Conformal Field Theories

Tutorial 1

Higgs School 2022

Exercise 1.1. *The purpose of this exercise is to find the most general infinitesimal transformation $x'^{\mu} \simeq x^{\mu} + \epsilon^{\mu}(x)$ that defines conformal transformations as the set of diffeomorphisms that leave the metric unchanged up to a overall scale factor, which in general can be coordinate dependent.*

Let us consider the metric tensor $g_{\mu\nu}(x)$ of a d-dimensional space-time. Show that the above definition of "conformal transformations" requires:

$$g'_{\mu\nu}(x') = \frac{\partial x^{\rho}}{\partial x'^{\mu}} \frac{\partial x^{\sigma}}{\partial x'^{\nu}} g_{\rho\sigma}(x) = \Omega(x)^2 g_{\mu\nu}(x) \,. \tag{1.1}$$

Another way to phrase this in the flat case is that the Jacobian of the transfomation is an orhogonal metric times a coordinate dependent prefactor.

$$\frac{\partial x^{\rho}}{\partial x'^{\mu}} = \Omega(x) R^{\rho}_{\mu} \tag{1.2}$$

Expand Eq. (1.1) at linear order in $\epsilon^{\mu}(x)$ (assuming $\Omega \simeq 1 - O(\epsilon)$) and obtain the "Killing equation" (we restrict to constant metric $\gamma_{\mu\nu} = \eta_{\mu\nu}$))

$$\partial_{\rho}\epsilon_{\mu} + \partial_{\mu}\epsilon_{\rho} = \frac{2}{d}(\partial^{\sigma}\epsilon_{\sigma})\eta_{\mu\rho} \,. \tag{1.3}$$

Deriving a second time, permuting the indices and taking linear combinations show that

$$\partial_{\rho}\partial_{\nu}\epsilon_{\mu} = \frac{1}{d}\left(\partial_{\nu}\partial^{\sigma}\epsilon_{\sigma}\eta_{\mu\rho} - \partial_{\mu}\partial^{\sigma}\epsilon_{\sigma}\eta_{\nu\rho} + \partial_{\rho}\partial^{\sigma}\epsilon_{\sigma}\eta_{\mu\nu}\right) \,. \tag{1.4}$$

Finally, contracting the indices and applying ∂_{ν} to the resultinh equation and $\Box$ to eq. (1.3), obtain:

$$\begin{aligned} \partial_{\nu}\Box\epsilon_{\mu} &= \frac{2-d}{d}\partial_{\mu}\partial_{\nu}(\partial^{\sigma}\epsilon_{\sigma}) \,, \\ \partial_{\nu}\Box\epsilon_{\mu} + \partial_{\mu}\Box\epsilon_{\nu} &= \frac{2}{d}\Box\partial^{\sigma}\epsilon_{\sigma}\eta_{\mu\nu} \end{aligned} \tag{1.5}$$

Symmetrizing the first equation show that

$$(2-d)\partial_{\mu}\partial_{\nu}(\partial^{\sigma}\epsilon_{\sigma}) = g_{\mu\nu}\Box\partial^{\sigma}\epsilon_{\sigma} \,, \tag{1.6}$$

and taking the trace show that $f(x) \doteq (\partial^\sigma \epsilon_\sigma)$ satisfies the following second order differential equation:

$$(d-1)\Box f(x) = 0\,. \tag{1.7}$$

Inserting the above constraint in eq.(1.6) argue that for $d > 2$ it must be $f(x) = A + B_\mu x^\mu$, which translates in the general expression

$$\epsilon^\mu = c^\mu + a_{\mu\nu}x^\nu + b_{\mu\nu\rho}x^\nu x^\rho \tag{1.8}$$

Plugging the general solution into eq. (1.4) we observe that the coefficient $b_{\mu\nu\rho}$ can be expressed in terms of only one vector $b_\mu \equiv b_{\sigma\ \mu}^{\ \sigma}$:

$$b_{\mu\nu\rho} = \frac{1}{d}\left(b_{\sigma\ \nu}^{\ \sigma}\eta_{\mu\rho} + b_{\sigma\ \rho}^{\ \sigma}\eta_{\mu\nu} - b_{\sigma\ \mu}^{\ \sigma}\eta_{\nu\rho}\right) \tag{1.9}$$

Finally, using eq. (1.3) show that the symmetric part of $a_{\mu\nu}$ is proportional to the matrix $\eta_{\mu\nu}$, while the antisymmetric one is completely unconstrained.

Recognize the transformations associated to the above parameters:

$$\begin{aligned}
x'^\mu &= x^\mu + c^\mu : \quad \textit{translations} \\
x'^\mu &= x^\mu + \lambda x^\mu : \quad \textit{dilatations} \\
x'^\mu &= x^\mu + a^\mu_{\ \nu}x^\nu : \quad \textit{Lorentz rotations} \\
x'^\mu &= x^\mu + 2(b_\rho x^\rho)x^\mu - x^2 b^\mu : \quad \textit{Conformal boosts}
\end{aligned} \tag{1.10}$$

Exercise 1.2. *Show that in $d = 2$, given the coordinates (z^0, z^1) with a flat metric on the plane, and the transfomation $z_i \to w_i(z_j)$, the condition*

$$g'_{\mu\nu}(x') = \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu}g_{\rho\sigma}(x) = \Omega(x)^2 g_{\mu\nu}(x)\,. \tag{1.11}$$

corresponds to

$$\begin{aligned}
&\left(\frac{\partial w^0}{\partial z^0}\right)^2 + \left(\frac{\partial w^0}{\partial z^1}\right)^2 = \left(\frac{\partial w^1}{\partial z^0}\right)^2 + \left(\frac{\partial w^1}{\partial z^1}\right)^2 \\
&\frac{\partial w^0}{\partial z^0}\frac{\partial w^1}{\partial z^0} + \frac{\partial w^0}{\partial z^1}\frac{\partial w^1}{\partial z^1} = 0
\end{aligned} \tag{1.12}$$

Show that the above conditions are equivalent to the Cauchy-Riemann equations for holomorphic (or anti-holomorphic) functions:

$$\frac{\partial w^1}{\partial z^0} = \pm\frac{\partial w^0}{\partial z^1}\,, \qquad \frac{\partial w^0}{\partial z^0} = \mp\frac{\partial w^1}{\partial z^1} \tag{1.13}$$

Conclude that conformal transformations in $d = 2$ are those for which the function $w(z)$ is holomorphic, where

$$z = z^0 + iz^1\,, \quad \overline{z} = z^0 - iz^1\,, \qquad w = w^0 + iw^1\,, \quad \overline{w} = w^0 - iw^1 \tag{1.14}$$

Exercise 1.3. *Starting from the infinitesimal transformations we can define the differential form of the generators acting on functions. Given a coordinate transformation* $x \to x' = \xi(x)$ *(therefore* $x = \xi^{-1}(x')$*), we have* $f(x) \to f'(x) = f(\xi^{-1}(x))$*. The implementation of function can be implemented through differential generators* J *such that* $f'(x) = e^J f(x)$*. In the case in exam we get:*

$$
\begin{array}{lll}
\textit{Translations}: & f'(x) = f(x^\mu - c^\mu) = f(x) - c^\mu \partial_\mu f(x) & \Rightarrow P_\mu = \partial_\mu \\
\textit{Lorentz}: & f'(x) = f(x^\mu) - w^\mu_\nu x^\nu \partial_\mu f(x) & \Rightarrow M_{\mu\nu} = -(x_\mu \partial_\nu - x_\nu \partial_\mu) \\
\textit{Dilatations}: & f'(x) = f(x^\mu) - \lambda x^\mu \partial_\mu f(x) & \Rightarrow D = x^\mu \partial_\mu \\
\textit{Conf. boosts}: & f'(x) = f(x) - (2b^\nu x_\nu x^\mu \partial_\mu - x^2 b^\mu \partial_\mu) f(x) & \Rightarrow K_\mu = (2x_\mu x^\rho \partial_\rho - x^2 \partial_\mu)
\end{array}
\tag{1.15}
$$

Using the above representation compute all the commutation relations among the conformal algebra generators.

Exercise 1.4 (DIFFICULT)**.** *Recalling the definition of conserved charge*

$$
Q_\epsilon = -\int_\Sigma dS_\mu \epsilon_\nu T^{\mu\nu} \tag{1.16}
$$

where Σ *is the boundary of some region* $\Sigma = \partial B$ *and* dS_μ *is the normal to the surface. The commutator with the Stress Tensor itself is fixed by symmeties:*

$$
[Q_\epsilon, T^{\mu\nu}] = \epsilon^\rho \partial_\rho T^{\mu\nu} + (\partial_\rho \epsilon^\rho) T^{\mu\nu} - \partial_\rho \epsilon^\mu T^{\rho\nu} + \partial^\mu \epsilon_\rho T^{\rho\nu} \tag{1.17}
$$

Contracting with $\epsilon^{'\mu}$ *and integrating over a surface* Σ' *shaw that :*

$$
[Q_\epsilon, Q_{\epsilon'}] = Q_{-[\epsilon,\epsilon']} \tag{1.18}
$$

where $[\epsilon, \epsilon'] = ((\partial \cdot \epsilon)\epsilon' - (\partial \cdot \epsilon')\epsilon)$ *is the Lie bracket. [Hint: choose a smart surface, for instance a cube in flat coordinates and show that the integral of* $(\epsilon \cdot \partial)(T_{\mu\nu}\epsilon^{'\nu}) - (\partial^\rho \epsilon_\mu)(T_{\rho\nu}\epsilon^{'\nu})$ *vanishes. To do this integrate by part and recall that* ϵ, ϵ' *satisfy the Killing equations].*

Exercise 1.5. *Using the definitions of the previous exercise, compute the algebra of conformal transformations by applying the Lie bracket. We write down the explicit form of* ϵ^μ*. We strip out the parameters of the transformation, which we collectively call* f^a*. Then* $\epsilon^\mu = f^a \epsilon^\mu_a$*. The nature of the index a depends on the kind of transformation considered:*

- *translations:* $\epsilon^\mu_a \to \epsilon^\mu_\nu = \delta^\mu_\nu$
- *rotations:* $\epsilon^\mu_a \to \epsilon^\mu_{\nu\rho} = \delta^\mu_\nu x_\rho - \delta^\mu_\rho x_\nu$
- *dilatations:* $\epsilon^\mu_a \to \epsilon^\mu = x^\mu$
- *SCT:* $\epsilon^\mu_a \to \epsilon^\mu_\nu = 2x^\mu x_\rho - \delta^\mu_\nu x^2$

Let us see an example. Consider the ϵ *the killing vector associated to dilatations and* ϵ' *the one associated to translations. Then*

$$
[Q_D, Q_{P_\rho}] \to \frac{1}{d}(\partial_\mu x^\mu)\delta^\nu_\rho = \delta^\nu_\rho \to [D, P_\rho] = P_\rho \tag{1.19}
$$

Compute all the other commutation relations.