# The DHMZ methodology for the data-driven HVP determination with realistic uncertainties

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*Based (mainly) on*: 1908.00921

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# Content of the talk

- Introduction: the HVP contribution to $(g\text{-}2)_\mu$ and $\alpha_{QED}(m_Z)$
- Data on $e^+e^- \to$ hadrons
- Combination of all $e^+e^-$ data:

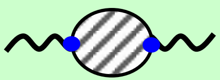
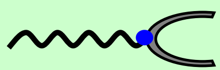
  focus on the combination procedure

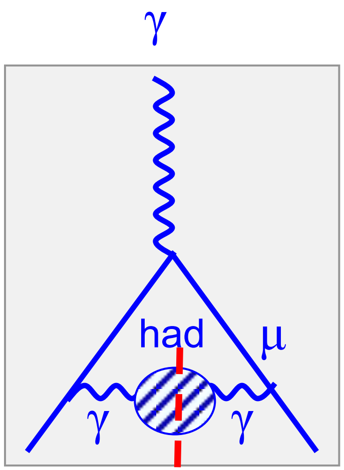
  (HVPTools and fit based on analyticity & unitarity)
- Indications of uncertainties on uncertainties and on correlations & their implications for combinations
- Results on $a_\mu$ and $\alpha_{QED}(m_Z)$
- Conclusions

Dominant uncertainty for the theoretical prediction: from lowest-order HVP piece
Cannot be calculated from QCD (low mass scale), but one can use experimental data on $e^+e^- \to$ hadrons cross section

Born: $\sigma^{(0)}(s) = \sigma(s)\left(\alpha/\alpha(s)\right)^2$

$$12\pi\ \mathrm{Im}\Pi_\gamma(s) = \frac{\sigma^0\,[e^+e^- \to \mathrm{hadrons}\ (\gamma_{FSR})]}{\sigma_{pt}} \equiv R(s)$$

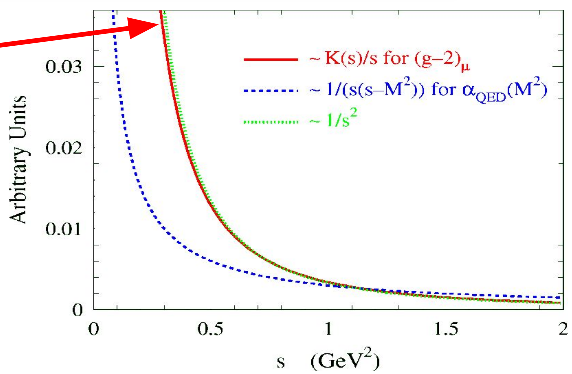
Im[  ] ∝ |  hadrons $|^2$




$$a_\mu^{\mathrm{had}} = \frac{\alpha^2}{3\pi^2}\int_{4m_\pi^2}^{\infty} ds\, \frac{K(s)}{s} R(s)$$

**Dispersion relation**

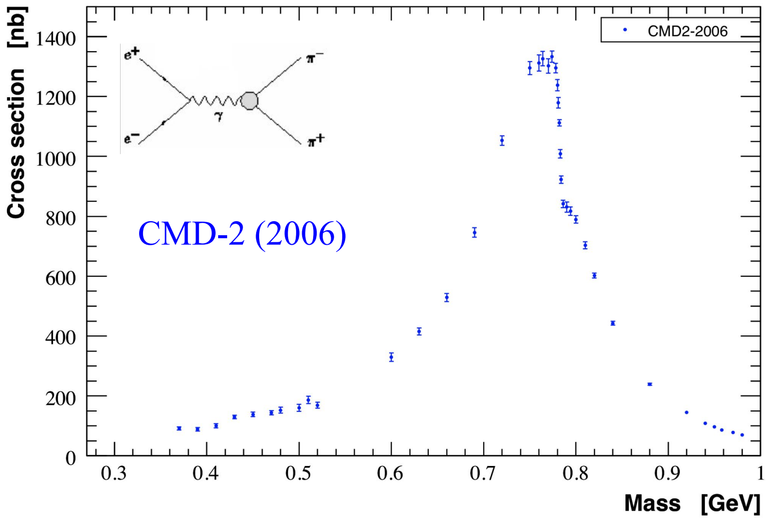
**Bouchiat and Michel, 1961**

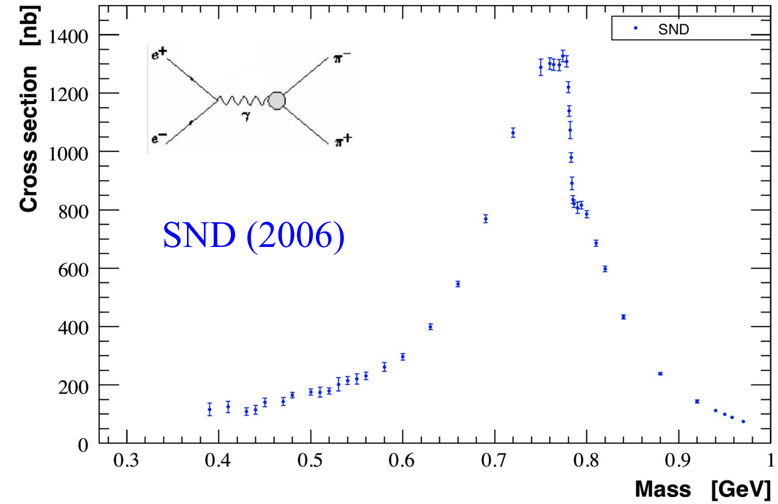


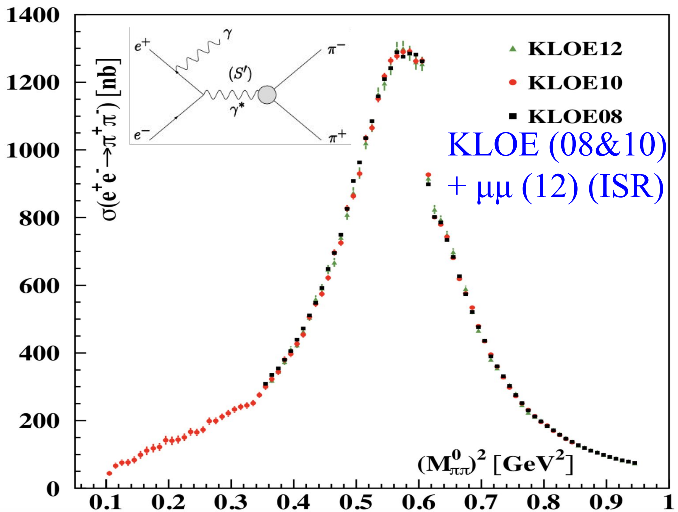

→ Precise $\sigma(e^+e^- \to \mathrm{hadrons})$ measurements at low energy are very important

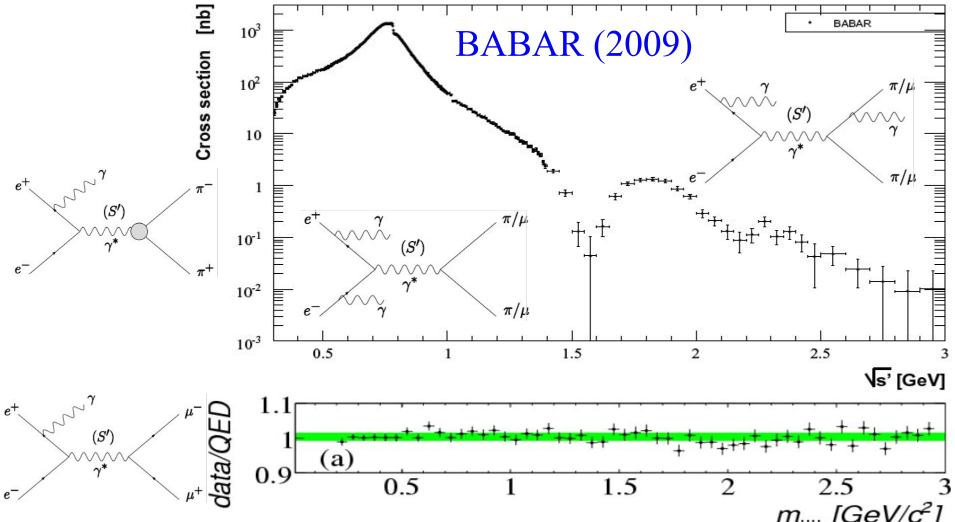
→ Do not use hadronic τ decays data anymore (less precise + theory uncertainties)

# HVP: Data on $e^+e^- \rightarrow$ hadrons

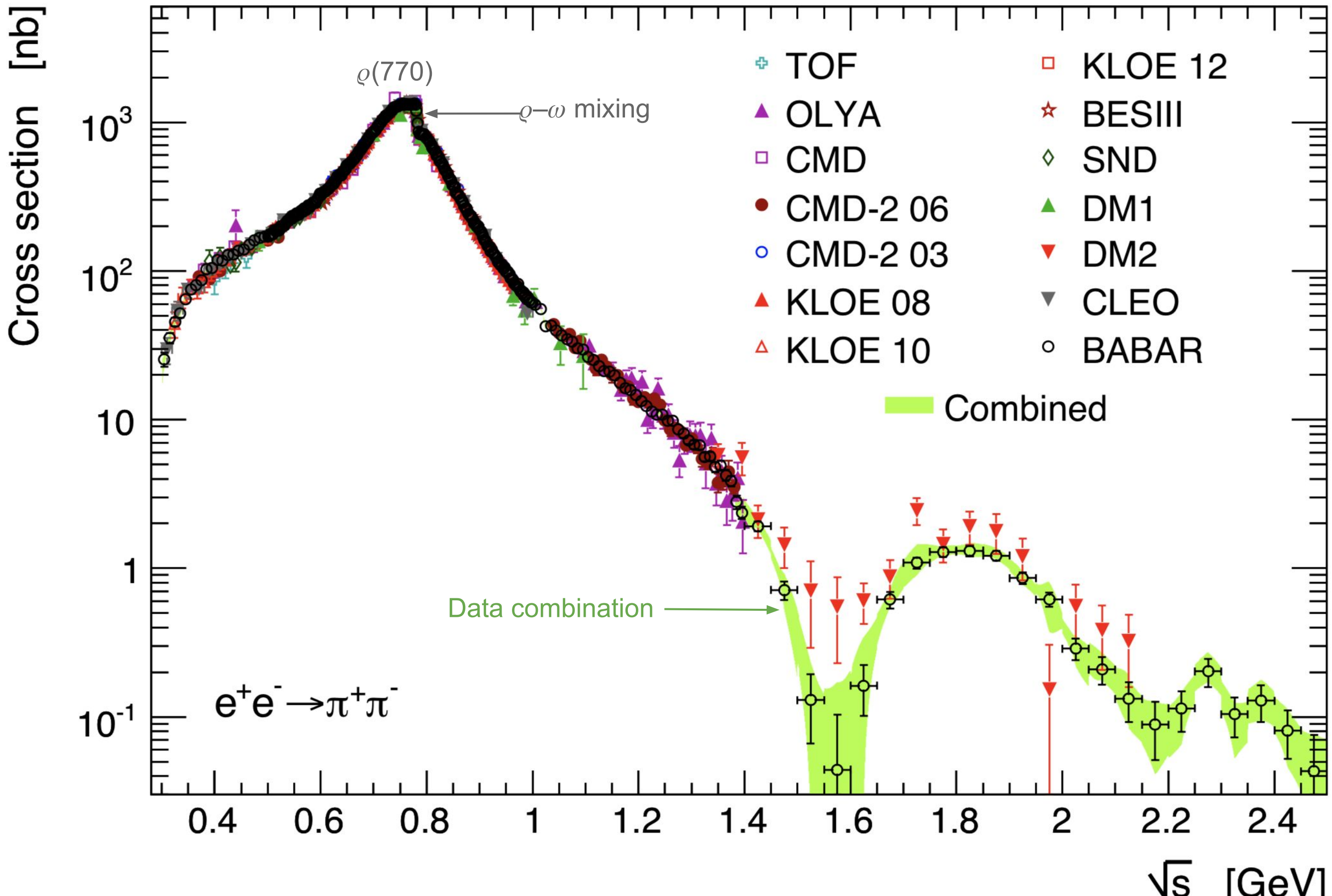


# Combination for the $e^+e^- \to \pi^+\pi^-$ channel




Procedure and software (*HVPTools - Since 2009*) for combining cross section data with arbitrary point spacing/binning

# Combine cross section data: goal and requirements

→ Goal: combine experimental spectra with arbitrary point spacing / binning

→ Requirements:

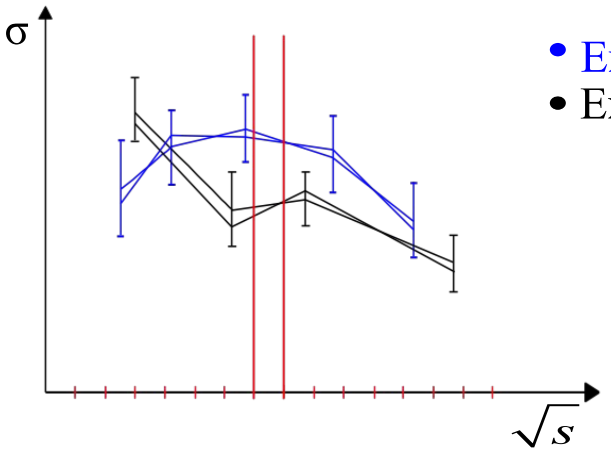
- Properly propagate uncertainties and correlations

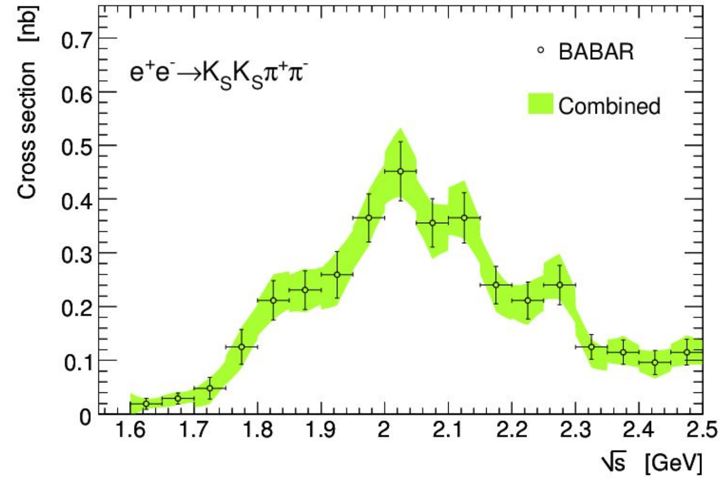
  - *Between measurements (data points/bins) of a given experiment*
    (covariance matrices and/or detailed split of uncertainties in sub-components)
  - *Between experiments* (common systematic uncertainties, e.g. VP)
    based on detailed information provided in publications
  - *Between different channels* – motivated by understanding of the meaning of systematic uncertainties and identifying the common ones
    BABAR luminosity (ISR or BhaBha), efficiencies (photon, Ks, Kl, modeling);
    BABAR radiative corrections; $4\pi 2\pi^0 - \eta\omega$
    CMD2 $\eta\gamma - \pi^0\gamma$; CMD2/3 luminosity; SND luminosity;
    FSR; hadronic VP (old experiments)
    *( $1^{st}$ motivation for using DHMZ uncertainties as "baseline" in the g-2 TI White Paper 2006.04822)*

- Minimize biases

- Optimize g-2 integral uncertainty
  (*without overestimating the precision with which the uncertainties of the measurements are known*)

# Combination procedure implemented in HVPTools software




→ Define a (fine) final binning (to be filled and used for integrals etc.)

→ Linear/quadratic splines to interpolate between the points/bins of each experiment
- for binned measurements: preserve integral inside each bin
- closure test: replace nominal values of data points by Gounaris-Sakurai model and re-do the combination

  → *(non-)negligible bias for (linear)quadratic interpolation*

→ Fluctuate data points taking into account correlations & re-do the splines for each (pseudo-)experiment
- each uncertainty fluctuated coherently for all the points/bins that it impacts
- eigenvector decomposition for (statistical) covariance matrices

# Combination procedure implemented in HVPTools software

For each final bin:

→ Compute an average value for each measurement and its uncertainty

→ Compute correlation matrix between experiments

→ Minimize $\chi^2$ and get average coefficients (weights)

→ Compute average between experiments and its uncertainty

Evaluation of integrals and propagation of uncertainties:

→ Integral(s) evaluated for nominal result and for each set of toy pseudo-experiments; uncertainty of integrals from RMS of results for all toys

→ The pseudo-experiments also used to derive (statistical & systematic) covariance matrices of combined cross sections → Integral evaluation

→ Uncertainties also propagated through $\pm 1\sigma$ shifts of each uncertainty:
- allows to account for correlations between different channels (for integrals and spectra)

→ *Checked consistency between the different approaches*

# Combination procedure: weights of various measurements

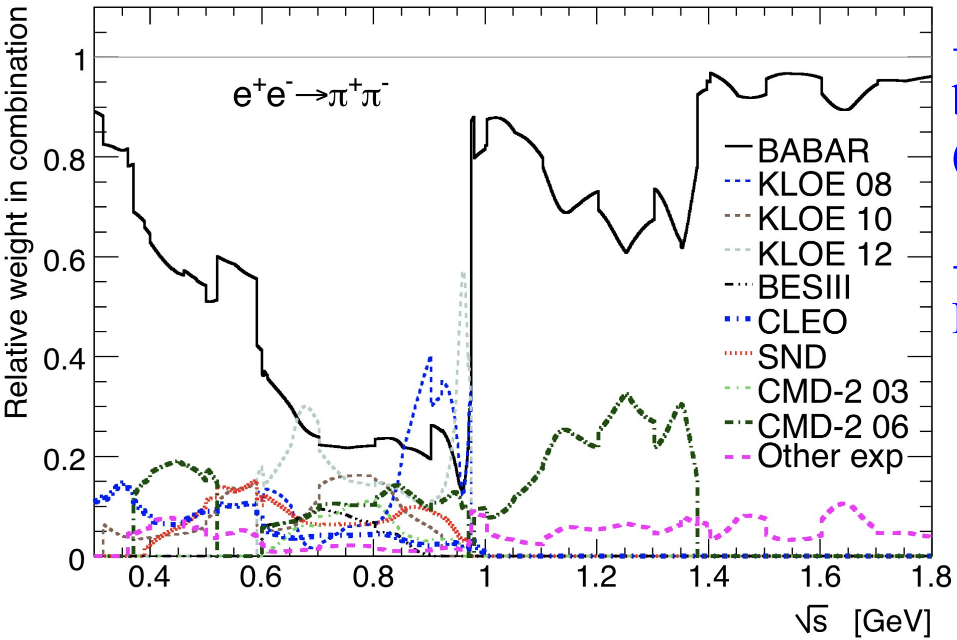
For each final bin:

→ Minimize $\chi^2$ and get average coefficients

Note: average weights must account for bin sizes / point spacing of measurements
(do not over-estimate the weight of experiments with large bins)

→ weights in fine bins evaluated using a common (large) binning for measurements + interpolation

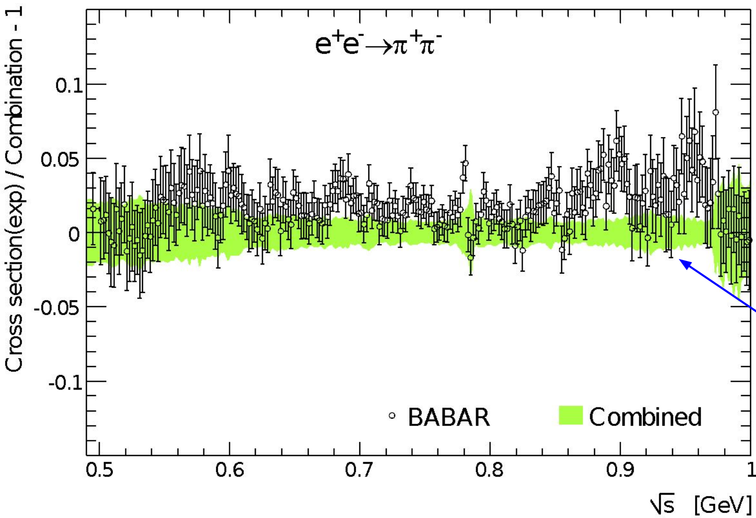
→ compare the precisions on the same footing

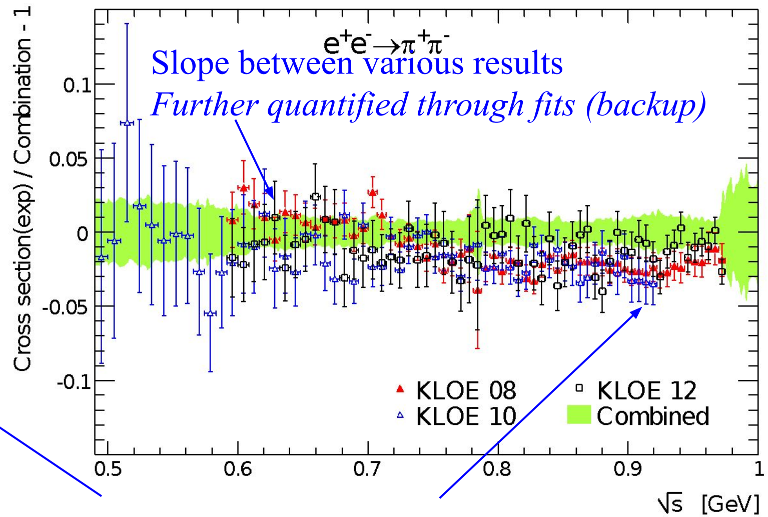



→ Bins used by KLOE larger than the ones by BABAR in ρ-ω interference region (factor ~3)

→ Average dominated by BaBar and KLOE, BaBar covering full range

# More on the combination for the $e^+e^- \to \pi^+\pi^-$ channel



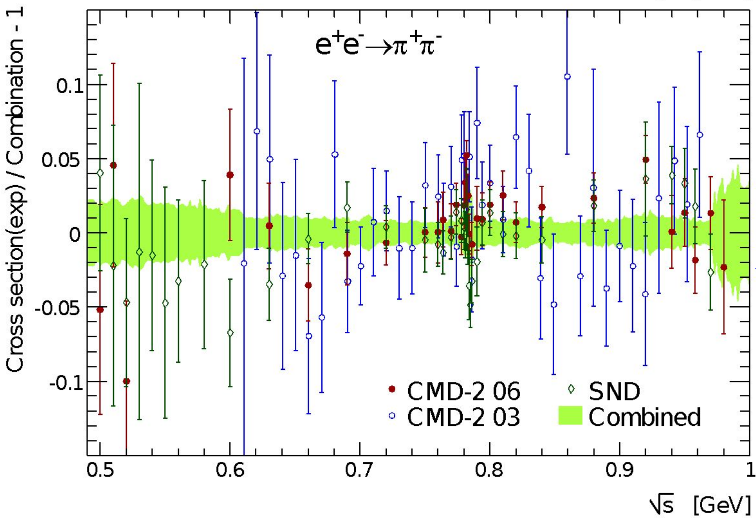




Slope between various results

*Further quantified through fits (backup)*

Local tension & systematic trends

*Indication of "uncertainties on uncertainties" (i.e. unaccounted biases)*




Other experiments not yet precise enough to discriminate

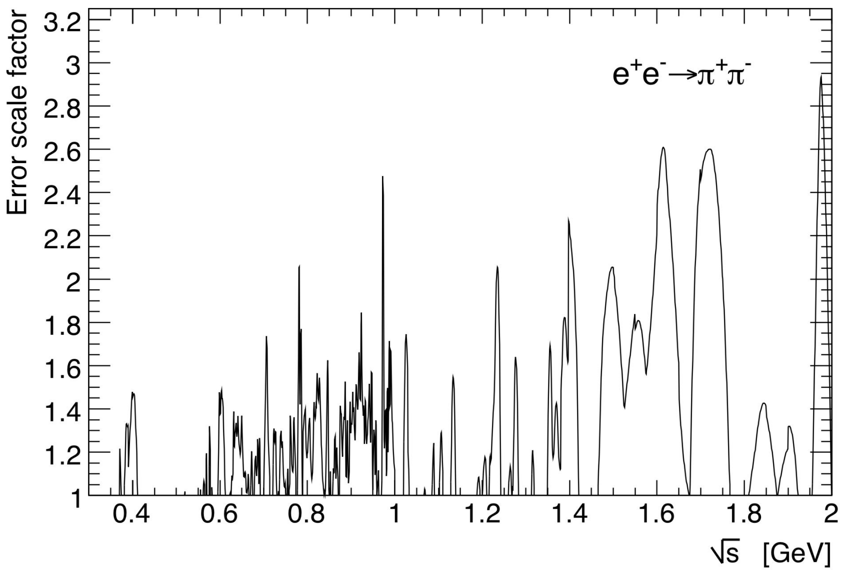
(see however recent update from SND (*backup*): ~significant tension with KLOE above 720 MeV)

# Combination procedure: compatibility between measurements

For each final bin:

→ $\chi^2$ /ndof: test locally the level of agreement between input measurements, *taking into account the correlations*

→ Scale uncertainties in bins with $\chi^2$ /ndof > 1 (PDG): *locally* conservative; Adopted by KNT since ’17




→ Tension between measurements:
*indication of underestimated uncertainties*
Motivates conservative uncertainty treatment in combination fit (evaluation of weights)

→ Observed (systematic) tension between BABAR and KLOE measurements

→ (*Since 2019*) Included extra (dominant) uncertainty: 1/2 difference between integrals w/o either BABAR or KLOE *( 2nd motivation for using DHMZ uncertainties as “baseline” in the TI White Paper)*

Extra uncertainty starts to be adopted in other studies (2205.12963)

# Improving $a_\mu$ through fits for the $e^+e^- \to \pi^+\pi^-$ channel (*Since 2019*)

→ Fit bare form-factor using 6 param. model based on *analyticity* and *unitarity*

$$|F_\pi^0|^2 = |R(s) \times J(s)|^2$$

$$R(s) = 1 + \alpha_V s + \frac{\kappa s}{m_\omega^2 - s - i m_\omega \Gamma_\omega}$$ (1611.09359, C. Hanhart et al.)

$$J(s) = e^{1-\frac{\delta_1(s_0)}{\pi}} \left(1 - \frac{s}{s_0}\right)^{\left[1-\frac{\delta_1(s_0)}{\pi}\right]\frac{s_0}{s}} \left(1 - \frac{s}{s_0}\right)^{-1} e^{\frac{s}{\pi}\int_{4m_\pi^2}^{s_0} dt \frac{\delta_1(t)}{t(t-s)}}$$

Omnès integral

(hep-ph/0402285, F.J. Yndurain et al.)

$$\cot \delta_1(s) = \frac{\sqrt{s}}{2k^3}\left(m_\rho^2 - s\right)\left[\frac{2m_\pi^3}{m_\rho^2\sqrt{s}} + B_0 + B_1\omega(s)\right]$$

(1102.2183, F.J. Yndurain et al.)

$$k = \frac{\sqrt{s - 4m_\pi^2}}{2}$$

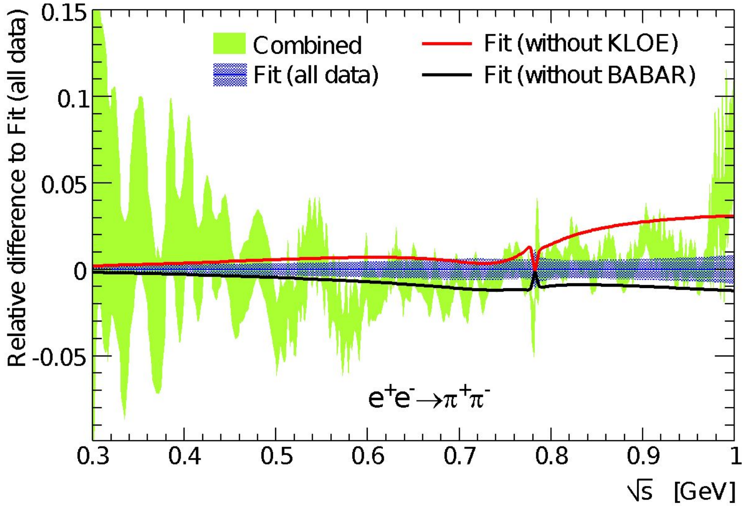
$$\omega(s) = \frac{\sqrt{s} - \sqrt{s_0 - s}}{\sqrt{s} + \sqrt{s_0 - s}} \qquad \sqrt{s_0} = 1.05\ \text{GeV}$$

→ Conservative $\chi^2$ (diagonal matrix) & local rescaling of input uncertainties

→ Full propagation of uncertainties & correlations using pseudo-experiments

DHMZ - 1908.00921

# Fit performed up to 1 GeV, Result used up to 0.6 GeV




→ Use fit only below 0.6 GeV for $a_\mu$ integral:

- where data is less precise and scarce
- less impacted by potential uncertainties of inelastic effects

| $\sqrt{s}$ range [GeV] | $a_\mu^{had}$ $[10^{-10}]$ Fit | $a_\mu^{had}$ $[10^{-10}]$ Data Integration |
|---|---|---|
| 0.3 - 0.6 | $109.80 \pm 0.37_{exp} \pm 0.36_{para*}$ | $109.6 \pm 1.0_{exp}$ |

→ The difference $0.2 \pm 0.9$ (72% correlation accounted for)

→ The fit improves the precision by a factor ~2

(*) Parameter uncertainty corresponds to variations with/without the $B_1$ term in the phase shift formula and $\sqrt{s_0}$ varied from 1.05 GeV to 1.3 GeV (absolute values summed linearly), *checked to be statistically significant*

# Combined results: Fit [<0.6GeV] + Data[0.6-1.8GeV]

→ Full uncertainty propagation using the same pseudo-experiments as for the spline-based combination: 62% correlation among the two contributions

| $\sqrt{s}$ range [GeV] | $a_\mu^{had}$ [$10^{-10}$] All data | $a_\mu^{had}$ [$10^{-10}$] All but BABAR | $a_\mu^{had}$ [$10^{-10}$] All but KLOE |
|---|---|---|---|
| threshold - 1.8 | $506.9 \pm 1.9_{total}$ | $505.0 \pm 2.1_{total}$ | $510.6 \pm 2.2_{total}$ |

→ The difference "All but BABAR" and "All but KLOE" = 5.6, to be compared with 1.9 uncertainty with "All data"

- The local error inflation is not sufficient to amplify the uncertainty
- Global tension (normalisation/shape) not previously accounted for
- Potential underestimated uncertainty in at least one of the measurements?
- Other measurements not precise enough to discriminate BABAR / KLOE

→ Given the fact we do not know which dataset is problematic, we decide to:

- Add half of the discrepancy ($2.8 \times 10^{-10}$) as an uncertainty (corrected local PDG inflation to avoid double counting)
- Take ("All but BABAR" + "All but KLOE") / 2 as central value

| Channel | $a_\mu^{\mathrm{had,LO}}$ $[10^{-10}]$ | $\Delta\alpha_{\mathrm{had}}(m_Z^2)$ $[10^{-4}]$ |
|---|---|---|
| $\pi^+\pi^-$ | $507.85 \pm 0.83 \pm \boxed{3.23} \pm 0.55$ | $34.50 \pm 0.06 \pm \boxed{0.20} \pm 0.04$ |

→ *Potential precision improvement* for $a_\mu$ ; less important for $\Delta\alpha_{had}(m_Z^2)$ , BABAR-KLOE syst. ~16% of total uncertainty

# Uncertainties on uncertainties and on correlations

*Topic relevant in other fields too (see backup)*

1908.00921(DHMZ), 2006.04822(WP g-2 Theory Initiative)

# Two different approaches for combining ($e^+e^-$) data

DHMZ:

→ $\chi^2$ computed locally (in each fine bin), taking into account correlations between measurements (see previous slides)

→ Used to determine the weights on the measurements in the combination and their level of agreement

→ Uncertainties and correlations propagated using pseudo-experiments or ±1σ shifts of each uncertainty component

KNT:

→ $\chi^2$ computed globally (for full mass range)

$$\chi^2_I = \sum_{i=1}^{N_{\rm tot}} \sum_{j=1}^{N_{\rm tot}} \left(R_i^{(m)} - \mathcal{R}_m^{i,I}\right) \mathbf{C}_I^{-1}\left(i^{(m)}, j^{(n)}\right)\left(R_j^{(n)} - \mathcal{R}_n^{j,I}\right)$$

KNT (1802.02995)

$$\chi^2 = \sum_{i=1}^{195} \sum_{j=1}^{195} \left(\sigma^0_{\pi\pi(\gamma)}(i) - \bar{\sigma}^0_{\pi\pi(\gamma)}(m)\right) \mathbf{C}^{-1}\left(i^{(m)}, j^{(n)}\right)\left(\sigma^0_{\pi\pi(\gamma)}(j) - \bar{\sigma}^0_{\pi\pi(\gamma)}(n)\right)$$

KLOE-KMT (1711.03085)

→ relies on description of correlations on long ranges

→ *One of the main sources of differences for the uncertainty on* $a_\mu$

# Evaluation of uncertainties and correlations ($e^+e^-$)

| | $\sigma_{\pi\pi\gamma}$ | $\sigma^0_{\pi\pi}$ | $F_\pi$ | $\Delta^{\pi\pi} a_\mu$ |
|---|---|---|---|---|
| Reconstruction Filter | negligible | | | |
| Background subtraction | Tab. [1] | | | 0.3% |
| Trackmass | 0.2% | | | |
| Pion cluster ID | negligible | | | |
| Tracking efficiency | 0.3% | | | |
| Trigger efficiency | 0.1% | | | |
| Acceptance | Tab. [2] | | | 0.2% |
| Unfolding | Tab. [3] | | | negligible |
| L3 filter | 0.1% | | | |
| $\sqrt{s}$ dependence of $H$ | - | Tab. [4] | | 0.2% |
| Luminosity | 0.3% | | | |
| Experimental systematics | | | | 0.6% |
| FSR resummation | - | 0.3% | | |
| Radiator function $H$ | - | 0.5% | | |
| Vacuum Polarization | - | 0.1% | - | 0.1% |
| Theory systematics | | | | 0.6% |

| $M^2_{\pi\pi}$ range ( GeV$^2$) | Systematic error (%) |
|---|---|
| $0.35 \leq M^2_{\pi\pi} < 0.39$ | 0.6 |
| $0.39 \leq M^2_{\pi\pi} < 0.43$ | 0.5 |
| $0.43 \leq M^2_{\pi\pi} < 0.45$ | 0.4 |
| $0.45 \leq M^2_{\pi\pi} < 0.49$ | 0.3 |
| $0.49 \leq M^2_{\pi\pi} < 0.51$ | 0.2 |
| $0.51 \leq M^2_{\pi\pi} < 0.64$ | 0.1 |
| $0.64 \leq M^2_{\pi\pi} < 0.95$ | - |

KLOE 08 (0809.3950)

KLOE 10 (1006.5313)

| | $\sigma_{\pi\pi\gamma}$ | $\sigma^{\rm bare}_{\pi\pi}$ | $\lvert F_\pi\rvert^2$ | $\Delta a^{\pi\pi}_\mu$ |
|---|---|---|---|---|
| | threshold ; $\rho$-peak | | | (0.1 - 0.85 GeV$^2$) |
| Background Filter | 0.5% ; 0.1% | | | negligible |
| Background subtraction | 3.4% ; 0.1% | | | 0.5% |
| $f_0 + \rho\pi$ bkg. | 6.5% ; negl. | | | 0.4% |
| $\Omega$ cut | 1.4% ; negl. | | | 0.2% |
| Trackmass cut | 3.0% ; 0.2% | | | 0.5% |
| $\pi$-e PID | 0.3% ; negl. | | | negligible |
| Trigger | 0.3% ; 0.2% | | | 0.2% |
| Acceptance | 1.9% ; 0.3% | | | 0.5% |
| Unfolding | negl. ; 2.0% | | | negligible |
| Tracking | 0.3% | | | |
| Software Trigger (L3) | 0.1% | | | |
| Luminosity | 0.3% | | | |
| Experimental syst. | | | | 1.0% |
| FSR treatment | - | 7% ; negl. | | 0.8% |
| Radiator function $H$ | - | 0.5% | | |
| Vacuum Polarization | - | Ref. [34] | - | 0.1% |
| Theory syst. | | | | 0.9% |

→ Systematics *evaluated* in ~wide mass ranges with sharp transitions

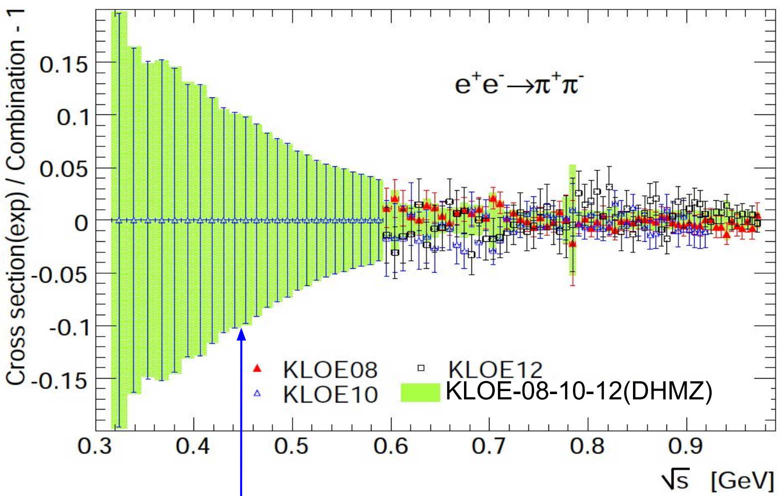
# Evaluation of uncertainties and correlations ($e^+e^-$)

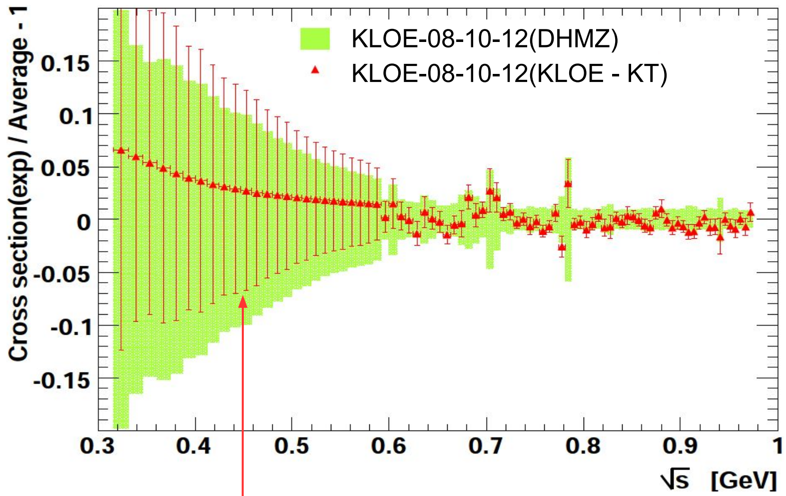
BABAR (1205.2228)

| Sources | 0.3-0.4 | 0.4-0.5 | 0.5-0.6 | 0.6-0.9 | 0.9-1.2 | 1.2-1.4 | 1.4-2.0 | 2.0-3.0 |
|---|---|---|---|---|---|---|---|---|
| trigger/ filter | 5.3 | 2.7 | 1.9 | 1.0 | 0.7 | 0.6 | 0.4 | 0.4 |
| tracking | 3.8 | 2.1 | 2.1 | 1.1 | 1.7 | 3.1 | 3.1 | 3.1 |
| $\pi$-ID | 10.1 | 2.5 | 6.2 | 2.4 | 4.2 | 10.1 | 10.1 | 10.1 |
| background | 3.5 | 4.3 | 5.2 | 1.0 | 3.0 | 7.0 | 12.0 | 50.0 |
| acceptance | 1.6 | 1.6 | 1.0 | 1.0 | 1.6 | 1.6 | 1.6 | 1.6 |
| kinematic fit ($\chi^2$) | 0.9 | 0.9 | 0.3 | 0.3 | 0.9 | 0.9 | 0.9 | 0.9 |
| correl $\mu\mu$ ID loss | 3.0 | 2.0 | 3.0 | 1.3 | 2.0 | 3.0 | 10.0 | 10.0 |
| $\pi\pi/\mu\mu$ non-cancel. | 2.7 | 1.4 | 1.6 | 1.1 | 1.3 | 2.7 | 5.1 | 5.1 |
| unfolding | 1.0 | 2.7 | 2.7 | 1.0 | 1.3 | 1.0 | 1.0 | 1.0 |
| ISR luminosity | 3.4 | 3.4 | 3.4 | 3.4 | 3.4 | 3.4 | 3.4 | 3.4 |
| sum (cross section) | 13.8 | 8.1 | 10.2 | 5.0 | 6.5 | 13.9 | 19.8 | 52.4 |

→ Systematics *evaluated* in ~wide mass ranges with sharp transitions (statistics limitations when going to narrow ranges)

# Combining the 3 KLOE measurements




Local combination (DHMZ)

Information propagated between mass regions, through shifts of systematics - relying on correlations, amplitudes and shapes of systematics (KLOE-KT)

# Combining the 3 KLOE measurements - $a_\mu^{\pi\pi}$ contribution

KLOE08 $a_\mu$[ 0.6 ; 0.9 ] : 368.3 ± 3.2 [$10^{-10}$]
KLOE10 $a_\mu$[ 0.6 ; 0.9 ] : 365.6 ± 3.3
KLOE12 $a_\mu$[ 0.6 ; 0.9 ] : 366.8 ± 2.5
→ Correlation matrix:

| | 08 | 10 | 12 |
|---|---|---|---|
| 08 | 1 | 0.70 | 0.35 |
| 10 | 0.70 | 1 | 0.19 |
| 12 | 0.35 | 0.19 | 1 |

→ Amount of independent information provided by each measurement

→ KLOE-08-10-12(DHMZ) - $a_\mu$[0.6 ; 0.9] : 366.5 ± 2.8 (Without $\chi^2$ rescaling: ± 2.2)
→ Conservative treatment of uncertainties and correlations (*not perfectly known*) in weight determination

→ KLOE-08-10-12(KLOE-KT) - $a_\mu$[0.6 ; 0.9]GeV : 366.9 ± 2.2 (Includes $\chi^2$ rescaling)

→ Assuming perfect knowledge of the correlations to minimize average uncertainty

# Uncertainties on uncertainties and correlations

*Numerous indications of uncertainties on uncertainties and on correlations, with a direct impact on combination fits*

→ Shapes of systematic uncertainties *evaluated* in ~wide mass ranges with sharp transitions

→ One standard deviation is statistically not well defined for systematic uncertainties

→ Systematic uncertainties like acceptance, tracking efficiency, background etc. not necessarily fully correlated between low and high mass

→ Are all systematic uncertainty components fully independent between each-other? (e.g. tracking and trigger)

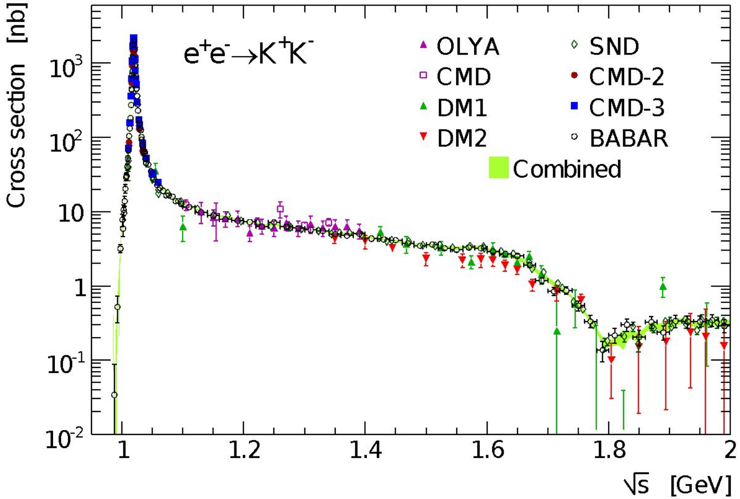
→ Y*ield uncertainties on uncertainties and on correlations*

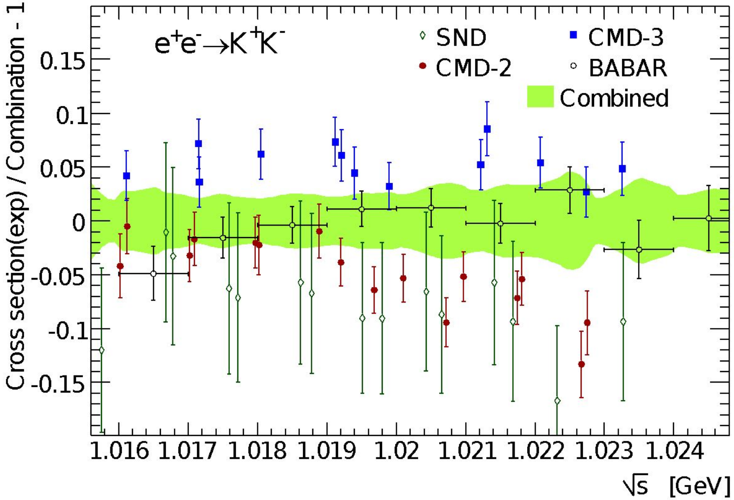
→ Tensions between measurements (BABAR/KLOE; 3 KLOE results etc.): *experimental indications of underestimated uncertainties*

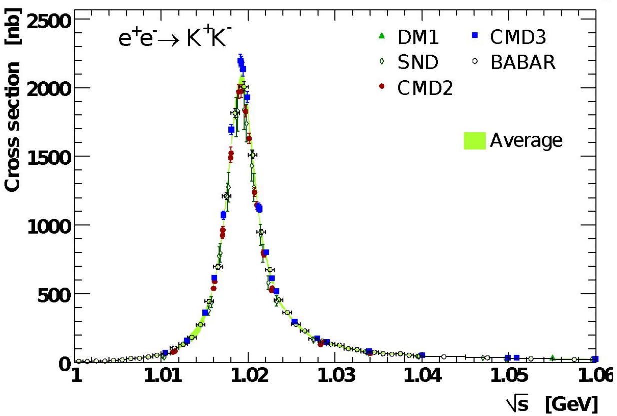
→ *Statistical methods* ($\chi^2$ with correlations, likelihood fits, ratios of measured quantities etc.) *should not over-exploit the information on the amplitude and correlations of uncertainties*

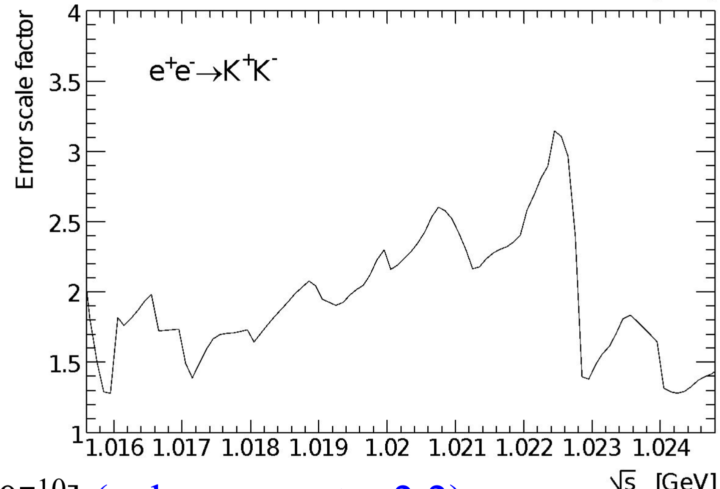
# Combination of measurements for various channels and total HVP contribution

# Combination for the $e^+e^- \to K^+K^-$ channel



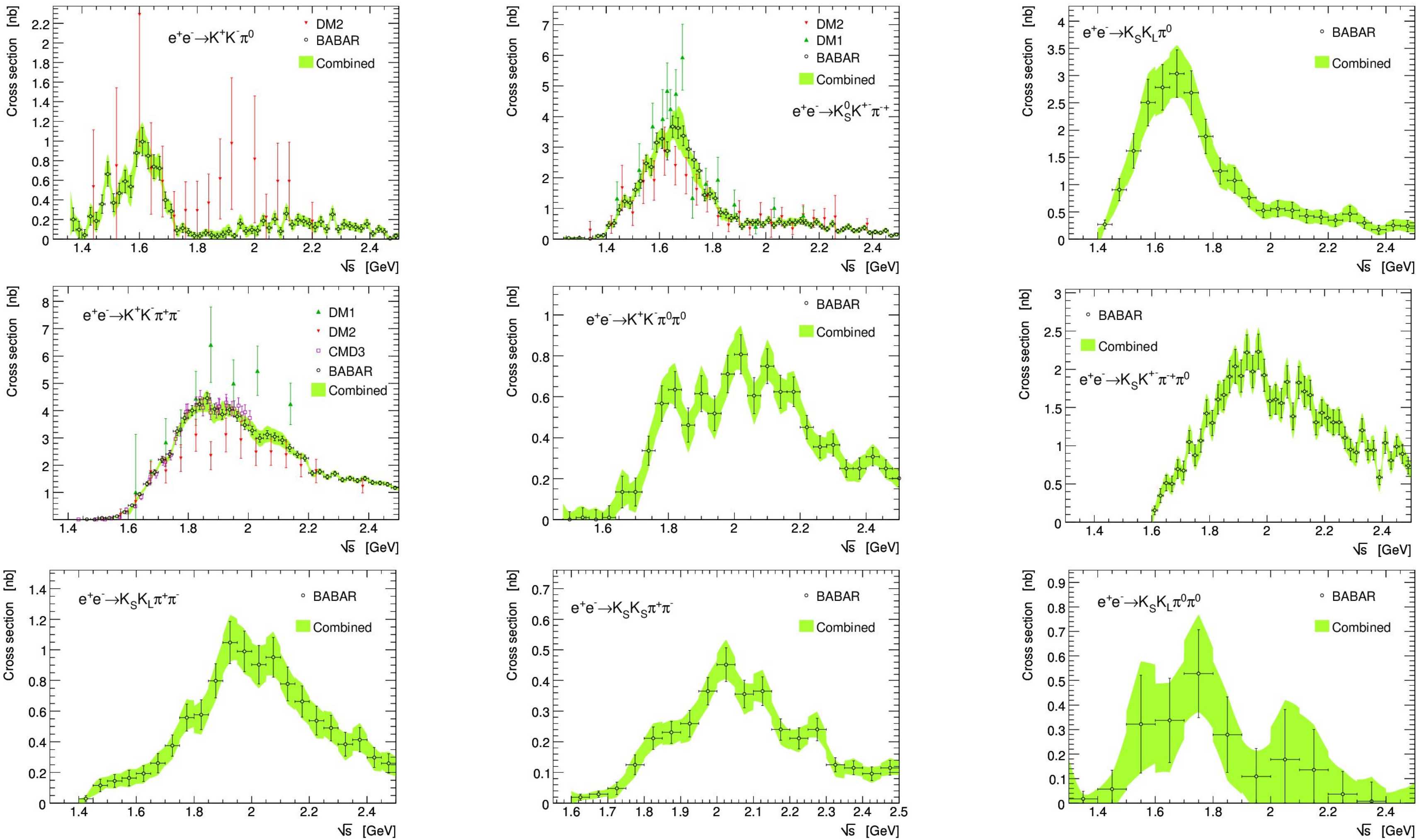



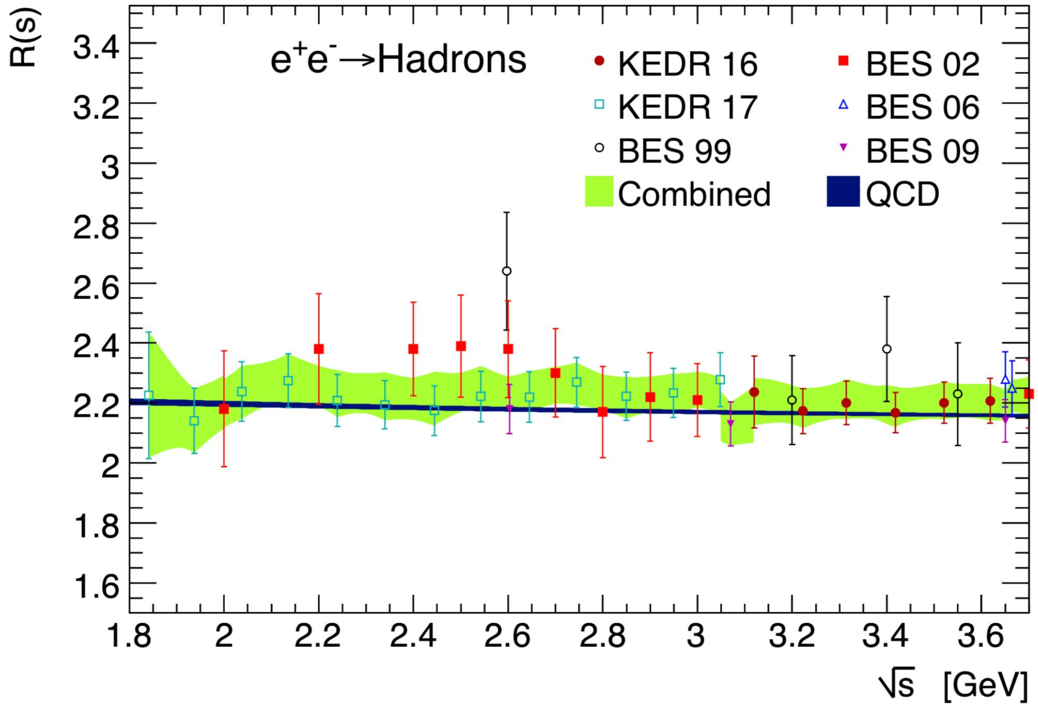

→ Tension between measurements

→ $a_\mu$[→1.8GeV]: 23.08 ± 0.20 (stat.) ± 0.40 (syst.) [$10^{-10}$] (enhancement x 2.2)

# Combination for the $e^+e^- \rightarrow KK\pi$ and $KK2\pi$ channels



# Contributions from the 1.8 – 3.7 GeV region




→ Contribution evaluated from pQCD (4 loops) + O($\alpha_s^2$) quark mass corrections

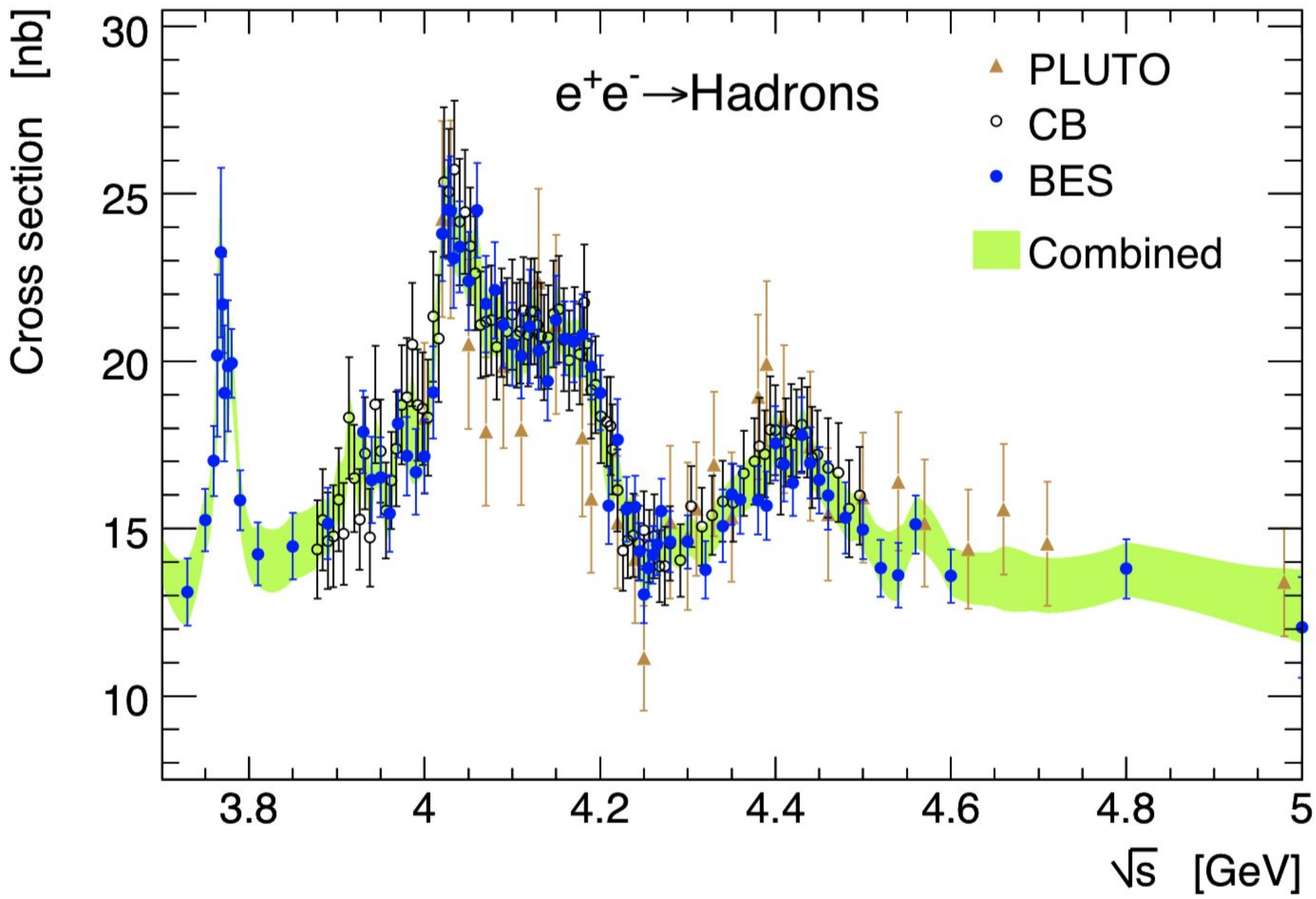
→ Uncertainties: $\alpha_s$, truncation of perturbative series, CIPT/FOPT, $m_q$

→ 1.8-2.0 GeV: 7.65±0.31(data excl.); 8.30±0.09(QCD); added syst. 0.65 [$10^{-10}$]

→ 2.0-3.7 GeV: 25.82±0.61(data); 25.15 ± 0.19(QCD); agreement within 1σ

→ BES III results to be included: ~tension with pQCD and with KEDR 16 (*backup*)

# Contributions from the charm resonance region



# Situation in arXiv:1908.00921 (EPJC)

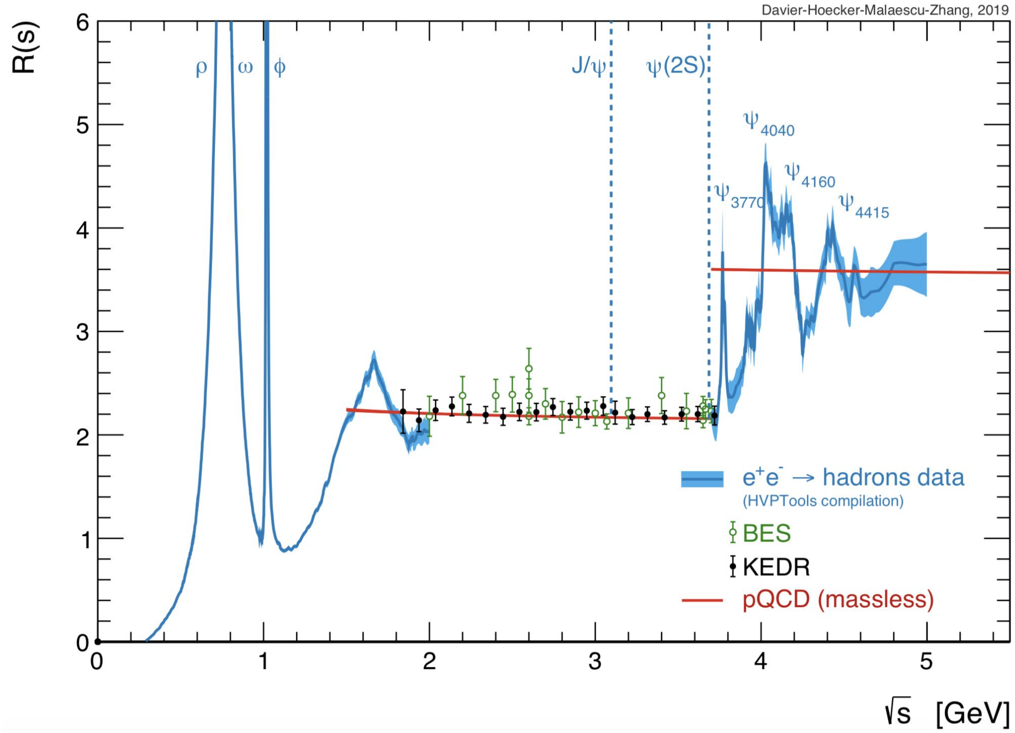
| Channel | $a_\mu^{\mathrm{had,LO}}$ $[10^{-10}]$ | $\Delta\alpha_{\mathrm{had}}(m_Z^2)$ $[10^{-4}]$ |
|---|---|---|
| $\pi^0\gamma$ | $4.41 \pm 0.06 \pm 0.04 \pm 0.07$ | $0.35 \pm 0.00 \pm 0.00 \pm 0.01$ |
| $\eta\gamma$ | $0.65 \pm 0.02 \pm 0.01 \pm 0.01$ | $0.08 \pm 0.00 \pm 0.00 \pm 0.00$ |
| $\pi^+\pi^-$ | $507.85 \pm 0.83 \pm 3.23 \pm 0.55$ | $34.50 \pm 0.06 \pm 0.20 \pm 0.04$ |
| $\pi^+\pi^-\pi^0$ | $46.21 \pm 0.40 \pm 1.10 \pm 0.86$ | $4.60 \pm 0.04 \pm 0.11 \pm 0.08$ |
| $2\pi^+2\pi^-$ | $13.68 \pm 0.03 \pm 0.27 \pm 0.14$ | $3.58 \pm 0.01 \pm 0.07 \pm 0.03$ |
| $\pi^+\pi^-2\pi^0$ | $18.03 \pm 0.06 \pm 0.48 \pm 0.26$ | $4.45 \pm 0.02 \pm 0.12 \pm 0.07$ |
| $2\pi^+2\pi^-\pi^0$ ($\eta$ excl.) | $0.69 \pm 0.04 \pm 0.06 \pm 0.03$ | $0.21 \pm 0.01 \pm 0.02 \pm 0.01$ |
| $\pi^+\pi^-3\pi^0$ ($\eta$ excl.) | $0.49 \pm 0.03 \pm 0.09 \pm 0.00$ | $0.15 \pm 0.01 \pm 0.03 \pm 0.00$ |
| $3\pi^+3\pi^-$ | $0.11 \pm 0.00 \pm 0.01 \pm 0.00$ | $0.04 \pm 0.00 \pm 0.00 \pm 0.00$ |
| $2\pi^+2\pi^-2\pi^0$ ($\eta$ excl.) | $0.71 \pm 0.06 \pm 0.07 \pm 0.14$ | $0.25 \pm 0.02 \pm 0.02 \pm 0.05$ |
| $\pi^+\pi^-4\pi^0$ ($\eta$ excl., isospin) | $0.08 \pm 0.01 \pm 0.08 \pm 0.00$ | $0.03 \pm 0.00 \pm 0.03 \pm 0.00$ |
| $\eta\pi^+\pi^-$ | $1.19 \pm 0.02 \pm 0.04 \pm 0.02$ | $0.35 \pm 0.01 \pm 0.01 \pm 0.01$ |
| $\eta\omega$ | $0.35 \pm 0.01 \pm 0.02 \pm 0.01$ | $0.11 \pm 0.00 \pm 0.01 \pm 0.00$ |
| $\eta\pi^+\pi^-\pi^0$(non-$\omega,\phi$) | $0.34 \pm 0.03 \pm 0.03 \pm 0.04$ | $0.12 \pm 0.01 \pm 0.01 \pm 0.01$ |
| $\eta 2\pi^+2\pi^-$ | $0.02 \pm 0.01 \pm 0.00 \pm 0.00$ | $0.01 \pm 0.00 \pm 0.00 \pm 0.00$ |
| $\omega\eta\pi^0$ | $0.06 \pm 0.01 \pm 0.01 \pm 0.00$ | $0.02 \pm 0.00 \pm 0.00 \pm 0.00$ |
| $\omega\pi^0$ ($\omega \to \pi^0\gamma$) | $0.94 \pm 0.01 \pm 0.03 \pm 0.00$ | $0.20 \pm 0.00 \pm 0.01 \pm 0.00$ |
| $\omega 2\pi$ ($\omega \to \pi^0\gamma$) | $0.07 \pm 0.00 \pm 0.00 \pm 0.00$ | $0.02 \pm 0.00 \pm 0.00 \pm 0.00$ |
| $\omega$ (non-$3\pi, \pi\gamma, \eta\gamma$) | $0.04 \pm 0.00 \pm 0.00 \pm 0.00$ | $0.00 \pm 0.00 \pm 0.00 \pm 0.00$ |
| $K^+K^-$ | $23.08 \pm 0.20 \pm 0.33 \pm 0.21$ | $3.35 \pm 0.03 \pm 0.05 \pm 0.03$ |
| $K_S K_L$ | $12.82 \pm 0.06 \pm 0.18 \pm 0.15$ | $1.74 \pm 0.01 \pm 0.03 \pm 0.02$ |
| $\phi$ (non-$K\overline{K}, 3\pi, \pi\gamma, \eta\gamma$) | $0.05 \pm 0.00 \pm 0.00 \pm 0.00$ | $0.01 \pm 0.00 \pm 0.00 \pm 0.00$ |
| $K\overline{K}\pi$ | $2.45 \pm 0.05 \pm 0.10 \pm 0.06$ | $0.78 \pm 0.02 \pm 0.03 \pm 0.02$ |
| $K\overline{K}2\pi$ | $0.85 \pm 0.02 \pm 0.05 \pm 0.01$ | $0.30 \pm 0.01 \pm 0.02 \pm 0.00$ |
| $K\overline{K}\omega$ | $0.00 \pm 0.00 \pm 0.00 \pm 0.00$ | $0.00 \pm 0.00 \pm 0.00 \pm 0.00$ |
| $\eta\phi$ | $0.33 \pm 0.01 \pm 0.01 \pm 0.00$ | $0.11 \pm 0.00 \pm 0.00 \pm 0.00$ |
| $\eta K\overline{K}$ (non-$\phi$) | $0.01 \pm 0.01 \pm 0.01 \pm 0.00$ | $0.00 \pm 0.00 \pm 0.01 \pm 0.00$ |
| $\omega 3\pi$ ($\omega \to \pi^0\gamma$) | $0.06 \pm 0.01 \pm 0.01 \pm 0.01$ | $0.02 \pm 0.00 \pm 0.00 \pm 0.00$ |
| $7\pi$ ($3\pi^+3\pi^-\pi^0$ + estimate) | $0.02 \pm 0.00 \pm 0.01 \pm 0.00$ | $0.01 \pm 0.00 \pm 0.00 \pm 0.00$ |
| $J/\psi$ (BW integral) | $6.20 \pm 0.11$ | $7.00 \pm 0.13$ |
| $\psi(2S)$ (BW integral) | $1.56 \pm 0.05$ | $2.48 \pm 0.08$ |
| $R$ data $[3.7 - 5.0]$ GeV | $7.29 \pm 0.05 \pm 0.30 \pm 0.00$ | $15.79 \pm 0.12 \pm 0.66 \pm 0.00$ |
| $R_{\mathrm{QCD}}$ $[1.8 - 3.7$ GeV$]_{uds}$ | $33.45 \pm 0.28 \pm 0.65_{\mathrm{dual}}$ | $24.27 \pm 0.18 \pm 0.28_{\mathrm{dual}}$ |
| $R_{\mathrm{QCD}}$ $[5.0 - 9.3$ GeV$]_{udsc}$ | $6.86 \pm 0.04$ | $34.89 \pm 0.18$ |
| $R_{\mathrm{QCD}}$ $[9.3 - 12.0$ GeV$]_{udscb}$ | $1.20 \pm 0.01$ | $15.53 \pm 0.04$ |
| $R_{\mathrm{QCD}}$ $[12.0 - 40.0$ GeV$]_{udscb}$ | $1.64 \pm 0.00$ | $77.94 \pm 0.13$ |
| $R_{\mathrm{QCD}}$ $[> 40.0$ GeV$]_{udscb}$ | $0.16 \pm 0.00$ | $42.70 \pm 0.05$ |
| $R_{\mathrm{QCD}}$ $[> 40.0$ GeV$]_{t}$ | $0.00 \pm 0.00$ | $-0.72 \pm 0.01$ |
| **Sum** | $694.0 \pm 1.0 \pm 3.5 \pm 1.6 \pm 0.1_\psi \pm 0.7_{\mathrm{QCD}}$ | $275.29 \pm 0.15 \pm 0.72 \pm 0.23 \pm 0.15_\psi \pm 0.55_{\mathrm{QCD}}$ |

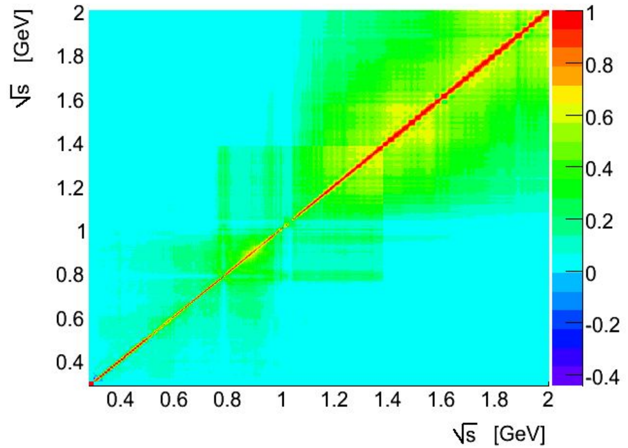
→ 32 exclusive channels are integrated up to 1.8 GeV

Relative contributions to $a_\mu$ from missing channels (estimated based on isospin symmetry)

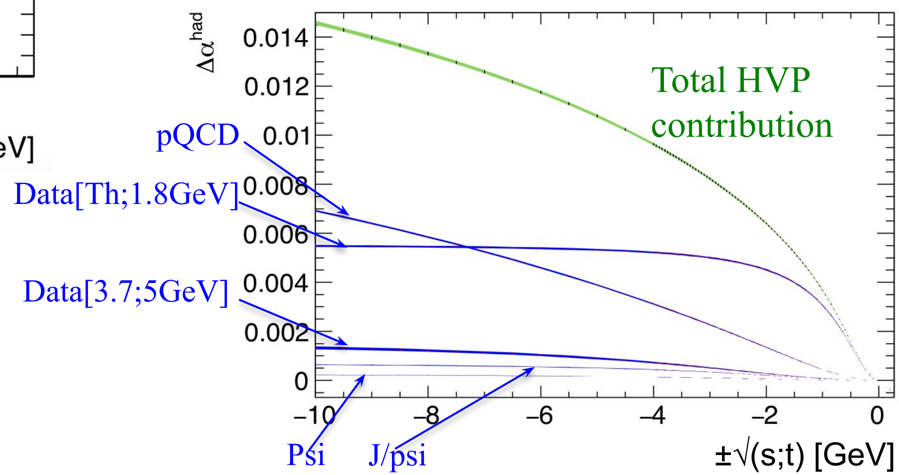
→ 0.87 ± 0.15 % (DEHZ 2003)
→ 0.69 ± 0.07 % (DHMZ 2010)
→ 0.09 ± 0.02 % (DHMZ 2017)
→ 0.016 ± 0.016 % (DHMZ 2019)
(*Nearly complete set of exclusive measurements from BABAR*)

Estimation procedures also adopted by KNT



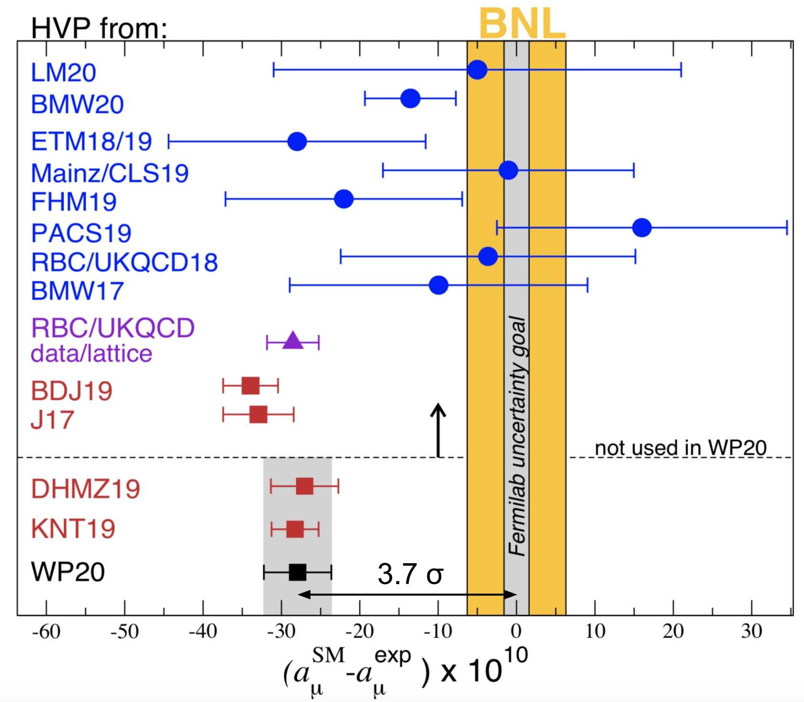

Sum of *32 exclusive channels* with *full propagation of correlations*



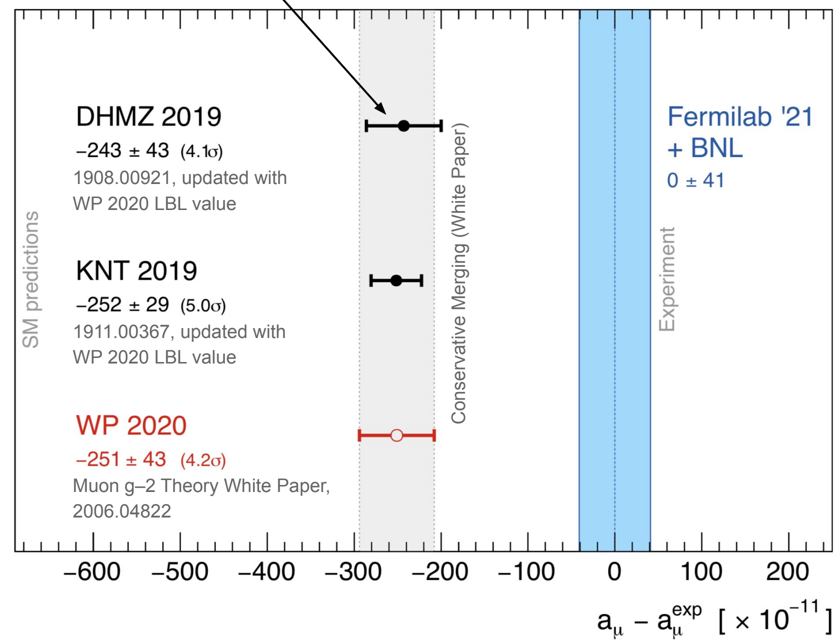

→ Performed non-trivial check: $a_\mu$ and $\Delta\alpha_{had}$ from sum of individual channels and from Ree integral < 1.8 GeV

→ Enables the determination of the various HVP contributions to the "running" of $\alpha_{QED}$

# Status of $a_\mu$ before/with Fermilab result




*Important to account for BABAR-KLOE diff. & inter-channel correlations*



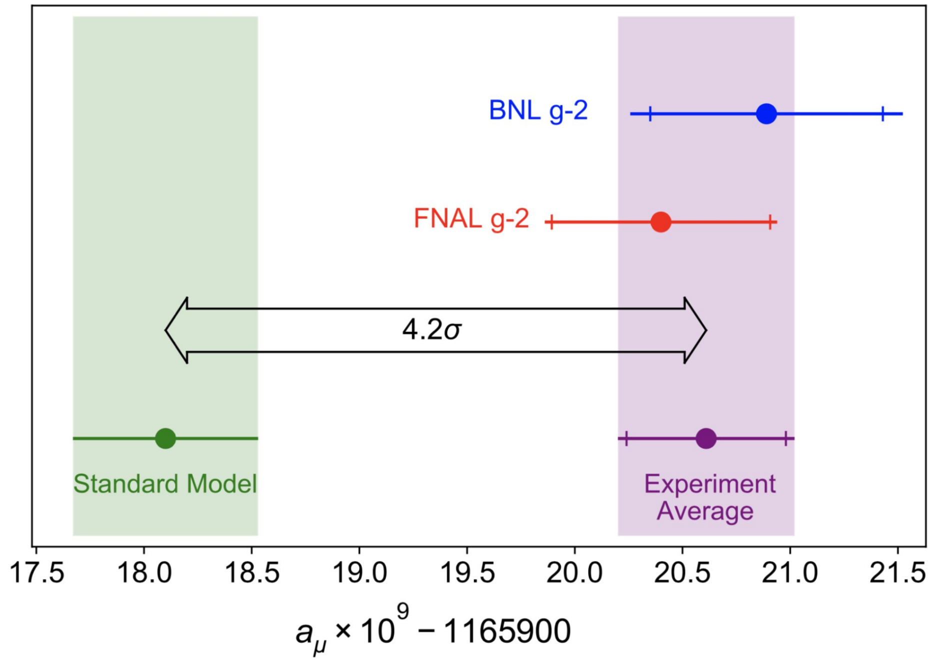

→ Caution about significance:
statistics-dominated measurement; prediction uncertainty limited by non-Gaussian systematic effects

→ Nevertheless, large discrepancy between measurement and reference SM prediction
(to be significantly improved in view of the forthcoming updates of the Fermilab measurement)

→ Tension significantly smaller when using BMW20 for the LO HVP (TBC by other lattice groups), *not* incompatible with the EW fit (see backup)

# Conclusion

*We have an interesting, long standing, multifaceted problem to solve...*




???

Guiding ideas:

→ Need *rigorous* and *realistic* treatment of uncertainties and correlations at all levels
(Underestimated uncertainties do not bring scientific progress & can put studies on wrong path)

→ Studies for understanding differences between data-driven and Lattice QCD approaches need to follow similar standards as the g-2 experiment: *double-blinding*

# Backup

# Lepton Magnetic Anomaly: from Dirac to QED

- Magnetic dipole moment of a charged lepton: $\vec{\mu} = g \frac{e}{2m} \vec{s}$

  Dirac (1928) $g_e=2$ $a_e=0$

- "anomaly" = deviation w.r.t. Dirac's prediction: $a = \frac{g-2}{2}$

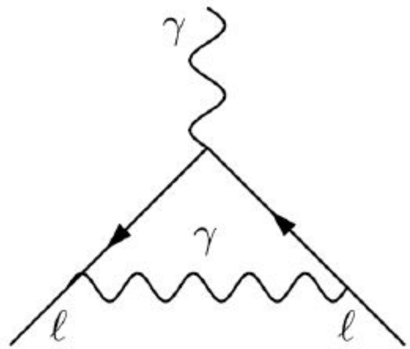
anomaly discovered:

Kusch-Foley (1948) $a_e = (1.19 \pm 0.05)\ 10^{-3}$

and explained by $O(\alpha)$ QED contribution:

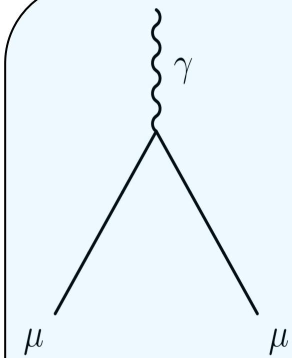
Schwinger (1948) $a_e = \alpha/2\pi = 1.16\ 10^{-3}$

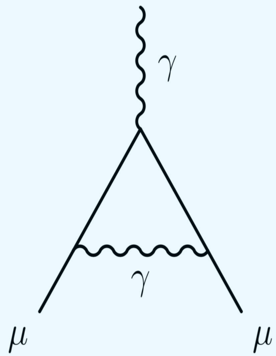
first triumph of QED

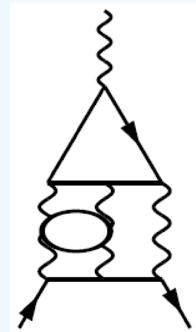



$\Rightarrow$ $a_e$ sensitive to quantum fluctuations of fields

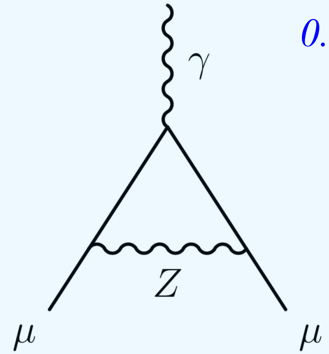
Why is it (so) complicated to compute one number ? (*very precisely*)



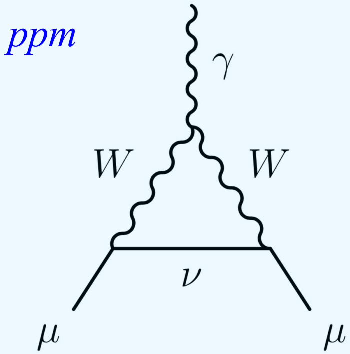



*0.001 ppm*



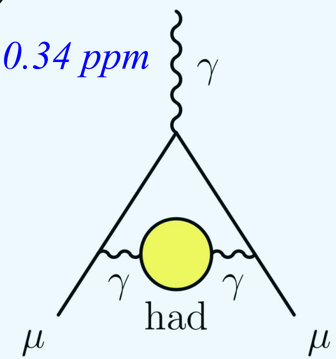

*0.01 ppm*



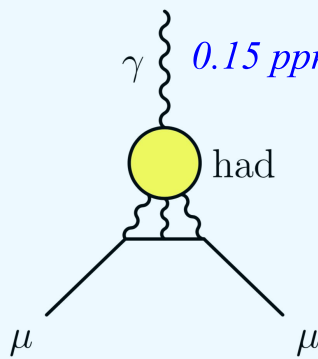

+ Many other diagrams at higher orders…

QED up to O($\alpha^5$) (Kinoshita et al.)

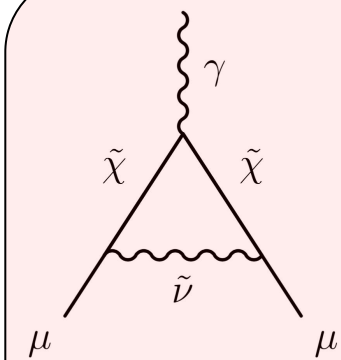
*0.34 ppm*

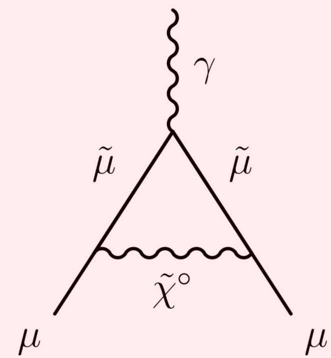



*0.15 ppm*

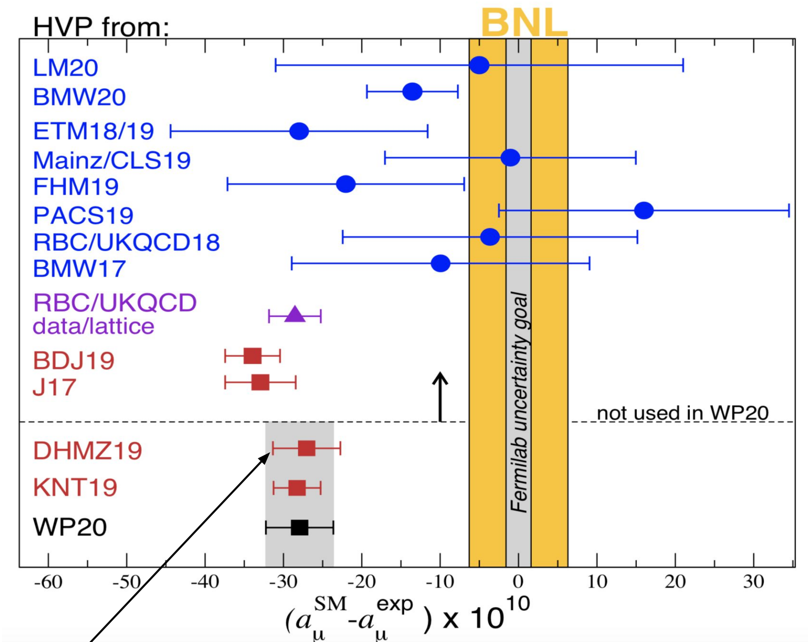



Dominant uncertainties: non-perturbative...

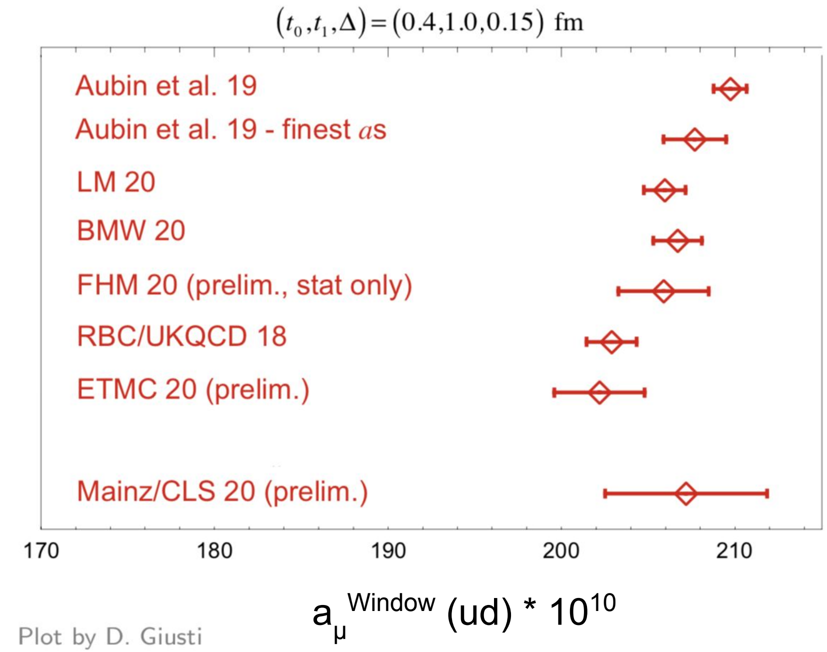



? ? ?




*Important to account for BABAR-KLOE diff. & inter-channel correlations*




→ *HVP(WP20)*: *Merging* of model independent results: DHMZ and KNT (and CHHKS for $\pi^+\pi^-$ & $\pi^+\pi^-\pi^0$) Central value from simple average; BABAR-KLOE tension & correlations between channels from DHMZ; Max(DHMZ & KNT uncertainties) in each channel

→ Excellent progress on the Lattice QCD (+QED) calculations; *Precision of BMW20* (*to be cross-checked* by other lattice groups) became *similar to the one of dispersive approaches*; *Ongoing cross-checks* using Euclidean time windows (related to HVP with suppression of very low and high energies) for which various groups achieved similar precision; *If BMW20 result is confirmed, the difference w.r.t. dispersive results* to be understood.

# Theory initiative white paper executive summary & new results

| Contribution | Section | Equation | Value $\times 10^{11}$ | References |
|---|---|---|---|---|
| Experiment (E821) | | Eq. (8.13) | 116 592 089(63) | Ref. [1] |
| HVP LO ($e^+e^-$) | Sec. 2.3.7 | Eq. (2.33) | 6931(40) | Refs. [2–7] |
| HVP NLO ($e^+e^-$) | Sec. 2.3.8 | Eq. (2.34) | −98.3(7) | Ref. [7] |
| HVP NNLO ($e^+e^-$) | Sec. 2.3.8 | Eq. (2.35) | 12.4(1) | Ref. [8] |
| HVP LO (lattice, *udsc*) | Sec. 3.5.1 | Eq. (3.49) | 7116(184) | Refs. [9–17] |
| HLbL (phenomenology) | Sec. 4.9.4 | Eq. (4.92) | 92(19) | Refs. [18–30] |
| HLbL NLO (phenomenology) | Sec. 4.8 | Eq. (4.91) | 2(1) | Ref. [31] |
| HLbL (lattice, *uds*) | Sec. 5.7 | Eq. (5.49) | 79(35) | Ref. [32] |
| HLbL (phenomenology + lattice) | Sec. 8 | Eq. (8.10) | 90(17) | Refs. [18–30, 32] |
| QED | Sec. 6.5 | Eq. (6.30) | 116 584 718.931(104) | Refs. [33, 34] |
| Electroweak | Sec. 7.4 | Eq. (7.16) | 153.6(1.0) | Refs. [35, 36] |
| HVP ($e^+e^-$, LO + NLO + NNLO) | Sec. 8 | Eq. (8.5) | 6845(40) | Refs. [2–8] |
| HLbL (phenomenology + lattice + NLO) | Sec. 8 | Eq. (8.11) | 92(18) | Refs. [18–32] |
| Total SM Value | Sec. 8 | Eq. (8.12) | 116 591 810(43) | Refs. [2–8, 18–24, 31–36] |
| Difference: $\Delta a_\mu := a_\mu^{\rm exp} - a_\mu^{\rm SM}$ | Sec. 8 | Eq. (8.14) | 279(76) | |

→ Dominant uncertainty: HVP LO → Based on *merging of model-independent methods*

→ HLbL also has an important uncertainty

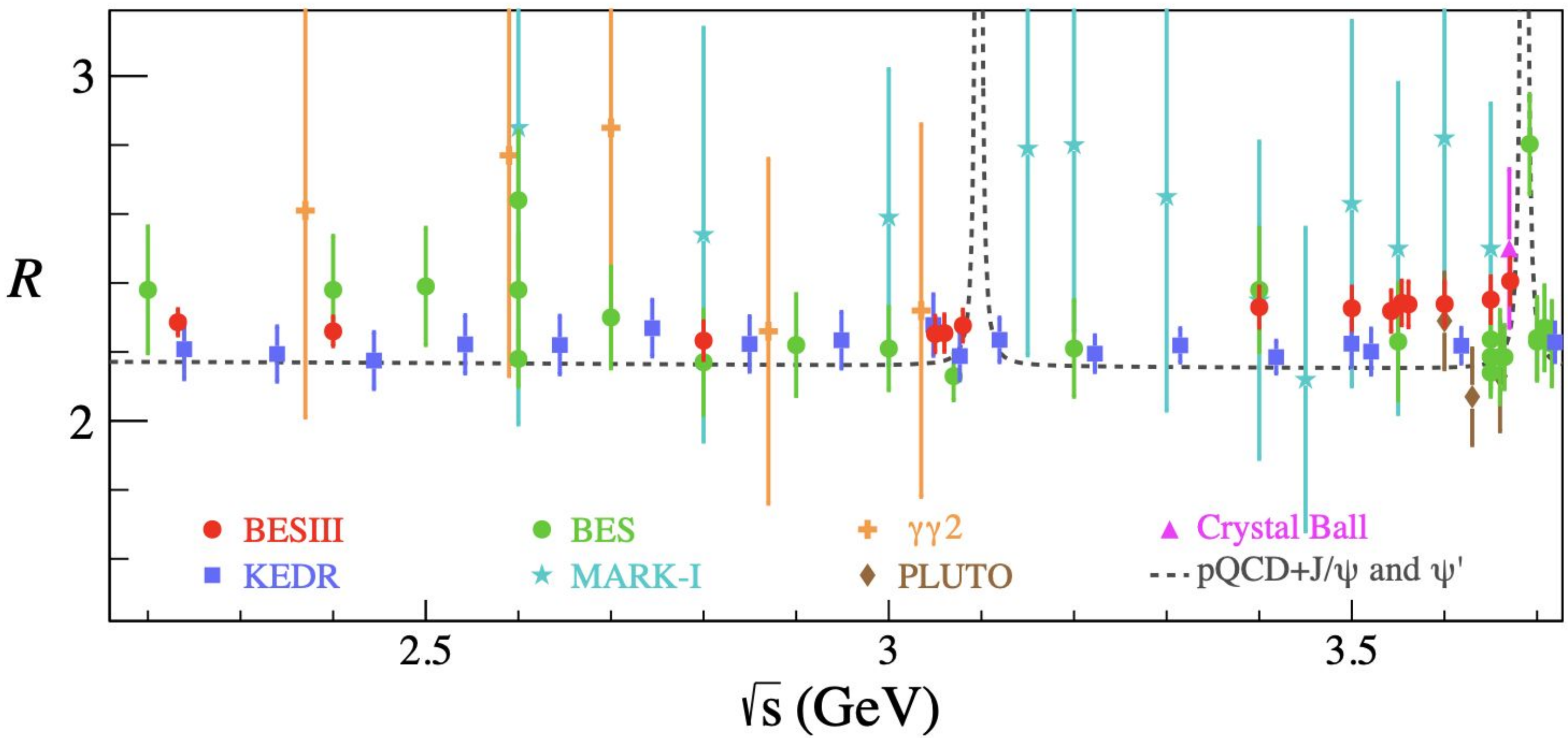
→ Lattice results become more and more interesting

→ A tension between the BNL measurement and the reference SM prediction: ~ 3.7 σ (~ 4.2 σ including FNAL)

→ Tension significantly smaller when using BMW20 for the LO HVP (TBC by other lattice groups)

# Comparison of inclusive measurements with pQCD

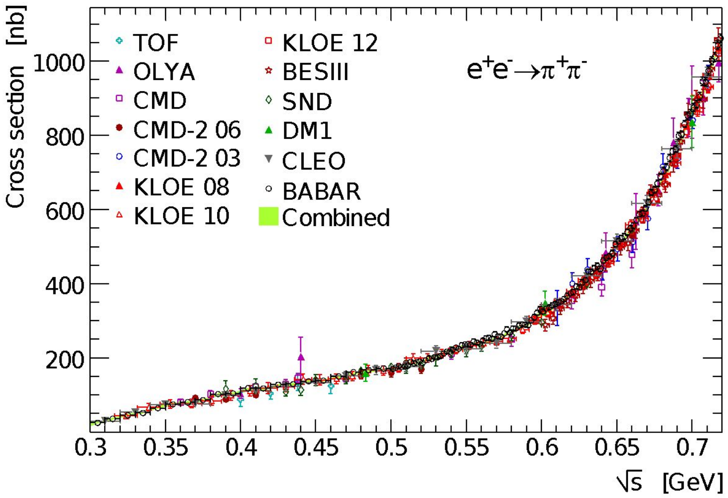
arXiv:2112.11728

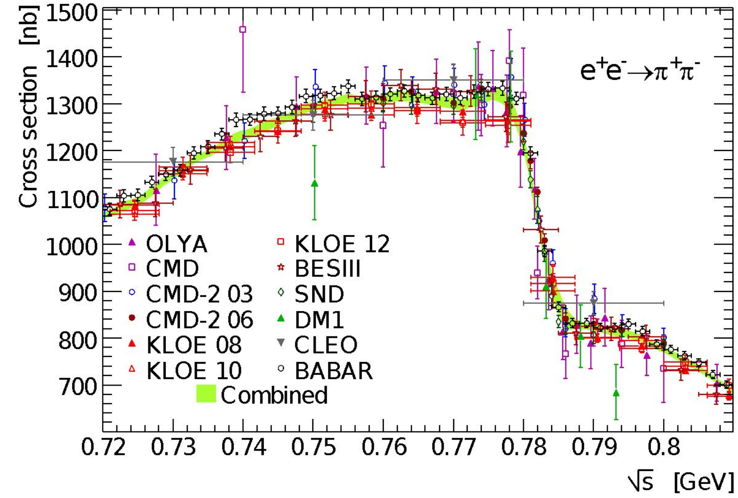


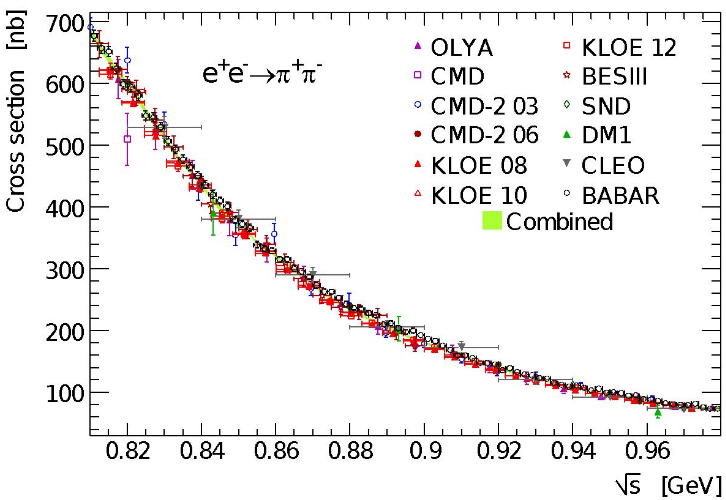

→ BES III results to be included: ~tension with pQCD and with KEDR 16

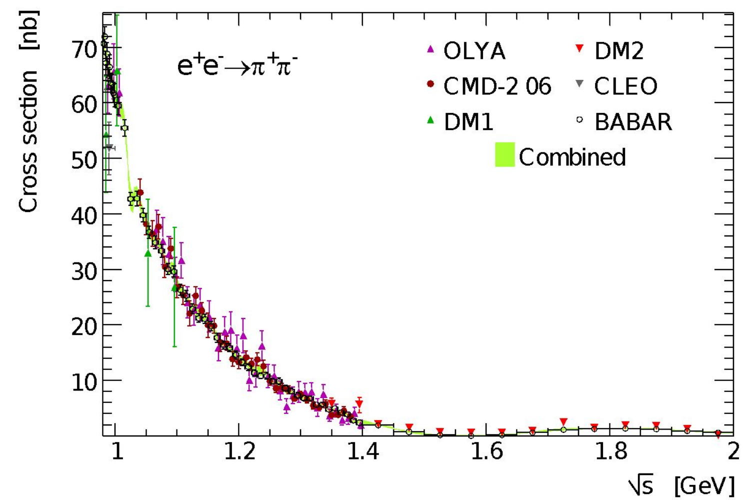
→ Another example of "*uncertainties on the uncertainties*" / *systematic effects to be understood* at the level of precision that is claimed

# More on the combination for the $e^+e^- \rightarrow \pi^+\pi^-$ channel



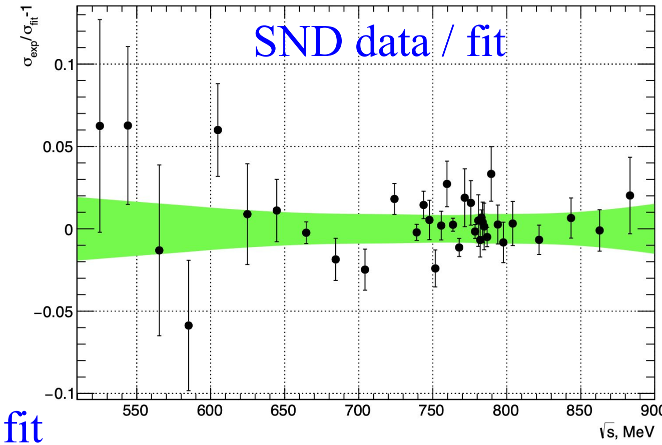



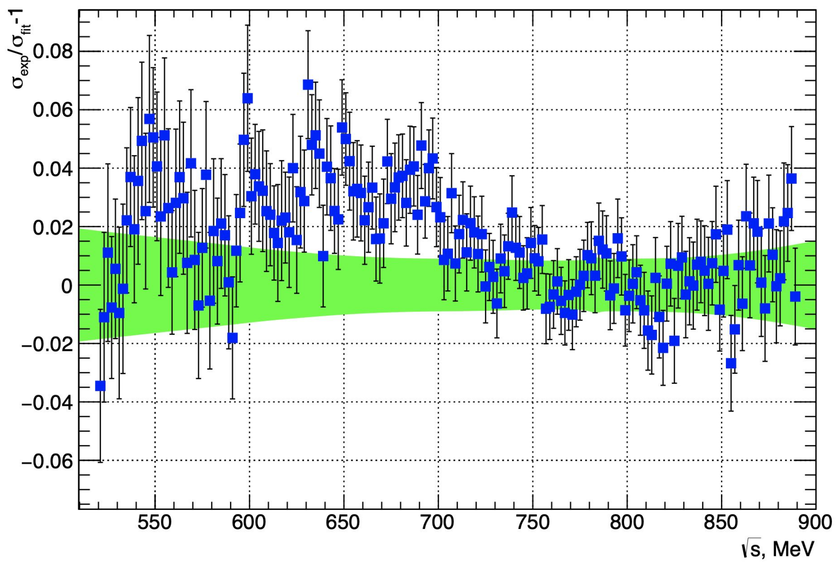



# Comparison of SND measurement with BABAR and KLOE

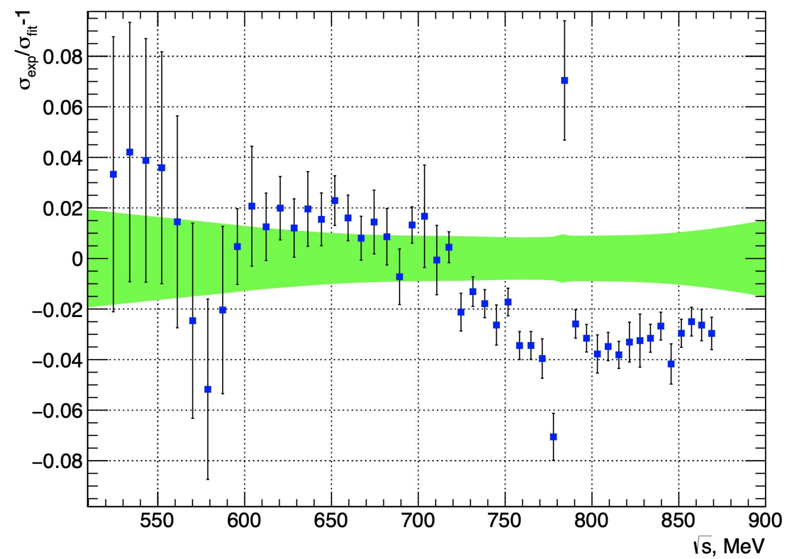
2004.00263



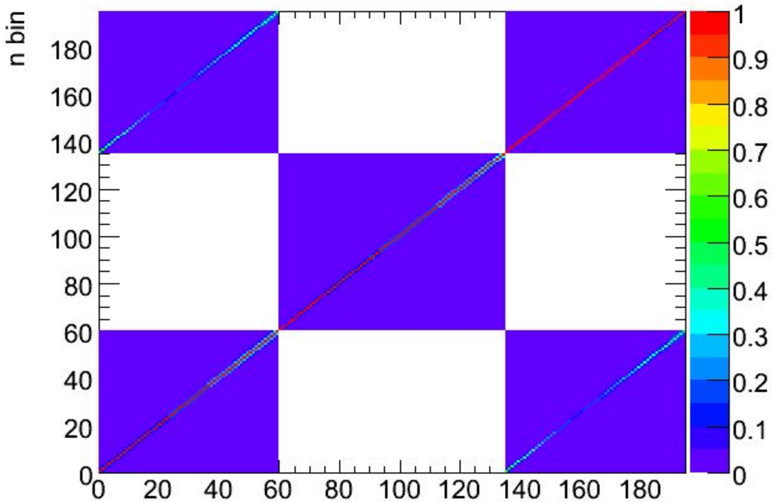

BABAR / SND fit



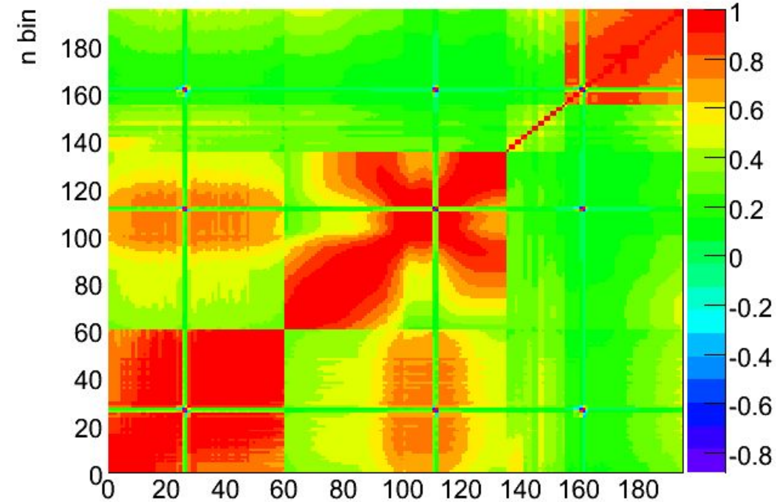

KLOE / SND fit



# Treatment of the KLOE correlation matrices



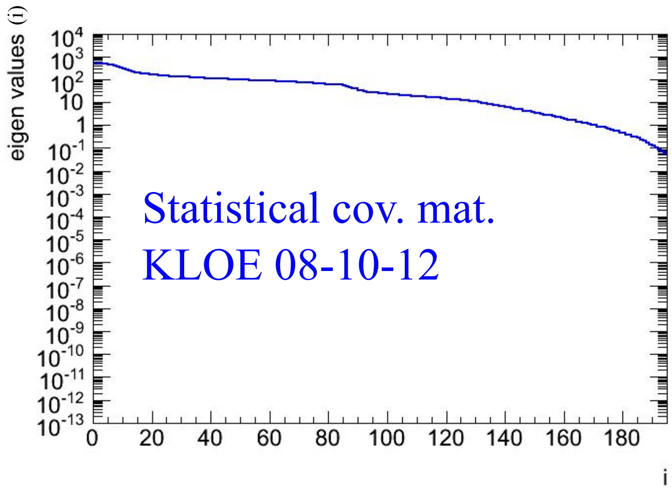

*KLOE: 08 10 12* n bin

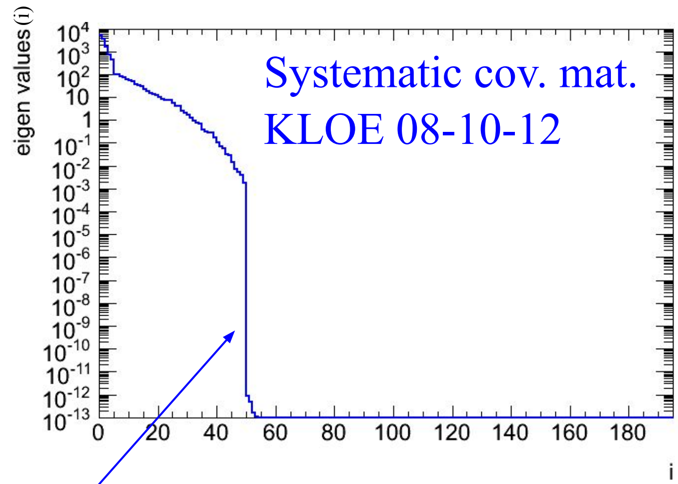



*KLOE: 08 10 12* n bin

→ Statistical and systematic correlation matrices among the 3 measurements

# Treatment of the KLOE data – eigenvector decomposition



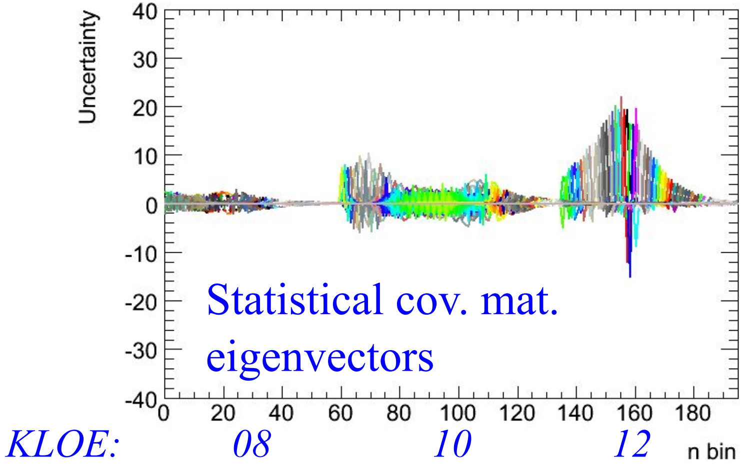

$$C = S \cdot D \cdot S^T$$

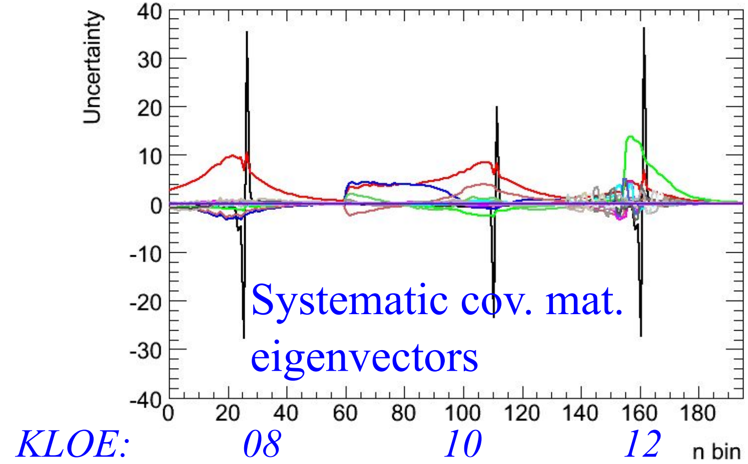
$$D = \begin{pmatrix} \ddots & 0 & 0 \\ 0 & \sigma_i^2 & 0 \\ 0 & 0 & \ddots \end{pmatrix}$$

$$S = \begin{pmatrix} V_1 & \dots & V_n \end{pmatrix}$$

→ "counting" the number of independent components (50) used to build the covariance matrix

→ Problem of negative eigenvalues for previous systematic covariance matrix solved
(informed KLOE collaboration about the problem in summer 2016)



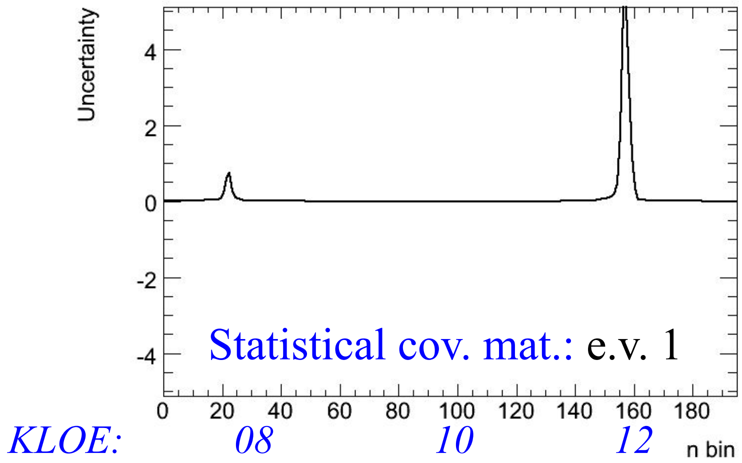

→ Each normalized eigenvector ($\sigma_i$*$V_i$) treated as an uncertainty fully correlated between the bins

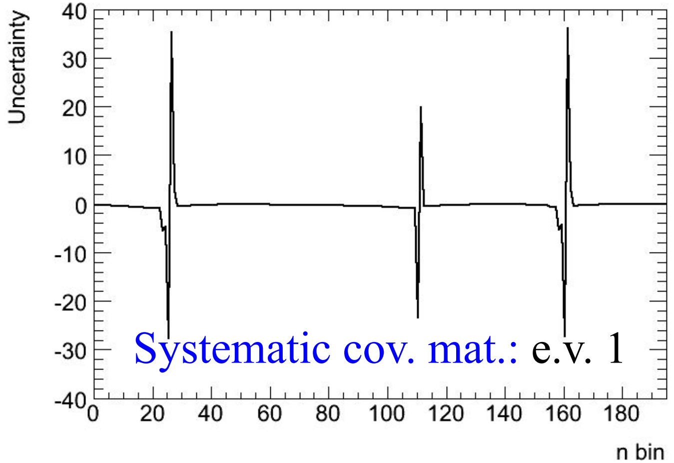
→ All these uncertainties are independent between each-other

$$C=\sum_{i=1}^{N_{bins}} \sigma_i^2 \cdot C\left(V_i\right)$$

→ Checked exact matching with the original matrices + with all $a_\mu$ integrals and uncertainties published by KLOE

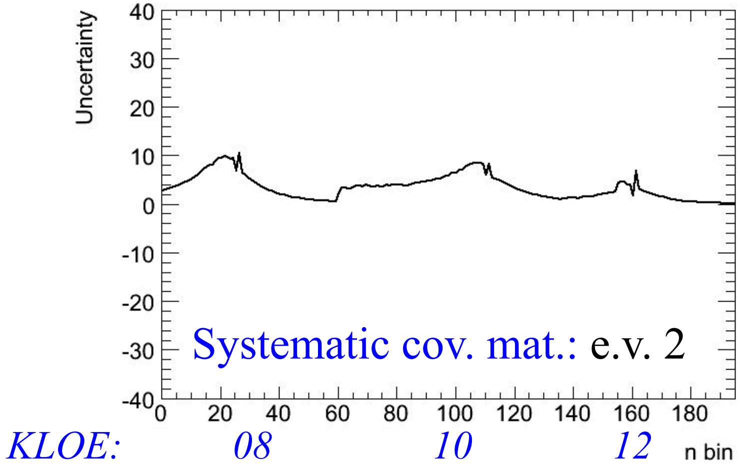
# Treatment of the KLOE data – eigenvector decomposition



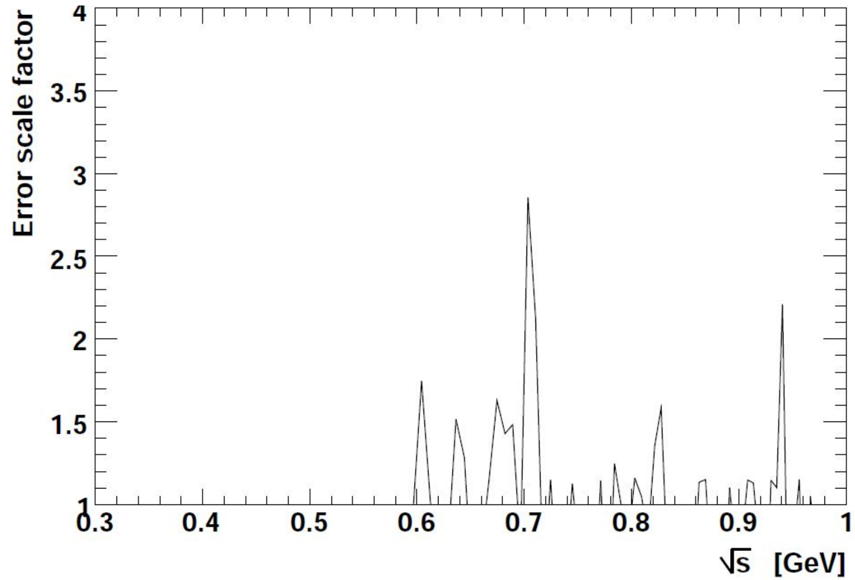




→ Eigenvectors carry the general features of the correlations:

- long-range for systematics
- ~short-range for statistical uncertainties + correlations between KLOE 08 & 12



# Local comparison of the 3 KLOE measurements

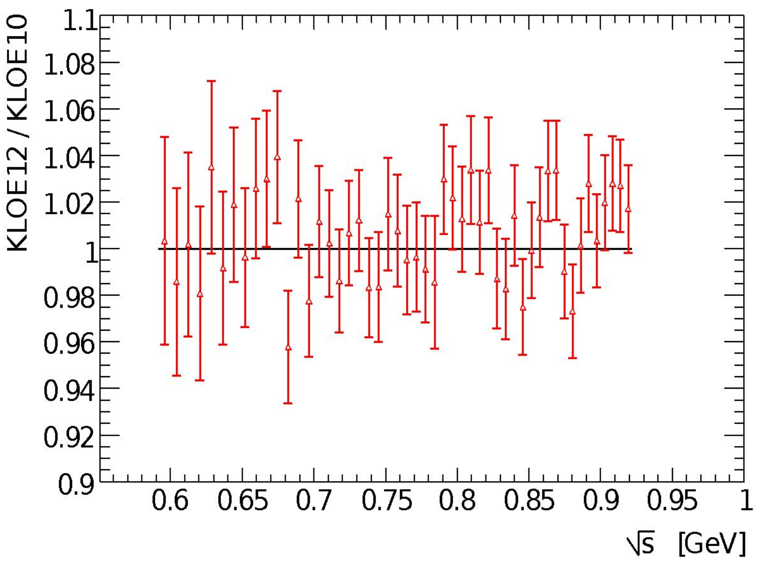


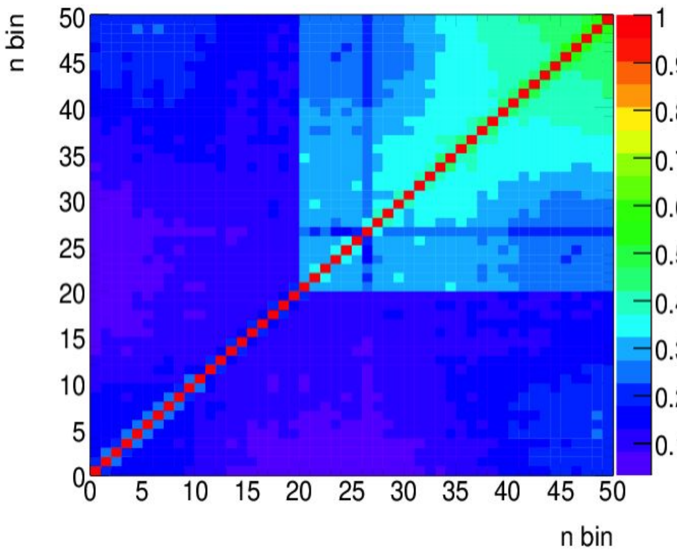

→ Local $\chi^2$ /ndof test of the local compatibility between KLOE 08 & 10 & 12, taking into account the correlations: some tensions observed

→ Does not probe general trends of the difference between the measurements (e.g. slopes in the ratio)

# Ratios between measurements

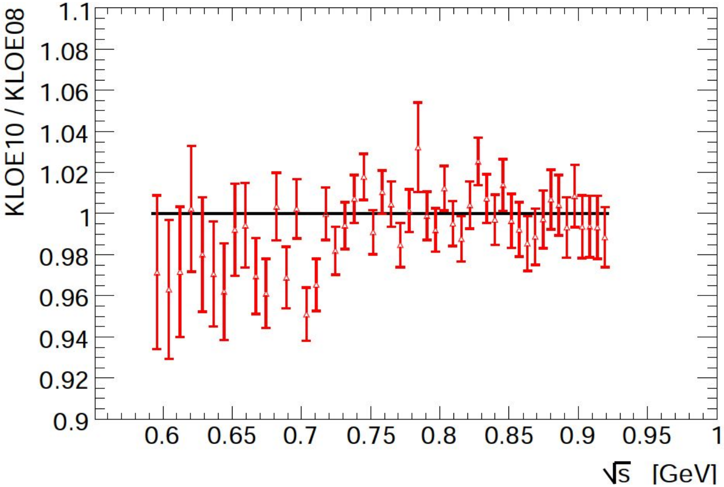
→ Compute ratio between pairs of KLOE measurements
→ Full propagation of uncertainties and correlations using pseudo-experiments
(agreement with analytical linear uncertainty propagation)

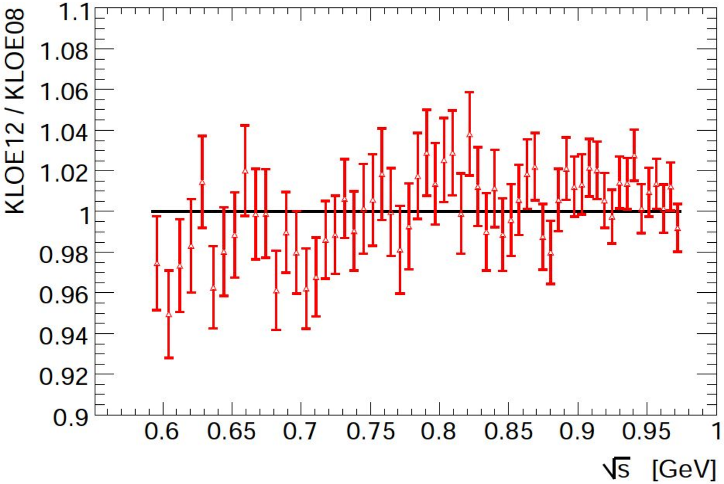


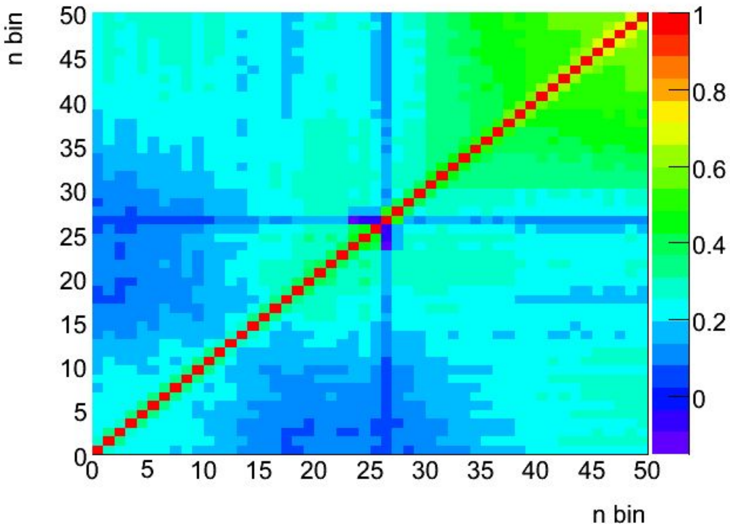



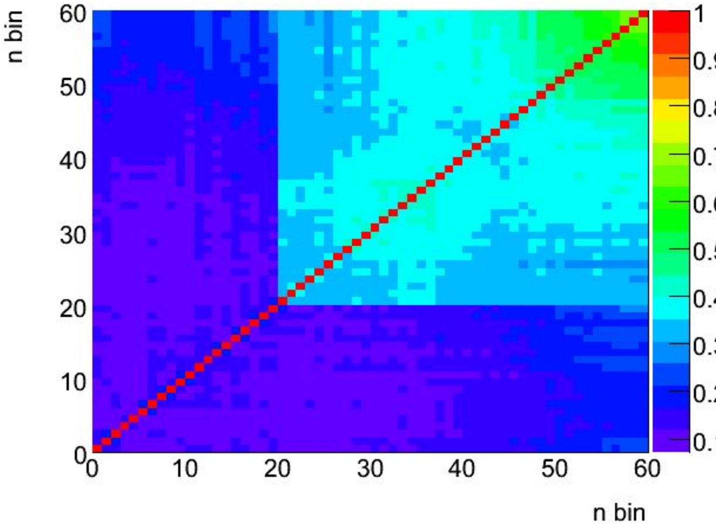

→ Good agreement between KLOE 10 and KLOE 12

# Ratios between measurements



# Direct comparison of the 3 KLOE measurements

→ Quantitative comparison between the ratios and unity, taking into account correlations

KLOE 10 / KLOE 08

$\chi^2$ [0.35;0.85] GeV$^2$ : 79.0 / 50(DOF)
p-value= 0.0056

$\chi^2$ [0.35;0.58] GeV$^2$ : 46.2 / 23(DOF)
p-value= 0.0028

$\chi^2$ [0.58;0.85] GeV$^2$ : 29.7 / 27(DOF)
p-value= 0.33

$\chi^2$ [0.64;0.85] GeV$^2$ : 20.7 / 21(DOF)
p-value= 0.47

KLOE 12 / KLOE 08

$\chi^2$ [0.35;0.95] GeV$^2$ : 73.7 / 60(DOF)
p-value= 0.11

$\chi^2$ [0.35;0.58] GeV$^2$ : 21.8 / 23(DOF)
p-value= 0.53

$\chi^2$ [0.35;0.64] GeV$^2$ : 27.5 / 29(DOF)
p-value= 0.55

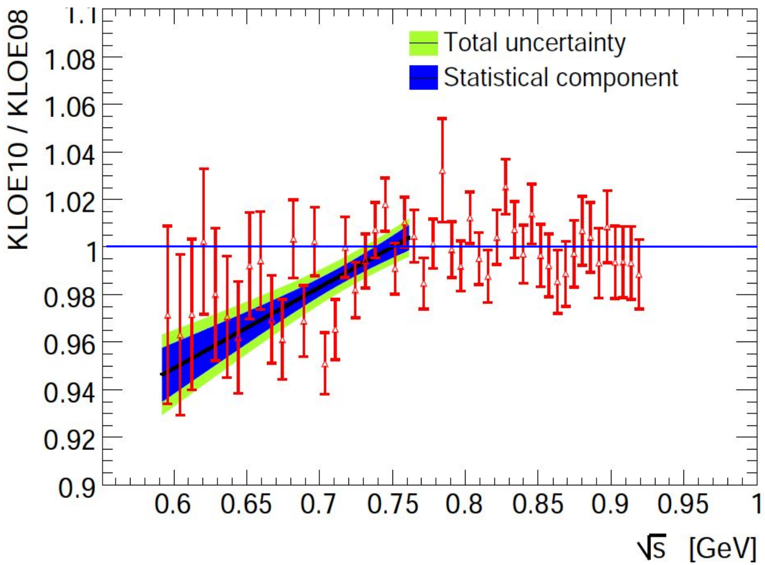
$\chi^2$ [0.64;0.95] GeV$^2$ : 39.4 / 31(DOF)
p-value= 0.14

# Quantitative comparisons of the KLOE measurements

→ Quantitative comparison between the ratios and unity, taking into account correlations

→ Fitting the ratio taking into account correlations

→ Full propagation of uncertainties and correlations – 3 methods yielding consistent results: ±1σ shifts of each uncertainty, pseudo-experiments and fit uncertainties from Minuit




Comparison with Unity:

$\chi^2$ [0.35;0.85] GeV$^2$ : 79.0 / 50(DOF)

p-value= 0.0056

$\chi^2$ [0.35;0.58] GeV$^2$ : 46.2 / 23(DOF)

p-value= 0.0028

$\chi^2$ [p0 + p1$\sqrt{s}$]: 36.1 / 21(DOF)

p-value= 0.02

p0 : 0.745 ± 0.085

p1 : 0.341 ± 0.117

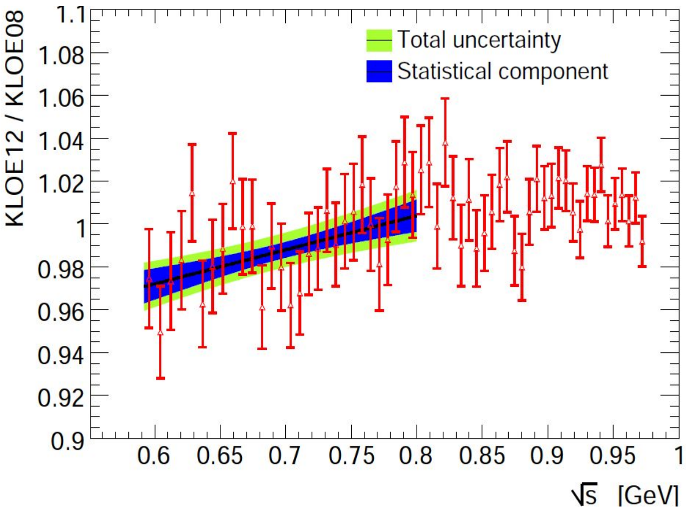
→ Significant shift & slope (~2.5-3σ) at low √s, no significant shift at high √s
Similar shift & slope for KLOE 12 / KLOE 08 (*see below*)

→ Should motivate conservative treatment of uncertainties and correlations in combination

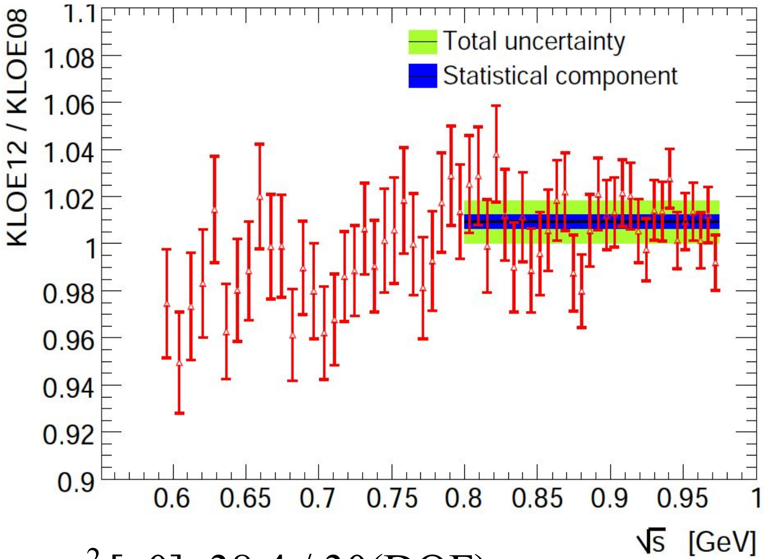
# Direct comparison of the 3 KLOE measurements

→ Fitting the ratio taking into account correlations

→ Full propagation of uncertainties and correlations – 3 methods yielding consistent results: ±1σ shifts of each uncertainty, pseudo-experiments and fit uncertainties from Minuit



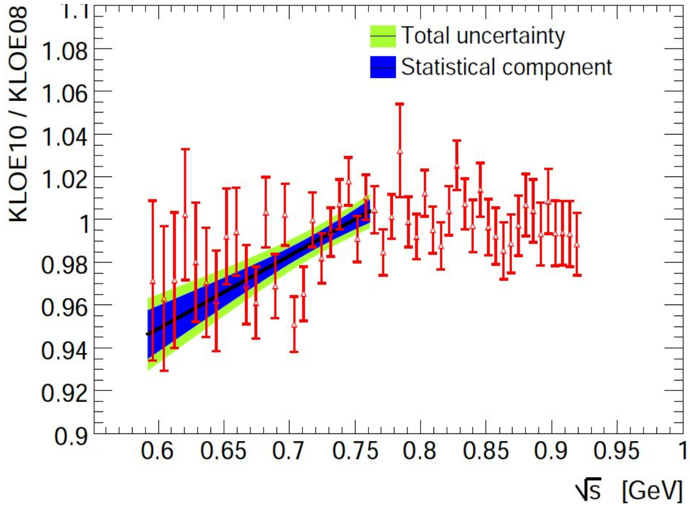

$\chi^2$ [p0 + p1√s]: 20.7 / 27(DOF)
p-value= 0.80
p0 : 0.876 ± 0.056
p1 : 0.159 ± 0.081



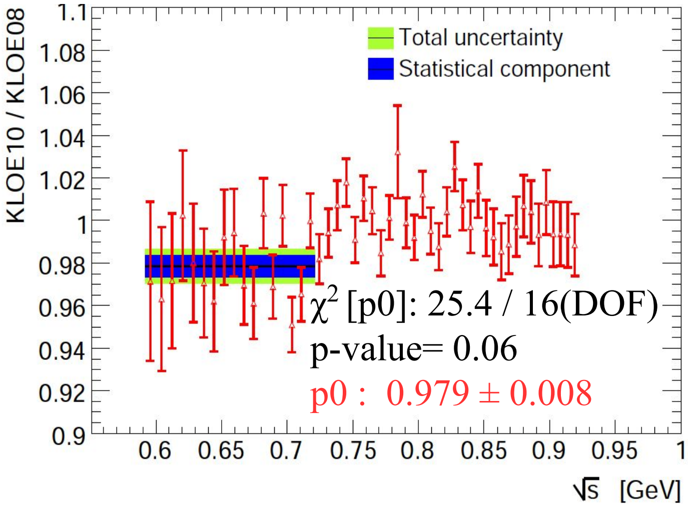

$\chi^2$ [p0]: 38.4 / 30(DOF)
p-value= 0.14
p0 : 1.009 ± 0.009

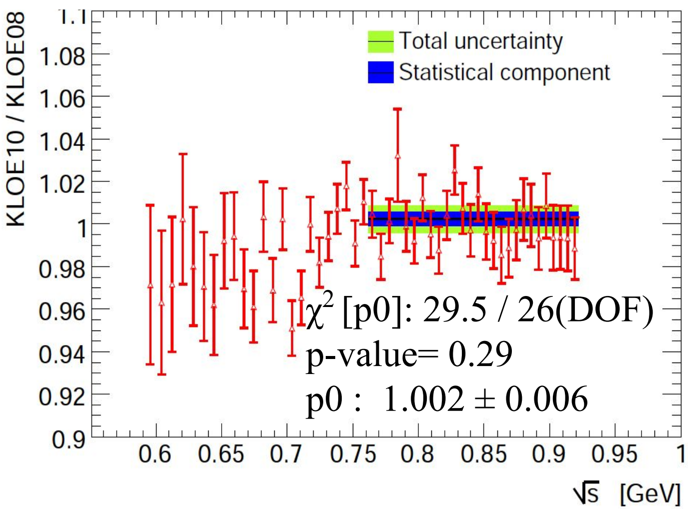
→ Significant shift and slope (~2σ) at low √s, no significant shift at high √s

# Direct comparison of the 3 KLOE measurements




$\chi^2$ [p0 + p1√s]: 36.1 / 21(DOF)
p-value= 0.02
p0 :  0.745 ± 0.085
p1 :  0.341 ± 0.117

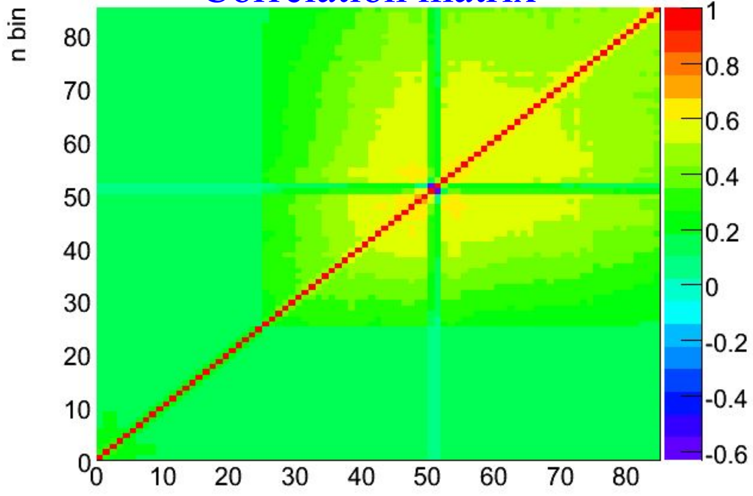


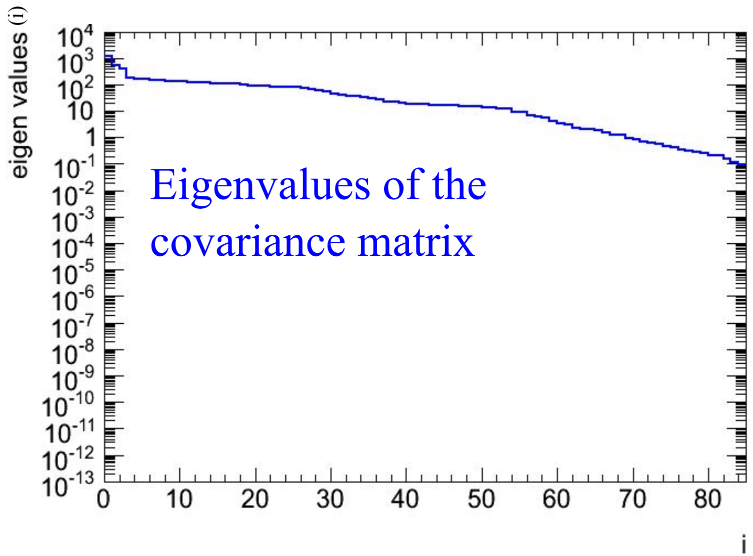




→ Significant shift and slope (~2.5-3σ) at low √s,
no significant shift at high √s

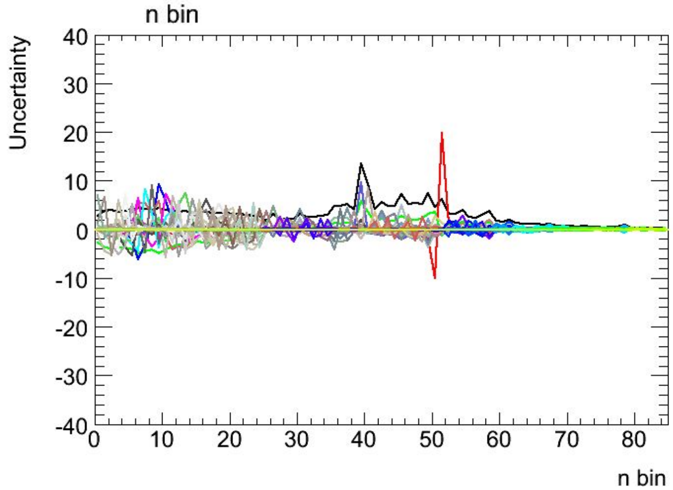
# Treatment of the combined KLOE data

Correlation matrix

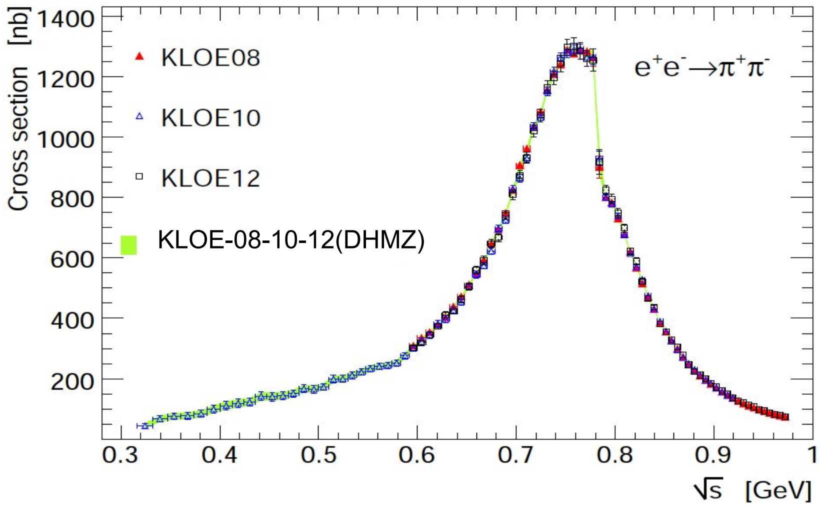


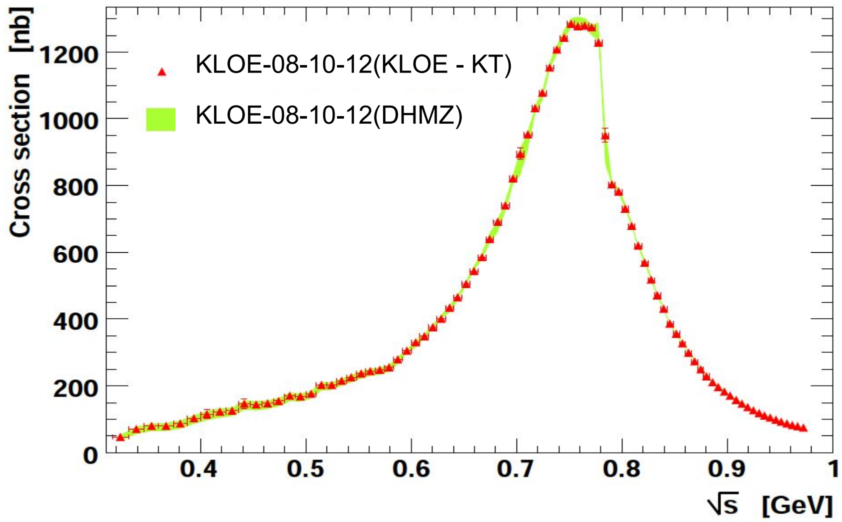



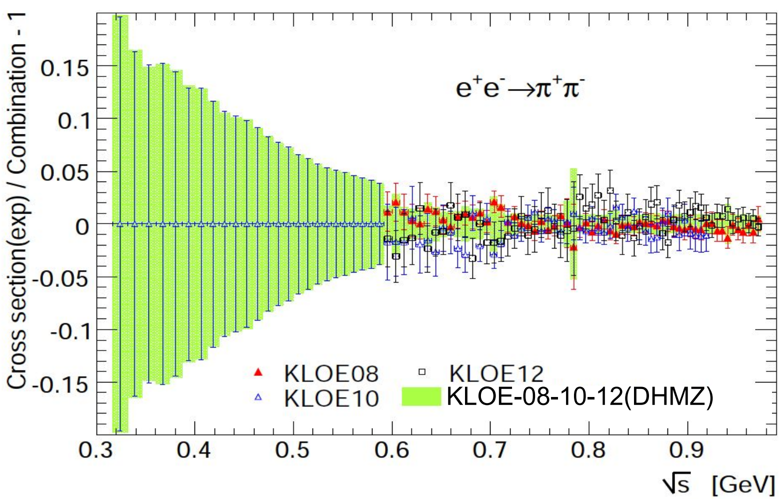

Eigenvalues of the covariance matrix

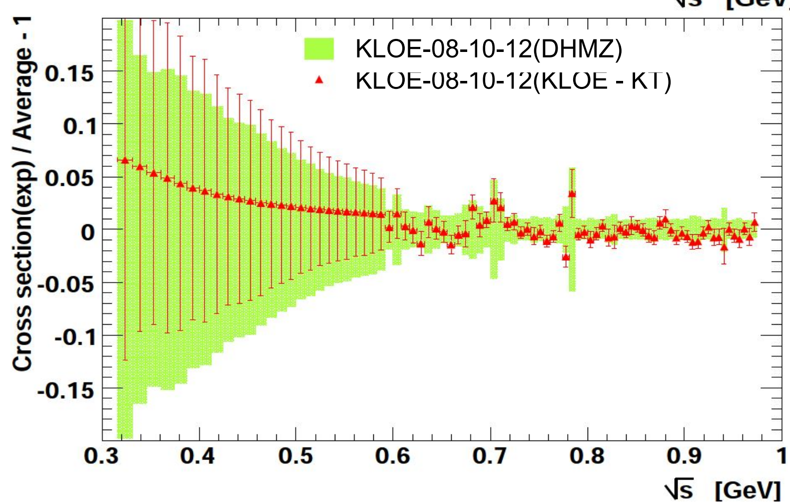


# Combining the 3 KLOE measurements



# $a_\mu^{\pi\pi}$ contribution [0.28; 1.8] GeV – spline-based (2018)

→ Updated result:

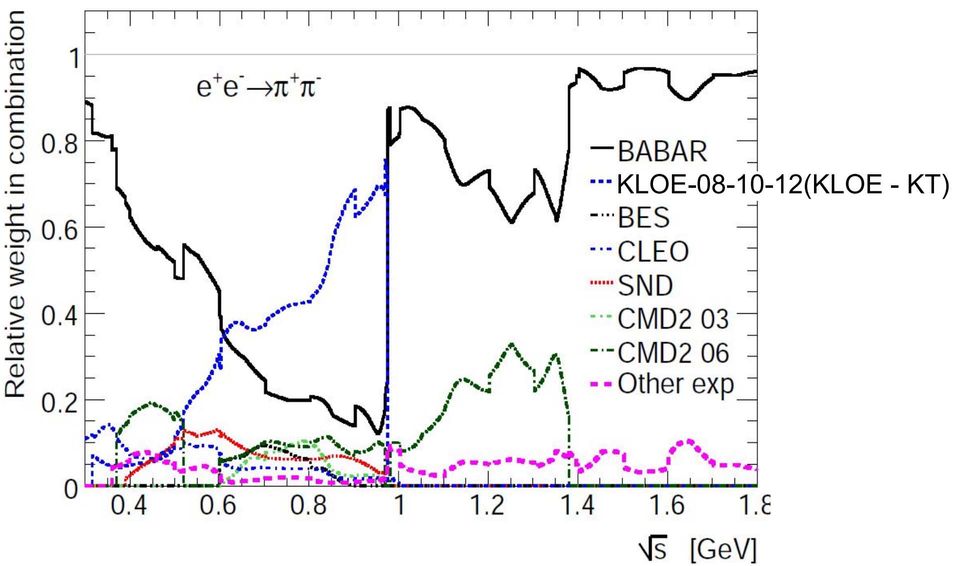
506.70 ± 2.32 ( ± 1.01 (stat.) ± 2.08 (syst.) ) [$10^{-10}$]

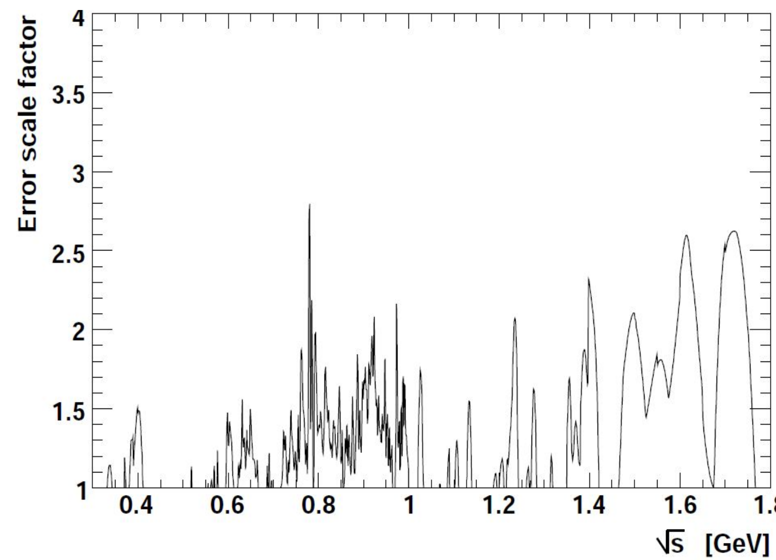
(after uncertainty enhancement by ~14% caused by the tension between inputs, taken into account through a local rescaling)

Total uncertainty: 5.9 (2003) → 2.8 (2011) → 2.6 (2017) → 2.3 (2018)

# $a_\mu^{\pi\pi}$ contribution [0.28; 1.8] GeV – spline-based (2018)

→ with KLOE-08-10-12 (KLOE-KT) used as input: 506.55 ± 2.38 [$10^{-10}$]
(after uncertainty enhancement by 18% caused by the tension between inputs, taken into account through a local rescaling)
→ Compensation between uncertainty reduction for KLOE-08-10-12 (KLOE-KT), inducing a change of weights in DHMZ combination, and tension enhancement



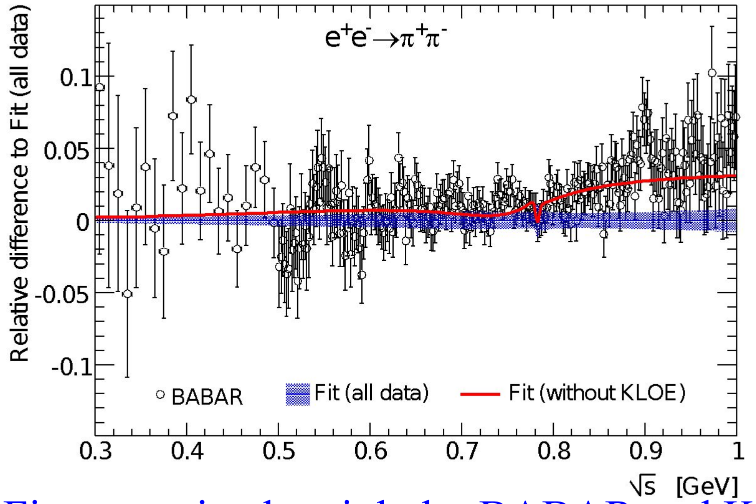



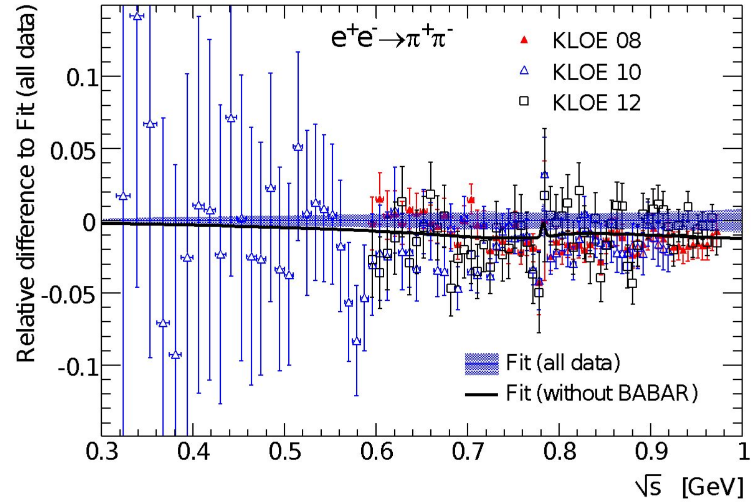
| | $\alpha_V$ | $\kappa[10^{-4}]$ | $B_0$ | $B_1$ | $m_\rho$ [MeV] | $m_\omega$ [MeV] |
|---|---|---|---|---|---|---|
| $\alpha_V$ | $0.133 \pm 0.020$ | 0.52 | −0.45 | −0.97 | 0.90 | −0.25 |
| $\kappa[10^{-4}]$ | | $21.6 \pm 0.5$ | −0.33 | −0.57 | 0.64 | −0.08 |
| $B_0$ | | | $1.040 \pm 0.003$ | 0.40 | −0.40 | 0.29 |
| $B_1$ | | | | $-0.13 \pm 0.11$ | −0.96 | 0.20 |
| $m_\rho$ [MeV] | | | | | $774.5 \pm 0.8$ | −0.17 |
| $m_\omega$ [MeV] | | | | | | $782.0 \pm 0.1$ |

→ κ corresponds to a Br ($\omega \to \pi^+\pi^-$) of $(2.09 \pm 0.09) \cdot 10^{-2}$, in agreement with the result extracted from the fit of arXiv:1810.00007, $(1.95 \pm 0.08) \cdot 10^{-2}$. Both values disagree with the PDG average $(1.51 \pm 0.12) \cdot 10^{-2}$, dominated by the result of arXiv:1611.09359 which uses fits to essentially the same data.

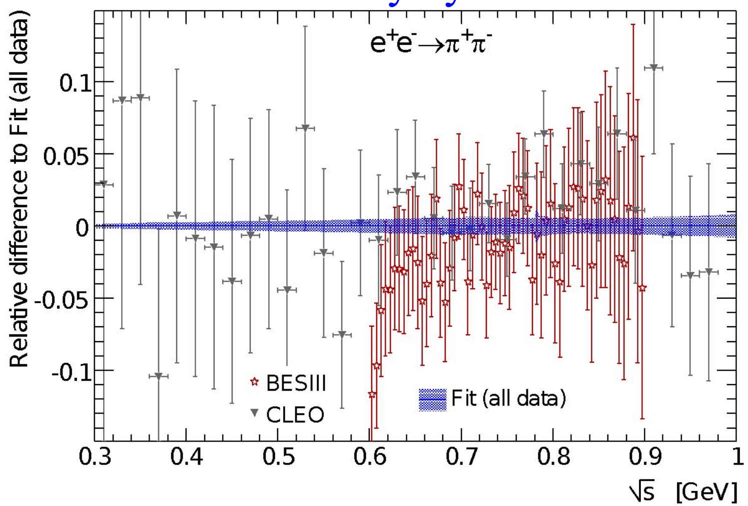
→ The fitted ω mass is found to be lower than the PDG average obtained from 3π decays by $(0.65 \pm 0.12 \pm 0.12_{PDG})$ MeV, in agreement with previous fits of the $\rho - \omega$ interference in the 2π spectrum (see e.g. arXiv:1205.2228 and arXiv:1810.00007).

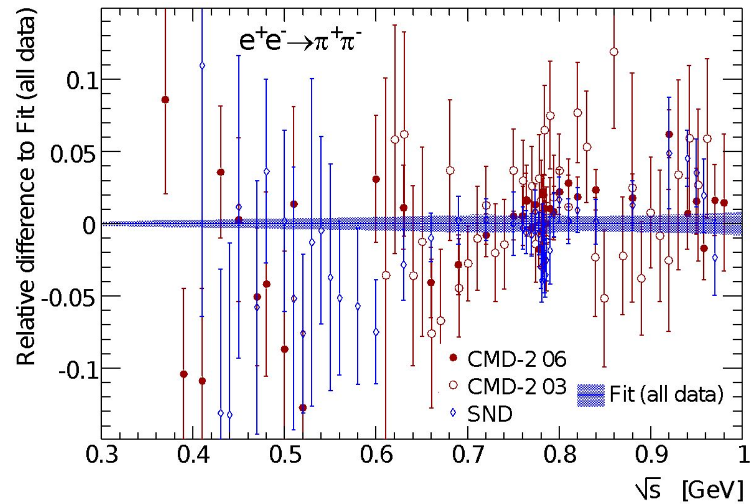
# Fit performed up to 1 GeV: comparison with data



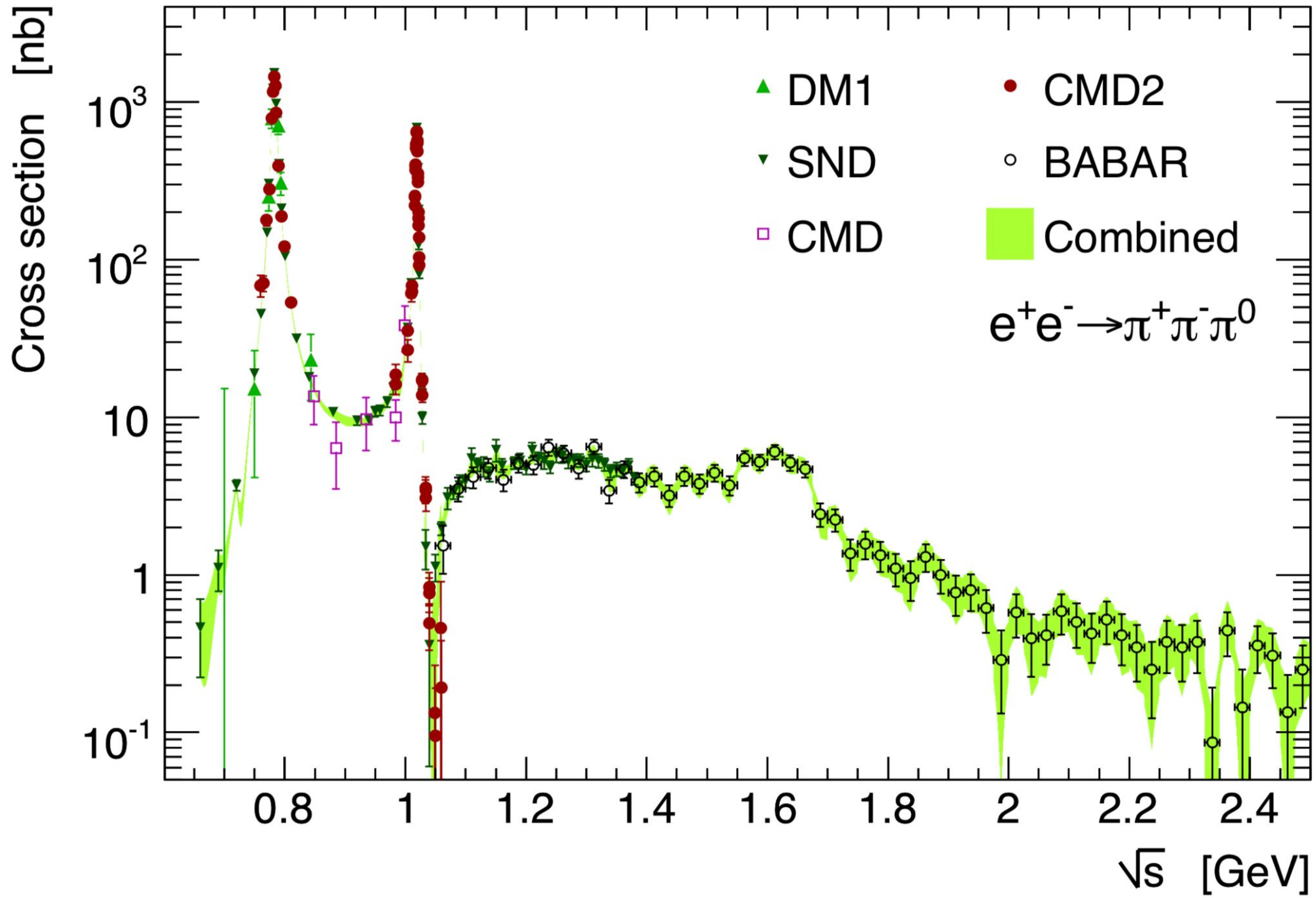




→ Fit constrained mainly by BABAR and KLOE measurements

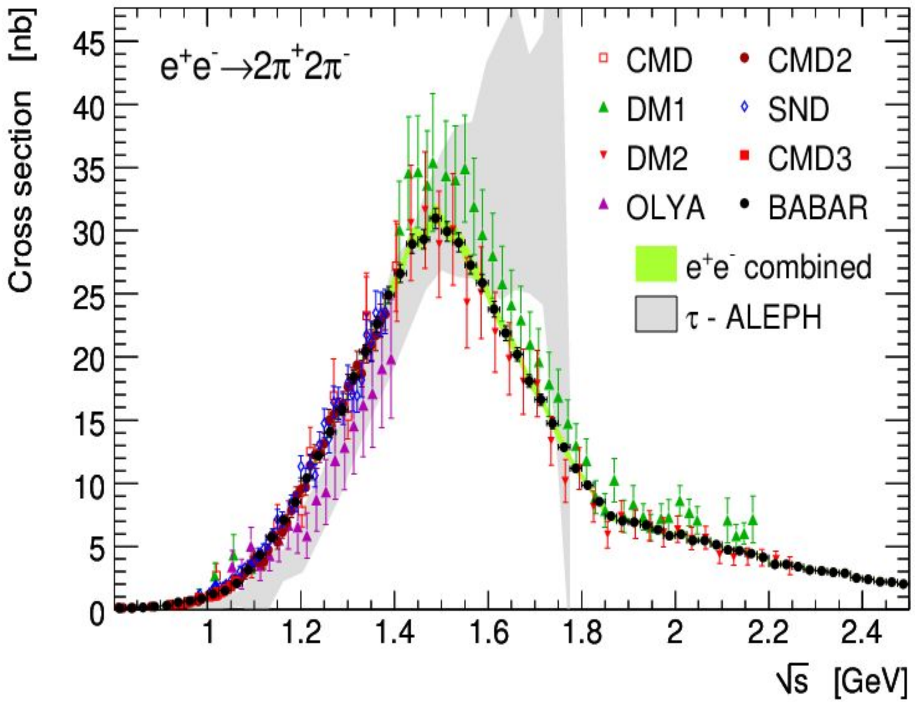


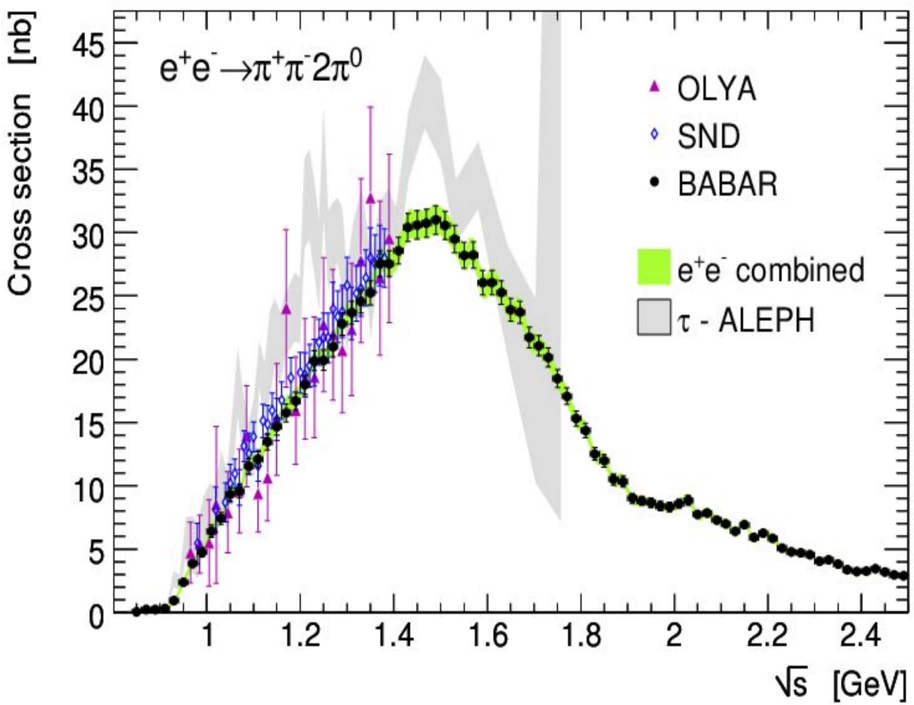



# Combination for the $e^+e^- \to \pi^+\pi^-\pi^0$ channel



# $e^+e^- \to \pi^+\pi^-\pi^+\pi^-$, $e^+e^- \to \pi^+\pi^-\pi^0\pi^0$



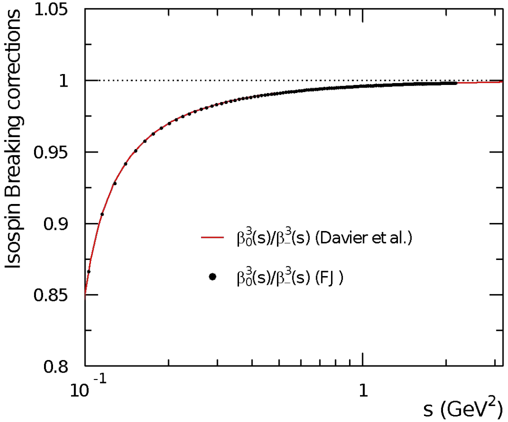



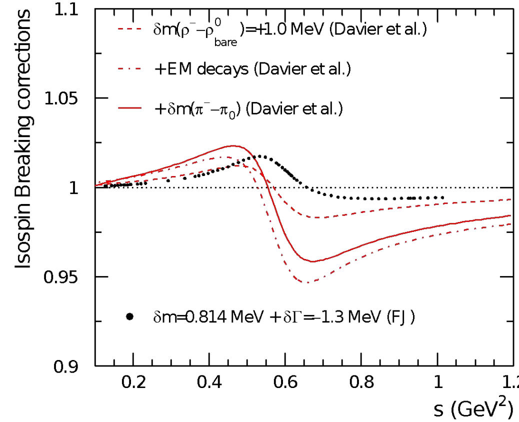

→ Essentially normalization differences w.r.t. $\tau$ data: *cross-checks very desirable*

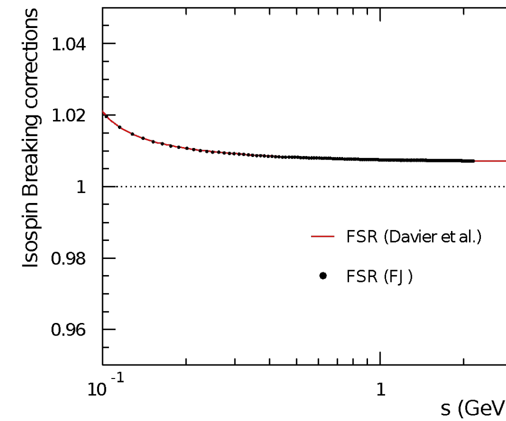
# Comparison with IB-corrected τ data

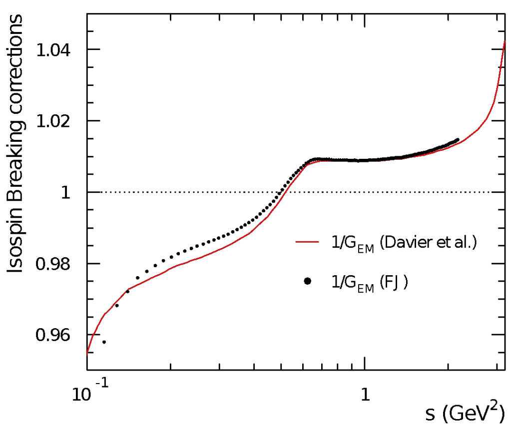
$$v_{1,X^-}(s) = \frac{m_\tau^2}{6|V_{ud}|^2}\frac{\mathcal{B}_{X^-}}{\mathcal{B}_e}\frac{1}{N_X}\frac{dN_X}{ds}\times\left(1-\frac{s}{m_\tau^2}\right)^2\left(1+\frac{2s}{m_\tau^2}\right)^{-1}\frac{R_{\rm IB}(s)}{S_{\rm EW}} \qquad R_{\rm IB}(s) = \frac{{\rm FSR}(s)}{G_{\rm EM}(s)}\frac{\beta_0^3(s)}{\beta_-^3(s)}\left|\frac{F_0(s)}{F_-(s)}\right|^2$$

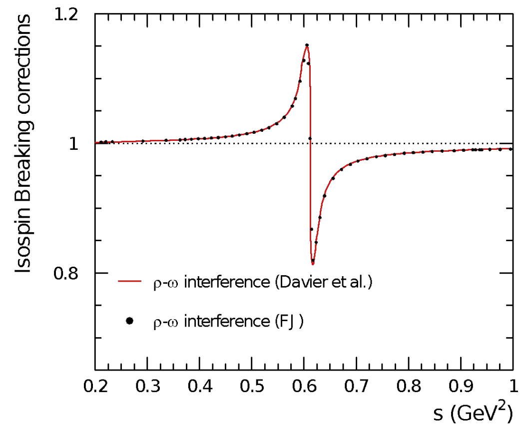
→ Comparing corrections used by Davier et al. with the ones by F. Jegerlehner



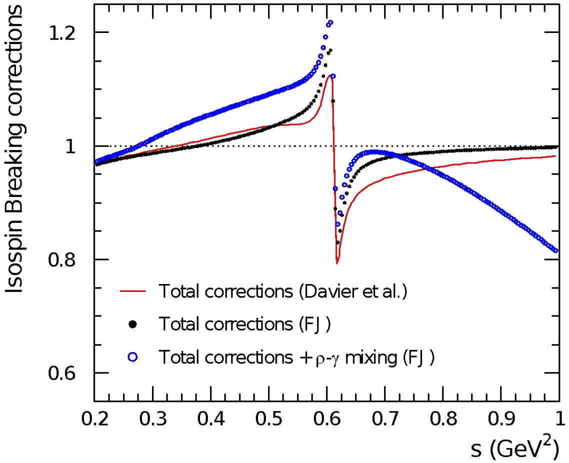



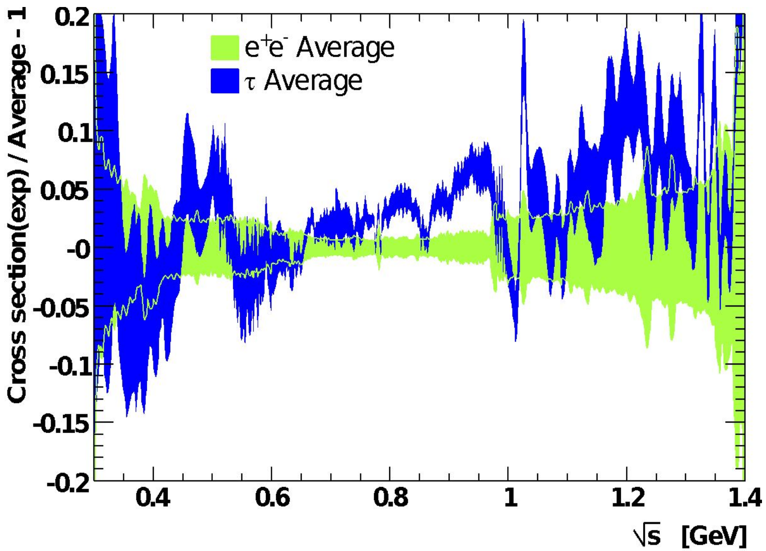

Plots by Z. Zhang, based on private communication with F. Jegerlehner

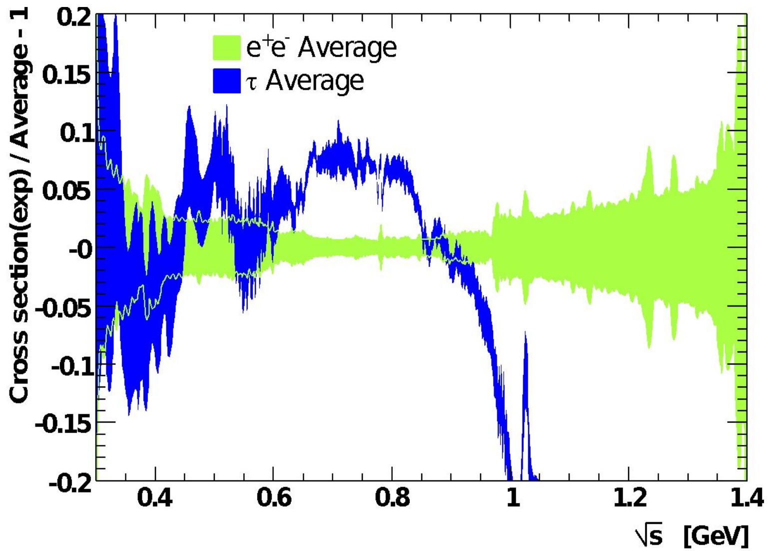
# Comparison with IB-corrected τ data

→ for $a_\mu$, $e^+e^- - \tau$ difference of 2.2 σ (Davier et al.)

→ the ρ−γ mixing correction proposed in arXiv:1101.2872 (FJ) seems to over-estimate the $e^+e^- - \tau$ difference



# $\chi^2$ definitions and properties

$$\chi^2(\mathbf{d};\mathbf{t}) = \sum_{i,j} (d_i - t_i) \cdot \left[C^{-1}(\mathbf{t})\right]_{ij} \cdot (d_j - t_j) \qquad C_{ij} = C_{ij}^{stat} + \sum_k s_i^k \cdot s_j^k$$

$$\chi^2(\mathbf{d};\mathbf{t}) = \min_{\beta_a} \left\{ \sum_{i,j} \left[ d_i - \left( 1 + \sum_a \beta_a \cdot \left(\boldsymbol{\epsilon}_a^{\pm}(\beta_a)\right)_i \right) t_i \right] \cdot \left[C_{\mathrm{su}}^{-1}(\mathbf{t})\right]_{ij} \cdot \left[ d_j - \left( 1 + \sum_a \beta_a \cdot \left(\boldsymbol{\epsilon}_a^{\pm}(\beta_a)\right)_j \right) t_j \right] + \sum_a \beta_a^2 \right\},$$

→ Two $\chi^2$ definitions, with systematic uncertainties included in covariance matrix or treated as fitted "nuisance parameters"

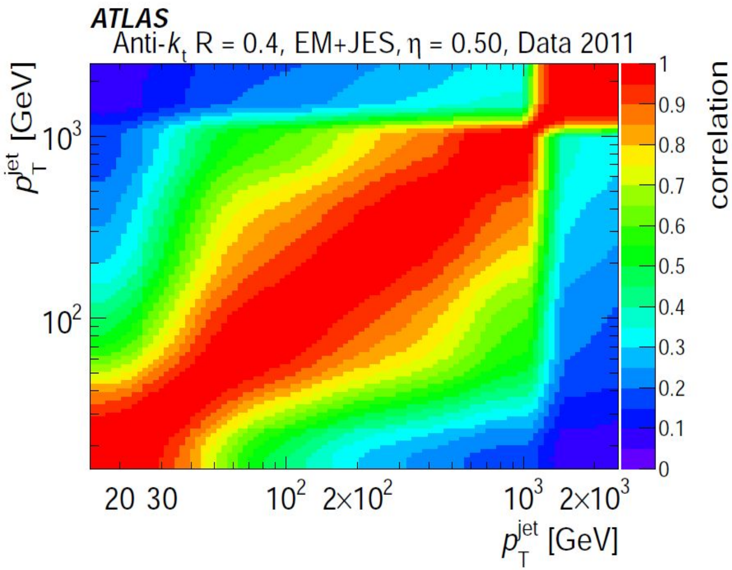
→ Equivalent for symmetric Gaussian uncertainties

(1312.3524 - ATLAS)

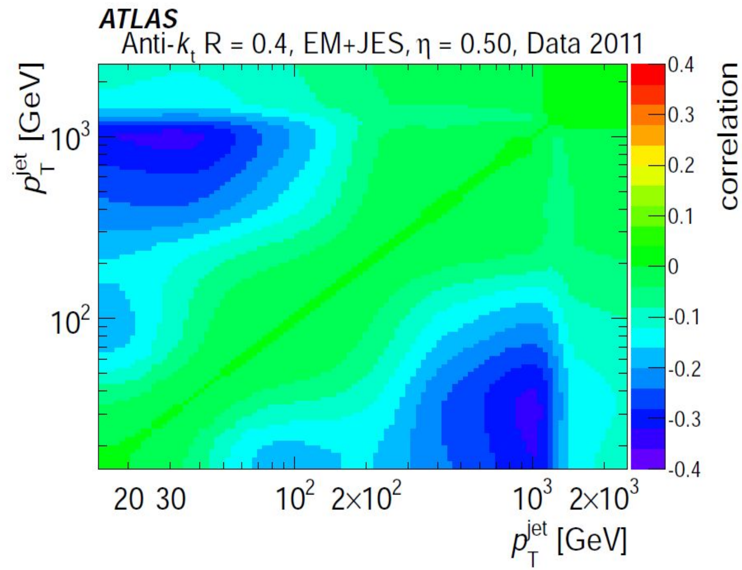
→ *Both approaches assume the knowledge of the amplitude, shape (phase-space dependence) and correlations of systematic uncertainties*

# Example: published uncertainties on correlations

1406.0076 – ATLAS jet energy scale uncertainties




*Nominal correlation scenario*

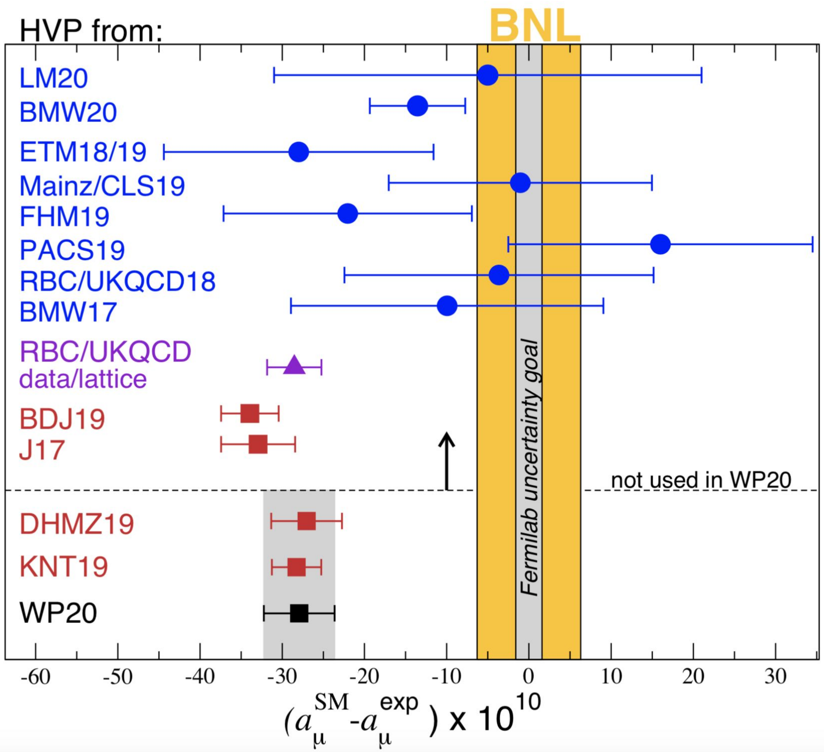



*Weaker - stronger correlation scenarios*

# Impact of correlations between $a_\mu$ and $\alpha_{QED}$ on the EW fit

2008.08107(BM, Matthias Schott)

**See also**: Crivellin et al, 2003.04886; Keshavarzi et al., 2006.12666 ;de Rafael, 2006.13880; Colangelo et al, 2010.07943



# Approaches considered for treating the $a_\mu$ - $\alpha_{QED}$ correlations

Studied approaches probing different hypotheses concerning the possible source(s) of the $a_\mu$ tension(s) :

(0) Scaling factor applied to the HVP contribution from some energy range of the hadronic spectrum

→ Approaches taking into account (*for the first time*) the full correlations between the uncertainties of the HVP contributions to $a_\mu$ and $\alpha_{QED}$ , based on input from DHMZ 19 (arXiv:1908.00921):
correlations between points/bins of a measurement in a given channel, between different measurements in the same channel, between different channels; full treatment of the BABAR-KLOE tension in the $\pi^+\pi^-$ channel

| Computation (Energy range) | $a_\mu^{\text{HVP, LO}}$ $[10^{-10}]$ | $\Delta\alpha_{\text{had}}(M_Z^2)$ $[10^{-4}]$ | $\rho$ |
|---|---|---|---|
| Phenomenology (Full HVP) | $694.0 \pm 4.0$ | $275.3 \pm 1.0$ | 44% |
| Phenomenology ([Th.; 1.8 GeV]) | $635.5 \pm 3.9$ | $55.4 \pm 0.4$ | 86% |
| Phenomenology ([Th.; 1 GeV]) | $539.8 \pm 3.8$ | $36.3 \pm 0.3$ | 99.5% |
| Lattice (Full HVP)BMW 20 (v1) | $712.4 \pm 4.5$ | - | - |

(1) Cov. matrix of $a_\mu$ and $\alpha_{QED}$ (Pheno) described by a nuisance parameter ($NP_1$) impacting both quantities (used to shift $a_\mu$ to some "target" value - coherent shift applied to $\alpha_{QED}$) and another one ($NP_2$) impacting only $\alpha_{QED}$ (used in the EW fit)
Note: "target" values chosen in order to reach agreement with the BMW 20 prediction / Experimental $a_\mu$ ($\pm 1\sigma$)

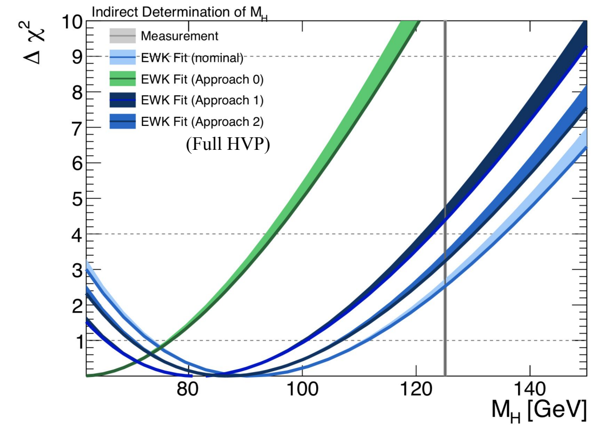
| Uncertainty components | $a_\mu^{\text{HVP, LO}}$ | $\Delta\alpha_{\text{had}}(M_Z^2)$ |
|---|---|---|
| $NP_1$ | $\sigma(a_\mu^{\text{HVP, LO}})$ | $\sigma(\Delta\alpha_{\text{had}}(M_Z^2)) \cdot \rho$ |
| $NP_2$ | 0 | $\sigma(\Delta\alpha_{\text{had}}(M_Z^2)) \cdot \sqrt{1-\rho^2}$ |

(2) Include the HVP contribution to $a_\mu$ as extra parameter in the EW fit, constrained by the Pheno & BMW 20 values
Note: Also accounted for the coherent impact of $\alpha_S$ on the HVP contribution and on the EW fit

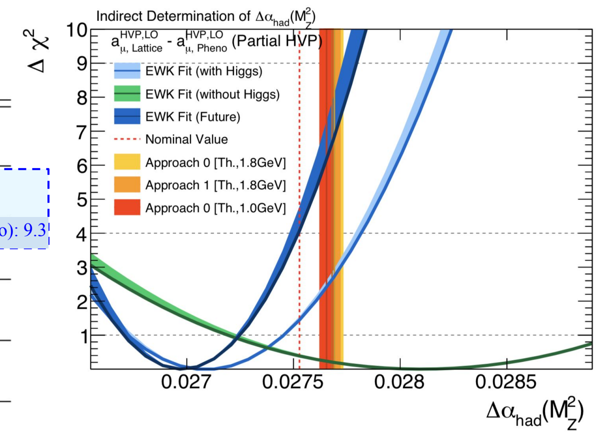
# Results: comparing the Phenomenology & BMW 20 values

| $a_\mu^{\text{HVP, LO}}$ shift (Energy range) | *Approach* 0 | | *Approach* 1 | | |
|---|---|---|---|---|---|
| | Scaling factor | $\Delta'\alpha_{\text{had}}(M_Z^2)$ | Shift NP$_1$ | $\sigma'\left(\Delta\alpha_{\text{had}}(M_Z^2)\right)$ | $\Delta'\alpha_{\text{had}}(M_Z^2)$ |
| $a_{\mu\,(\text{Lattice})}^{\text{HVP, LO}} - a_{\mu\,(\text{Pheno})}^{\text{HVP, LO}}$ (Full HVP) | 1.027 | 0.02826 | 4.6 | $9.0 \cdot 10^{-5}$ | 0.02774 |
| $(a_{\mu\,(\text{Lattice})}^{\text{HVP, LO}} - 1\sigma) - a_{\mu\,(\text{Pheno})}^{\text{HVP, LO}}$ (Full HVP) | 1.020 | 0.02808 | 3.5 | $9.0 \cdot 10^{-5}$ | 0.02769 |
| $a_{\mu\,(\text{Lattice})}^{\text{HVP, LO}} - a_{\mu\,(\text{Pheno})}^{\text{HVP, LO}}$ ([Th.; 1.8 GeV]) | 1.029 | 0.02769 | 4.7 | $9.5 \cdot 10^{-5}$ | 0.02768 |
| $(a_{\mu\,(\text{Lattice})}^{\text{HVP, LO}} - 1\sigma) - a_{\mu\,(\text{Pheno})}^{\text{HVP, LO}}$ ([Th.; 1.8 GeV]) | 1.022 | 0.02765 | 3.5 | $9.5 \cdot 10^{-5}$ | 0.02764 |
| $a_{\mu\,(\text{Lattice})}^{\text{HVP, LO}} - a_{\mu\,(\text{Pheno})}^{\text{HVP, LO}}$ ([Th.; 1 GeV]) | 1.034 | 0.02765 | - | - | - |
| $(a_{\mu\,(\text{Lattice})}^{\text{HVP, LO}} - 1\sigma) - a_{\mu\,(\text{Pheno})}^{\text{HVP, LO}}$ ([Th.; 1 GeV]) | 1.026 | 0.02762 | - | - | - |

→ Large scaling factors (w.r.t. exp. uncertainties) & significant shifts of NP$_1$




| $a_\mu^{\text{HVP, LO}}$ shift (Energy range) | Nominal | | *Approach* 0 | | *Approach* 1 | | *Approach* 2 | |
|---|---|---|---|---|---|---|---|---|
| | $\Delta'\alpha_{\text{had}}(M_Z^2)$ | $\chi^2/\text{ndf}$ | $\Delta'\alpha_{\text{had}}(M_Z^2)$ | $\chi^2/\text{ndf}$ | $\Delta'\alpha_{\text{had}}(M_Z^2)$ | $\chi^2/\text{ndf}$ | $\Delta'\alpha_{\text{had}}(M_Z^2)$ | $\chi^2/\text{ndf}$ |
| | 0.02753 | 18.6/16 (p=0.29) | - | - | - | - | 0.02753 | 28.1/17 (p=0.04) $\chi^2$(BMW20-Pheno): 9.3 |
| $a_{\mu\,(\text{Lattice})}^{\text{HVP, LO}} - a_{\mu\,(\text{Pheno})}^{\text{HVP, LO}}$ (Full HVP) | - | - | 0.02826 | 27.6/16 (p=0.04) | 0.02774 | 20.3/16 (p=0.21) | - | |
| $a_{\mu\,(\text{Lattice})}^{\text{HVP, LO}} - a_{\mu\,(\text{Pheno})}^{\text{HVP, LO}}$ ([Th.; 1.8 GeV]) | - | - | 0.02769 | 19.9/16 (p=0.22) | 0.02768 | 19.8/16 (p=0.23) | - | - |
| $a_{\mu\,(\text{Lattice})}^{\text{HVP, LO}} - a_{\mu\,(\text{Pheno})}^{\text{HVP, LO}}$ ([Th.; 1.0 GeV]) | - | - | 0.02765 | 19.6/16 (p=0.24) | - | - | - | - |




→ Addressing the BMW 20 - Pheno difference for $a_\mu$ has little impact on the EW fit, except for the unrealistic scenario rescaling the full HVP contribution

Note: Similar conclusions for the comparison with the Experimental $a_\mu$ value (see next slides)

# Scaling factors and NP shifts

| $a_\mu^{\text{HVP, LO}}$ shift (Energy range) | *Approach 0* | | *Approach 1* | | |
|---|---|---|---|---|---|
| | Scaling factor | $\Delta'\alpha_{\text{had}}(M_Z^2)$ | Shift NP$_1$ | $\sigma'\left(\Delta\alpha_{\text{had}}(M_Z^2)\right)$ | $\Delta'\alpha_{\text{had}}(M_Z^2)$ |
| $a_{\mu\,(\text{Lattice})}^{\text{HVP, LO}} - a_{\mu\,(\text{Pheno})}^{\text{HVP, LO}}$ (Full HVP) | 1.027 | 0.02826 | 4.6 | $9.0 \cdot 10^{-5}$ | 0.02774 |
| $(a_{\mu\,(\text{Lattice})}^{\text{HVP, LO}} - 1\sigma) - a_{\mu\,(\text{Pheno})}^{\text{HVP, LO}}$ (Full HVP) | 1.020 | 0.02808 | 3.5 | $9.0 \cdot 10^{-5}$ | 0.02769 |
| $a_{\mu\,(\text{Lattice})}^{\text{HVP, LO}} - a_{\mu\,(\text{Pheno})}^{\text{HVP, LO}}$ ([Th.; 1.8 GeV]) | 1.029 | 0.02769 | 4.7 | $9.5 \cdot 10^{-5}$ | 0.02768 |
| $(a_{\mu\,(\text{Lattice})}^{\text{HVP, LO}} - 1\sigma) - a_{\mu\,(\text{Pheno})}^{\text{HVP, LO}}$ ([Th.; 1.8 GeV]) | 1.022 | 0.02765 | 3.5 | $9.5 \cdot 10^{-5}$ | 0.02764 |
| $a_{\mu\,(\text{Lattice})}^{\text{HVP, LO}} - a_{\mu\,(\text{Pheno})}^{\text{HVP, LO}}$ ([Th.; 1 GeV]) | 1.034 | 0.02765 | - | - | - |
| $(a_{\mu\,(\text{Lattice})}^{\text{HVP, LO}} - 1\sigma) - a_{\mu\,(\text{Pheno})}^{\text{HVP, LO}}$ ([Th.; 1 GeV]) | 1.026 | 0.02762 | - | - | - |
| $a_\mu^{\text{Exp}} - a_\mu^{\text{SM (Pheno)}}$ (Full HVP) | 1.037 | 0.02856 | 6.6 | $9.0 \cdot 10^{-5}$ | 0.02782 |
| $(a_\mu^{\text{Exp}} - 1\sigma) - a_\mu^{\text{SM (Pheno)}}$ (Full HVP) | 1.028 | 0.02831 | 5.0 | $9.0 \cdot 10^{-5}$ | 0.02775 |
| $a_\mu^{\text{Exp}} - a_\mu^{\text{SM (Pheno)}}$ ([Th.; 1.8 GeV]) | 1.041 | 0.02776 | 6.6 | $9.5 \cdot 10^{-5}$ | 0.02774 |
| $(a_\mu^{\text{Exp}} - 1\sigma) - a_\mu^{\text{SM (Pheno)}}$ ([Th.; 1.8 GeV]) | 1.031 | 0.02770 | 5.0 | $9.5 \cdot 10^{-5}$ | 0.02769 |
| $a_\mu^{\text{Exp}} - a_\mu^{\text{SM (Pheno)}}$ ([Th.; 1 GeV]) | 1.048 | 0.02771 | - | - | - |
| $(a_\mu^{\text{Exp}} - 1\sigma) - a_\mu^{\text{SM (Pheno)}}$ ([Th.; 1 GeV]) | 1.036 | 0.02766 | - | - | - |

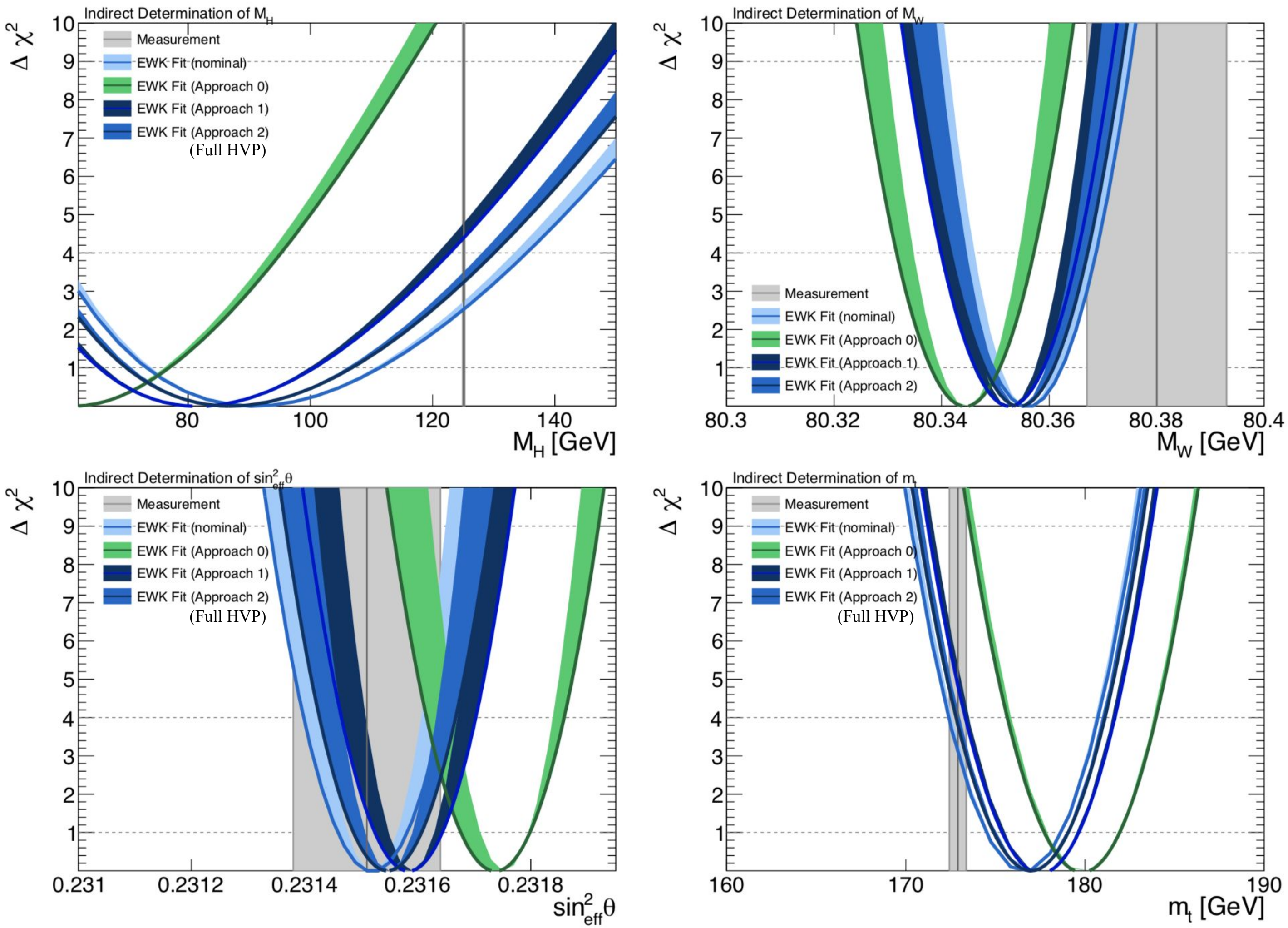
→ Large scaling factors (w.r.t. uncertainties) & significant shifts of NP$_1$

# EW fit inputs and $\chi^2$ results

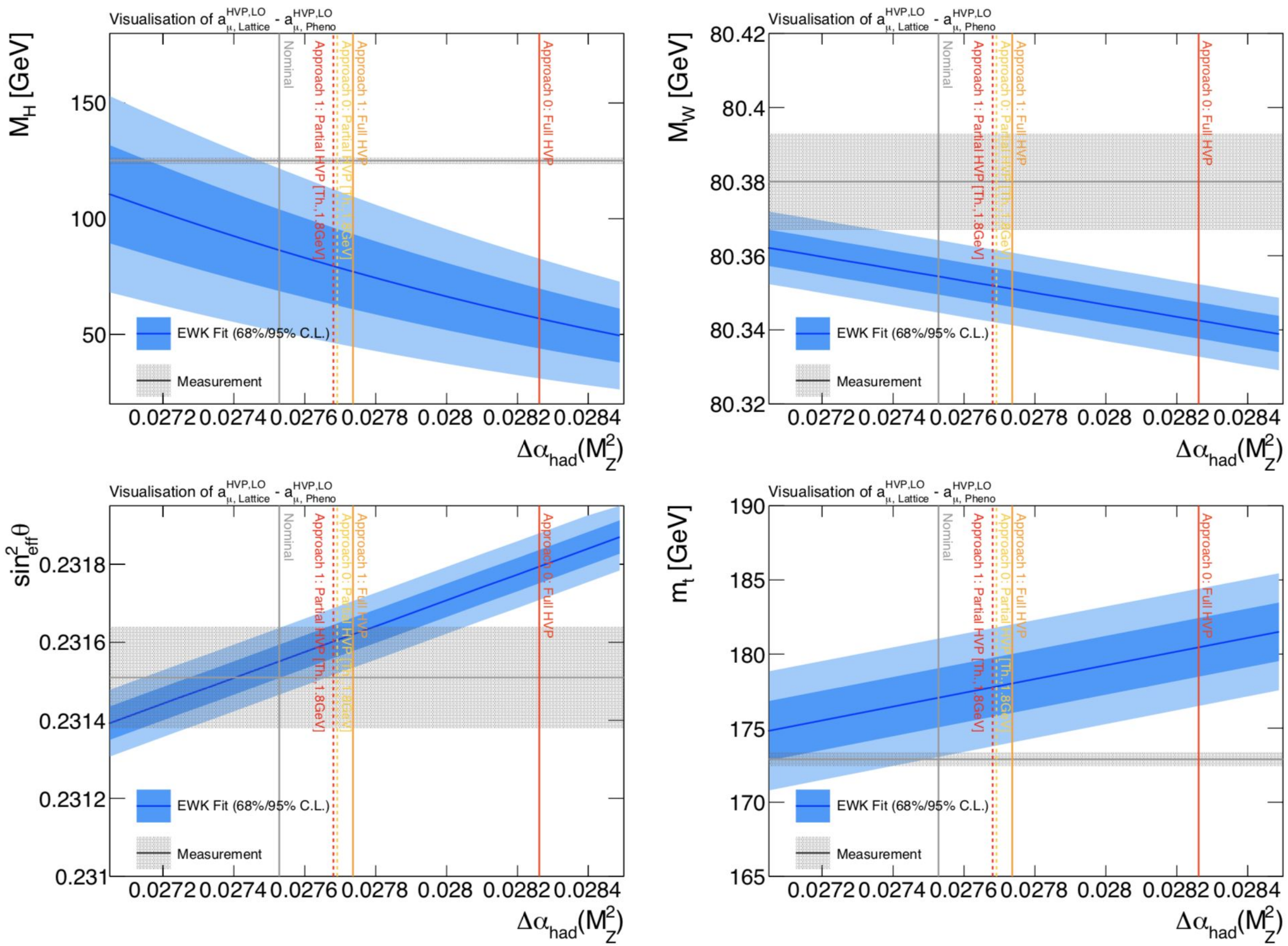
| **LEP/LHC/Tevatron** | | | | | |
|---|---|---|---|---|---|
| $M_Z$ [GeV] | $91.188 \pm 0.002$ | $R_c^0$ | $0.1721 \pm 0.003$ | $M_H$ [GeV] | $125.09 \pm 0.15$ |
| $\sigma^0_{\mathrm{had}}$ [nb] | $41.54 \pm 0.037$ | $R_b^0$ | $0.21629 \pm 0.00066$ | $M_W$ [GeV] | $80.380 \pm 0.013$ |
| $\Gamma_Z$ [GeV] | $2.495 \pm 0.002$ | $A_c$ | $0.67 \pm 0.027$ | $m_t$ [GeV] | $172.9 \pm 0.5$ |
| $A_l$ (SLD) | $0.1513 \pm 0.00207$ | $A_l$ (LEP) | $0.1465 \pm 0.0033$ | $\sin^2\theta^l_{\mathrm{eff}}$ | $0.2314 \pm 0.00023$ |
| $A^l_{\mathrm{FB}}$ | $0.0171 \pm 0.001$ | $m_c$ [GeV] | $1.27^{+0.07}_{-0.11}$ GeV | **After HL-LHC** | |
| $A^c_{\mathrm{FB}}$ | $0.0707 \pm 0.0035$ | $m_b$ [GeV] | $4.20^{+0.17}_{-0.07}$ GeV | $M_W$ [GeV] | $80.380 \pm 0.008$ |
| $A^b_{\mathrm{FB}}$ | $0.0992 \pm 0.0016$ | $\alpha_s(M_Z)$ | $0.1198 \pm 0.003$ | $\sin^2\theta^l_{\mathrm{eff}}$ | $0.2314 \pm 0.00012$ |
| $R_l^0$ | $20.767 \pm 0.025$ | $\Delta\alpha^{(5)}_{\mathrm{had}}(M_Z^2)$ [$10^{-5}$] | $2760 \pm 9$ | $m_t$ [GeV] | $172.9 \pm 0.3$ |

| $a_\mu^{\mathrm{HVP,\,LO}}$ shift (Energy range) | Nominal | | *Approach* 0 | | *Approach* 1 | | *Approach* 2 | |
|---|---|---|---|---|---|---|---|---|
| | $\Delta'\alpha_{\mathrm{had}}(M_Z^2)$ | $\chi^2/\mathrm{ndf}$ | $\Delta'\alpha_{\mathrm{had}}(M_Z^2)$ | $\chi^2/\mathrm{ndf}$ | $\Delta'\alpha_{\mathrm{had}}(M_Z^2)$ | $\chi^2/\mathrm{ndf}$ | $\Delta'\alpha_{\mathrm{had}}(M_Z^2)$ | $\chi^2/\mathrm{ndf}$ |
| | 0.02753 | 18.6/16 (p=0.29) | - | - | - | - | 0.02753 | 28.1/17 (p=0.04) $\chi^2$(BMW20-Pheno): 9.3 |
| $a_\mu^{\mathrm{HVP,\,LO}}{}_{\mathrm{(Lattice)}} - a_\mu^{\mathrm{HVP,\,LO}}{}_{\mathrm{(Pheno)}}$ (Full HVP) | - | - | 0.02826 | 27.6/16 (p=0.04) | 0.02774 | 20.3/16 (p=0.21) | - | |
| $a_\mu^{\mathrm{HVP,\,LO}}{}_{\mathrm{(Lattice)}} - a_\mu^{\mathrm{HVP,\,LO}}{}_{\mathrm{(Pheno)}}$ ([Th.; 1.8 GeV]) | - | - | 0.02769 | 19.9/16 (p=0.22) | 0.02768 | 19.8/16 (p=0.23) | - | - |
| $a_\mu^{\mathrm{HVP,\,LO}}{}_{\mathrm{(Lattice)}} - a_\mu^{\mathrm{HVP,\,LO}}{}_{\mathrm{(Pheno)}}$ ([Th.; 1.0 GeV]) | - | - | 0.02765 | 19.6/16 (p=0.24) | - | - | - | - |
| $a_\mu^{\mathrm{Exp}} - a_\mu^{\mathrm{SM\ (Pheno)}}$ (Full HVP) | - | - | 0.02856 | 33.6/16 (p=0.01) | 0.02782 | 21.2/16 (p=0.17) | - | - |
| $a_\mu^{\mathrm{Exp}} - a_\mu^{\mathrm{SM\ (Pheno)}}$ ([Th.; 1.8 GeV]) | - | - | 0.02776 | 20.6/16 (p=0.19) | 0.02774 | 20.4/16 (p=0.20) | - | - |
| $a_\mu^{\mathrm{Exp}} - a_\mu^{\mathrm{SM\ (Pheno)}}$ ([Th.; 1.0 GeV]) | - | - | 0.02771 | 20.1/16 (p=0.22) | - | - | - | - |

# EW fit results: $\chi^2$ scans



# EW fit results: parameter scans for varying $\Delta\alpha_{had}(M_Z^2)$



# EW fit results: indirect determination of $\Delta\alpha_{had}(M_Z^2)$