Leading isospin breaking effects in the HVP contribution to a_μ and to the running of α - an update of the Mainz effort

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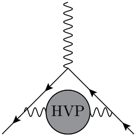
LO-HVP contribution to the muon anomalous magnetic moment a_μ

- a_μ^{HVP} in time-momentum representation (TMR)[1]:

$$a_\mu^{\mathrm{HVP}} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty \mathrm{d}t\, \widetilde{K}(t, m_\mu)\, C(t)$$

$$C(x^0)\,\delta^{\mu_2\mu_1} = \int \mathrm{d}x^3 \langle \mathcal{V}_{\mathrm{R}}^{\gamma x \mu_2} \mathcal{V}_{\mathrm{R}}^{\gamma 0 \mu_1} \rangle$$



- Intermediate window contribution of $a_{\mu,\mathrm{win}}^{\mathrm{HVP}}$ in TMR:

$$a_{\mu,\mathrm{win}}^{\mathrm{HVP}} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty \mathrm{d}t\, \omega_{\mathrm{win}}(t)\, \widetilde{K}(t, m_\mu)\, C(t)$$

$$\omega_{\mathrm{win}}(t) = \Theta(t, t_0, \Delta) - \Theta(t, t_1, \Delta) \qquad \Theta(t, t', \Delta) = \frac{1}{2}\left(1 + \tanh\left(\frac{t - t'}{\Delta}\right)\right)$$

$$t_0 = 0.4\,\mathrm{fm} \qquad t_1 = 1.0\,\mathrm{fm} \qquad \Delta = 0.15\,\mathrm{fm}$$

- Goal: Precise Lattice QCD determination with sub-percent error
  $\Rightarrow$ Isospin breaking effects become relevant!

[1] Bernecker and Meyer 2011; Francis et al. 2013; Della Morte et al. 2017.

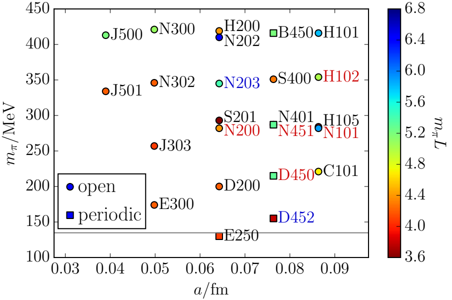
QCD+QED on $\mathrm{QCD_{iso}}$ gauge ensembles

- Non-compact $\mathrm{QED_L}$-action[2] for IR regularisation, Coulomb gauge
- QCD+QED quark action from $\mathrm{QCD_{iso}}$:
  $$(m_\mathrm{u}^{(0)}, m_\mathrm{d}^{(0)}, m_\mathrm{s}^{(0)}) \to (m_\mathrm{u}, m_\mathrm{d}, m_\mathrm{s}) \qquad m_\mathrm{u}^{(0)} = m_\mathrm{d}^{(0)}$$
  $$U^{x\mu} \to W^{x\mu} = U^{x\mu} e^{\mathrm{i}aeQA^{x\mu}} \qquad Q = \mathrm{diag}\left(\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}\right)$$
- QCD+QED on $\mathrm{QCD_{iso}}$ gauge ensembles:
  - QCD+QED parametrised by $\varepsilon = (m_\mathrm{u}, m_\mathrm{d}, m_\mathrm{s}, \beta, e^2)$
  - $\mathrm{QCD_{iso}}$ parametrised by $\varepsilon^{(0)} = (m_\mathrm{u}^{(0)}, m_\mathrm{d}^{(0)}, m_\mathrm{s}^{(0)}, \beta^{(0)}, 0)$
  - Reweighting and perturbative expansion[3] in $\Delta\varepsilon = \varepsilon - \varepsilon^{(0)}$ around $\varepsilon^{(0)}$
- $\mathrm{QCD_{iso}}$ $O(a)$-improved, isospin breaking introduces $O(a)$ lattice artefacts
- Investigate leading order isospin breaking effects
  $\Rightarrow \alpha_\mathrm{em}$ does not renormalise
- Neglect IB effects in lattice spacing a and $\Delta\beta = 0$

[2] Hayakawa and Uno 2008.
[3] Divitiis et al. 2013.

QCD+QED on QCD_{iso} gauge ensembles

- QCD_{iso} ensembles (CLS)[4]:
  - Tree-level improved Lüscher-Weisz gauge action
  - $N_f = 2 + 1$ $O(a)$-improved Wilson fermion action
  - Periodic/open temporal boundary conditions
  - $\text{tr}(M) = const.$



	$(\frac{L}{a})^3 \times \frac{T}{a}$	a [fm]	m_π [MeV]	m_K [MeV]	$m_\pi L$	L [fm]
H102	$32^3 \times 96$	0.08636(10)	354(5)	438(4)	5.0	2.8
N101	$48^3 \times 128$	0.08636(10)	282(4)	460(4)	5.9	4.1
N451	$48^3 \times 128$	0.07634(97)	287(4)	462(5)	5.3	3.7
D450	$64^3 \times 128$	0.07634(97)	217(3)	476(6)	5.4	4.9
D452	$64^3 \times 128$	0.07634(97)	157(2)	483(5)	3.8	4.9
N203	$48^3 \times 128$	0.06426(76)	346(4)	442(5)	5.4	3.1
N200	$48^3 \times 128$	0.06426(76)	282(3)	463(5)	4.4	3.1

processed, work in progress

[4]Bruno et al. 2015; Bruno et al. 2017.

Hadronic renormalisation scheme for QCD+QED

- Fix bare parameters of lattice QCD+QED am_{u}, am_{d}, am_{s}, β, e^2 by (hadronic) renormalisation scheme

- Pseudo-scalar meson masses in χPT + QED[5]:

  $$\hat{m} = \tfrac{1}{2}(m_{\mathrm{u}} + m_{\mathrm{d}}) \qquad \varepsilon = \tfrac{\sqrt{3}}{4}\tfrac{m_{\mathrm{d}} - m_{\mathrm{u}}}{m_{\mathrm{s}} - \hat{m}} \quad \pi^0\text{-}\eta \text{ mixing angle}$$

  At $O(e^2 p^0)$ and $O(\varepsilon)$:

  $$m_{\pi^0}^2 = 2B\hat{m} \qquad m_{K^+}^2 = B\Big((m_{\mathrm{s}} + \hat{m}) - \tfrac{2\varepsilon}{\sqrt{3}}(m_{\mathrm{s}} - \hat{m})\Big) + 2e^2 Z F^2$$

  $$m_{\pi^+}^2 = 2B\hat{m} + 2e^2 Z F^2 \qquad m_{K^0}^2 = B\Big((m_{\mathrm{s}} + \hat{m}) + \tfrac{2\varepsilon}{\sqrt{3}}(m_{\mathrm{s}} - \hat{m})\Big)$$

  In chiral limit: F pion decay constant, B vacuum condensate parameter, Z dimensionless coupling constant

- Proxies for bare parameters $m_{\mathrm{u}} + m_{\mathrm{d}}$, m_{s}, $m_{\mathrm{u}} - m_{\mathrm{d}}$ and e^2:

  $$m_{\pi^0}^2 = B(m_{\mathrm{u}} + m_{\mathrm{d}}) \qquad m_{K^+}^2 + m_{K^0}^2 - m_{\pi^+}^2 = 2Bm_{\mathrm{s}}$$

  $$m_{K^+}^2 - m_{K^0}^2 - m_{\pi^+}^2 + m_{\pi^0}^2 = B(m_{\mathrm{u}} - m_{\mathrm{d}}) \qquad m_{\pi^+}^2 - m_{\pi^0}^2 = 2e^2 Z F^2$$

- Define scheme by equating proxies in theory and experiment

[5] Neufeld and Rupertsberger 1996.

Hadronic renormalisation scheme for QCD+QED

- Scheme for $\mathrm{QCD_{iso}}$:

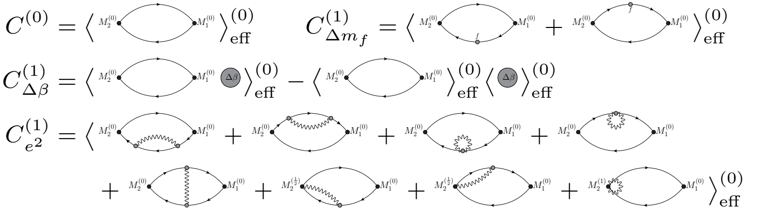
$$m_{\pi^0}^2 \propto m_\mathrm{u} + m_\mathrm{d} \qquad m_{K^+}^2 + m_{K^0}^2 - m_{\pi^+}^2 \propto m_\mathrm{s}$$
$$m_\mathrm{u} = m_\mathrm{d} \qquad \alpha_\mathrm{em} = 0$$

- Scheme for QCD+QED:

$$m_{\pi^0}^2 \propto m_\mathrm{u} + m_\mathrm{d} \qquad m_{K^+}^2 + m_{K^0}^2 - m_{\pi^+}^2 \propto m_\mathrm{s}$$
$$m_{K^+}^2 - m_{K^0}^2 - m_{\pi^+}^2 + m_{\pi^0}^2 \propto m_\mathrm{u} - m_\mathrm{d} \qquad \alpha_\mathrm{em}$$

- Advantages:
  - Physical point of $\mathrm{QCD_{iso}}$ is defined directly via experimental hadronic input
  - $\mathrm{QCD_{iso}}$ scheme applicable without full QCD+QED computation
  - No need to compute renormalised quark masses
  - α_em in principle replaceable by $m_{\pi^+}^2 - m_{\pi^0}^2$
- Disadvantage:
  - Scheme depends on m_{π^0}, but $\pi^0 \to 2\gamma$ in QCD+QED
  - Potentially sizable NLO χPT effects
  - Scheme deviates from GRS scheme (quark mass scheme)

- Diagrammatic expansion of mesonic two-point function $C = \langle \mathcal{M}_2 \mathcal{M}_1 \rangle$ (only quark-connected 0th and 1st order contributions):



- Lattice discretisations of vector current: $\mathcal{V}_\mathrm{l}$ (local), $\mathcal{V}_\mathrm{c}$ (conserved)

$$\mathcal{V}_\mathrm{l}^{x\mu i} = \overline{\Psi}^x \Lambda^i \gamma^\mu \Psi^x \qquad\qquad W^{x\mu} = U^{x\mu} e^{iaeQA^{x\mu}}$$

$$\mathcal{V}_\mathrm{c}^{x\mu i} = \tfrac{1}{2}\left(\overline{\Psi}^{x+a\hat{\mu}} (W^{x\mu})^\dagger \Lambda^i (\gamma^\mu + \mathbb{1}) \Psi^x + \overline{\Psi}^x \Lambda^i (\gamma^\mu - \mathbb{1}) W^{x\mu} \Psi^{x+a\hat{\mu}}\right)$$

- Flavour-diagonal vector currents $\mathcal{V} = (\mathcal{V}^0, \mathcal{V}^3, \mathcal{V}^8)$ with $\Lambda^0 = \frac{1}{\sqrt{6}}\mathbb{1}$, $\Lambda^3 = \frac{1}{2}\lambda^3$ and $\Lambda^8 = \frac{1}{2}\lambda^8$ may mix under renormalisation:

$$\mathcal{V}_\mathrm{l,R} = Z_{\mathcal{V}_\mathrm{l,R}\mathcal{V}_\mathrm{l}} \mathcal{V}_\mathrm{l} \qquad \mathcal{V}_\mathrm{c,R} = Z_{\mathcal{V}_\mathrm{c,R}\mathcal{V}_\mathrm{c}} \mathcal{V}_\mathrm{c} \quad \text{Assumption: } Z_{\mathcal{V}_\mathrm{c,R}\mathcal{V}_\mathrm{c}} = \mathbb{1}$$

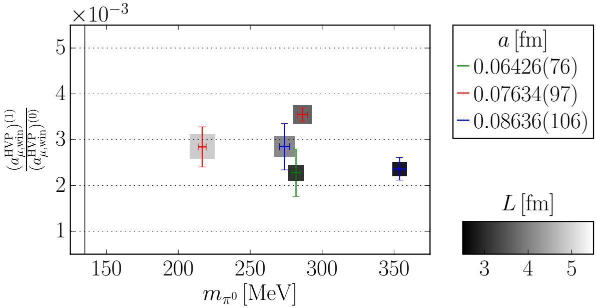
- Renormalisation condition[6] $\langle 0 | \mathcal{V}_\mathrm{c,R} | V \rangle = \langle 0 | \mathcal{V}_\mathrm{l,R} | V \rangle$:

$$\langle \mathcal{V}_\mathrm{c}^{t_2} \mathcal{V}_\mathrm{l}^{t_1} \rangle \to Z_{\mathcal{V}_\mathrm{l,R}\mathcal{V}_\mathrm{l}} \langle \mathcal{V}_\mathrm{l}^{t_2} \mathcal{V}_\mathrm{l}^{t_1} \rangle \qquad \text{for } T \gg t_2 \gg t_1 \gg 0$$

[6] Maiani and Martinelli 1986.

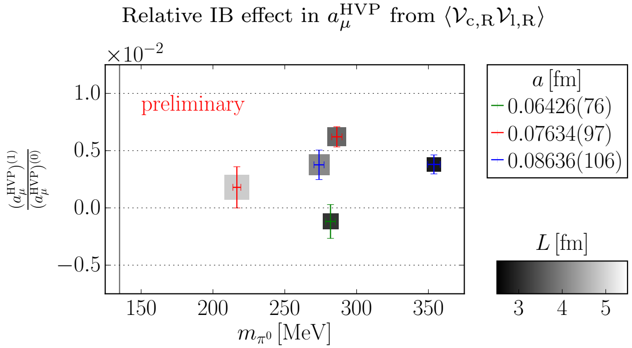
Intermediate window contribution $a_{\mu,\mathrm{win}}^{\mathrm{HVP}}$

Relative IB effect in $a_{\mu,\mathrm{win}}^{\mathrm{HVP}}$ from $\langle \mathcal{V}_{\mathrm{c,R}} \mathcal{V}_{\mathrm{l,R}} \rangle$



- $\frac{(a_{\mu,\mathrm{win}}^{\mathrm{HVP}})^{(1)}}{(a_{\mu,\mathrm{win}}^{\mathrm{HVP}})^{(0)}} = 0.3(1)\%$
- Compatible results for local-local and conserved-local lattice discretisations
- IB in scale setting and QED-FV effects to be included
- Uncertainty on relative correction doubled in final results for $a_{\mu,\mathrm{win}}^{\mathrm{HVP}}$ [7] cf. Simon Kuberski's talk

[7] Cè et al. 2022.



- $\langle \mathcal{V}_{\mathrm{R}}^{\gamma t_2} \mathcal{V}_{\mathrm{R}}^{\gamma t_1} \rangle$ exhibits noise problem for large $t_2 - t_1$:
  $\Rightarrow$ Single state reconstruction via fit
- To be replaced by bounding method[8]
- IB in scale setting and QED-FV effects to be included

[8] Borsanyi et al. 2017; Blum et al. 2018.

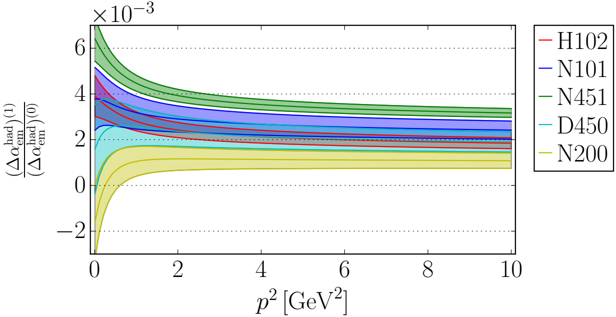
Hadronic contribution to the running of α_{em}

- Running of α_{em}: $\alpha_{\mathrm{em}}(p^2) = \frac{\alpha_{\mathrm{em}}}{1-\Delta\alpha_{\mathrm{em}}(p^2)}$
- LO hadronic contribution to $\alpha_{\mathrm{em}}(-p^2)$ in TMR:

$$\Delta\alpha_{\mathrm{em}}^{\mathrm{had}}(-p^2) = 4\pi\alpha \int_0^\infty \mathrm{d}t\, K(p^2,t)\, C(t)$$

$$C(x^0)\,\delta^{\mu_2\mu_1} = \int \mathrm{d}x^3 \langle \mathcal{V}_{\mathrm{R}}^{\gamma x \mu_2} \mathcal{V}_{\mathrm{R}}^{\gamma 0 \mu_1} \rangle \quad K(\omega^2,t) = -\frac{1}{\omega^2}\left(\omega^2 t^2 - 4\sin^2\left(\frac{\omega t}{2}\right)\right)$$

cf. Marco Cè's talk[9]



Relative IB effect in $\Delta\alpha_{\mathrm{em}}^{\mathrm{had}}(p^2)$ from $\langle \mathcal{V}_{\mathrm{c,R}} \mathcal{V}_{\mathrm{l,R}} \rangle$

[9] Cè et al. 2022.

Outlook

- Include LO QED finite volume corrections
- Determination of a_μ^{HVP} using bounding method
- Determine isospin-breaking effects in octet and decouplet baryons[10]
  $\Rightarrow$ find QCD+QED scale setting candidate
- Determination of $Z_{\mathcal{V}_{\mathrm{l,R}}\mathcal{V}_{\mathrm{l}}}$ by means of vector Ward identity including LO isospin breaking effects
- Analyse further ensembles to explore parameter space
- Include quark-disconnected and sea-quark contributions

[10] with A. Segner, A. Hanlon and H.Wittig

QCD+QED on $\mathrm{QCD_{iso}}$ gauge ensembles

- QCD+QED on $\mathrm{QCD_{iso}}$ gauge ensembles by Monte-Carlo reweighting[11]:
  - $\mathrm{QCD_{iso}}$ ensembles generated with $S_{\mathrm{eff}}^{(0)}[U] = S_{\mathrm{g}}^{(0)}[U] - \log(Z_{\mathrm{q}}^{(0)}[U])$:

$$\langle \mathcal{O}[U] \rangle_{\mathrm{eff}}^{(0)} = \frac{\int DU\, e^{-S_{\mathrm{eff}}^{(0)}[U]}\, \mathcal{O}[U]}{\int DU\, e^{-S_{\mathrm{eff}}^{(0)}[U]}} \qquad Z_{\mathrm{q}}^{(0)}[U] = \int D\Psi D\overline{\Psi}\, e^{-S_{\mathrm{q}}^{(0)}[U,\Psi,\overline{\Psi}]}$$

  - QED in fixed $\mathrm{QCD_{iso}}$ background field U:

$$\langle \mathcal{O}[U, A, \Psi, \overline{\Psi}] \rangle_{\mathrm{q}\gamma} = \frac{1}{Z_{\mathrm{q}\gamma}[U]} \int DAD\Psi D\overline{\Psi}\, e^{-S_\gamma[A] - S_{\mathrm{q}}[U,A,\Psi,\overline{\Psi}]}\, \mathcal{O}[U, A, \Psi, \overline{\Psi}]$$

$$Z_{\mathrm{q}\gamma}[U] = \int DAD\Psi D\overline{\Psi}\, e^{-S_\gamma[A] - S_{\mathrm{q}}[U,A,\Psi,\overline{\Psi}]}$$

  - Full expectation value with reweighting factor $R[U]$:

$$\langle \mathcal{O}[U, A, \Psi, \overline{\Psi}] \rangle = \frac{\langle R[U]\, \langle \mathcal{O}[U, A, \Psi, \overline{\Psi}] \rangle_{\mathrm{q}\gamma} \rangle_{\mathrm{eff}}^{(0)}}{\langle R[U] \rangle_{\mathrm{eff}}^{(0)}} \qquad R[U] = \frac{e^{-S_{\mathrm{g}}[U]}\, Z_{\mathrm{q}\gamma}[U]}{e^{-S_{\mathrm{g}}^{(0)}[U]}\, Z_{\mathrm{q}}^{(0)}[U]}$$

- Leading-order perturbative expansion:

$$(\langle \mathcal{O} \rangle)^{(0)} = \langle (\langle \mathcal{O} \rangle_{\mathrm{q}\gamma})^{(0)} \rangle_{\mathrm{eff}}^{(0)}$$

$$(\langle \mathcal{O} \rangle)_l^{(1)} = \langle (\langle \mathcal{O} \rangle_{\mathrm{q}\gamma})^{(1)} \rangle_{\mathrm{eff}}^{(0)} + \langle R_l^{(1)}\, (\langle \mathcal{O} \rangle_{\mathrm{q}\gamma})^{(0)} \rangle_{\mathrm{eff}}^{(0)} - \langle R_l^{(1)} \rangle_{\mathrm{eff}}^{(0)}\, \langle (\langle \mathcal{O} \rangle_{\mathrm{q}\gamma})^{(0)} \rangle_{\mathrm{eff}}^{(0)}$$

[11] Divitiis et al. 2013.

David Bernecker and Harvey B. Meyer. "Vector Correlators in Lattice QCD: Methods and applications". In: *Eur. Phys. J. A* 47 (2011), p. 148. DOI: 10.1140/epja/i2011-11148-6. arXiv: 1107.4388 [hep-lat].

T. Blum et al. "Calculation of the hadronic vacuum polarization contribution to the muon anomalous magnetic moment". In: *Phys. Rev. Lett.* 121.2 (2018), p. 022003. DOI: 10.1103/PhysRevLett.121.022003. arXiv: 1801.07224 [hep-lat].

Sz. Borsanyi et al. "Slope and curvature of the hadronic vacuum polarization at vanishing virtuality from lattice QCD". In: *Phys. Rev. D* 96.7 (2017), p. 074507. DOI: 10.1103/PhysRevD.96.074507. arXiv: 1612.02364 [hep-lat].

Mattia Bruno, Tomasz Korzec, and Stefan Schaefer. "Setting the scale for the CLS $2+1$ flavor ensembles". In: *Phys. Rev. D* 95.7 (2017), p. 074504. DOI: 10.1103/PhysRevD.95.074504. arXiv: 1608.08900 [hep-lat].

Mattia Bruno et al. "Simulation of QCD with $N_f = 2+1$ flavors of non-perturbatively improved Wilson fermions". In: *JHEP* 02 (2015), p. 043. DOI: 10.1007/JHEP02(2015)043. arXiv: 1411.3982 [hep-lat].

References II

- Marco Cè et al. "The hadronic running of the electromagnetic coupling and the electroweak mixing angle from lattice QCD". In: *JHEP* 08 (2022), p. 220. DOI: 10.1007/JHEP08(2022)220. arXiv: 2203.08676 [hep-lat].
- Marco Cè et al. "Window observable for the hadronic vacuum polarization contribution to the muon $g-2$ from lattice QCD". In: (June 2022). arXiv: 2206.06582 [hep-lat].
- M. Della Morte et al. "The hadronic vacuum polarization contribution to the muon $g-2$ from lattice QCD". In: *JHEP* 10 (2017), p. 020. DOI: 10.1007/JHEP10(2017)020. arXiv: 1705.01775 [hep-lat].
- G.M. de Divitiis et al. "Leading isospin breaking effects on the lattice". In: *Phys. Rev. D* 87.11 (2013), p. 114505. DOI: 10.1103/PhysRevD.87.114505. arXiv: 1303.4896 [hep-lat].
- Anthony Francis et al. "A new representation of the Adler function for lattice QCD". In: *Phys. Rev. D* 88 (2013), p. 054502. DOI: 10.1103/PhysRevD.88.054502. arXiv: 1306.2532 [hep-lat].
- Masashi Hayakawa and Shunpei Uno. "QED in finite volume and finite size scaling effect on electromagnetic properties of hadrons". In: *Prog. Theor. Phys.* 120 (2008), pp. 413–441. DOI: 10.1143/PTP.120.413. arXiv: 0804.2044 [hep-ph].

L. Maiani and G. Martinelli. "Current Algebra and Quark Masses from a Monte Carlo Simulation with Wilson Fermions". In: *Phys. Lett. B* 178 (1986), pp. 265–271. DOI: 10.1016/0370-2693(86)91508-X.

H. Neufeld and H. Rupertsberger. "The Electromagnetic interaction in chiral perturbation theory". In: *Z. Phys. C* 71 (1996), pp. 131–138. DOI: 10.1007/s002880050156. arXiv: hep-ph/9506448.