epsilon\omega}$ will ensure this limit is well defined for any negative integer Δ.

free of poles in $\Delta + n$. We conclude that the $\Delta \to -n$ limit of a celestial operator for $n = -1, 0, 1, ...$ picks out the $\mathcal{O}(\omega^n)$ term in an expansion around $\omega = 0$. For example, a subleading soft photon will correspond to the celestial operator

$$\lim_{\Delta \to 0} \Delta \mathcal{O}^+_\Delta(z, \bar{z}). \tag{7.33}$$

This infinity of soft currents has been studied in [20, 31, 87]. There exists a complementary tower of positive integer-dimension operators (also known as Goldstone modes), canonically conjugate to the conformally soft modes above [88]. Their combined Ward identities are expected to constrain celestial amplitudes, but a complete understanding of these symmetries and their implications remains an important open problem. In the next section we describe some instances in which soft celestial symmetries were used to derive non-trivial properties of celestial amplitudes.

7.4 Applications

We conclude this section with a brief overview of recent work on soft constraints on celestial amplitudes [19] and the infinite tower of soft currents [20, 21].

7.4.1 Celestial operator products of gluons

We start by assuming that positive helicity gluons admit the holomorphic collinear expansion

$$\mathcal{O}^{+,a}_{\Delta_1}(z_1, \bar{z}_1)\mathcal{O}^{+,b}_{\Delta_2}(z_2, \bar{z}_2) \sim \frac{if^{abc}}{z_{12}} C(\Delta_1, \Delta_2)\mathcal{O}^{+,c}_{\Delta_1+\Delta_2-1}(z_2, \bar{z}_2) + ..., \tag{7.34}$$

where ... include contributions from $\mathrm{SL}(2,\mathbb{C})$ descendants. Here and in the next section, $z, \bar{z}$ are treated as real independent variables in which case the CCFT becomes Lorentzian and $\mathrm{SL}(2,\mathbb{C})$ is replaced by $\mathrm{SL}(2,\mathbb{R})_\mathrm{L} \times \mathrm{SL}(2,\mathbb{R})_\mathrm{R}$. The form of the OPE is fixed by the leading soft theorem and $\mathrm{SL}(2,\mathbb{C})$ up to a coefficient $C(\Delta_1, \Delta_2)$. We now show that the subleading conformally soft gluon theorem determines this leading OPE coefficient up to a normalization fixed by the leading soft gluon theorem [19].

The subleading soft gluon theorem can be recast as a "symmetry"[26] under which gluons transform as follows

$$\begin{aligned}\delta_b \mathcal{O}^{\pm,a}_\Delta(z, \bar{z}) &= -(\Delta - 1 \pm 1 + z\partial_z) i f^a_{\ bc} \mathcal{O}^{\pm,c}_{\Delta-1},\\ \bar{\delta}_b \mathcal{O}^{\pm,a}_\Delta(z, \bar{z}) &= -(\Delta - 1 \mp 1 + \bar{z}\partial_{\bar{z}}) i f^a_{\ bc} \mathcal{O}^{\pm,c}_{\Delta-1}.\end{aligned} \tag{7.35}$$

Acting with $\bar{\delta}$ on both sides of (7.34) and comparing the two sides, we deduce that $C(\Delta_1, \Delta_2)$ obey the recursion relation

$$(\Delta_1 - 2)C(\Delta_1 - 1, \Delta_2) = (\Delta_1 + \Delta_2 - 3)C(\Delta_1, \Delta_2). \tag{7.36}$$

(7.36) has the unique[27] solution

$$C(\Delta_1, \Delta_2) = B(\Delta_1 - 1, \Delta_2 - 1). \tag{7.37}$$

[26] These have not been shown to be associated with asymptotic charges.

[27] By Wieland's theorem, see appendix E of [19]. The normalization is fixed by the leading soft theorem.

Exercise 7.4. *Write down the action of $\bar{\delta}$ on* (7.34) *and deduce* (7.36).

Similar recursion relations are also implied by the subsubleading soft graviton theorem and can be shown to completely fix the leading OPE coefficients in Einstein-Yang-Mills theory.

7.4.2 Holographic symmetry algebras

(7.34) can be generalized to include contributions from $\mathrm{SL}(2,\mathbb{R})_R$ descendants. One finds

$$\mathcal{O}^{+,a}_{\Delta_1}(z_1,\bar{z}_1)\mathcal{O}^{+,b}_{\Delta_2}(z_2,\bar{z}_2) \sim \frac{-if^{ab}{}_c}{z_{12}}\sum_{n=0}^{\infty} B(\Delta_1-1+n,\Delta_2-1)\frac{\bar{z}_{12}^n}{n!}\bar{\partial}^n\mathcal{O}^{+,c}_{\Delta_1+\Delta_2-1}(z_2,\bar{z}_2). \quad (7.38)$$

(7.38) follows by resuming contributions from the right-moving descendants through the OPE block [20]

$$\mathcal{O}^{+,a}_{\Delta_1}(z_1,\bar{z}_1)\mathcal{O}^{+,b}_{\Delta_2}(z_2,\bar{z}_2) \sim \frac{-if^{ab}{}_c}{z_{12}}\int_0^1 dt\frac{\mathcal{O}^{+,c}_{\Delta_P}(z_2,\bar{z}_2+t\bar{z}_{12})}{t^{2-\Delta_1}(1-t)^{2-\Delta_2}}. \quad (7.39)$$

It is interesting to study the algebra of soft operators discussed in section 7.3. First notice that if $\Delta_1,\Delta_2\in\{1,0,-1,...\}$, $\Delta_1+\Delta_2-1\in\{1,0,-1,...\}$ and the algebra of soft operators closes. Then mode expanding such an operator on the right, one finds for $k=1,0,-1,...$

$$\lim_{\epsilon\to 0}\epsilon\mathcal{O}^{+,a}_{k+\epsilon}(z,\bar{z}) = \lim_{\epsilon\to 0}\sum_n\frac{\epsilon\mathcal{O}^{+,a}_{k+\epsilon,n}(z)}{\bar{z}^{n+\frac{k-1}{2}}}. \quad (7.40)$$

Defining

$$R^{k,a}(z,\bar{z}) = \lim_{\epsilon\to 0}\epsilon\mathcal{O}^{+,a}_{k+\epsilon}(z,\bar{z}),\quad R^{k,a}_n(z) = \lim_{\epsilon\to 0}\epsilon\mathcal{O}^{+,a}_{k+\epsilon,n}(z), \quad (7.41)$$

we see that for

$$\frac{k-1}{2}\le n\le\frac{1-k}{2}, \quad (7.42)$$

$R^{k,a}_n(z)$ organize into $(2-k)$-dimensional $\mathrm{SL}(2,\mathbb{R})_R$ representations as

$$\bar{\partial}^{2-k}R^{k,a}(z,\bar{z}) = 0. \quad (7.43)$$

Using (7.38) and (7.41), the OPE of soft currents is found to be

$$R^{k,a}(z_1,\bar{z}_1)R^{l,b}(z_2,\bar{z}_2) \sim \frac{-if^{ab}{}_c}{z_{12}}\sum_{n=0}^{1-k}\binom{2-k-l-n}{1-l}\frac{\bar{z}_{12}^n}{n!}\bar{\partial}^n R^{k+l-1,c}(z_2,\bar{z}_2). \quad (7.44)$$

This follows from setting $\Delta_1=k+\epsilon,\Delta_2=l+\epsilon$ in (7.38) and using

$$\lim_{\epsilon\to 0}\epsilon\frac{\Gamma(k+\epsilon-1+n)\Gamma(l+\epsilon-1)}{\Gamma(k+l+2\epsilon+n-2)} = \frac{1}{(1-l)!}\frac{\Gamma(3-k-l-n)}{\Gamma(2-k-n)}. \quad (7.45)$$

Finally, (7.44) can be used to derive the algebra of the soft operators [20]

$$[R^{k,a}_n,R^{l,b}_{n'}] = -if^{ab}{}_c\binom{\frac{1-k}{2}-n+\frac{1-l}{2}-n'}{\frac{1-k}{2}-n}\binom{\frac{1-k}{2}+n+\frac{1-l}{2}+n'}{\frac{1-k}{2}+n}R^{k+l-1,c}_{n+n'}, \quad (7.46)$$

which follows from

$$R_n^{k,a}(z) = \oint \frac{d\bar{z}}{2\pi i} \bar{z}^{n+\frac{k-1}{2}-1} R^{k,a}(z,\bar{z}) \tag{7.47}$$

and the commutator of holomorphic operators

$$[A,B](z) = \oint_z \frac{dw}{2\pi i} A(w)B(z) \tag{7.48}$$

applied to (7.44).

Notice that upon redefining [21]

$$\hat{R}_n^{k,a} \equiv \left(\frac{1-k}{2} - n\right)! \left(\frac{1-k}{2} + n\right)! R_n^{k,a}, \tag{7.49}$$

(7.46) simplifies to

$$[\hat{R}_n^{k,a}, \hat{R}_{n'}^{l,b}] = -i f^{ab}{}_c \hat{R}_{n+n'}^{k+l-1,c}. \tag{7.50}$$

A similar analysis can be done for gravitons, where the analog of (7.50) was identified with a $w_{1+\infty}$ algebra in [21]

$$[w_m^p, w_n^q] = [m(q-1) - n(p-1)] w_{m+n}^{p+q-2}, \tag{7.51}$$

with p, q running over positive, half-integral values $p, q = 1, \frac{3}{2}, \cdots$. Working out the implications of (7.51) for gravity in AFS remains a fascinating open problem.

8 New lessons about spacetime from CCFT

Lecture 5

- New charges, new algebras
- Solution space labeled by charges
- Comments on practical applications.

What is the spacetime interpretation of the tower of symmetries (7.51)? First notice that $p, q = \frac{3}{2}, 2$ are associated with the leading and subleading soft symmetries. Recall from **??** that provided the geometry relaxes to vacuum at $\mathcal{I}_\pm^\pm$ the corresponding asymptotic charges

$$Q_f = \int_{S^2} d^2 z f m_B(z,\bar{z}), \quad Q_Y = \int d^2 z Y^A N_A(z,\bar{z}) \tag{8.1}$$

can be re-expressed in terms of the Einstein constraint equations (5.21), (5.22) expanded to $\mathcal{O}(r^{-2})$

$$G_{uu} = 0, \quad G_{uA} = 0. \tag{8.2}$$

These govern the time evolution of the Bondi mass and angular momentum aspects m_B, N_A respectively. One more "constraint" exists at the same order in the large$-r$

expansion, namely $G_{AB} = 0$. A brute force computation show this constraint reduces to an evolution equation []

$$\partial_u E_{AB} = \frac{1}{2} m_B C_{AB} + \frac{1}{3} D_{(A} N_{B)} - \frac{1}{6}\gamma_{AB} D_C N^C + \frac{1}{4} C_{AB} N_{CD} C^{CD} - \frac{1}{8}\epsilon_A{}^C C_{CB} \epsilon_{DE} D^E D_C C^{CD}, \tag{8.3}$$

where E_{AB} appears at subleading order in an expansion of the transverse metric as

$$g_{AB} = r^2 \gamma_{AB} + r C_{AB} + D_{AB} + \frac{1}{r} E_{AB} + \mathcal{O}(r^{-2}) \tag{8.4}$$

and ϵ_{AB} is the Levi-Civita tensor on the sphere. By analogy with (8.1) one can define a new charge

$$Q_t = \int_{S^2} d^2 z t_{AB}(z, \bar{z}) E^{AB} \tag{8.5}$$

parameterized by a tensor field t_{AB} on the sphere.

As before we use the evolution equation to extend (**??**) as an integral over $\mathcal{I}$ and (hopefully) identify the linear contribution with the sub-subleading soft graviton and the action of the quadratic contribution on asymptotic states (or equivalently the phase space) with the sub-subleading soft factor. Note that since (8.3) depends explicitly on N_A, the direct calculation appears very tedious.

Potential issues:

- Boundary conditions needed to integrate (8.3) not clear;
- Naively, Q_t takes us out of the phase space;
- There are qubic contributions to Q_t which will lead to corrections to the soft theorem!

Proceed by noting that there exsists a field redefinition $m_B \to \mathcal{Q}_0,\ N_z \to \mathcal{Q}_1,\ E_{zz} \to \mathcal{Q}_2$ that turns the mess (8.3) into a simple recursive equation

$$\partial_u Q_2 = D_z Q_1 + \frac{3}{2} C_{zz} Q_0. \tag{8.6}$$

We should understand Q_2 and Q_1 as carrying two and one helicity indices respectively so that the equation makes sense. A more covariant way of writing this is by introducing a spacetime tetrad with $m^A, \bar{m}^A$ tangent vectors to the sphere at infinity. In these notes, we will be focussing on the round sphere with metric

$$ds^2 = \frac{2dzd\bar{z}}{P^2}, \quad P = \frac{1 + z\bar{z}}{\sqrt{2}} \tag{8.7}$$

and in which case for an arbitrary tensor S_{AB}

$$S_{AB} m^A m^B \leftrightarrow P^{-2} S_{zz}, \quad S_{AB} \bar{m}^A \bar{m}^B \leftrightarrow P^{-2} S_{\bar{z}\bar{z}}. \tag{8.8}$$

Explicitly, we have

$$\begin{aligned} Q_0 &= \\ Q_1 &= \\ Q_2 &= \end{aligned} \tag{8.9}$$

where

$$\begin{aligned} \hat{N} &= N_{AB}\bar{m}^A\bar{m}^B, \\ C &= C_{AB}m^Am^B, \\ DO_S &\equiv m^Am^{A_1}\cdots m^{A_s}D_AO_{A_1\cdots A_s}. \end{aligned} \tag{8.10}$$

Note that we define $\hat{N}$ and C such that they are canonically conjugate variables

$$\{\partial_u\hat{N}(u,z), C(u',z')\} = \frac{\kappa^2}{2}\partial_u\delta(u-u')\delta(z,z'), \tag{8.11}$$

with

$$\delta(z,z') = P^{-2}\delta(z-z'). \tag{8.12}$$

The transformation properties of these objects under $\mathrm{SL}(2,\mathbb{C})$ are inherited from the transformations of the frame fields

$$m^A \to \mathit{write} \tag{8.13}$$

and the field redefinitions are determined by symmetry considerations. In particular, Q_s are found by requiring that on the $u=0$ cut of $\mathcal{I}^+$ they transform like "primaries" under the action of the homogeneous part of the extended BMS (ie under superrotations)

$$\delta_{(Y,W)}O_{(\Delta,s)} = (\mathcal{L}_Y + (\Delta-s)W)O_{(\Delta,s)}. \tag{8.14}$$

For the extended BMS, $W = \frac{1}{2}D^AY_A$.

Exercise 8.1. *Show that Q_0, Q_1 are primaries in the sense that they obey* (8.14)*.*

After a lot of work (see []), one finds that in terms of $Q_s,\ s=0,1,2$, the G_{uu}, G_{uz}, G_{AB} constraints reduce to

$$\dot{\mathcal{Q}}_s = D\mathcal{Q}_{s-1} + \frac{(1+s)}{2}C\mathcal{Q}_{s-2}, \qquad \mathcal{Q}_{-1} = \frac{1}{2}D\hat{N}, \quad \mathcal{Q}_{-2} = \frac{1}{2}\dot{\hat{N}}. \tag{8.15}$$

These equations can be integrated provided that

$$\hat{N} = \mathcal{O}(|u|^{-1-s-\epsilon}), \quad u\to\pm\infty, \quad \epsilon>0 \tag{8.16}$$

and

$$\lim_{u\to\infty}\mathcal{Q}_s = 0. \tag{8.17}$$

The solutions can be shown to take the form

$$\mathcal{Q}_s = \sum_{k=1}^{s+1} Q_s^{(k)}, \tag{8.18}$$

where

$$
\begin{aligned}
Q_s^{(1)} &= D\left(\partial_u^{-1}D\right)^{s+1} Q_{-2}\\
Q_s^{(2)} &= \sum_{n=0}^{s}\frac{1+n}{2}(\partial_u^{-1}D)^{s-n}C(\partial_u^{-1}D)^n Q_{-2}.
\end{aligned}
\tag{8.19}
$$

These will be sufficient in establishing the connection to soft theorems. The higher non-linear contributions can be worked out, and lead to contact term contributions to the soft theorems that need to be better understood (cf. dressings?).

Following **??** and using the canonical commutation relations (8.11) we can now compute the action of the $s=2$ charge on the shear (the $s=0,1$ charges generate supertranslations and superrotations respectively and can be shown give rise to the leading and subleading soft graviton theorems discussed at the beginning)

$$
\begin{aligned}
\frac{8}{\kappa^2}\{\mathcal{Q}_2(u,z,\bar{z}),C(u',z',\bar{z}')\} = &-\frac{3}{2}C^2\delta(z,z')\\
&+\frac{u'^2}{2}\{D^2\mathcal{M}_{\mathbb{C}},C\}+2u'D_z\left[CD_z\delta(z,z')+\frac{1}{2}D_z(C\delta(z,z'))\right]\\
&-\frac{3}{2}\int_u^{u'}du''C(u'')\{\mathcal{M}_{\mathbb{C}},C\}+u\{D_z\mathcal{P},C\}-\frac{u^2}{2}\{D^2\mathcal{M}_{\mathbb{C}},C\}.
\end{aligned}
\tag{8.20}
$$

Note!! For $s\geq 1$ and $u\to-\infty$ this action is not well defined! Need to find a prescriptions to subtract the $u\to-\infty$ divergences. From the expression above, we see that in this limit the divergences arise manifestly from the terms in the last line. We can therefore define a new operator

$$
q_2(z,\bar{z}) = \frac{8}{\kappa^2}\frac{1}{3}\lim_{u\to-\infty}\left(\mathcal{Q}_2-\frac{3}{2}\int_\infty^u du''C(u'')\mathcal{Q}_0(u)-uD_z\mathcal{Q}_1+\frac{u^2}{2}D^2\mathcal{Q}_0\right). \tag{8.21}
$$

Then

$$
\begin{aligned}
\{q_2(z,\bar{z}),C(u,z')\} = &-\frac{u^2}{3}\partial_z^4\delta+\left[\frac{u^2}{6}\partial_u^2+u\partial_u+1\right]\int^u C\partial_z^2\delta\\
&+\left(-\frac{2}{3}u\partial_u-2\right)\int^u\partial_{z'}C\partial_z\delta+\delta\partial_{z'}^2\int^u C-\frac{1}{2}\partial_u\left(C\int^u C\right).
\end{aligned}
\tag{8.22}
$$

Remarkably, upon defining

$$
2h_k \equiv u\partial_u+3, \tag{8.23}
$$

and dropping quadratic terms on the RHS (which amounts to truncating the $s=2$ charge to quadratic order in the fields) we find

$$
\begin{aligned}
&\lim_{u\to-\infty}[q_2(u),C_{z'z'}(u')] = -\frac{u'^2}{12}D_z^4\delta^{(2)}(z-z')\\
&+\frac{1}{24}\left[D_z^2\delta^{(2)}(z-z')2h_k(2h_k-1)-4D_z\delta^{(2)}(z-z')2h_k+6\delta^{(2)}(z-z')\right]\\
&\times\int_{-\infty}^{\infty}du''C_{z'z'}(u'',z',\bar{z}')\Theta(u'-u'').
\end{aligned}
\tag{8.24}
$$

By the convolution theorem, with $\mathcal{F}$ denoting the Fourier transform, we find that

$$\mathcal{F}[\int_{-\infty}^{\infty} du'' C_{z'z'}(u'', z', \bar{z}')\Theta(u' - u'')] = \mathcal{F}[C_{z'z'}]\mathcal{F}[\Theta] \propto \frac{1}{\omega}\widetilde{C}(\omega). \tag{8.25}$$

It remains to find the Fourier transform of $u\partial_u + 3$ namely

$$\int du e^{i\omega u}(u\partial_u + 3)f(u) = (-i\partial_\omega(-i\omega) + 3)\widetilde{f}(\omega) = (2 - \omega\partial_\omega)\widetilde{f}(\omega) = 2h_k\widetilde{f}(\omega), \tag{8.26}$$

which is indeed $2h_k$ for a field of helicity $s_k = 2$ like C. One can show that (8.24) precisely reproduces the sub-subleading soft theorem.

8.1 Renormalized charges for all spins

8.1.1 Linear

Generalizing (8.21) for all $s \in \mathbb{Z}_+$, we define the renormalized charges

$$q_s(z) = \lim_{u\to-\infty} \sum_{l=0}^{s} \frac{(-u)^{s-l}}{(s-l)!} D_z^{s-l} \mathcal{Q}_l(s). \tag{8.27}$$

The soft component of the renormalized charge takes the form (see appendix **??**)

$$q_s^1(z) = \frac{(-1)^{s+1}}{2} \int_{-\infty}^{\infty} \left(\frac{u^s}{s!} D_z^{s+2} \hat{N} \right). \tag{8.28}$$

It will be convenient to write this as

$$\hat{Q}_s^1(z) = \frac{(-1)^{s+1}}{2} D_z^{s+2} N^{(s)}, \tag{8.29}$$

where

$$N^{(s)} = \frac{1}{s!} \int du u^s N = -\frac{\kappa}{8\pi s!}(-i\partial_\omega)^s \left(\omega a_-^{\text{out}} + (-1)^n \omega a_+^{\text{out}\dagger} \right). \tag{8.30}$$

8.1.2 Quadratic

The renormalized quadratic charge takes the form

$$\begin{aligned} q_s^2(u, z) &= \sum_{n=0}^{s} \frac{(-u)^{s-n}}{(s-n)!} Q_n^2 \\ &= \frac{1}{4} \sum_{n=0}^{s} \sum_{l=0}^{n} \frac{(-u)^{s-n}}{(s-n)!} (l+1) D_z^{s-l} (\partial_u^{-1})^{n-l+1} \left[C(\partial_u^{-1} D)^l \mathcal{N} \right], \end{aligned} \tag{8.31}$$

where is the last line we used (8.19). The action of (10.31) on C is computed in appendix **??** and we find

$$\{\hat{\mathcal{Q}}_s^2(z), C(u', z')\} = \frac{\kappa^2}{8} \sum_{n=0}^{s} (-1)^{s+n} \frac{(n+1)(\Delta+2)_{s-n}}{\Gamma(s-n+1)} (\partial_{u'}^{-1})^{s-1} D_{z'}^n C(u', z') D_z^{s-n} \delta(z, z'). \tag{8.32}$$

8.2 From conservation law to soft theorem for all spins

8.2.1 Soft insertion

Using the results of the previous section we find

$$\begin{aligned}\langle \text{out}|[\hat{Q}^1_s, S]|\text{in}\rangle &= \langle \text{out}|\hat{Q}^1_s S - S\hat{Q}^1_s|\text{in}\rangle \\ &= \frac{\kappa}{8\pi}\frac{1}{2}D^{2+s}\lim_{\omega\to 0}\left[\frac{i^s}{s!}\partial^s_\omega\langle \text{out}|(\omega a^{\text{out}}_-(\omega\hat{x}))S|\text{in}\rangle\right]\times 2 \\ &= \frac{\kappa}{8\pi}\frac{i^s}{s!}\lim_{\omega\to 0}\left[\partial^s_\omega\langle \text{out}|(\omega a^{\text{out}}_-(\omega\hat{x}))S|\text{in}\rangle\right].\end{aligned} \tag{8.33}$$

8.2.2 Hard charge actions on modes

At the quantum level, $[.,.] = -i\{.,.\}$ so

$$[\partial_u N(u,z), C(u',z')] = -i\frac{\kappa^2}{2}\partial_u\delta(u-u')\delta(z,z'). \tag{8.34}$$

This bracket is consistent with the standard mode commutator

$$[a^{\text{out}}_-(\omega\hat{x}), a^{\text{out}\dagger}_-(\omega'\hat{x}')] = 2(2\pi)^3\omega^{-1}\delta(\omega-\omega')\delta(z,z'). \tag{8.35}$$

To see this, note that

$$\begin{aligned}\int du e^{i\omega u}\int du' e^{-i\omega' u'}[\bar{N}(u,z), C(u',z')] &= -\frac{\kappa}{4\pi}\omega\times\frac{i\kappa}{4\pi}[a^{\text{out}}_-(\omega\hat{x}), a^{\text{out}\dagger}_-(\omega'\hat{x}')] \\ &= -i\frac{16\pi G}{2(2\pi)^2}\omega[a^{\text{out}}_-(\omega\hat{x}), a^{\text{out}\dagger}_-(\omega'\hat{x}')]\end{aligned} \tag{8.36}$$

since $\kappa^2 = 32\pi G$, while from (8.34), the LHS is

$$\int du e^{i\omega u}\int du' e^{-i\omega' u'}[\bar{N}(u,z), C(u',z')] = -i16\pi G\delta(\omega-\omega')2\pi\delta(z,z'). \tag{8.37}$$

We can now compute the bracket of q_2 with the modes starting with their canonical action on C

$$\{\hat{\mathcal{Q}}^2_s(z), C(u',z')\} = \frac{\kappa^2}{8}\sum_{l=0}^{s}(-1)^{s+l}\frac{(1+l)(\Delta+2)_{s-l}}{\Gamma(1-l+s)}(\partial_{u'}^{-1})^{s-1}D^l_{z'}C(u',z')D^{s-l}_z\delta(z,z'), \tag{8.38}$$

where

$$\Delta + 2 \equiv u\partial_u + 3. \tag{8.39}$$

This directly implies upon Fourier transform the commutator

$$[q^2_s(z), a^{\text{out}}_+] = -i\frac{\kappa^2}{8}\sum_{l=0}^{s}(-1)^{s+l}\frac{(1+l)(\Delta+2)_{s-l}}{\Gamma(1-l+s)}(-i\omega)^{-s+1}D^l_{z'}a^{\text{out}}_+(\omega\hat{x}')D^{s-l}_z\delta(z,z'). \tag{8.40}$$

or, upon insertion into the S-matrix,

$$\langle \text{out}|[q_s^2, \mathcal{S}]|\text{in}\rangle = i^s \frac{\kappa^2}{8} \sum_{k=1}^{n} \sum_{\ell=0}^{s} (-1)^{s+\ell} \frac{(1+\ell)(2h_k)_{s-\ell}}{(s-\ell)!} (\epsilon_k \omega_k)^{-s+1} D_z^{s-\ell} \delta(z, z_k) D_{z_k}^{\ell} \langle \text{out}|\mathcal{S}|\text{in}\rangle \,. \tag{8.41}$$

The conservation law

$$\langle \text{out}|[q_s, \mathcal{S}]|\text{in}\rangle \tag{8.42}$$

truncated to quadratic order in the charges then implies

$$\frac{1}{\pi} \lim_{\omega\to 0} (\partial_\omega)^s D^{2+s} \omega \langle \text{out}|a_-^{\text{out}}(\omega\hat{x}) S|\text{in}\rangle + \kappa \sum_{k=1}^{n} \sum_{l=0}^{s} (-1)^{s+l} (1+l)(2h_k)_{s-l}(s)_l (\epsilon_k \omega_k)^{-s+1} \\ \times D_z^{s-l} \delta(z, z_k) D_{z_k}^{l} \langle \text{out}|S|\text{in}\rangle = 0. \tag{8.43}$$

Comments:

- Can show that the quadratic charge bracket agrees with that computed from the celestial OPE (including L *or* R descendants). In this sense, conformal symmetry in the CCFT implies a dynamical evolution equation for higher spin charges at $\mathcal{I}^{\pm}$.
- In the previous lessons we saw that (upon rescaling and redefinition) the conformally soft currents obey a $w_{1+\infty}$ algebra. One can show that the charges q_s also satisfy $w_{1+\infty}$. To linear order, no restriction to the wedge is necessary.
- The charges can be extracted from the retarded time evolution of the $\Psi_0 \propto C_{rArB}$ Weyl tensor components to all order in a large-r expansion (this is implied by Einstein's equations). The Ψ_0 components can be identified in the $s \geq 3$ celestial diamonds as the shadow transforms of the conformally soft gravitons, while the charges are their light transforms.

9 Open problems

- "Good" basis for CCFT and bulk interpretation: light/shadow transforms
- Massive particles
- Loops; quantum symmeries ?
- Relation to AdS/CFT; string worldsheet; intrinsic, non-perturbative definition?
- Entropy: one dimension lower than suggested by BH area
- IR divergences in QCD, observables?
- Bootstrap; unitarity, causality;

10 Discussion Sessions

10.1 Properties of coherent states

Define the following state

$$|\alpha_i\rangle = e^{-\frac{1}{2}|\alpha_i|^2} e^{\alpha_i a_i^\dagger}|0\rangle = e^{-\frac{1}{2}|\alpha_i|^2} \sum_n \frac{(\alpha_i a_i^\dagger)^n}{n!}|0\rangle, \qquad a_i^\dagger = \int d^3k f_i(k) a^\dagger(k), \tag{10.1}$$

where $\{f_i(k)\}$ is a complete, orthonormal set of functions defined over some region of momentum space (or all of momentum space)

$$\int d^3k f_i(k) f_j^*(k) = \delta_{ij}. \tag{10.2}$$

Using the canonical commutation relations for $a(k), a^\dagger(k)$, together with (10.2), one finds that a_i, $a_i^\dagger$ obey the commutation relations

$$[a_i, a_j^\dagger] = \int d^3k \int d^3k' f_i^*(k) f_j(k')[a(k), a^\dagger(k')] = \int d^3k f_i^*(k) f_j(k) = \delta_{ij} \tag{10.3}$$

and

$$[a(k), a_i^\dagger] = [a(k), \int d^3k' f_i(k') a^\dagger(k')] = f_i(k). \tag{10.4}$$

Note that $|\alpha_i\rangle$ is an eigenstate of $a(k)$ with eigenvalue $\alpha_i f(k)$,

$$a(k)|\alpha_i\rangle = e^{-\frac{1}{2}|\alpha_i|^2}[a(k), e^{\alpha_i a_i^\dagger}]|0\rangle = e^{-\frac{1}{2}|\alpha_i|^2} \sum_n \frac{\alpha_i^n}{n!} n f(k) a_i^{n-1}|0\rangle = \alpha_i f(k)|\alpha_i\rangle. \tag{10.5}$$

From (10.5) and (10.2), the expectation value of the number operator in a coherent state is

$$\langle\alpha_i|\hat{N}|\alpha_i\rangle = \int d^3k \langle\alpha_i|a^\dagger(k)a(k)|\alpha_i\rangle = |\alpha_i|^2. \tag{10.6}$$

Alternatively, we can build the coherent state by acting with a unitary operator on the vacuum as follows

$$|\alpha_i\rangle = U(\alpha_i)|0\rangle = e^{\alpha_i a_i^\dagger - \alpha_i^* a_i}|0\rangle. \tag{10.7}$$

We check this by using the BCH formula

$$e^{\hat{A}+\hat{B}} = e^{-\frac{1}{2}[\hat{A},\hat{B}]} e^{\hat{A}} e^{\hat{B}}, \qquad \text{if} \qquad [\hat{A}, \hat{B}] \sim c, \tag{10.8}$$

where c is a number. Then

$$e^{\alpha_i a_i^\dagger - \alpha_i a_i} = e^{-\frac{1}{2}|\alpha_i|^2} e^{\alpha_i a_i^\dagger} e^{-\alpha_i^* a_i} \tag{10.9}$$

and $e^{-\alpha_i^* a_i}|0\rangle = |0\rangle$ which confirms (10.7). Note that coherent states are not orthogonal

$$\langle\alpha_i|\beta_i\rangle = e^{-|\alpha_i - \beta_i|^2} \tag{10.10}$$

but nevertheless obey

$$\frac{1}{\pi}\int d^2\alpha_i |\alpha_i\rangle\langle\alpha_i| = \sum_{n_i} |n_i\rangle\langle n_i| = I, \tag{10.11}$$

implying that they form an overcomplete set of states.

10.2 Coherent states and memory observables

We consider the FK state associated with a single massless particle []

$$|p\rangle_{FK} = \exp\left\{\frac{ieQ_)}{2\pi}\int d^2wG(z_0,w)D\cdot A_w(v_0,w)\right\}. \tag{10.12}$$

The action of the soft charge

$$Q = \int duF_{uz} \tag{10.13}$$

on this state is

$$Q|p\rangle_{FK} = \frac{eQ_0}{z-z_0}|p\rangle_{FK}. \tag{10.14}$$

Then this state is such that

$$\langle p|Q|p\rangle_{FK} = \frac{eQ_0}{z-z_0} \tag{10.15}$$

so this state carries memory.

On the other hand,

$$\langle p|F_{uz}F_{u\bar{z}}|p\rangle_{FK} \propto \delta(u)\delta(u)\frac{e^2Q_0^2}{(z-z_0)(\bar{z}-\bar{z}_0)} \tag{10.16}$$

and so

$$\int du\langle p|F_{uz}F_{u\bar{z}}|p\rangle_{FK} \propto \delta(0)\frac{e^2Q_0^2}{(z-z_0)(\bar{z}-\bar{z}_0)}. \tag{10.17}$$

This seems to imply that such FK states have infinite energy even before integrating over the sphere. Put it differently, upon regulating the UV and IR divergences,

$$\delta(0) \to \ln\frac{\Lambda}{\lambda}. \tag{10.18}$$

This calculation assumes that F_{uz} annihilates the vacuum, which is not true. We instead repeat it for the modes. We have

$$|\vec{p}\rangle_{F.K.} = \mathcal{N}e^{-Q\int d\omega d^2z\frac{p^\mu}{\hat{q}\cdot p}a^\dagger_\mu(\omega,z,\bar{z})}|p\rangle \tag{10.19}$$

where $\mathcal{N}$ is such that $\langle\vec{p}|\vec{p}\rangle_{F.K.} = 1$. Then using the near $\mathcal{I}^+$ mode expansion we have that

$$F_{uz} = N_F\int_0^\infty d\omega\omega\left[a_+e^{-i\omega u} + a_-^\dagger e^{i\omega u}\right]. \tag{10.20}$$

This implies that

$$\begin{aligned} E \equiv \int duF_{uz}F_{u\bar{z}} &= |N_F|^2\int du\int_0^\infty d\omega d\omega'\omega\omega'\left[a_+e^{-i\omega u} + a_-^\dagger e^{i\omega u}\right]\left[a_+^\dagger e^{i\omega' u} + a_-e^{-i\omega' u}\right] \\ &= |N_F|^2\int_0^\infty d\omega\omega^2\left[a_+^\dagger a_+ + a_-^\dagger a_-\right] + E_0. \end{aligned} \tag{10.21}$$

Therefore

$$\langle\vec{p}|E|\vec{p}\rangle_{F.K.} = |N_F|^2\int_0^\infty d\omega\left[\frac{|p\cdot\varepsilon^+|^2}{|p\cdot\hat{q}|^2} + \frac{|p\cdot\varepsilon^-|^2}{|p\cdot\hat{q}|^2}\right] \to \infty \tag{10.22}$$

since the integrand is independent on ω.

10.3 BMS(W) primaries

Show that m_B doesn't obey (8.14). Using the Bondi expansion of an asymptotically flat metric (**??**), show that

$$\mathcal{M} \equiv m_B + \frac{1}{8} C_{AB} N^{AB} \tag{10.23}$$

does. What are Δ, s? Argue this is the unique combination that obeys (**??**) for these values of Δ, s.

10.4 Linear and quadratic charges

Derive (8.19) from (8.15).

10.4.1 Linear

The soft component of the renormalized charge takes the form

$$\begin{aligned}
\hat{Q}^1_s(z) &= \lim_{u\to-\infty} \sum_{l=0}^{s} \frac{(-u)^{s-l}}{(s-l)!} D_z^{s-l} Q^1_l(s) \\
&= \frac{1}{2} \lim_{u\to-\infty} \sum_{l=0}^{s} \frac{(-u)^{s-l}}{(s-l)!} D_z^{s-l} (\partial_u^{-1} D)^{l+2} \partial_u \hat{N} \\
&= \frac{1}{2} \sum_{l=0}^{s} \frac{(-u)^{s-l}}{(s-l)!} (\partial_u^{-1})^{l+1} D_z^{s+2} \hat{N} = \frac{(-1)^s}{2} \partial_u^{-1} \left(\frac{u^s}{s!} D_z^{s+2} \hat{N} \right) = \frac{(-1)^{s+1}}{2} \int_{-\infty}^{\infty} \left(\frac{u^s}{s!} D_z^{s+2} \hat{N} \right).
\end{aligned} \tag{10.24}$$

From the mode expansions near $\mathcal{I}^+$ we have

$$\begin{aligned}
-\frac{\kappa}{4\pi}(-i\partial_\omega)^n (\omega a_+^{\text{out}}(\omega\hat{x})) &= (-i\partial_\omega)^n \int_{-\infty}^{\infty} du e^{i\omega u} \bar{N}(u,\hat{x}) = \int_{-\infty}^{\infty} du u^n \bar{N}(u,\hat{x}) \\
&= -\partial_u^{-1}(u^n \bar{N}(u,\hat{x})).
\end{aligned} \tag{10.25}$$

$$\begin{aligned}
-\frac{\kappa}{4\pi}(-i\partial_\omega)^n (\omega a_-^{\text{out}}(\omega\hat{x})) &= (-i\partial_\omega)^n \int_{-\infty}^{\infty} du e^{i\omega u} N(u,\hat{x}) = \int_{-\infty}^{\infty} du u^n N(u,\hat{x}) \\
&= -\partial_u^{-1}(u^n N(u,\hat{x})),
\end{aligned} \tag{10.26}$$

where $N = \partial_u \bar{C}$. Recall that C and N are canonically conjugate variables and hence N corresponds to a negative helicity graviton. Then taking half of a linear combination of the above and the hermitian conjugate of (10.25), we find

$$-\frac{\kappa}{8\pi}(-i\partial_\omega)^n \left(\omega a_-^{\text{out}} + (-1)^n \omega a_+^{\text{out}\dagger} \right) = \frac{1}{2}(-i\partial_\omega)^n \left(N^\omega + (-1)^n N^{-\omega} \right) = \int_{-\infty}^{\infty} du u^n N. \tag{10.27}$$

We defined

$$N^\omega = \int e^{i\omega u} N. \tag{10.28}$$

Finally, comparing with (10.24), we find

$$\hat{Q}^1_s(z) = \frac{(-1)^{s+1}}{2} D_z^{s+2} N^{(s)}, \tag{10.29}$$

where

$$N^{(s)} = \frac{1}{s!}\int du u^s N. \tag{10.30}$$

10.5 Quadratic

The renormalized quadratic charge takes the form

$$\begin{aligned}\hat{Q}^2_s(u,z) &= \sum_{n=0}^{s} \frac{(-u)^{s-n}}{(s-n)!} Q^2_n \\ &= \frac{1}{4}\sum_{n=0}^{s}\sum_{l=0}^{n} \frac{(-u)^{s-n}}{(s-n)!}(l+1) D_z^{s-l} (\partial_u^{-1})^{n-l+1}\left[C(\partial_u^{-1}D)^l \mathcal{N}\right],\end{aligned} \tag{10.31}$$

where is the last line we used (**??**). We can now compute the action of (10.31) on C.

$$\begin{aligned}\{\hat{Q}^2_s, C(u',z')\} &= \frac{\kappa^2}{8}\sum_{n=0}^{s}\sum_{l=0}^{n}\frac{(-u)^{s-n}}{(s-n)!}(l+1) D_z^{s-l}(\partial_u^{-1})^{n-l+1}\left[C(u,z)(\partial_u^{-1}D)^l \partial_u \delta(u-u')\delta(z,z')\right] \\ &= \frac{\kappa^2}{8}\sum_{n=0}^{s}\sum_{l=0}^{n}\frac{(-u)^{s-n}}{(s-n)!}(l+1) D_z^{s-l}(\partial_u^{-1})^{n-l+1}(-\partial_{u'}^{-1})^{l-1}\left[C(u',z)\delta(u-u')D^l\delta(z,z')\right] \\ &= -\frac{\kappa^2}{8}\sum_{n=0}^{s}\sum_{l=0}^{n}(l+1) D_z^{s-l}(-\partial_{u'}^{-1})^{l-1}\left[C(u',z)\frac{(u-u')^{n-l}}{(n-l)!}\frac{(-u)^{s-n}}{(s-n)!}D^l\delta(z,z')\right] \\ &= \frac{\kappa^2}{8}\sum_{l=0}^{s}(-1)^l(l+1) D_z^{s-l}(\partial_{u'}^{-1})^{l-1}\left[C(u',z)\frac{(-u')^{s-l}}{(s-l)!}D^l\delta(z,z')\right],\end{aligned} \tag{10.32}$$

where we have used

$$\begin{aligned}\left(\partial_u^{-1}\right)^a \delta(u-u') &= (-1)^a(\partial_{u'})^a\delta(u-u'), \\ \left(\partial_u^{-1}\right)^b \delta(u-u') &= \frac{(u-u')^b}{b!}\end{aligned} \tag{10.33}$$

as well as

$$\begin{aligned}\sum_{n=l}^{s}\frac{(-u)^{s-n}(u-u')^{n-l}}{(s-n)!(n-l)!} &= \sum_{n=0}^{s-l}\frac{1}{(z-n-l)!n!}(-u)^{s-n-l}(u-u')^n \\ &= \frac{1}{(s-l)!}(-u')^{s-l}.\end{aligned} \tag{10.34}$$

We also swapped the sums,

$$\sum_{n=0}^{s}\sum_{l=0}^{n} = \sum_{l=0}^{s}\sum_{n=l}^{s}. \tag{10.35}$$

The regularization is now done, we see that the expression is independent on u and we can therefore safely take $u \to -\infty$.

Now using the identity

$$\begin{aligned} C(z)D^l\delta^{(2)}(z-z') &= (-D')^l \left(C(z)\delta^{(2)}(z-z')\right) \\ &= \sum_{n=0}^{l}(-1)^l \binom{l}{n} D^n_{z'}C(z')D^{l-n}_{z'}\delta^{(2)}(z-z') \\ &= \sum_{n=0}^{l}(-1)^n \binom{l}{n} D^n_{z'}C(z')D^{l-n}_{z}\delta^{(2)}(z-z'). \end{aligned} \tag{10.36}$$

we have

$$\begin{aligned} \{Q_s^{\text{quadr.}}, C(u',z')\} = \frac{\kappa^2}{8}\sum_{l=0}^{s}\sum_{n=0}^{l}(-1)^{l+n}\binom{l}{n}(l+1)(\partial_{u'}^{-1})^{l-1}D^n_{z'}\left(C(u',z')\frac{(-u')^{s-l}}{(s-l)!}\right) \\ \times D_z^{s-n}\delta^{(2)}(z-z'), \end{aligned} \tag{10.37}$$

Finally

$$\partial_{u'}^{\alpha}\left(\frac{u'^k}{k!}C(u')\right) = \sum_{n=0}^{k}\frac{(\alpha)_n}{n!}\frac{u'^{(k-n)}}{(k-n)!}\partial_{u'}^{\alpha-n}C(u') = \frac{1}{k!}(\Delta+\alpha-1)_k(\partial_{u'})^{\alpha-k}C(u'). \tag{10.38}$$

The last equality can be proven by noting that

$$u^{k-n}\partial_u^{k-n} = u^{k-n-1}u\partial_u\partial_u^{k-n-1} = (u\partial_u - k+n+1)u^{k-n-1}\partial_u^{k-n-1} = \ldots = (u\partial_u)_{k-n} \tag{10.39}$$

and

$$\sum_{n=0}^{k}\frac{(\alpha)_n(u\partial_u)_{k-n}}{n!(k-n)!} = \frac{(u\partial_u+\alpha-1)_k}{k!}, \tag{10.40}$$

which was checked in the file “Bulk sum” in mathematica. In proving (10.39), we have used that

$$[u, u\partial_u] = -u. \tag{10.41}$$

We can thus write

$$\{\hat{\mathcal{Q}}_s^2(z), C(u',z')\} = \frac{\kappa^2}{8}\sum_{\ell=0}^{s}\sum_{n=0}^{\ell}(-)^{s+n}\frac{(\ell+1)!}{n!(\ell-n)!}\frac{(\Delta-\ell)_{s-\ell}}{(s-\ell)!}(\partial_{u'}^{-1})^{s-1}D^n_{z'}C(u',z')D_z^{s-n}\delta(z,z'). \tag{10.42}$$

We can swap the sums and use that

$$\sum_{\ell=n}^{s}\frac{(\ell+1)!}{(\ell-n)!}\frac{(\Delta-\ell)_{s-\ell}}{(s-\ell)!} = \frac{(\Delta+2)_{s-n}(n+1)}{\Gamma(s-n+1)} = \frac{(\Delta+2)_{s-n}(n+1)!}{\Gamma(s-n+1)}. \tag{10.43}$$

We arrive at the final result

$$\{\hat{\mathcal{Q}}_s^2(z), C(u', z')\} = \frac{\kappa^2}{8}\sum_{n=0}^{s}(-1)^{s+n}\frac{(n+1)(\Delta+2)_{s-n}}{\Gamma(s-n+1)}(\partial_{u'}^{-1})^{s-1}D_{z'}^{n}C(u', z')D_z^{s-n}\delta(z, z'). \tag{10.44}$$

Alternatively, after (10.37) one can go to the conformal primary basis (although one can also use the pseudodifferential calculus - it's equivalent) and replace

$$(\partial_{u'}^{-1})^{s-k-1}\left(C(u')\frac{u'^k}{k!}\right) \to \frac{1}{k!}(\Delta - s + k)_k e^{-(s-1)\partial_\Delta}\widetilde{C}(\Delta). \tag{10.45}$$

Plugging back into (10.37), swapping the sums and evaluating the sum over k

$$\begin{aligned}\sum_{k=0}^{s-l}\frac{(s-k+1)(s-k)!(\Delta-s+k)_k}{(s-k-l)!k!} &= -\frac{(2+\Delta)(1+l)\pi\csc(\pi s)\,{}_2F_1^R(1+\Delta-s, l-s; -s; 1)}{(2+\Delta+l-s)\Gamma(1-l+s))}\\ &= l!\frac{1+l}{\Gamma(1-l+s)}(\Delta+2)_{s-l}\end{aligned} \tag{10.46}$$

where the last line holds for integer $l \leq s$. We conclude

$$[Q_s^{\text{quadr.}}, C(u', z')] = \frac{1}{2}\sum_{l=0}^{s}(-1)^{s+l-1}\frac{(1+l)(\Delta+2)_{s-l}}{\Gamma(1-l+s)}e^{-(s-1)\partial_\Delta}D_{z'}^{l}\widetilde{C}(\Delta, z')D_z^{s-l}\delta^{(2)}(z-z'), \tag{10.47}$$

which agrees with (10.39).

10.6 Light-transform in CCFT

The light transforms of an operator in conformal field theory is a transformation that takes primaries O of weights $(h, \bar{h})$ to primaries of weights $(1-h, \bar{h})$ or $(h, 1-\bar{h})$, namely

$$\begin{aligned}L^+[O](w, \bar{w}) &= \int dz\frac{1}{(w-z)^{2-2h}}O(z, \bar{w}),\\ L^-[O](w, \bar{w}) &= \int d\bar{z}\frac{1}{(\bar{w}-\bar{z})^{2-2\bar{h}}}O(w, \bar{z}).\end{aligned} \tag{10.48}$$

Exercise: Show that for half-integral negative weights (or $2-2h>0$)

$$L^+[O](w, \bar{w}) = \frac{2\pi i}{(1-2h)!}\lim_{z\to w}\frac{\partial^{1-2h}}{\partial z^{1-2h}}O(z, \bar{w}). \tag{10.49}$$

Solution: It follows immediately from the definition upon integration by parts and using the residue theorem

$$\begin{aligned}L^+[O](w, \bar{w}) &= \int dz\frac{1}{(w-z)^{2-2h}}O(z, \bar{w}) = \frac{1}{(1-2h)!}\int dz\frac{1}{(w-z)}\partial_z^{1-2h}O(z, \bar{w})\\ &= \frac{1}{(1-2h)!}\lim_{z\to w}\frac{\partial^{1-2h}}{\partial z^{1-2h}}O(z, \bar{w}).\end{aligned} \tag{10.50}$$

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A Charge commutators

In this section we compute the commutators of the charges (5.33) with C_{ww}. Upon integration by parts the hard charges can be rewritten as

$$Q_H^+ = \frac{1}{4}\int_{\mathcal{I}^+} dud^2z\gamma_{z\bar{z}}\Big(uD_zY^{+z}N_{zz}N^{zz} + uD_{\bar{z}}Y^{+\bar{z}}N_{\bar{z}\bar{z}}N^{\bar{z}\bar{z}} + D_zY^{+z}C_{zz}N^{zz} + D_{\bar{z}}Y^{+\bar{z}}C_{\bar{z}\bar{z}}N^{\bar{z}\bar{z}} + 2D_z(Y^{+z}C_{zz})N^{zz} + 2Y^{+\bar{z}}N_{\bar{z}\bar{z}}D_{\bar{z}}C^{\bar{z}\bar{z}}\Big) + \text{matter}. \tag{A.1}$$

Then using the canonical commutation relations

$$[N_{\bar{z}\bar{z}}(u,z,\bar{z}), C_{ww}(u',w,\bar{w})] = 2i\gamma_{z\bar{z}}\delta^{(2)}(z-w)\delta(u-u'), \tag{A.2}$$

one derives

$$\begin{aligned}[Q_H^+, C_{ww}(u',w,\bar{w})] &= \frac{2i}{4}\Big(uD\cdot Y^+N_{ww} + (D_wY^{+w} - D_{\bar{w}}Y^{+\bar{w}})C_{ww} + 2D_w(Y^{+w}C_{ww}) \\ &\quad + 2Y^{+\bar{w}}D_{\bar{w}}C_{ww}\Big) = i\Big(\frac{u}{2}D\cdot Y^+N_{ww} - \frac{1}{2}D\cdot Y^+C_{ww} \\ &\quad + \underbrace{2D_wY^{+w}C_{ww} + Y^{+w}D_wC_{ww} + Y^{+\bar{w}}D_{\bar{w}}C_{ww}}_{\mathcal{L}_{Y^+}C_{ww}}\Big) = i\delta^H_{Y^+}C_{ww}.\end{aligned} \tag{A.3}$$

Similarly,

$$[Q_S^+, C_{ww}] = -iuD_w^3Y^{+w} = i\delta^S_{Y^+}C_{ww}. \tag{A.4}$$

The commutator of Q^+ with N_{ww} is derived analogously.

B Conformal primaries on Milne slices

The relation between the Lorentz generators J_i, K_i and the $\mathrm{SL}(2,\mathbb{C})$ generators $L_i, \bar{L}_i$ is

$$\begin{aligned}
&L_0 = -\frac{i}{2}(J_3 + iK_3), \quad L_1 = -\frac{i}{2}(J_1 + iK_1 + i(J_2 + iK_2)), \quad L_{-1} = \frac{i}{2}(J_1 + iK_1 - i(J_2 + iK_2)), \\
&\bar{L}_0 = \frac{i}{2}(J_3 - iK_3), \quad \bar{L}_1 = \frac{i}{2}(J_1 - iK_1 - i(J_2 - iK_2)), \quad \bar{L}_{-1} = -\frac{i}{2}(J_1 - iK_1 + i(J_2 - iK_2)).
\end{aligned} \tag{B.1}$$

The Lorentz algebra (6.9) immediately implies that (B.1) obey the $\mathrm{SL}(2,\mathbb{C})$ algebra (6.10).

The form (6.23) of the Minkowski metric with $\rho = \sinh\eta$ can be obtained directly from

$$ds^2 = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2 \tag{B.2}$$

via the coordinate transform

$$\begin{aligned}
x^0 &= \tau\cosh\eta, \\
x^1 &= \tau\sin\theta\cos\varphi\sinh\eta, \\
x^2 &= \tau\sin\theta\sin\varphi\sinh\eta, \\
x^3 &= \tau\cos\theta\sinh\eta,
\end{aligned} \tag{B.3}$$

The isometries of (6.23) are inherited from isometries of (B.2) which preserve the slices of constant τ and hence coincide with the Lorentz transformations (6.7), (6.8). In $(\tau, \eta, \theta, \varphi)$ coordinates, the Lorentz generators generators take the form

$$\begin{aligned}
&J_3 = -\partial_\varphi, \quad J_1 = \sin\varphi\partial_\theta + \cos\varphi\cot\theta\partial_\varphi, \quad J_2 = -\cos\varphi\partial_\theta + \sin\varphi\cot\theta\partial_\varphi, \\
&K_3 = -(\cos\theta\partial_\eta - \sin\theta\coth\eta\partial_\theta), \\
&K_1 = -(\cos\varphi\sin\theta\partial_\eta + \cos\theta\cos\varphi\coth\eta\partial_\theta - \coth\eta\csc\theta\sin\varphi\partial_\varphi), \\
&K_2 = -(\sin\theta\sin\varphi\partial_\eta + \cos\theta\coth\eta\sin\varphi\partial_\theta + \coth\eta\cos\varphi\csc\theta\partial_\varphi).
\end{aligned} \tag{B.4}$$

In the limit $\eta \to \infty$, (B.4) reduce to

$$\begin{aligned}
&J_3 = -\partial_\varphi, \quad J_1 = \sin\varphi\partial_\theta + \cos\varphi\cot\theta\partial_\varphi, \quad J_2 = -\cos\varphi\partial_\theta + \sin\varphi\cot\theta\partial_\varphi, \\
&K_3 = \sin\theta\partial_\theta, \quad K_1 = -(\cos\theta\cos\varphi\partial_\theta - \csc\theta\sin\varphi\partial_\varphi), \quad K_2 = -(\cos\theta\sin\varphi\partial_\theta + \cos\varphi\csc\theta\partial_\varphi).
\end{aligned} \tag{B.5}$$

Equivalently in $(z, \bar{z})$ coordinates[28]

$$z = -\cot\frac{\theta}{2}e^{i\varphi}, \qquad \bar{z} = -\cot\frac{\theta}{2}e^{-i\varphi} \tag{B.6}$$

and using identities such as

$$\frac{1}{(\sin\theta/2)^2} = 1 + z\bar{z}, \tag{B.7}$$

[28] $z \to -z, \bar{z} \to -\bar{z}$ is an automorphism of the Lorentz algebra. (B.6) yield formulas for the Lorentz generators that match with [12] up to an overall sign. This overall sign is such that the standard Lorentz algebra (6.9) is obeyed.

(B.5) take the form

$$
\begin{aligned}
J_3 &= -i(z\partial_z - \bar{z}\partial_{\bar{z}}), & J_1 &= -\frac{i}{2}\left[(z^2-1)\partial_z - (\bar{z}^2-1)\partial_{\bar{z}}\right], \\
J_2 &= -\frac{1}{2}\left[(z^2+1)\partial_z + (\bar{z}^2+1)\partial_{\bar{z}}\right], & K_3 &= -(z\partial_z + \bar{z}\partial_{\bar{z}}), \\
K_1 &= -\frac{1}{2}\left[(z^2-1)\partial_z + (\bar{z}^2-1)\partial_{\bar{z}}\right], & K_2 &= \frac{i}{2}\left[(z^2+1)\partial_z - (\bar{z}^2+1)\partial_{\bar{z}}\right].
\end{aligned}
\tag{B.8}
$$

These precisely agree with (5.25) with $f = 0$ and Y^{+z} given in (5.28).

As before,

$$
\Psi_\Delta = \frac{f(\tau^2)}{(x^0+x^3)^\Delta} = \frac{f(\tau^2)}{(\tau(\cosh\eta + \cos\theta\sinh\eta))^\Delta} \tag{B.9}
$$

obeys

$$
(L_0 + \bar{L}_0)\Psi_\Delta = \Delta\Psi_\Delta, \quad (L_0 - \bar{L}_0)\Psi_\Delta = 0 \tag{B.10}
$$

and

$$
L_1\Psi_\Delta = \bar{L}_1\Psi_\Delta = 0. \tag{B.11}
$$

(B.9) diagonalizes boosts along the x^3 axis and obeys the highest weight condition (B.11).

C Celestial 3-point example

In this appendix we spell out the steps involved in evaluating the integral (6.62). We first notice that on the support of the momentum-conserving delta function,

$$
\begin{aligned}
\omega_2 &= \frac{m^2}{4\omega_1|z_{12}|^2}, \quad y = \frac{2m\omega_1|z_{12}|^2}{m^2+4\omega_1^2|z_{12}|^2}, \\
w &= \frac{m^2 z_2 + 4\omega_1^2 z_1|z_{12}|^2}{m^2+4\omega_1^2|z_{12}|^2}, \quad \bar{w} = \frac{m^2\bar{z}_2 + 4\omega_1^2\bar{z}_1|z_{12}|^2}{m^2+4\omega_1^2|z_{12}|^2}.
\end{aligned}
\tag{C.1}
$$

The Jacobian for the transformation from $(\omega_i\hat{q}_i, m\hat{p})$ to $(\omega_2, y, w, \bar{w})$ is

$$
|J| = \frac{m^3(y^2+|w-z_2|^2)}{2y^4}. \tag{C.2}
$$

We then find that

$$
\begin{aligned}
&\frac{1}{y^3}\frac{1}{|J|}\left(\frac{y}{y^2+|w-z_3|^2}\right)^{\Delta_3}\delta\left(\omega_2 - \frac{m^2}{4\omega_1|z_{12}|^2}\right)\delta\left(y - \frac{2m\omega_1|z_{12}|^2}{m^2+4\omega_1^2|z_{12}|^2}\right) \\
&\times\delta^{(2)}\left(w - \frac{m^2 z_2 + 4\omega_1^2 z_1|z_{12}|^2}{m^2+4\omega_1^2|z_{12}|^2}\right) = \frac{2}{m^3}\frac{m}{2\omega_1|z_{12}|^2}\left(\frac{2m\omega_1|z_{12}|^2}{4\omega_1^2|z_{12}|^2|z_{13}|^2+m^2|z_{23}|^2}\right)^{\Delta_3}\times\delta^{(4)}.
\end{aligned}
\tag{C.3}
$$

The integrals over $y, w, \bar{w}$ are now trivial and the celestial 3-point amplitude becomes

$$
\begin{aligned}
\widetilde{\mathcal{A}}(\Delta_i, z_i, \bar{z}_i) &= g\left(\frac{m^2}{4|z_{12}|^2}\right)^{\Delta_2-1}\frac{(2m|z_{12}|^2)^{\Delta_3}}{m^2|z_{12}|^2}\int_0^\infty d\omega_1\frac{\omega_1^{\Delta_1-\Delta_2+\Delta_3-1}}{(4\omega_1^2|z_{12}|^2|z_{13}|^2+m^2|z_{23}|^2)^{\Delta_3}} \\
&= \frac{gm^{2\Delta_2+\Delta_3-4}}{2^{2\Delta_2-\Delta_3-2}|z_{12}|^{2\Delta_2-2\Delta_3}}\int_0^\infty d\omega_1\frac{\omega_1^{\Delta_1-\Delta_2+\Delta_3-1}}{(4\omega_1^2|z_{12}|^2|z_{13}|^2+m^2|z_{23}|^2)^{\Delta_3}},
\end{aligned}
\tag{C.4}
$$

which precisely agrees with (6.64).

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