Conformal Field Theories

Tutorial 2

Higgs School 2022

Exercise 2.1. *Show that under inversion* $x^\mu \to x^\mu/x^2$ *the quantity* $x_{ij}^2 \equiv (x_i^\mu - x_j^\mu)(x_{i\mu} - x_{j\mu})$ *transform as follows:*

$$x_{ij}^2 \to \frac{x_{ij}^2}{x_i^2 x_j^2} \tag{2.1}$$

Use this result to show that in order to build conformal invariants one needs at least four distinct points. Show that the following combinations are invarinat

$$u \equiv \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}\,, \qquad v \equiv \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}\,. \tag{2.2}$$

Exercise 2.2. *Recall the transformation property of a scalar field under a conformal transformation* $x^\mu \to x'^\mu$ *such that* $\frac{\partial x'^\mu}{\partial x^\nu} = \Omega(x)\Lambda^\mu_\nu$, *with* Λ *a rotation of* $SO(d)$*:*

$$\phi'(x') = U\phi(x')U^\dagger = \Omega^{-\Delta}\phi(x(x')) \tag{2.3}$$

where Δ *is the scaling dimension of* ϕ*.[For instance under dilatations* $\Omega \equiv \lambda = const$*, and we have* $\phi'(x') = \lambda^{-\Delta}\phi(x'/\lambda)$*].*
Use invariance under translation, rotations, dilatations and inversion (use previous exercise) to find the most general form of the correlation function of three scalars consistent with conformal symmetry:

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\rangle = ? \tag{2.4}$$

Exercise 2.3. *Consider the correlation function of four identical scalars with dimension* Δ*. Show that the following ansatz has the right covariance properties under conformal transformations*

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \frac{1}{x_{12}^{2\Delta} x_{34}^{2\Delta}} g(u,v) \tag{2.5}$$

with g *any function of the conformal invariants* u, v *defined in the first exercise.*

Exercise 2.4. *Consider a theory of a free real scalars* ϕ*. This theory is clearly scale invariant at classical level and quantum level since there are no interactions. This can be shown by computing the stress tensor:*

$$T_\mu\nu =: \partial_\mu\phi\partial_\nu\phi : -\frac{1}{2}\eta_{\mu\nu} : (\partial_\mu\phi)^2 : \tag{2.6}$$

Here the notation :: means normal ordered. If we take the trace of the above expression we see that it doesn't vanish, however it is proportional to

$$T_\mu^\mu = (1 - d/2) : (\partial_\mu \phi)^2 : \tag{2.7}$$

which is a total derivatives on the equations of motion $\Box\phi = 0$. *Indeed* $: (\partial_\mu \phi)^2 :\sim \partial_\mu : \phi \partial^\mu \phi := \partial_\mu V^\mu$. *Here* V^μ *is the Virial current. This is one of the cases in which the Energy momentum tensor can indeed be modified to obtain a symmetric, conserved and traceless operator. This because the Virial current is in fact itself a divergence:* $V^\mu = \frac{1}{2}\partial^\mu : \phi^2 := \partial_\nu \sigma^{\mu\nu}$, *with* $\sigma^{\mu\nu} = \frac{1}{2}\eta^{\mu\nu} : \phi^2 :$. *Then one can show that an improved tensor* T', *defined as*

$$\begin{aligned} \sigma_+^{\mu\nu} &= \frac{1}{2}\left(\sigma^{\mu\nu} + \sigma^{\nu\mu}\right) \\ X^{\lambda\rho\mu\nu} &= \frac{2}{d-2}\left(\eta^{\lambda\rho}\sigma_+^{\mu\nu} - \eta^{\lambda\mu}\sigma_+^{\rho\nu} - \eta^{\rho\mu}\sigma_+^{\lambda\nu} + \eta^{\mu\nu}\sigma_+^{\lambda\rho} + \frac{1}{d-1}\left(\eta^{\mu\nu}\eta^{\lambda\rho} - \eta^{\rho\nu}\eta^{\lambda\mu}\right)\sigma_{+\gamma}^{\gamma}\right) \\ T'_{\mu\nu} &= T_{\mu\nu} + \partial^\lambda \partial^\rho X_{\lambda\rho\mu\nu} \end{aligned} \tag{2.8}$$

satisfies the required properties.

Show that In our case this becomes:

$$T'_{\mu\nu} =: \partial_\mu \phi \partial_\nu \phi : - \frac{1}{4(d-1)}\left((d-2)\partial^\mu \partial^\nu + \eta^{\mu\nu}\Box\right) : \phi^2 : \tag{2.9}$$

Now that we have established conformal invariance, we can compute a few correlation functions and check they indeed satisfy the general construction we made. The starting point is

$$\phi(x) = \int \frac{d^{d-1}k}{(2\pi)^{d-1} 2k_0}\left(a(\vec{k})e^{ikx} + a^\dagger(\vec{k})e^{-ikx}\right) \tag{2.10}$$

Show that the 2pt function is

$$\langle \phi(x)\phi(y)\rangle = \frac{1}{|x-y|^{d-2}} \tag{2.11}$$

This means that the scalar ϕ *behaves as an operator of dimensions* $\Delta_\phi = (d-2)/2$.

Using Wick theorem compute:

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \frac{1}{x_{12}^{2\Delta_\phi} x_{34}^{2\Delta_\phi}}\left(1 + u^{\Delta_\phi/2} + \left(\frac{u}{v}\right)^{\Delta_\phi/2}\right) \tag{2.12}$$

Similarly, introduce the composite operator $\mathcal{O}(x) \equiv \frac{:\phi^2(x):}{\sqrt{2}}$ *and compute:*

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\rangle \quad \langle \phi(x_1)\phi(x_2)\mathcal{O}(x_3)\rangle \quad \langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\rangle \quad \langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4)\rangle$$

Finally, compute the three point function $\langle \phi(x_1)T^{\mu\nu}(x_2)\phi(x_3)\rangle$ *and show that it verifies the Ward identities.*