The hadronic vacuum polarization (RBC/UKQCD)

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RBC/UKQCD status 2018

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Editors' Suggestion

Calculation of the Hadronic Vacuum Polarization Contribution to the Muon Anomalous Magnetic Moment

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We present a first-principles lattice QCD + QED calculation at physical pion mass of the leading-order hadronic vacuum polarization contribution to the muon anomalous magnetic moment. The total contribution of up, down, strange, and charm quarks including QED and strong isospin breaking effects is $a_\mu^{\mathrm{HVP\ LO}} = 715.4(18.7) \times 10^{-10}$. By supplementing lattice data for very short and long distances with R-ratio data, we significantly improve the precision to $a_\mu^{\mathrm{HVP\ LO}} = 692.5(2.7) \times 10^{-10}$. This is the currently most precise determination of $a_\mu^{\mathrm{HVP\ LO}}$.

Pure lattice result and dispersive result with reduced $\pi\pi$ dependence (window method)

Aaron Meyer (BNL $\to$ LBNL) & Mattia Bruno (BNL $\to$ CERN $\to$ Milano) joined since this 2018 paper

Lattice QCD – Time-Moment Representation

Starting from the vector current $J_\mu(x) = i\sum_f Q_f \overline{\Psi}_f(x)\gamma_\mu \Psi_f(x)$ we may write

$$a_\mu^{\mathrm{HVP\ LO}} = \sum_{t=0}^{\infty} w_t C(t)$$

with

$$C(t) = \frac{1}{3}\sum_{\vec{x}}\sum_{j=0,1,2}\langle J_j(\vec{x},t)J_j(0)\rangle$$

and w_t capturing the photon and muon part of the HVP diagrams (Bernecker-Meyer 2011).

The correlator $C(t)$ is computed in lattice QCD+QED at physical pion mass with non-degenerate up and down quark masses including up, down, strange, and charm quark contributions. The missing bottom quark contributions are computed in pQCD.

Window method (introduced in RBC/UKQCD 2018)

We also consider a window method. Following Meyer-Bernecker 2011 and smearing over t to define the continuum limit we write

$$a_\mu = a_\mu^{\mathrm{SD}} + a_\mu^{\mathrm{W}} + a_\mu^{\mathrm{LD}}$$

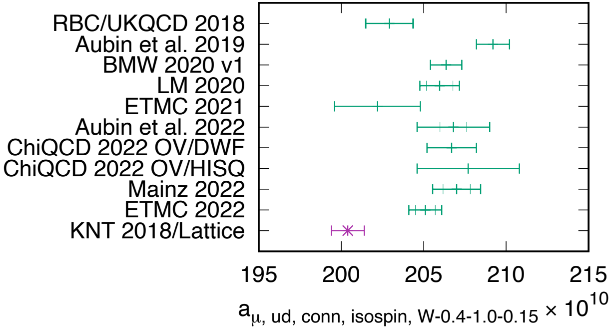
with

$$\begin{aligned}
a_\mu^{\mathrm{SD}} &= \sum_t C(t) w_t [1 - \Theta(t, t_0, \Delta)]\,, \\
a_\mu^{\mathrm{W}} &= \sum_t C(t) w_t [\Theta(t, t_0, \Delta) - \Theta(t, t_1, \Delta)]\,, \\
a_\mu^{\mathrm{LD}} &= \sum_t C(t) w_t \Theta(t, t_1, \Delta)\,, \\
\Theta(t, t', \Delta) &= [1 + \tanh[(t - t')/\Delta]]\,/2\,.
\end{aligned}$$

All contributions are well-defined individually and can be computed from lattice or R-ratio via $C(t) = \frac{1}{12\pi^2}\int_0^\infty d(\sqrt{s}) R(s) s e^{-\sqrt{s}t}$ with $R(s) = \frac{3s}{4\pi\alpha^2}\sigma(s, e^+e^- \to \mathrm{had})$.

a_μ^{W} has small statistical and systematic errors on lattice!

- In last few years, we reported on our progress for the complete result (improved bounding method, $(\pi\pi)_{I=1}$ phase shift study, improvements for disconnected/QED/SIB diagrams), this talk focuses entirely on progress on the Euclidean time window (RBC/UKQCD 2018) in the isospin symmetric limit with $t_0 = 0.4$ fm, $t_1 = 1.0$ fm, $\Delta = 0.15$ fm.
- This quantity promises reduced systematic lattice uncertainties, however, currently exhibits tensions between different lattice and R-ratio results:



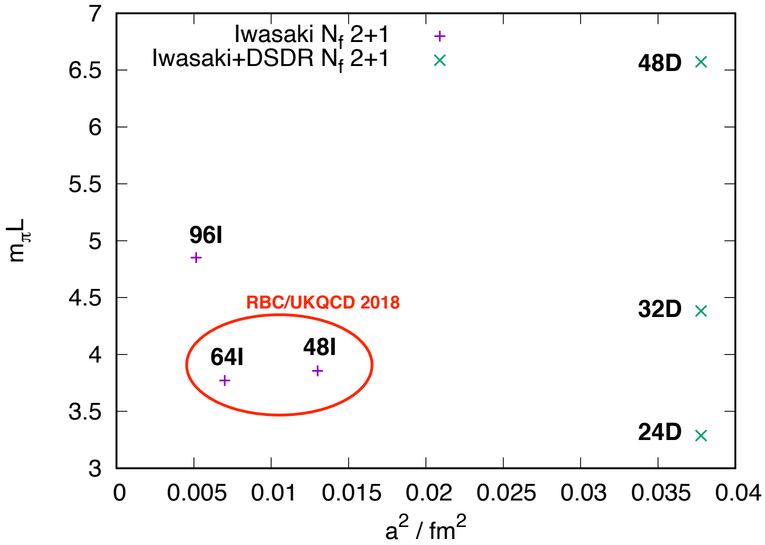
What will we calculate in our next update:

- $a_\mu^{\rm SD}$ for $t_0 = 0.1, 0.2, 0.3, \ldots, 2.5$ fm
- a_μ^W for $t_0 = 0.1, 0.2, 0.3, \ldots, 2.5$ fm and $t_1 = t_0 + 0.1$ fm
- a_μ^W for all combinations of $t_0 = 0.3, 0.4, 0.5$ fm and $t_1 = 1.0, 1.3, 1.6, 1.9, 2.2, 2.5$ fm
- $\Delta = 0.15$ fm for all of the above

Calculate in two definitions of the isospin symmetric world:

- World 1 (RBC/UKQCD 2018): $m_\pi = 0.135$ GeV, $m_K = 0.4957$ GeV, $m_\Omega = 1.67225$ GeV
- World 2 (BMW 2020): $m_\pi = 0.13497$ GeV, $m_{ss*} = 0.6898$ GeV, $w_0 = 0.17236$ fm

Our extended list of ensembles, all with $m_\pi = 135 \pm 5$ MeV:



New Mobius ensembles tuned to precision HVP (including $N_f = 2+1+1$ ensembles):

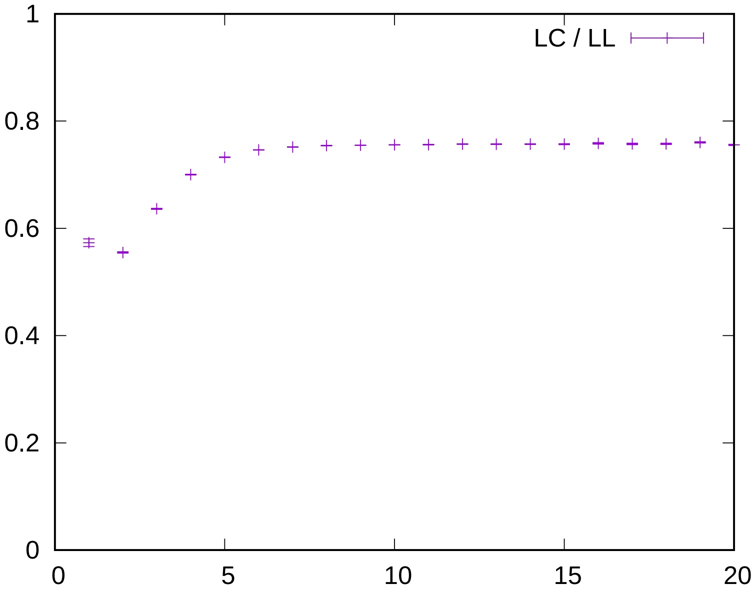
id	a^{-1} / GeV	m_π / GeV	m_K / GeV	m_{D_s} / GeV	$m_\pi L$	L_s
1	1.73	0.210	0.530	–	3.8	24
3	1.73	0.210	0.600	–	3.8	24
4	1.73	0.280	0.530	–	3.8	24
2	1.73	0.280	0.530	–	3.8	32
A	1.73	0.280	0.530	–	3.8	8
5	1.73	0.280	0.530	1.9	3.8	24
7	1.73	0.280	0.530	1.3	3.8	24
8	2.359	0.280	0.530	1.9	3.8	12
B	1.73	0.140	0.500	–	2.5	24
C	1.73	0.140	0.500	–	5.0	24
D	1.73	0.280	0.500	–	5.0	24
E	3.5	0.280	0.530	–	3.8	12
48I	1.73	0.140	0.500	–	3.8	24
64I	2.359	0.140	0.500	–	3.8	12
96I	2.7	0.135	0.500	–	4.8	12

New ensembles and HVP running on Booster (Germany), Summit & Perlmutter (US); Just now data complete for next update

Overview of improvements:

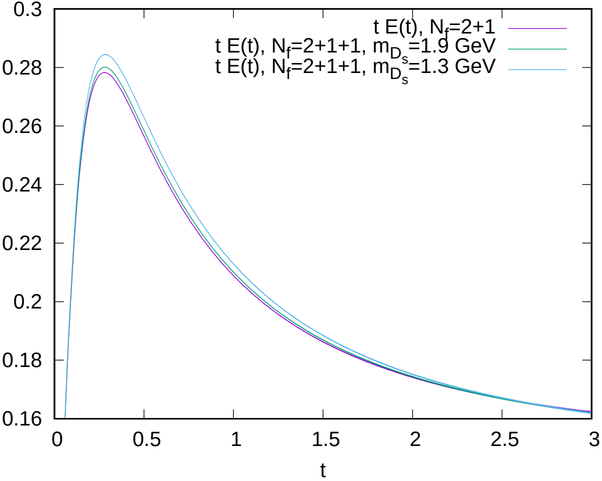
- 4x statistics on 48I and 64I
- Add third, finer lattice spacing ($a^{-1} = 2.7$ GeV) at physical pion mass; fourth at $a^{-1} = 3.5$ GeV is in progress
- Add local-conserved correlators in addition to local-local correlators (check for consistent continuum limit)
- Explicit calculation of parametric derivatives at physical point (master field)
- Study of missing charm determinant ($2{+}1 \to 2{+}1{+}1$) and $m_{\rm res}$ effects from first principles
- 5d (space-time+Markov) master-field statistical error analysis

Ratio of local-local to local-conserved correlators (here 96I):



Separate local-local (LL) and Local-conserved (LC) continuum limits

The $N_f = 2+1$ and $N_f = 2+1+1$ ensembles are matched to the same pion and kaon masses and the Wilson-Flowed energy density at long-dinstance. Clear signal of charm effects in energy density at shorter distances.



Then measure the sea charm effects to the HVP (in particular for short-distance windows). Multiple m_c and a to disentangle discretization errors from real charm effects.

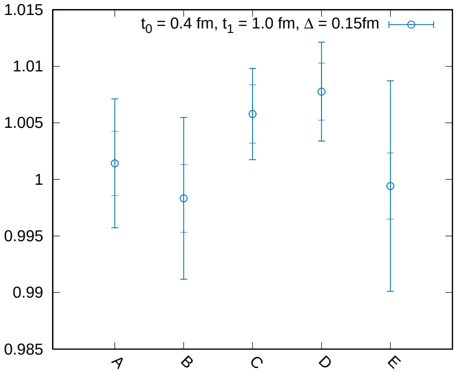
Blinding

- 2 analysis groups for ensemble parameters (not blinded)
- 5 analysis groups for vector-vector correlators (blinded, to avoid bias towards other lattice/R-ratio results)
- Blinded vector correlator $C_b(t)$ relates to true correlator $C_0(t)$ by

$$C_b(t) = (b_0 + b_1 a^2 + b_2 a^4) C_0(t) \quad (1)$$

with appropriate random b_0, b_1, b_2, different for each analysis group. This prevents complete unblinding based on previously shared data on coarser ensembles.

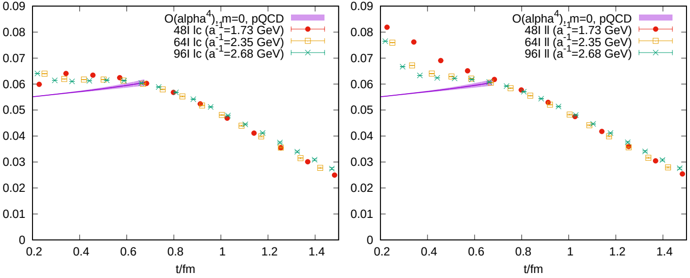
Relative unblinding (standard window)



- Inner error bar is statistical, outer error bar is statistical and systematic added in quadrature.
- Blinding procedure allows a^4 term to affect result by up to ± 0.0025 (eliminated after unblinding)
- Different groups use different methods (e.g., continuum, FV)

Convergence to RBC/UKQCD22 prescription

- Local-local (LL) has much larger a^2/t^2 errors in $C(t)t^3$ that affect also main window:



- Mean-field improved LPT finds $\approx 2\times$ size of discretization errors in LL versus local-conserved (LC)

- Size of a^4 coefficient in $a^2 + a^4$ fit for SD window much larger for LL compared to LC

- Continuum limit is sensitive to FV corrections

RBC/UKQCD22 prescription

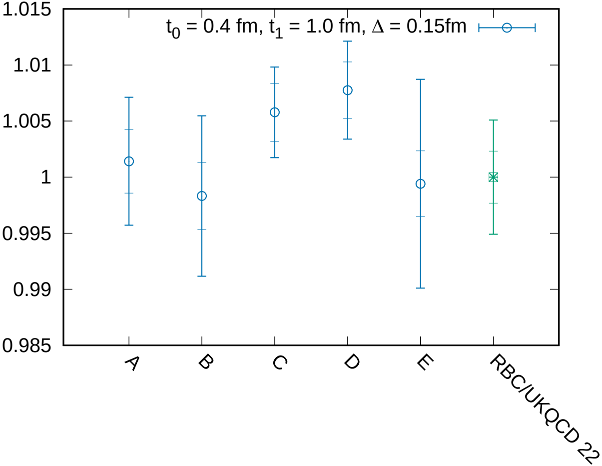
- Decision 1: when using the local-local, always add a a^4 term to the fits
- Decision 2: Use Hansen-Patella FV corrections instead of data-driven approaches fit to $e^{-m_\pi L}$ ansatz

Resulting prescription:

- Use $Z_V = Z_V^\pi$ (pion charge normalization) and $Z_V = Z_V^\star$ (lc to ll ratio at $t = 1$ fm)
- Use p and $\hat{p} = 2\sin(p/2)$ Bernecker-Meyer prescriptions for Euclidean time weights
- Perform universality-constrained fits to ll ($a^2 + a^4$) and lc (a^2) as well as fits to ll ($a^2\alpha_s(\mu = 1/a) + a^4$) and lc ($a^2\alpha_s(\mu = 1/a)$)

This gives 8 fits. Take the average of minimum and maximum as central value. Take difference of central value to maximum as systematic error.

Relative unblinding (standard window)



Note: full unblinding could change values at ± 0.0025 level due to artificial a^4 terms in our blinding procedure

Next slide: full unblinding

Full unblinding (standard window)

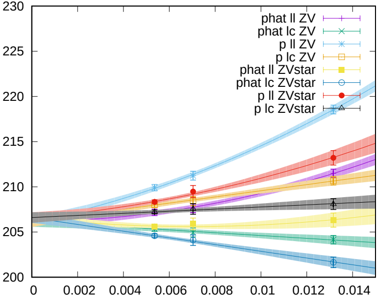
Result for $a_\mu^{\mathrm{W,iso,ud,0.4,1.0,0.15}}$ in BMW20 world

$$206.36(44)_{\mathrm{S}}(42)_{\mathrm{C}}(01)_{\mathrm{FV}}(00)_{m_\pi\ \mathrm{FV}}(08)_{\partial_m\ \mathrm{C}}(00)_{\mathrm{WF\ order}}(03)_{m_{\mathrm{res}}} \times 10^{-10}$$

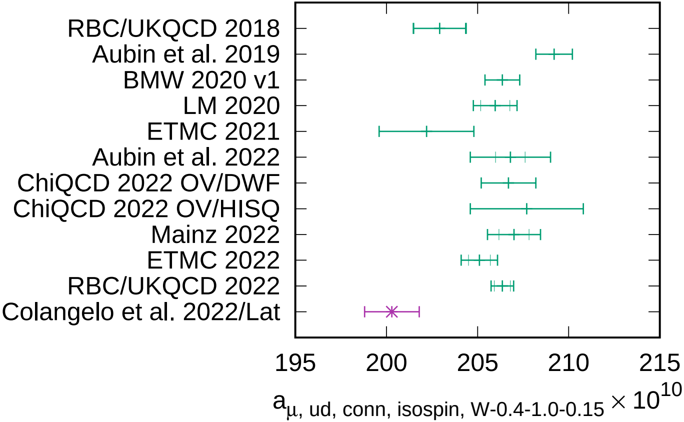
and RBC/UKQCD18 world

$$206.46(53)_{\mathrm{S}}(43)_{\mathrm{C}}(01)_{\mathrm{FV}}(01)_{m_\pi\ \mathrm{FV}}(09)_{\partial_m\ \mathrm{C}}(00)_{\mathrm{WF\ order}}(03)_{m_{\mathrm{res}}} \times 10^{-10}\,.$$

Fits ll ($a^2 + a^4$) and lc (a^2):



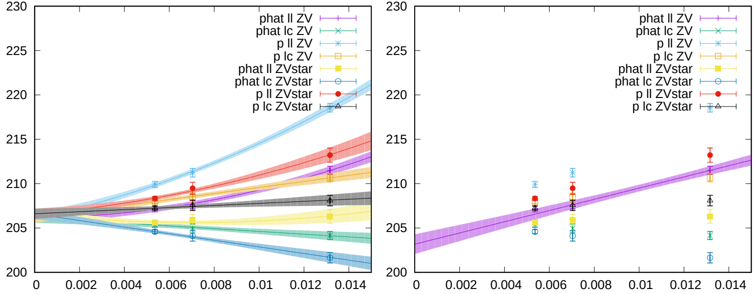
The RBC/UKQCD22 result in context



- 3.9σ tension of RBC/UKQCD22 with Colangelo et al. 22/Lattice
- More on RBC/UKQCD18 on next slide

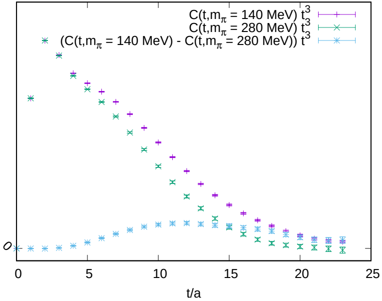
Comparison to RBC/UKQCD18

- We increased the basis for our continuum extrapolation from 2 data points to 24 data points
- If we repeat a fit of the original 2 data points, we find a result consistent with RBC/UKQCD18:



Short-distance windows (1/3)

Short-distance correlator is insensitive to quark mass



Therefore we generate pairs of ensembles with m_π and $2m_\pi$ to compute

$$a_\mu(m_\pi) = \underbrace{a_\mu(m_\pi) - a_\mu(2m_\pi)}_{\delta a_\mu} + a_\mu(2m_\pi)\,. \quad (2)$$

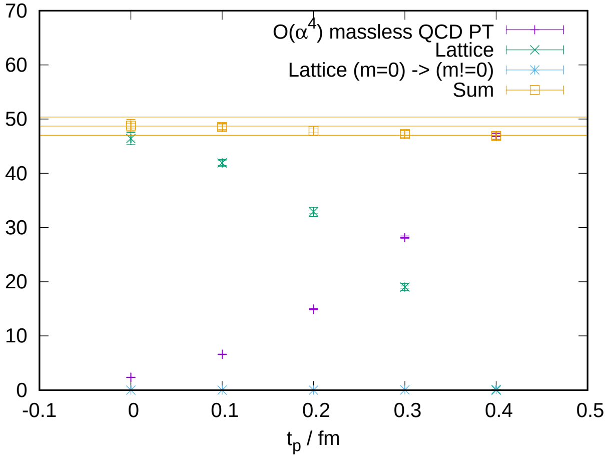
This allows the costly term δa_μ to be calculated at coarser lattice spacings compared to $a_\mu(2m_\pi)$. We proposed this in Snowmass 2021 LOI.

Short-distance windows (2/3)

SD windows can also be computed in perturbative QCD at 5 loops ($O(\alpha_s^4)$, Chetyrkin-Maier 2010).

Stability plot of

$$a_\mu^{\mathrm{SD}}(t_0 = 0.4fm) = a_\mu^{\mathrm{SD,pQCD}}(t_0 = t_p) + a_\mu^{\mathrm{W}}(t_0 = t_p, t_1 = 0.4fm) \quad (3)$$



Short-distance windows (3/3)

For the short-distance window, we find for pure lattice in the BMW20 world

$$a_\mu^{\mathrm{SD,iso,ud,0.4,0.15}} = 48.7(05)(16) \times 10^{-10} \tag{4}$$

and in the RBC/UKQCD18 world

$$a_\mu^{\mathrm{SD,iso,ud,0.4,0.15}} = 49.0(06)(14) \times 10^{-10}\,. \tag{5}$$

If we replace below $t_p = 0.1$ fm with continuum five-loop PT, we find

$$a_\mu^{\mathrm{SD,iso,ud,0.4,0.15}} = 48.51(43)(53) \times 10^{-10} \tag{6}$$

in the BMW20 world and

$$a_\mu^{\mathrm{SD,iso,ud,0.4,0.15}} = 48.70(52)(59) \times 10^{-10} \tag{7}$$

in the RBC/UKQCD18 world.

Quantifying the isospin-symmetric world ambiguity

Correlated difference of BMW20 and RBC/UKQCD18 world (BMW20 - RBC/UKQCD18) for the main window

$$\Delta a_\mu^{\mathrm{W,iso,ud,0.4,1.0,0.15}} = -0.10(24)(07) \times 10^{-10}\,, \tag{8}$$

for the SD window

$$\Delta a_\mu^{\mathrm{SD,iso,ud,0.4,0.15}} = -0.33(36)(36) \times 10^{-10} \tag{9}$$

using the pure lattice result.

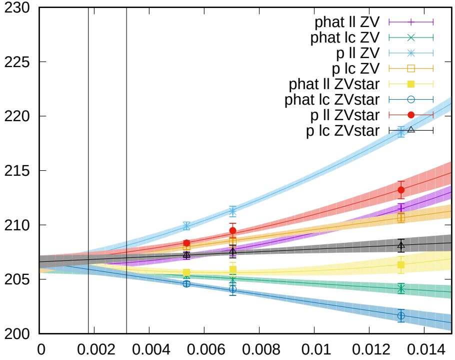
Differences between these two schemes negligible for direct comparison

Conclusions

- Fully unblinded, paper in preparation
- 4x statistics on 48I and 64I
- Add third, finer lattice spacing ($a^{-1} = 2.7$ GeV) at physical pion mass
- Add local-conserved correlators in addition to local-local correlators (check for consistent continuum limit).
- Continuum limit now based on 24 instead of 2 data points (RBC/UKQCD18)
- Explicit calculation of parametric derivatives at physical point (master field)
- Study of missing charm determinant ($2{+}1 \to 2{+}1{+}1$) and $m_{\rm res}$ effects from first principles
- 5d (space-time+Markov) master-field statistical error analysis

Outlook

- Two more lattice spacings planned (3.5 GeV ensemble thermalized, 4.6 GeV ensemble soon as well)



- For complete HVP analysis data set almost complete as well (still finishing distillation data on 96I, a lot of new data also on QCD+QED, see Mattia Bruno's talk)

Backup

Master-field calculation of gradients

For a local observable

$$O = \frac{1}{V}\sum_{y} O_y \tag{10}$$

we can define the truncated master-field covariance

$$\mathrm{Cov}_R(O,A) \equiv \frac{1}{V}\sum_{x,y,|y|\leq R}\big(\langle O_x A_{x+y}\rangle_\beta - \langle O_x\rangle_\beta \langle A_{x+y}\rangle_\beta\big) \tag{11}$$

such that, e.g., the β-derivative of O is given by

$$\frac{\langle O\rangle_{\beta+\varepsilon} - \langle O\rangle_\beta}{\varepsilon} = 6\lim_{R\to\infty}\mathrm{Cov}_R(O,A)\,. \tag{12}$$

In practice use exponential approach to plateau for $R\to\infty$.

We isolate the dependence on sea-quark mass m of an observable O by studying

$$\langle O \rangle_m \equiv \frac{\int \det(D(m)) O P}{\int \det(D(m)) P} \tag{13}$$

with Dirac matrix $D(m)$ and residual weight P. Can show that

$$\frac{\langle O \rangle_{m+\varepsilon} - \langle O \rangle_m}{\varepsilon} = \mathrm{Cov}(O, \mathrm{Tr}[D_{4d}^{-1}(m)]) + \mathcal{O}(\varepsilon)\,. \tag{14}$$

Finally, for DWF an additional flavor enters as

$$\det(D(m) D^{-1}(1)) \tag{15}$$

such that for $m = 1$ the factor is trivial and we can view adding an additional flavor as changing the sea-quark mass down from $m = 1$ to the target value.

Sea charm effects

Short distance based on five-loop, massless pQCD:

$$a_\mu^{\mathrm{SD,iso,ud,0.4,0.15,4f}} - a_\mu^{\mathrm{SD,iso,ud,0.4,0.15,3f}} = -0.19(1) \times 10^{-10}\,. \quad (16)$$

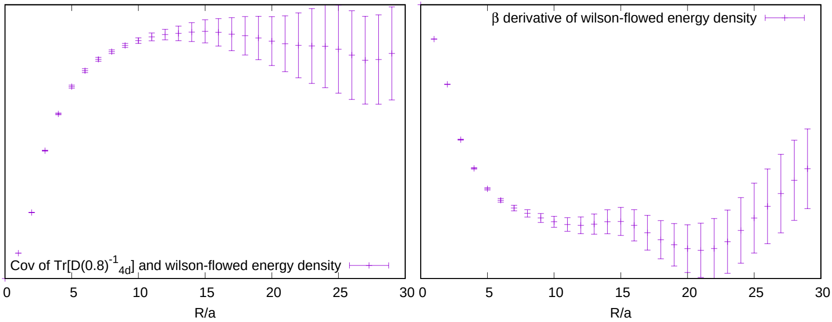
Using direct lattice calculation based on our new 2+1+1f ensembles:

$$a_\mu^{\mathrm{SD,iso,ud,0.4,0.15,4f}} - a_\mu^{\mathrm{SD,iso,ud,0.4,0.15,3f}} = -0.05(6)(1) \times 10^{-10}\,. \quad (17)$$

For the middle window, we find

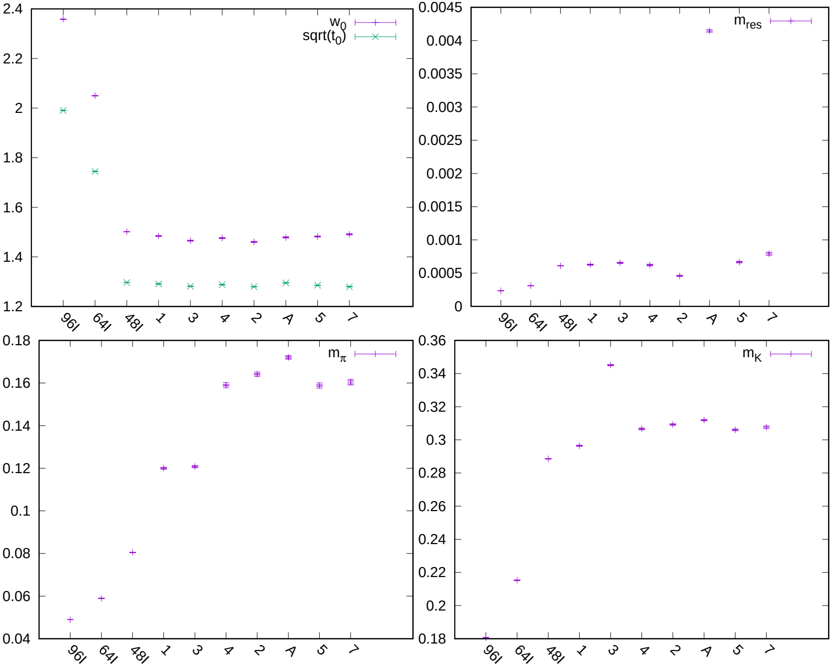
$$a_\mu^{\mathrm{W,iso,ud,0.4,1.0,0.15,4f}} - a_\mu^{\mathrm{W,iso,ud,0.4,1.0,0.15,3f}} = -0.32(71)(34) \times 10^{-10}\,. \quad (18)$$

Example for wilson-flowed energy density (96I, $t_0 \approx 2$)

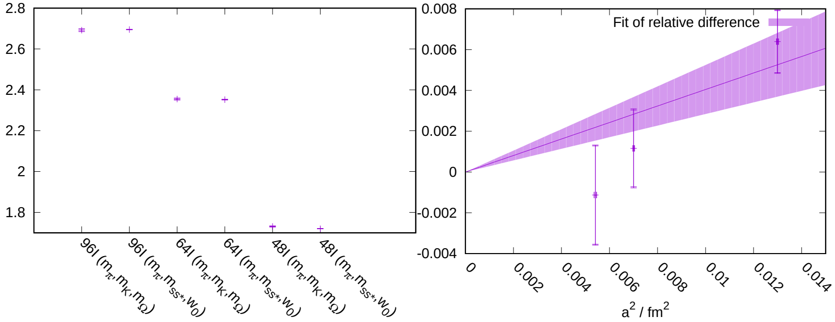


Computed in similar way also derivatives of, e.g., VV and PP correlators.

Ensemble parameters:

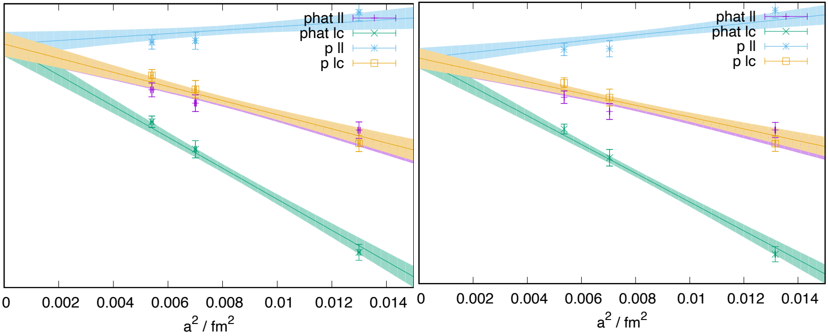


Lattice cutoff a^{-1}/GeV in isospin symmetric worlds:



Isospin limit 1: $m_\pi = 0.135$ GeV, $m_K = 0.4957$ GeV, $m_\Omega = 1.67225$ GeV
Isospin limit 2: $m_\pi = 0.13497$ GeV, $m_{ss*} = 0.6898$ GeV, $w_0 = 0.17236$ fm

Improved continuum extrapolation:



Statistical error in continuum 0.3% (2018 paper had 0.7%)

Left side: $m_\pi = 0.135$ GeV, $m_K = 0.4957$ GeV, $m_\Omega = 1.67225$ GeV
Right side: $m_\pi = 0.13497$ GeV, $m_{ss*} = 0.6898$ GeV, $w_0 = 0.17236$ fm
ll: local-local vector correlator
lc: local-conserved vector correlator
p: use continuum momentum in construction of w_t
phat: use lattice momentum $\hat{p} = 2\sin(p/2)$ in construction of w_t