Time windows of the muon HVP from twisted mass lattice QCD.

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(On behalf of the ETM Collaboration)

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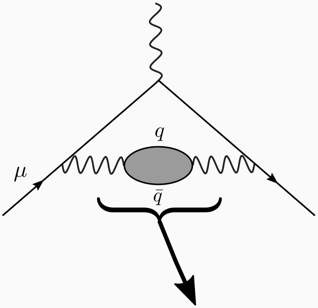
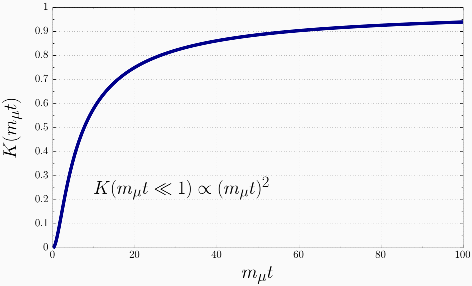
Fifth Plenary Workshop of the Muon g-2 Theory Initiative, September 5th, Edinburgh.

LO-HVP from Lattice QCD





$$\Pi_{\mu\nu}(Q) = \int d^4x\, e^{iQ\cdot x} \left\langle J_\mu(x) J_\nu(0) \right\rangle = \left(\delta_{\mu\nu} Q^2 - Q_\mu Q_\nu\right) \Pi(Q^2)$$

$$\Downarrow$$

$$a_\mu^{\mathrm{LO-HVP}} = 4\alpha_{em}^2 \int_0^\infty dQ^2 \frac{1}{m_\mu^2} f\left(\frac{Q^2}{m_\mu^2}\right) \cdot \left(\Pi(Q^2) - \Pi(0)\right).$$

Time-Momentum representation (Bernecker & Meyer, 2011)

$$a_\mu^{\mathrm{LO-HVP}} = 2\alpha_{em}^2 \int_0^\infty dt\; t^2 K(m_\mu t)\, V(t), \quad V(t) \equiv \frac{1}{3} \sum_{i=1,2,3} \int d\vec{x} \left\langle J_i(\vec{x},t) J_i(0) \right\rangle$$

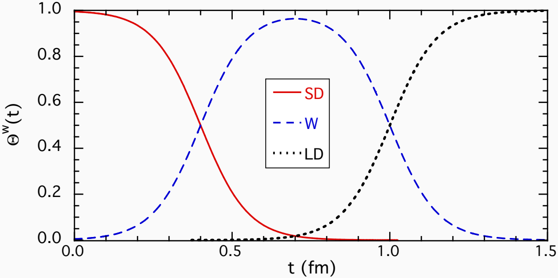
RBC/UKQCD windows

$a_\mu^{\mathrm{LO-HVP}}$ decomposed as a sum of three contributions that probe different (Euclidean) time regions:

$$a_\mu^{SD} = 2\alpha_{em}^2 \int_0^\infty dt\; t^2 K(m_\mu t)\, V(t) \cdot \Theta^{SD}(t)$$

$$a_\mu^{W} = 2\alpha_{em}^2 \int_0^\infty dt\; t^2 K(m_\mu t)\, V(t) \cdot \Theta^{W}(t)$$

$$a_\mu^{LD} = 2\alpha_{em}^2 \int_0^\infty dt\; t^2 K(m_\mu t)\, V(t) \cdot \Theta^{LD}(t)$$



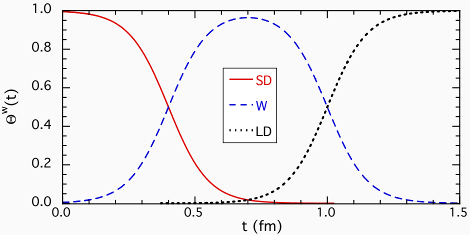
$$\Theta^{SD}(t) + \Theta^{W}(t) + \Theta^{LD}(t) = 1$$

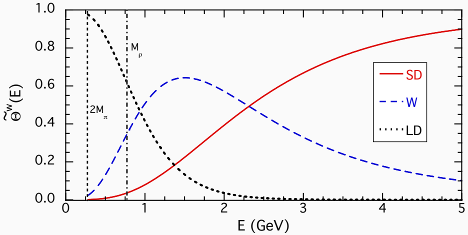
$$\Downarrow$$

$$a_\mu^{\mathrm{LO-HVP}} = a_\mu^{\mathrm{SD}} + a_\mu^{\mathrm{W}} + a_\mu^{\mathrm{LD}}$$

Window observables of $a_\mu^{\mathrm{LO-HVP}}$ as a probe of $R^{had}(E)$

$$\underbrace{2\alpha_{em}^2 \int_0^\infty dt\; t^2 K(m_\mu t)\, V(t)}_{\text{lattice, SM}} = a_\mu^{\mathrm{LO-HVP}} = \underbrace{\frac{2\alpha_{em}^2 m_\mu^2}{9\pi^2} \int_{2M_\pi}^\infty \frac{dE}{E^2}\, \tilde{K}(E)\, R^{had}(E)}_{\text{dispersive, experimental}}$$





$$2\alpha_{em}^2 \int_0^\infty dt\; t^2 K(m_\mu t) V(t) \underline{\Theta^w(t)} = a_\mu^w = \frac{2\alpha_{em}^2 m_\mu^2}{9\pi^2} \int_{2M_\pi}^\infty \frac{dE}{E^2} \tilde{K}(E) R^{had}(E) \underline{\tilde{\Theta}^w(E)}$$

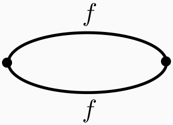
- a_μ^w, $w = \{SD, W, LD\}$, probe $R^{had}(E)$ more locally in energy E.
- Key test of SM (lattice QCD + QED) vs experimental $e^+e^- \to$ hadrons (independent of $g_\mu - 2$).
- Lattice data for $w = SD, W$ are very precise, no S/N problem.

Outline of our twisted-mass lattice QCD calculation

We computed the u, d, s, c, quark-line connected and disconnected contributions to a_μ^{SD} and a_μ^{W} in the isospin symmetric limit $m_u{=}m_d$, neglecting α_{em}^3 QED effects.

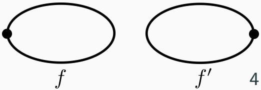
Connected contributions: $f = u, d, s, c$.

$$V_{conn}^{f}(t) \equiv -\frac{1}{3}\sum_{i=1,2,3}\int d^3x \langle J_i^f(\vec{x},t)\; J_i^f(0)\rangle = q_f^2 \times$$



Disconnected contributions: $f, f' = u, d, s, c$.

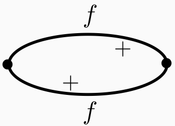
$$V_{disco}^{ff'}(t) \equiv -\frac{1}{3}\sum_{i=1,2,3}\int d^3x \langle J_i^f(\vec{x},t)\; J_i^{f'}(0)\rangle = -q_f q_{f'} \times$$

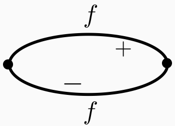


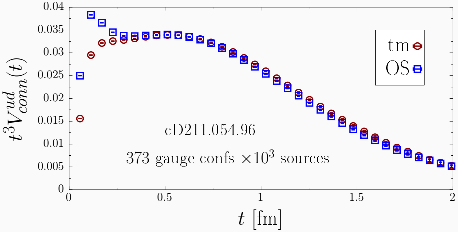
Twisted-mass (tm) and Osterwalder-Seiler (OS) currents

- For connected contributions, two different ways to approach the continuum limit.
- We computed $V^{f}_{conn}(t)$ employing two distinct lattice versions of the local e.m. current (peculiar to tm-LQCD):

$$J_\mu^{f,OS} \propto \bar{\psi}_f^+ \gamma_\mu \psi_f^+ \qquad J_\mu^{f,tm} \propto \bar{\psi}_f^+ \gamma_\mu \psi_f^-$$





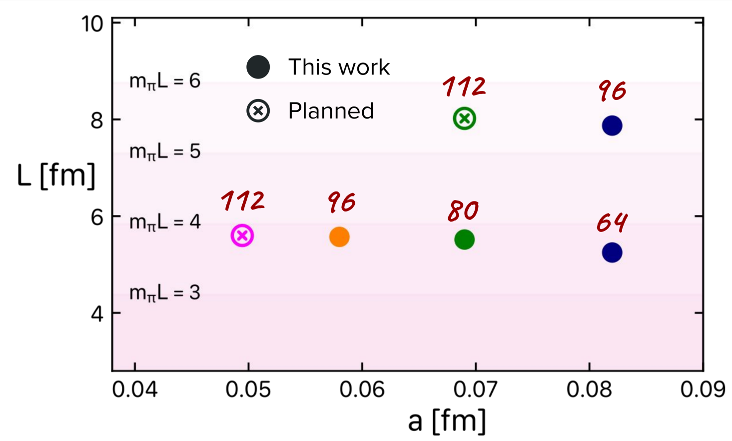


- $\pm$ is the sign of the twisted Wilson parameter.
- Connected $V^{f,OS}_{conn}(t)$ and $V^{f,tm}_{conn}(t)$ differ by $\mathcal{O}(a^2)$ cut-off effects, including a^2-dependent FSEs.

Simulations at the $\simeq$ physical point

Four ($\simeq$) physical point ensembles, with $a \in [0.057 \text{ fm} - 0.080 \text{ fm}]$.
$L \sim 5.1$ fm and $L \sim 7.6$ fm to control Finite Size Effects (FSEs).

$$M_\pi \in [136, 141] \text{ MeV}, \qquad M_\pi L > 3.5, \qquad V = L^3 \times T, \qquad T = 2L.$$



Planned simulations at $a < 0.05$ fm, and on larger volume at $a \sim 0.068$ fm.

Details of the ETMC ensembles

ensemble	β	V/a^4	a (fm)	$a\mu_\ell$	M_π (MeV)	L (fm)	$M_\pi L$
cB211.072.64	1.778	$64^3 \times 128$	0.0796 (1)	0.00072	140.2 (0.2)	5.10	3.62
cB211.072.96	1.778	$96^3 \times 192$	0.0796 (1)	0.00072	140.1 (0.2)	7.64	5.43
cC211.060.80	1.836	$80^3 \times 160$	0.0682 (1)	0.00060	136.6 (0.2)	5.46	3.78
cD211.054.96	1.900	$96^3 \times 192$	0.0569 (1)	0.00054	140.8 (0.3)	5.46	3.90

ensemble	Z_V	Z_A
cB211.072.64	0.706378 (16)	0.74284 (23)
cB211.072.96	0.706402 (15)	0.74274 (20)
cC211.060.80	0.725405 (13)	0.75841 (16)
cD211.054.96	0.744105 (11)	0.77394 (10)

	#inversions (conn)		
ensemble	ℓ	s	c
cB211.072.64	10^3	16	4
cB211.072.96	10^3	16	4
cC211.060.80	10^3	16	4
cD211.054.96	10^3	64	24

- RCs have $< 0.1\%$ uncertainties.
- Iwasaki action for gluons.
- Wilson-clover twisted mass fermions at maximal twist for quarks (automatic $\mathcal{O}(a)$ improvement).

Extended Twisted Mass Collaboration

General fit ansatz

For all connected contributions to a_μ^W and a_μ^{SD} we perform combined continuum fits employing both tm and OS lattice correlators.

- General structure of the fit ansatz ($w = \{SD, W\}$):

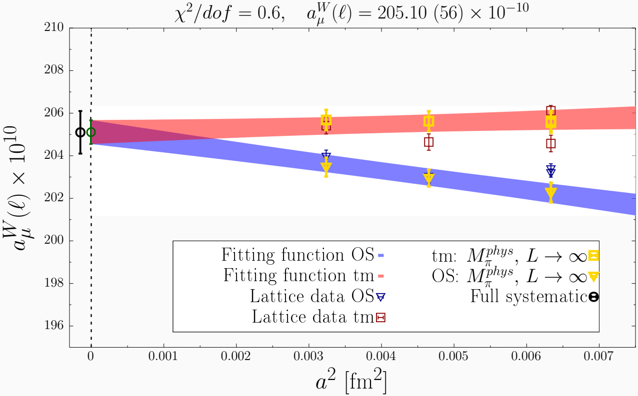
$$a_\mu^w(\ell) = \left[a_\mu^{w,cont}(\ell) + \Delta a_\mu^w(L) + F_1^r a^2 \frac{\partial}{\partial M_\pi} \Delta a_\mu^w(L) \right]$$

$$\times \left[1 + A\left(M_\pi - M_\pi^{phys}\right) + D_1^r \frac{a^2}{[\log(a^2/w_0^2)]^{n_r}} + D_2^r\, a^4 \right]$$

- $a_\mu^{w,cont}$,F_1^r,A,D_1^r and D_2^r are free fitting parameters ($n_r = \{0, 3\}$).
- $a_\mu^{w,cont}$,A do not depend upon the regularization $r = \{tm, OS\}$.
- $\Delta a_\mu^w(L)$: based on MLLGS modelization of Finite Size Effect, tuned to reproduce known lattice results (only for $u + d$ contribution).
- Term proportional to F_1^r describes a^2-dependent FSEs due to $\mathcal{O}(a^2)$ distortions of the pion spectrum.

Connected-light contribution to the intermdiate window: $a_\mu^W(\ell)$

Light $(u+d)$ connected contribution to a_μ^W



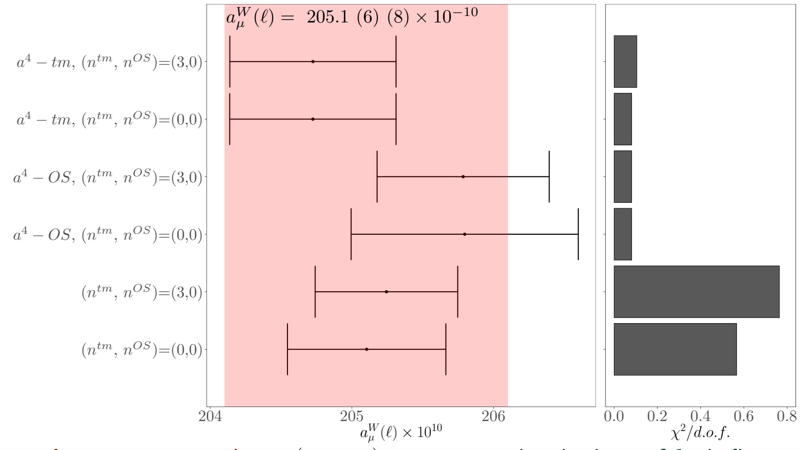
- Typical accuracy of $0.1 - 0.2\%$ for all ensembles and regularizations.
- At $a \sim 0.008$ fm evidence for a^2-dependent FSEs $(F_1^{tm} \neq F_1^{OS})$.
- Negligible for OS and:

$$\Delta a_\mu^{W;tm}(\ell, L \sim 7.6 \text{ fm} \ \ vs. \ \ L \sim 5.1 \text{ fm}) \sim 1.5\ (5) \times 10^{-10}$$

- We show continuum/infinite-volume extrapolation performed using MLLGS-like ansatz (slide 8) w.o. a^4 terms or Logs.

Analysis of the systematics

We selected all fits ($N \simeq \mathcal{O}(10)$) leading to $\chi^2/d.o.f. < 1.8$.



Averaging procedure: (x_k, σ_k) is mean and std. dev. of k-th fit

$$\bar{x} = \sum_{k=1}^{N} \omega_k x_k, \quad \sigma_{\bar{x}}^2 = \sum_{k=1}^{N} \omega_k \sigma_k^2 + \sum_{k=1}^{N} \omega_k (x_k - \bar{x})^2, \quad \omega_k = 1/N$$

Connected-light contribution to the short-distance: $a_\mu^{SD}(\ell)$

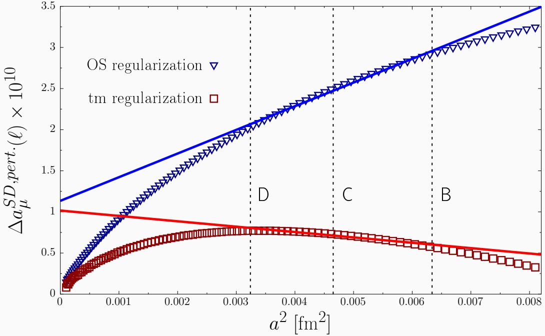
Light (u+d) connected contribution to a_μ^{SD}

Lattice evaluation of a_μ^{SD} suffers from dangerous $a^2 \log(a^2)$ artifacts generated by the short-times integration [Cè, Harris, Meyer et al. (2021)]

$$V(t \ll m^{-1}, a) \propto \frac{1}{t^3}\left[1 + \sum_{n=1}^{\infty} c_n \cdot \left(\frac{a}{t}\right)^{2n}\right], \qquad K(m_\mu t \ll 1) \propto t^2$$

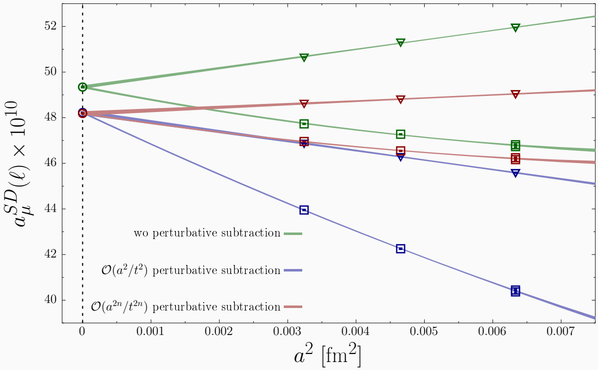
$$\implies a_\mu^{SD} \simeq \int_a^{t_0} dt\, V(t,a)\, t^2 K(m_\mu t) = A + D\, a^2 \log(a^2) + \mathcal{O}(a^2)$$

Naive continuum extrapolation of free theory cut-off effects



- $a^2 \log(a^2)$ cut-off effects already present in the free-theory correlator.
- $\Delta a_\mu^{SD,pert.}(\ell)$ are cut-off effects of the tm (OS) $\mathcal{O}(\alpha_s^0)$ massless correlator.

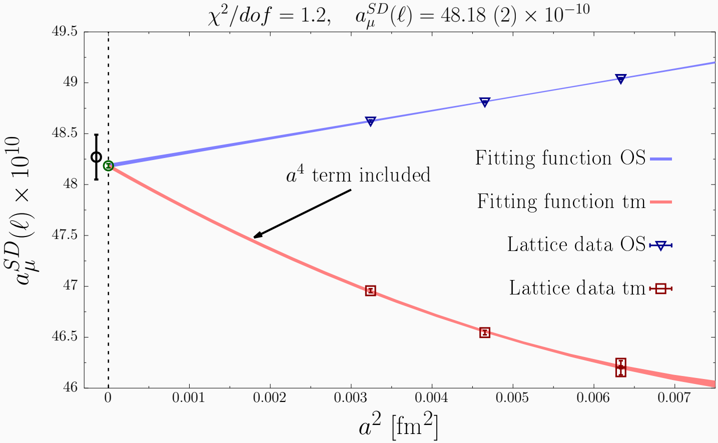
Perturbative $\mathcal{O}(\alpha_s^0)$ subtraction of cut-off effects



Three options:

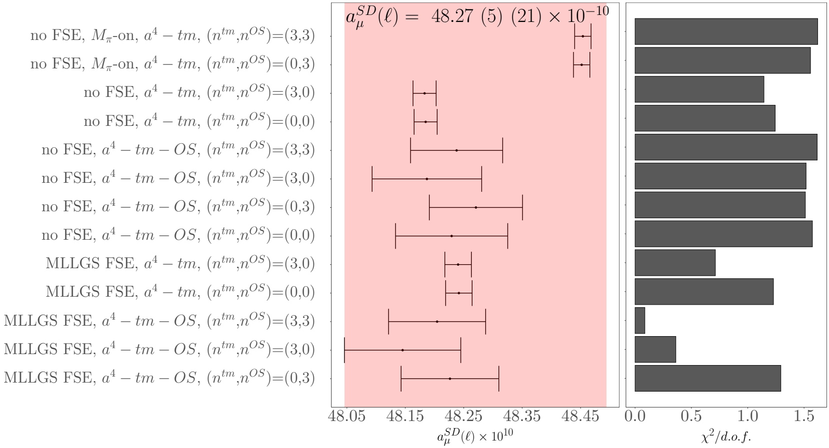
- No perturbative subtraction: continuum limit missed by $\simeq 1 \times 10^{-10}$ (effect larger than any other source of systematics).
- LO $\mathcal{O}(\frac{a^2}{t^2})$ subtraction: sufficient to get correct continuum limit.
- Full $\mathcal{O}(a^{2n}/t^{2n})$ subtraction: makes lattice data even flatter.

$a_\mu^{SD}(\ell)$



- Achieved a remarkable precision better than 0.1%.
- We subtracted from the raw lattice data the $\mathcal{O}(a^{2n}/t^{2n})$ free-theory cut-off effects for both regularizations.
- a^4 term on tm regularization necessary to have a good χ^2/dof.

Analysis of the systematics for $a_\mu^{SD}(\ell)$



- Final error entirely due to systematics in continuum extrapolation.
- $A\left(M_\pi - M_\pi^{phys}\right)$ term not visible.
- Alternative continuum limit extrapolation with ultra-short distance regulator (Backup).

Strange and charm connected contributions to a_μ^W and a_μ^{SD}

Calculation details

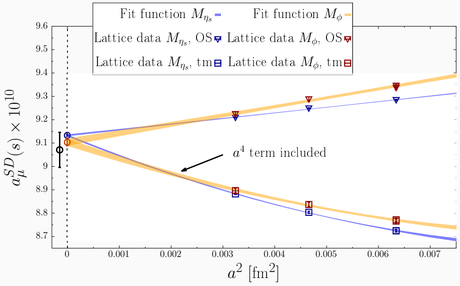
Strange contributions

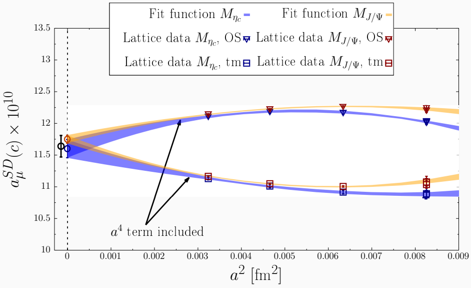
- Valence s quark mass tuned alternatively using M_{η_s} or M_ϕ as input.
- Both determinations included in final analysis of systematics.
- Subtraction of perturbative $\mathcal{O}(\alpha_s^0)$ cut-off effects in $a_\mu^{SD}(s)$.
- Finite size effects and M_π^{sea} mistuning effects not visible within accuracy.

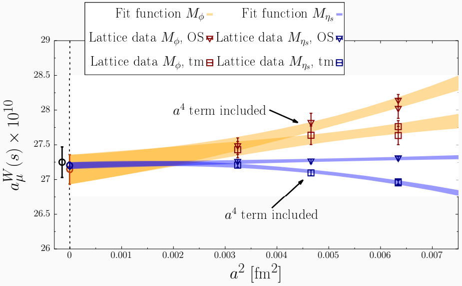
Charm contributions

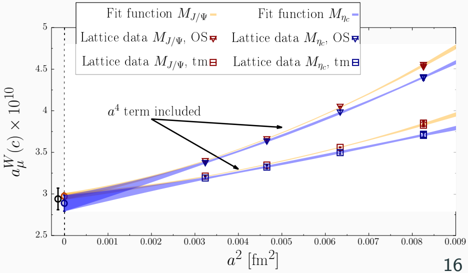
- Valence c quark mass tuned alternatively using M_{η_c} or $M_{J/\Psi}$ as input.
- Both determinations included in final analysis of systematics.
- Added a fourth (coarser) lattice spacing $a \sim 0.09$ fm with pion masses $M_\pi^{sea} \in [250 - 350]$ MeV to improve continuum limit extrapolation.
- No M_π^{sea} dependence observed, negligible finite size effects.
- Subtraction of perturbative $\mathcal{O}(\alpha_s^0)$ cut-off effects in $a_\mu^{SD}(c)$. More effective if evaluated with $m_q = m_c^{bare}$.

Strange and charm connected contributions



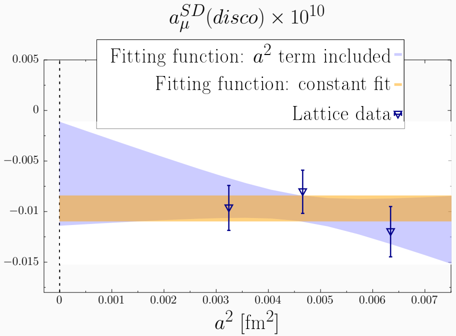


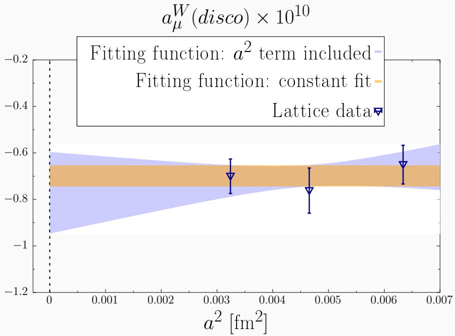




Disconnected contribution to a_μ^W and a_μ^{SD}

Disconnected contribution to a_μ^W and a_μ^{SD}

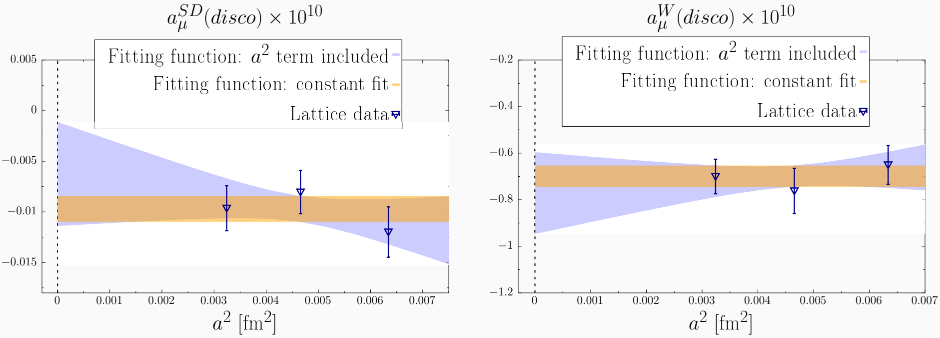




- Various noise-reduction techniques employed: one-end-trick, exact deflation of low-modes, hierarchical probing.
- Small cut-off effects within accuracy. No study of Finite Size Effect.
- $a_\mu^{SD}(disco)$ completely negligible, $a_\mu^W(disco) \sim 0.3\%\ a_\mu^W$.

$$a_\mu^{SD}(disco) = -0.006\ (5) \times 10^{-10}\ , \qquad a_\mu^{W}(disco) = -0.77\ (17) \times 10^{-10}$$

Disconnected contribution to a_μ^W and a_μ^{SD}



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	ETMC-22	BMW-20	CLS/MAINZ-22	RBC/UKQCD-18
$a_\mu^W(disco) \times 10^{10}$	$-0.77\ (17)$	$-0.85\ (6)$	$-0.81\ (9)$	$-1.00\ (10)$

Summary

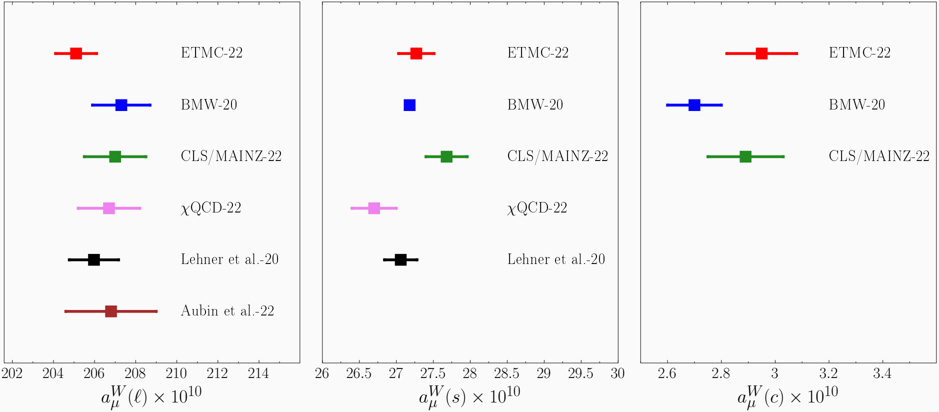
	$a_\mu^{SD} \times 10^{10}$	$a_\mu^{W} \times 10^{10}$
ℓ	48.27 (22)	205.1 (1.0)
s	9.071 (75)	27.27 (24)
c	11.64 (16)	2.95 (13)
disco	−0.006 (5)	−0.77 (17)
IB	0.03*	0.43 (4)**
b	0.32 (2)***	–
total	69.33 (29)	235.0 (1.1)

- * *rhad software package.* 0.04% *of the total* a_μ^{SD} *(or* 0.1σ*).*
- ** *From Borsanyi et al. (Nature, 2021).* 0.18% *of the total* a_μ^{W} *(or* 0.4σ*).*
- *** *rhad & lattice.* 0.46% *of total* a_μ^{SD} *(or* 1.1σ*).*

Precision achieved on a_μ^{W} and a_μ^{SD} is $\sim 0.5\%$.

Per-flavour lattice comparisons...

...include only results from at least 3 lattice spacings and 1 phys. point ensemble.

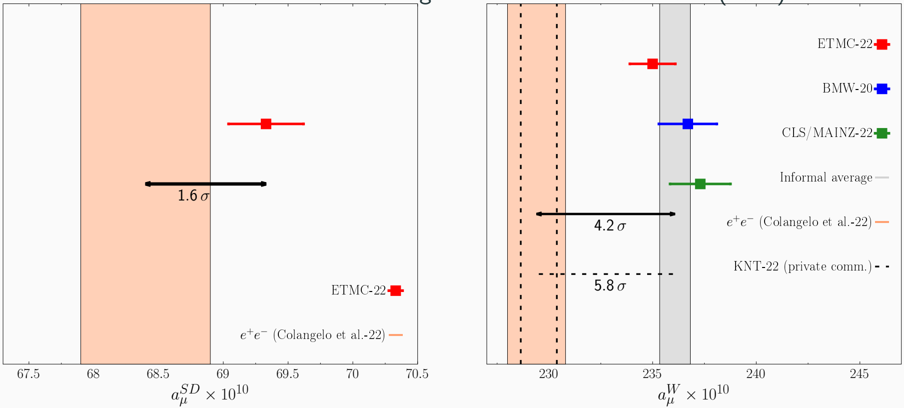


$[a_\mu^{SD} + a_\mu^W] \times 10^{10}$	ℓ	s	c	total, incl. disc., IB, b
ETMC-22*	253.4 (1.2)	36.3 (0.3)	14.6 (3)	304.3 (1.4)
Fermilab/HPQCD/MILC-22	253.5 (0.9)	36.3 (0.2)	14.63 (5)	303.8 (1.1)

*Preliminary: conservative error (Assuming 100% correlation between a_μ^{SD} and a_μ^W).

Comparison with $e^+e^- \to$ hadrons results

$e^+e^- \to$ hadrons from Colangelo et al. arXiv:2205.12963 (2022).



- Tension in a_μ^W rises to 4.2σ if we combine ETMC '22, BMW '20 and CLS/Mainz '22 (informal average → next WP).
- Deviation of $e^+e^- \to$ hadrons data w.r.t. the SM in the low and (possibly) intermediate energy regions, but not in the high energy region.

Towards the determination of $\bar{\Pi}(-Q^2)$

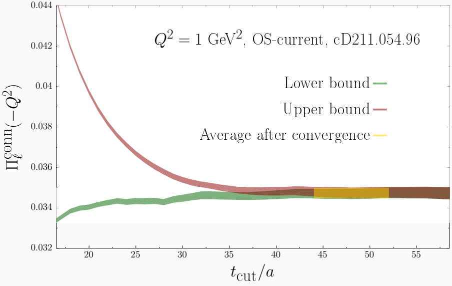
Light-connected contribution (Preliminary) (I)

$$\bar{\Pi}(-Q^2) = \int_0^{\infty} dt\ V(t) \cdot [t^2 - 4\frac{\sin^2(Qt/2)}{Q^2}]$$

- $\bar{\Pi}(-Q^2)$ will be directly measured by MUonE for $Q^2 < 0.14\ \mathrm{GeV}^2$.
- From $\bar{\Pi}(-Q^2)$ one can determine $\Delta\alpha_{\mathrm{had}}(M_Z^2)$ using the Adler function method.

We applied the bounding method to the light-connected correlator $V_\ell(t)$:

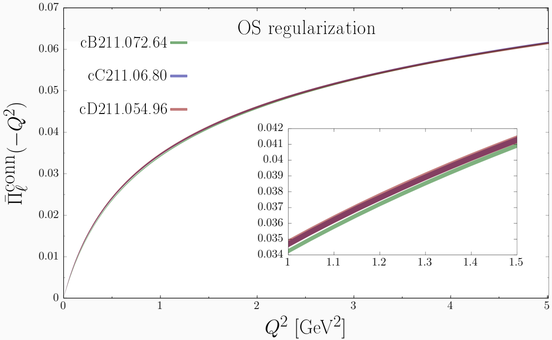
$$V_\ell(t_{\mathrm{cut}})\ e^{-E_{\mathrm{eff}}(t_{\mathrm{cut}})\cdot(t-t_{\mathrm{cut}})} \ \leq\ V_\ell(t)\ \leq\ V_\ell(t_{\mathrm{cut}})\ e^{-E_{\mathrm{low}}\cdot(t-t_{\mathrm{cut}})}$$



- $E_{\mathrm{eff}}(t_{\mathrm{cut}}) = \log\left(\frac{V_\ell(t_{\mathrm{cut}})}{V_\ell(t_{\mathrm{cut}}+a)}\right)$
- E_{low} is the energy of the lightest $I=1\ \pi\pi$ state, obtained from a a MLLGS fit to the tail of $V_\ell(t)$.

Light-connected contribution (Preliminary) (II)

We computed $\bar{\Pi}_\ell^{\rm conn}(-Q^2)$ on all the ETMC ($\simeq$) phys. point ensembles.



- Typical accuracy is $0.4\%,\ 0.8\%$ and 2% at $Q^2 = 5\ \mathrm{GeV}^2,\ 1\ \mathrm{GeV}^2$ and $0.1\ \mathrm{GeV}^2$. Increased statistics is needed at small Q^2.
- FSEs analysis and combined tm-OS continuum extrapolation in progress (perturbative subtraction of cut-off effects needed to suppress the $a^2 \log(a)$ terms).

Conclusions

- We performed a first-principle evaluation of the isosymmetric QCD contribution to a_μ^{SD} and a_μ^{W}.
- Thanks to our dedicated simulations at the $(\simeq)$ physical point, and to a high-statistics computation of the VV correlator, we achieved a relative precision better than 0.5% on both a_μ^{SD} and a_μ^{W}.
- For a_μ^{SD} our SM determination agrees with data-driven results at $1.6\,\sigma$ level (No NP at high E, consistent with EW precision tests).
- Our result for a_μ^{W} confirms the tension between lattice (SM) and dispersive (data-driven) $e^+e^- \to$ hadrons results. Current tension between lattice (informal average) and $R(E)$-driven results is $4.2\,\sigma$ (problems in data-driven results? NP at low/intermediate E?).

Work in progress

- Inclusion of $\mathcal{O}(\alpha_{em}^3)$ QED-effects and strong IB contribution.
- New dedicated ensembles to study FSEs, analysis of a_μ^{LD}.

Extended Twisted Mass Collaboration

Thanks for your attention!

Backup slides

Renormalization constants

Hadronic determination of Z_V

We define:

$$R_V(t) = 2\mu_q \frac{C_{PP}^{tm}(t)}{\partial_t C_{AP}^{tm}(t)}$$

P and A are pseudoscalar and axial local bare currents in C_{AP} and C_{PP}.

Owing to the exact flavor symmetry of massless Wilson fermions ($\mu_q = 0$):

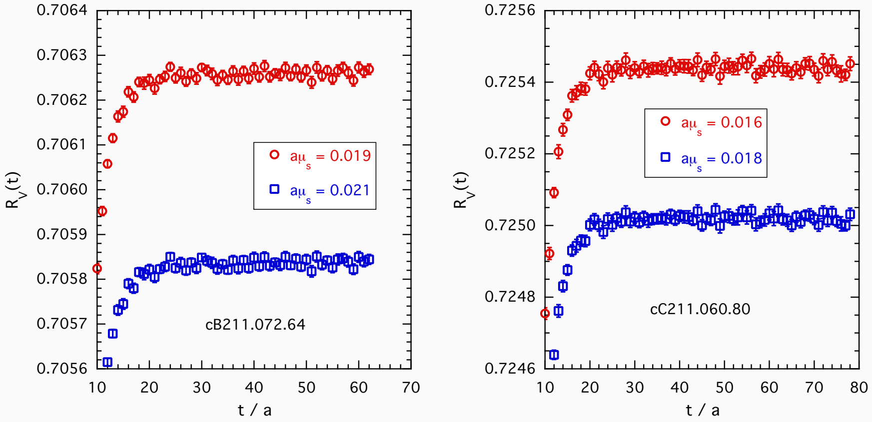
$$R_V(t) \stackrel{t/a \gg 1}{=} Z_V$$

- Freedom in the choice of μ_q (provided μ_q^R is kept constant at all β):

$$Z_V(\mu_q) = Z_V(0) + \mathcal{O}(a^2 \mu_q^R \Lambda_{QCD}, a^2 \mu_q^2)$$

- We choose μ_q^R such that $M_P^{tm} = M_{\eta_s}^{isoQCD}$.
- $M_{\eta_s}^{isoQCD} = 689.89\ (50)\ \mathrm{MeV}$, [*Borsanyi et al., Nature (2021)*].

R_V around the physical strange quark mass



- Remarkable precision better than 0.01%.

Hadronic determination of Z_A

We define:

$$R_A(t) = 2\mu_q \frac{C_{PP}^{OS}(t)}{\partial_t C_{AP}^{OS}(t)}$$

P and A are pseudoscalar and axial local bare currents in C_{AP} and C_{PP}. In the large time limit $t/a \gg 1$ ($X = tm, OS$):

$$C_{PP}^X(t) \to |G_P^X|^2 \frac{e^{-M_P^X t} + e^{-M_P^X (T-t)}}{2M_P^X}, \; R_A(t) \to 2a\mu_q \frac{Z_A}{f_P^{OS}} \frac{G_P^{OS}}{M_P^{OS} \sinh(aM_P^{OS})}$$

We impose (true up to $\mathcal{O}(a^2)$ lattice artifacts):

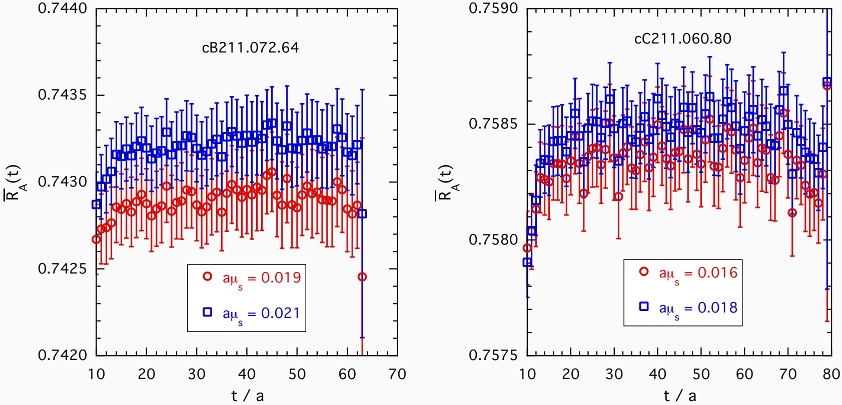
$$\boxed{f_P^{OS} = f_P^{tm} = 2a\mu_q \frac{G_P^{tm}}{M_P^{tm} \sinh(aM_P^{tm})}}$$

and using $\frac{Z_P}{Z_S} = \frac{G_P^{OS}}{G_P^{tm}}$, we can extract Z_A through

$$\boxed{\bar{R}_A(t) \equiv R_A(t) \frac{M_P^{OS} \sinh(aM_P^{OS})}{M_P^{tm} \sinh(aM_P^{tm})} \frac{Z_S}{Z_P} \to Z_A}$$

$Z_A(\mu_q) = Z_A(0) + \mathcal{O}(a^2 \mu_q^R \Lambda_{QCD}, (a\mu_q^R)^2) \implies$ any choice of μ_q^R legitimate.

$\bar{R}_A$ around the physical strange quark mass.



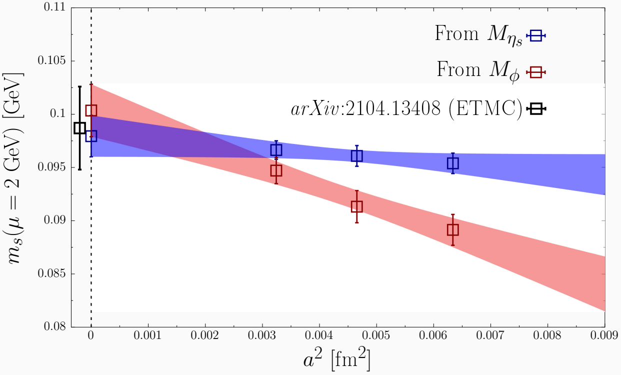
We interpolate $Z_A(\mu_q)$ at $Z_A(\mu_q = \mu_s^{phys})$ with $M_{\eta_s}(\mu_s^{phys}) = 689.89\ (50)\ \mathrm{MeV}$.

Strange and charm quark masses

Strange quark mass in the $\overline{\mathrm{MS}}$ scheme at $\mu = 2$ GeV.

We determine m_s^{phys} using two different hadronic inputs (both included in the a_μ^W and a_μ^{SD} analyses).

- From the mass of the *fictious* η_s meson: $M_{\eta_s}^{\text{isoQCD}} = 689.9(5)$ MeV.
- From the mass of the vector ϕ meson: $M_\phi^{PDG} = 1.01946(2)$ GeV.

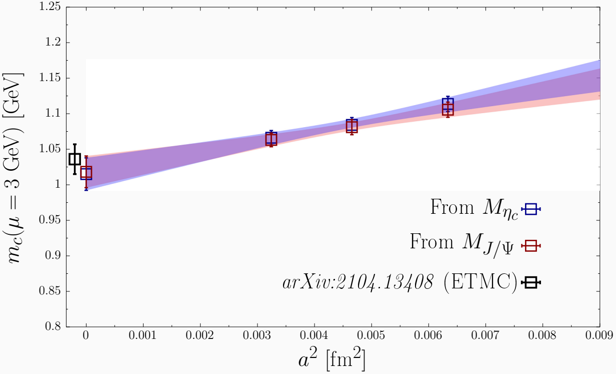


Good agreement with previous ETMC determination from m_K.

Charm quark mass in the $\overline{\mathrm{MS}}$ scheme at $\mu = 3$ GeV.

We determine m_c^{phys} using two different hadronic inputs (both included in the a_μ^W and a_μ^{SD} analyses).

- From the mass of the η_c meson: $M_{\eta_c} = 2.984\ (4)_{disco}$ GeV.
- From the mass of the J/Ψ resonance: $M_{J/\Psi} = 3.097\ (1)_{disco}$ GeV.

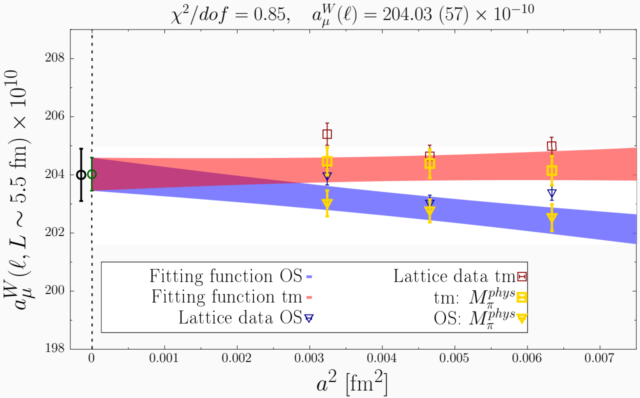


Good agreement with previous ETMC determination from m_D.

Cross-check for $a_\mu^W(\ell)$

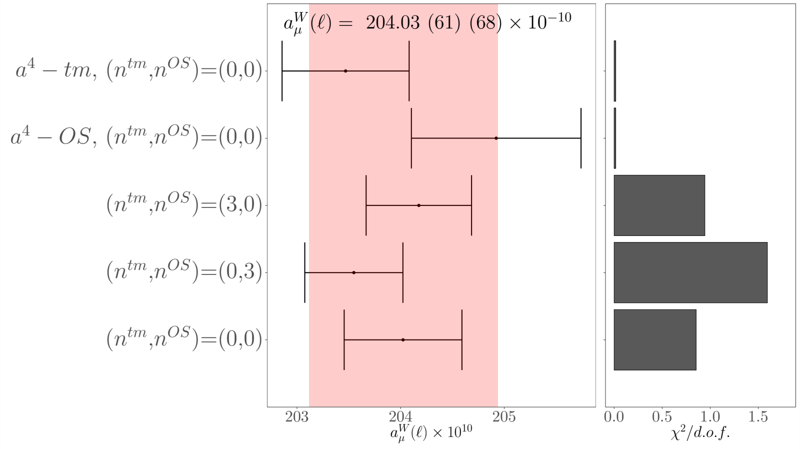
Cross-check: $a_\mu^W(\ell)$ at fixed physical volume

We performed the continuum extrapolation at fixed $L \sim 5.46$ fm, which corresponds to the cC.060.80 and cD.054.96 ensemble volumes.



- At $a \sim 0.08$ fm we interpolated $a_\mu^W(\ell)$ at $L \sim 5.46$ fm, linearly in $e^{-M_\pi L}$, using our results at $L \sim 5.10$ fm and $L \sim 7.64$ fm.

Analysis of the systematics at $L \sim 5.46$ fm



We add to $a_\mu^W(\ell, L \sim 5.46\ \mathrm{fm})$, the FSE estimated from the MLLGS-model multiplied by the correcting factor $\kappa = 1.25\ (25)$:

$$\Delta a_\mu^W(\ell, L \sim 5.46\ \mathrm{fm}) = 1.00(20) \times 10^{-10} \implies a_\mu^W(\ell) = 205.03(93) \times 10^{-10}$$

Cross-check for $a_\mu^{SD}(\ell)$

Cross-check for $a_\mu^{SD}(\ell)$ (I)

Introduce an UV regulator t_{min} and define:

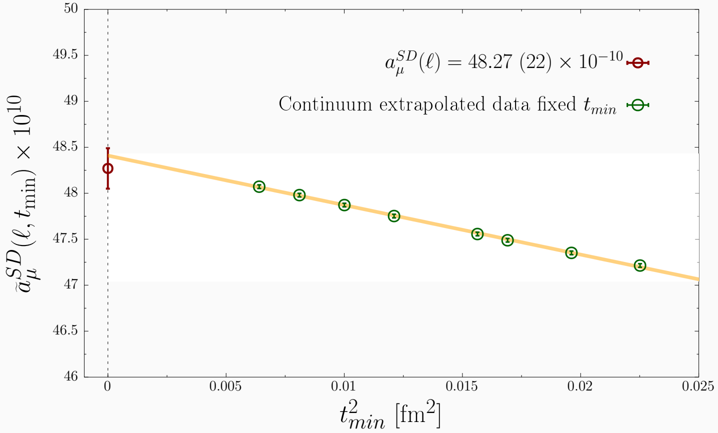
$$a_\mu^{SD}(\ell, t_{min}) \equiv 4\alpha_{em}^2 \int_{t_{min}}^{\infty} dt\; t^2 K(m_\mu t)\, V_\ell(t) \cdot [1 - \Theta(t, t_0, \Delta)]$$

- Switch the order in which the limit $a \to 0$ and $t_{\rm min} \to 0$ are performed (perform $a \to 0$ first).
- Residual Logs $a^2/[\log(a^2/w_0^2)]^n$ generated by integration at short-times become $a^2/[\log(t_{min}^2/w_0^2)]^n$.
- After performing cont. extr. at fixed t_{min}, take $t_{min} \to 0$ limit.
- To speed-up convergence we add continuum perturbative $\mathcal{O}(\alpha_s^0)$ contribution in $t \in [0, t_{min}]$:

$$\widetilde{a}_\mu^{SD}(\ell, t_{min}) = a_\mu^{SD}(\ell, t_{min}) + \int_0^{t_{min}} dt\; t^2 K(m_\mu t)\, V_\ell^{pert.}(t) \cdot [1 - \Theta(t, t_0, \Delta)]$$

- We use $t_{min} \in [0.08, 0.15]$ fm, kept fixed for all ensembles.

Cross-check for $a_\mu^{SD}(\ell)$ (II)



- For small t_{min}: $\tilde{a}_\mu^{SD}(\ell, t_{\rm min}) \propto t_{min}^2$.
- $\tilde{a}_\mu^{SD}(\ell, 0) = 48.41 \pm 0.03 \pm syst \times 10^{-10}$.
- Good agreement with our previous determination:

$$a_\mu^{SD}(\ell) = 48.27\ (22) \times 10^{-10}$$

Finite size effects

Finite Size Effects in the MLLGS-model (I)

Assuming the dominance of $\pi\pi$ states contributions in $V_\ell(t)$:

$$V_\ell(t,L) \sim V_{\pi\pi}(t,L) = \sum_n \nu_n |A_n|^2 e^{-\omega_n t}, \qquad \underbrace{\delta_{11}(k_n) + \phi\left(\frac{k_n L}{2\pi}\right) = n\pi}_{\text{Lellouch-Luscher FV quantisation}}$$

- A_n and δ_{11} depend on the time-like pion form factor $F_\pi(\omega)$.
- In th MLLGS model:

$$F_\pi^{(GS)}(\omega) = \frac{M_\rho^2 - A_{\pi\pi}(0)}{M_\rho^2 - \omega^2 - A_{\pi\pi}(\omega)}$$

- The (twice-subtracted) $\pi\pi$ amplitude $A_{\pi\pi}$ depends on three parameters M_ρ, M_π and $g_{\rho\pi\pi}$:

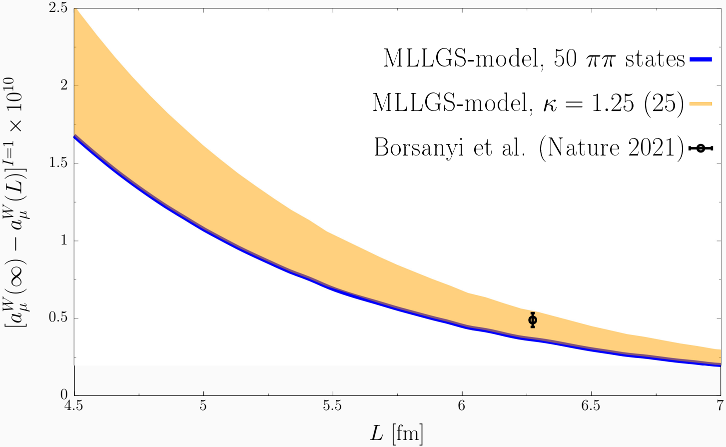
$$\Gamma_{\rho\pi\pi}(\omega) = \frac{g_{\rho\pi\pi}^2}{6\pi}\frac{k^3}{\omega^2}, \qquad k \equiv \sqrt{\omega^2/4 - M_\pi^2}$$

- FSE determined from $V_{\pi\pi}^\infty(t) - V_{\pi\pi}(t,L)$ with

$$V_{\pi\pi}^\infty(t) = \frac{1}{48\pi^2}\int_{2M_\pi}^\infty d\omega\ \omega^2 \left[1 - \frac{4M_\pi^2}{\omega^2}\right]^{3/2} |F_\pi(\omega)|^2 e^{-\omega t}$$

Finite Size Effects in the MLLGS-model (II)

$$M_\pi \sim 135 \text{ MeV} \;, \qquad M_\rho \sim 775 \text{ MeV} \;, \qquad \Gamma_{\rho\pi\pi} \sim 149 \text{ MeV}$$



- Correcting factor $\kappa = 1.25\ (25)$ to take into account deviation between the FSEs from the MLGGS-model and the BMW result.

Disconnected contributions

Disconnected contributions per flavor

ℓ, s, c are flavour-diagonal light, strange and charm contributions.

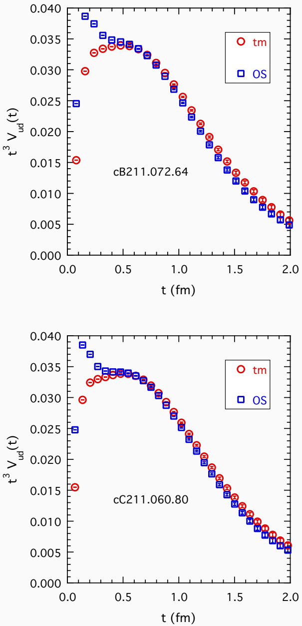
us, uc and sc are flavour off-diagonal light-strange, light-charm and strange-charm contributions.

$a_\mu^{SD}(disco) \times 10^{12}$						
Ensemble	ℓ	s	c	us	uc	sc
cB211.072.64	$-3.36\ (11)$	$-2.090\ (58)$	$-1.18\ (18)$	$+5.29\ (12)$	$-1.52\ (23)$	$+1.67\ (17)$
cC211.060.80	$-3.36\ (14)$	$-2.090\ (69)$	$-0.78\ (13)$	$+5.53\ (19)$	$-1.48\ (24)$	$+1.37\ (18)$
cD211.054.96	$-3.54\ (16)$	$-2.084\ (71)$	$-0.71\ (14)$	$+5.60\ (19)$	$-1.50\ (22)$	$+1.27\ (17)$

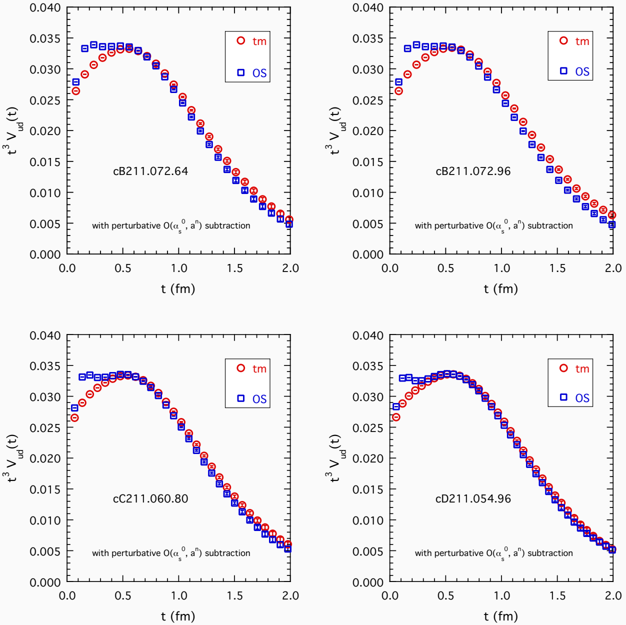
$a_\mu^{W}(disco) \times 10^{10}$						
Ensemble	ℓ	s	c	us	uc	sc
cB211.072.64	$-1.086\ (48)$	$-0.149\ (20)$	$-0.030\ (57)$	$+0.635\ (48)$	$+0.00\ (7)$	$-0.02\ (5)$
cC211.060.80	$-1.300\ (63)$	$-0.159\ (27)$	$-0.033\ (46)$	$+0.726\ (73)$	$-0.03\ (9)$	$+0.04\ (6)$
cD211.054.96	$-1.201\ (52)$	$-0.149\ (28)$	$+0.018\ (56)$	$+0.627\ (70)$	$+0.02\ (9)$	$-0.02\ (7)$

Effect of perturbative subtraction in vector correlator

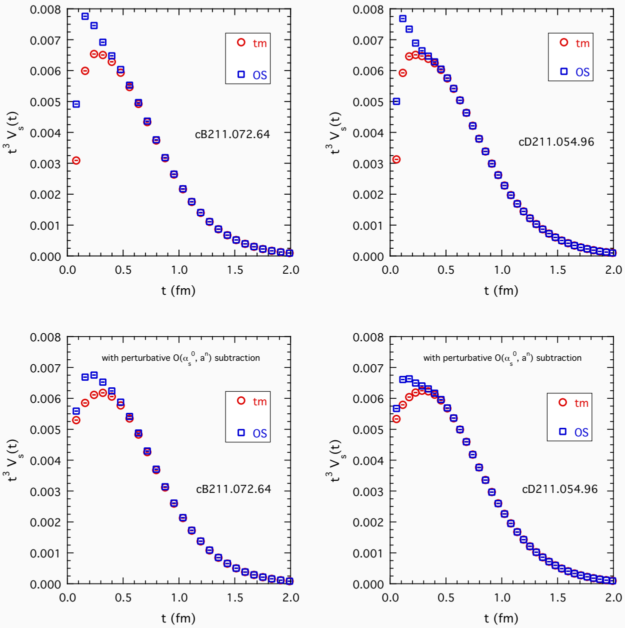
Light-connected (Unsubtracted)



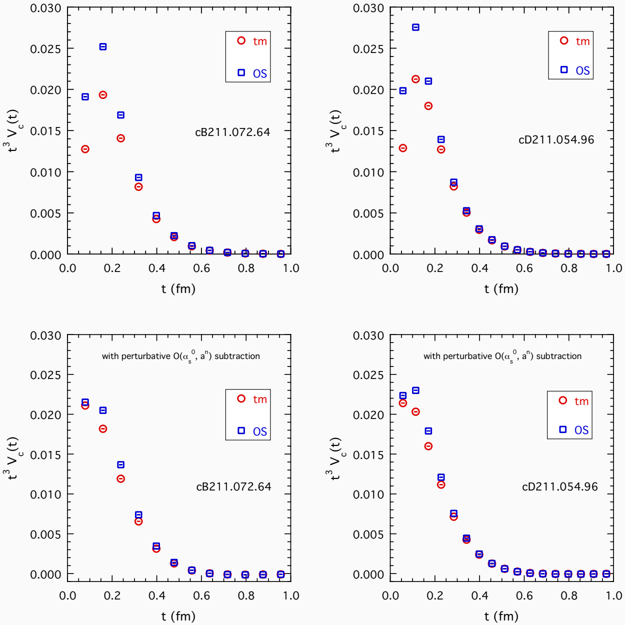
Light-connected (subtracted)



Strange-connected



Charm-connected



Charged, neutral and OS pion masses

Charged, neutral and OS pion masses

- In our twisted-mass mixed action setup, three different type of pions occur (charged, OS, neutral): masses differing from each other by $\mathcal{O}(a^2)$.
- Charged (or tm) pion is the only Goldstone ($M_{\pi^+}(m_q = 0) = 0$) at finite a: in the sea and in the valence.
- OS pion is the heaviest: only in the valence.

ensemble	$M_{\pi^+}^{tm}$ (MeV)	M_{π}^{OS} (MeV)
cB211.072.64	$140.2\ (0.2)$	$297.5\ (0.7)$
cB211.072.96	$140.1\ (0.2)$	$298.4\ (0.5)$
cC211.060.80	$136.6\ (0.2)$	$248.9\ (0.5)$
cD211.054.96	$140.8\ (0.3)$	$210.0\ (0.4)$

- Neutral pion the lightest: only in the sea.
- Neutral pion mass calculated on the cB211.072.64 ensemble (*ETMC (2018) arXiv:1807.00495*):

$$\delta M_\pi \equiv M_\pi^{\text{isoQCD}} - M_{\pi^0} = 26\ (22)\ \text{MeV}.$$

- δM_π estimated to be $\sim 13\ (11)\ \text{MeV} \left(\frac{a}{0.06}\right)^2$ on our finest lattice.
- Dominating contributions: in $V^{OS}(t)$ two tm π, in $V^{tm}(t)$ tm+neutral π.

tm mixed-action setup

Mixed action

Local and renormalizable mixed action employed
[Frezzotti and Rossi (2004)] :

$$S = S_{YM}[U] + S_{q,sea}[\Psi_\ell, \Psi_h, U] + S_{q,val}[q_f^\eta, U] + S_{q,ghost}[\phi_f^\eta, U]$$

- Gluonic sector: improved Iwasaki action $S_{YM}[U]$ (not detailed here).
- Fermionic sector: sea quark action $S_{g,sea}$ written in terms of degenerate quark doublet $\Psi_\ell^t = \{u_{sea}, d_{sea}\}$, and heavy non-degenerate doublet $\Psi_h^t = \{c_{sea}, s_{sea}\}$.
- Fermionic sector: valence quark action $S_{g,val}$ written in terms of quark fields q_f^η, where $f = u, d, s, c$, and the replica index η runs from 1 to 3 with $r_f^\eta = (-1)^{\eta+1}$.
- Ghost sector $S_{q,ghost}$ introduced to cancel contribution of $S_{q,val}$ to fermionic determinant.

$S_{q,sea}$ and $S_{q,val}$

Sea quark action

$$S_{q,sea} = a^4 \sum_x \left\{ \bar{\Psi}_\ell(x) \left[\gamma \cdot \tilde{\nabla} + \mu_\ell - i\gamma_5 \tau^3 W_{cr}^{clov.} \right] \Psi_\ell \right.$$

$$\left. + \bar{\Psi}_h(x) \left[\gamma \cdot \tilde{\nabla} + \mu_\sigma + \mu_\delta \tau^3 - i\gamma_5 \tau^1 W_{cr}^{clov.} \right] \Psi_h \right\}$$

Valence quark action

$$S_{q,val} = a^4 \sum_x \bar{q}_f^\eta(x) \left[\gamma \cdot \tilde{\nabla} + m_f - \underbrace{r_{f,\eta}}_{(-1)^{\eta+1}} i\gamma_5 W_{cr}^{clov.} \right] q_f^\eta(x)$$

Critical Wilson-clover operator

$$W_{cr}^{clov.} = -\frac{a}{2} \nabla^* \cdot \nabla + m_{cr} + \frac{c_{SW}}{32} \gamma_\mu \gamma_\nu \left[Q_{\mu\nu} - Q_{\nu\mu} \right]$$

Role of the replica fields (I)

- We want to make sure that any connected or disconnected single-flavour contribution ($G =$ connected ℓ, s or c ; sum of disconnected) we considered, admits (separately) a well defined continuum limit.

- For connected contributions this must be true for both tm and OS regularizations.

- Given the mixed action S, for each contribution $a_\mu^{w,G}$ to a_μ^w considered separately, suitable renormalized currents $J^{G,ren}, J'^{G,ren}$, exist such that:

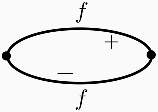
$$a_\mu^{w,G} = 4\alpha_{em}^2 \int \; dt \; K_\mu(t) \; M_w(t) \; \frac{1}{3} \int d^3x \; \langle J_i^{G,ren}(\vec{x},t) J_i'^{G,ren}(0) \rangle$$

$$M_w(t) = \begin{cases} 1 - \Theta(t, t_0, \Delta) & w = \mathrm{SD} \\ \Theta(t, t_0, \Delta) - \Theta(t, t_1, \Delta) & w = \mathrm{W} \end{cases}$$

Role of the replica fields (II)

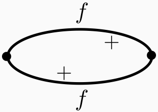
The mixed action S serves to this goal, and the QCD correlators of interest can be easily reconstructed as $a \to 0$.

- Connected, flavour f, contrib. in tm reg. (RC $= Z_A$):



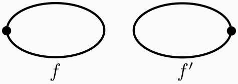
$$= \langle J_\mu^{f,tm}(x) J_\mu^{f,tm}(0) \rangle, \text{ with } J_\mu^{f,tm}(x) = \bar{q}_f^1(x)\gamma_\mu q_f^2(x)$$

- Connected, flavour f, contrib. in OS reg. (RC $= Z_V$):



$$= \langle J_\mu^{f,OS}(x) J_\mu^{f,OS}(0) \rangle, \text{ with } J_\mu^{f,OS}(x) = \bar{q}_f^1(x)\gamma_\mu q_f^3(x)$$

- Disconnected, flavours f, f', contrib. in OS reg. (RC $= Z_V$) :



$$= -\langle J_\mu^{f}(x) J_\mu^{f'}(0) \rangle, \text{ with } J_\mu^{f(f')}(x) = \bar{q}_{f(f')}^{1(3)}(x)\gamma_\mu q_{f(f')}^{1(3)}(x)$$

- For Wilson fermions $Z_V^0 = Z_V$ (known to 2-loops order, proof to appear in paper).

Comparison with e^+e^- and other lattice determinations

Comparison with e^+e^-

window (w)	$a_\mu^w(\text{LQCD})$	$a_\mu^w(e^+e^-)$ [1]	Δa_μ^w	$a_\mu^w(2\pi)$ [1]	$\Delta a_\mu^w / a_\mu^w(2\pi)$
SD	69.3 (0.3) [*]	68.4 (0.5)	0.9 (0.6)	13.7 (0.1)	0.066 (43)
W	235.0 (1.1) [*]	229.4 (1.4)	5.6 (1.9)	138.3 (1.2)	0.040 (14)
HVP	707.5 (5.5) [2]	693.0 (3.9)	14.5 (6.7)	494.3 (3.6)	0.029 (14)

[*] = this work.

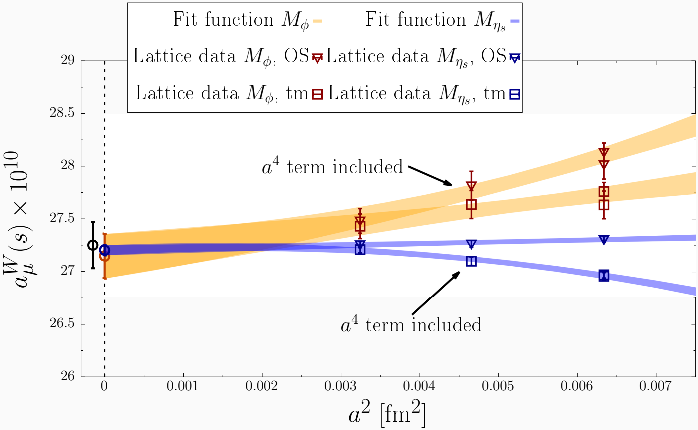
[1] = *Colangelo et al. arXiv:2205.12963 (2022).*

[2] = *Borsanyi et al. (Nature, 2021).*

- $a_\mu^W(2\pi)$ is $\pi - \pi$ states contribution below $1\ \text{GeV}$ from e^+e^- [1].
- Almost constant rescaling of $a_\mu^w(2\pi)$ ($\sim 4\%$) sufficient to reconcile all window contributions.

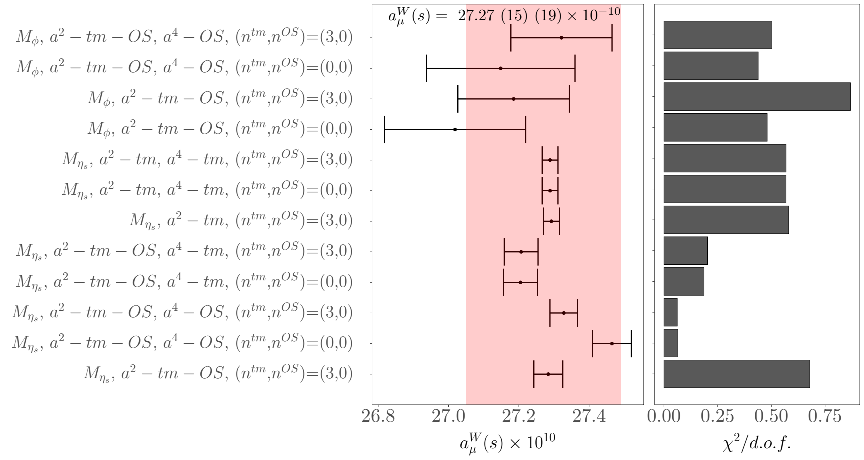
Connected-strange contribution to a_μ^W and a_μ^{SD}: $a_\mu^W(s)$ and $a_\mu^{SD}(s)$

Connected-strange contribution to a_μ^W



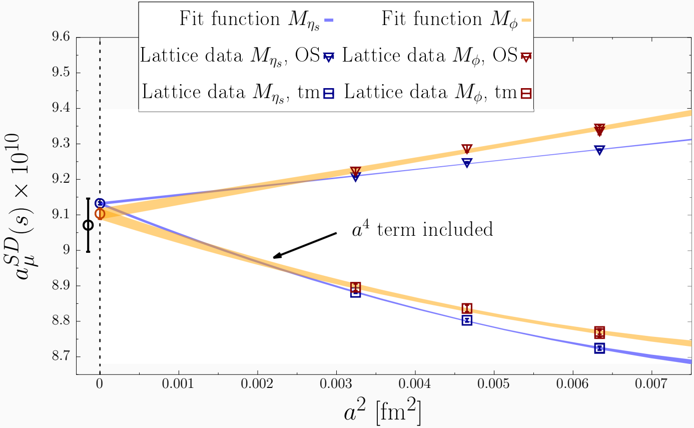
- Strange quark mass tuned alternatively using M_{η_s} or M_ϕ as input (Backup).
- Typical precision achieved for $a_\mu^W(s)$ is 0.1% (0.4% using M_ϕ).
- Finite Size, and M_π mistuning effects not visibile within accuracy.

Analysis of the systematics for $a_\mu^W(s)$



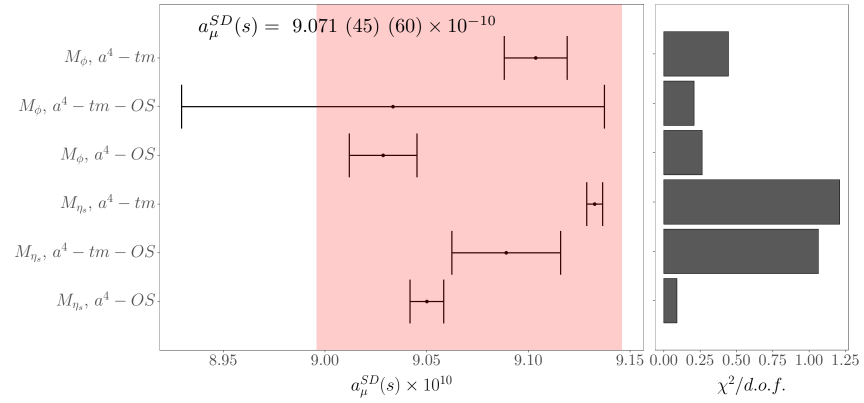
- Performed a total of about 30 different fits.
- Final estimate comes from combining the results obtained using M_{η_s} and M_ϕ to determine the physical strange mass m_s^{phys}.

Connected-strange contribution to a_μ^{SD}



- Subtraction of perturbative $\mathcal{O}(\alpha_s^0)$ cut-off effects evaluated with $m_q = m_s^{bare} \neq 0$.
- Typical precision achieved for $a_\mu^{SD}(s)$ is 0.1%.
- Finite Size, and M_π mistuning effects not visibile within accuracy.

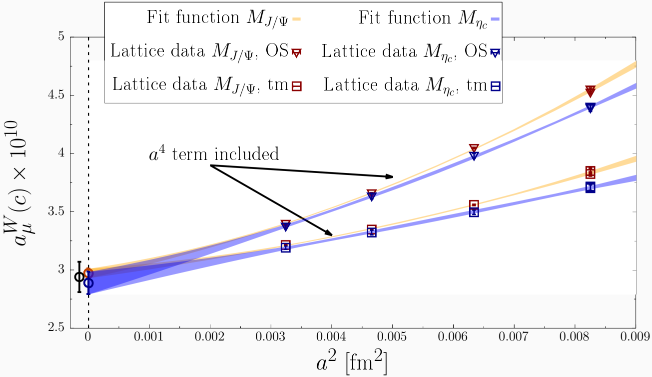
Analysis of the systematics for $a_\mu^{SD}(s)$



- Performed a total of about 30 different fits.
- Final estimate comes from combining the results obtained using M_{η_s} and M_ϕ to determine the physical strange mass m_s^{phys}.
- Total error dominated by continuum limit extrapolation.

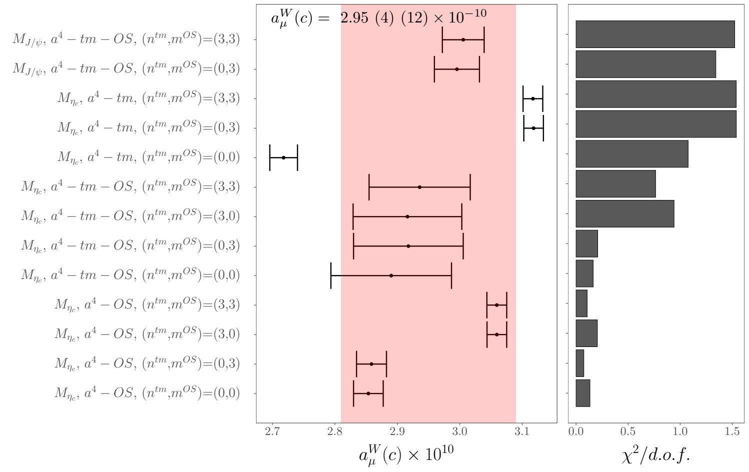
Connected-charm contribution to a_μ^W and a_μ^{SD}: $a_\mu^W(c)$ and $a_\mu^{SD}(c)$

Connected-charm contribution to a_μ^W



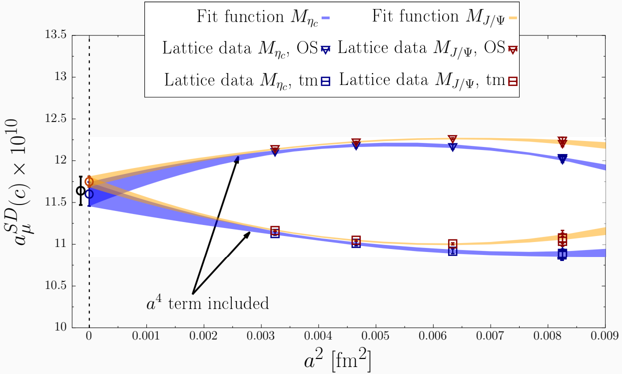
- Charm quark mass tuned alternatively via M_{η_c} or $M_{J/\Psi}$ as input (Backup). Visible a^4 cut-off effects.
- Added a fourth (coarse) lattice spacing $a \sim 0.09$ fm with $M_\pi \in [250, 350]$ MeV. No data on the cB211.072.96 ensemble.
- No M_π dependence or FSEs observed.

Analysis of the systematics for $a_\mu^W(c)$



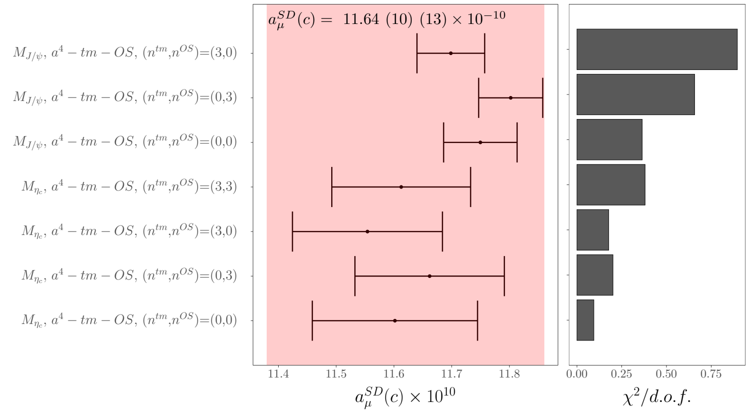
- Performed a total of about 60 different fits.
- Final estimate comes from combining the results obtained using M_{η_c} and $M_{J/\Psi}$ to determine the physical charm mass m_c^{phys}.

Connected-charm contribution to a_μ^{SD}



- Subtraction of $\mathcal{O}(\alpha_s^0)$ cut-off effects more effective if evaluated with $m_q = m_c^{bare} \neq 0$.
- Added the fourth (coarse) lattice spacing with $a \sim 0.09$ fm.
- Typical precision achieved for $a_\mu^{SD}(c)$ is $\sim 0.1\%$ on B, C, D ensembles ($\sim 0.5 - 0.8\%$ on the A ensembles).
- No M_π dependence or FSEs observed.

Analysis of the systematics for $a_\mu^{SD}(c)$



- Performed a total of about 60 different fits.
- Final estimate comes from combining the results obtaining using M_{η_c} and $M_{J/\Psi}$ to determine the physical charm mass m_c^{phys}.
- a^4 corrections on both tm and OS regularizations needed to have a good χ^2/dof.

Impact of scale setting uncertainty on a_μ^w

$$a_\mu^w = \int d\tau\ \tau^2\ V^L(\tau)\ K^w(a,\tau), \qquad \tau = t/a$$

- V^L is the vector correlator in lattice units, K^w is full kernel for each window.
- Uncertainty Δa on the lattice spacing propagates to a_μ^w only through dependence of K^w on a (kernel parameters m_μ, t_0, t_1, Δ must be converted in lattice units):

$$\frac{\Delta a_\mu^w}{a_\mu^w} = \left|\frac{\partial \log\left(a_\mu^w\right)}{\partial \log\left(a\right)}\right| \cdot \frac{\Delta a}{a} \equiv C^w \cdot \frac{\Delta a}{a}$$

- C^w can be computed from derivative dK^w/da.

	SD	W	LO-HVP
C^w	0.134 (1)	0.436 (5)	1.79 (3)

- Estimate for C^w obtained on the cB211.072.64 ensemble.