Analyticity and unitarity constraints on the two-pion contribution to HVP

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Colangelo, Hoferichter, Stoffer, JHEP **1902**, 006 (2019), PLB **814** (2021) 136073

Colangelo, El-Khadra, Hoferichter, Keshavarzi, Lehner, Stoffer, Teubner, PLB **833** (2022) 137313

Colangelo, Hoferichter, Kubis, Stoffer, arXiv:2208.08993 [hep-ph]

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Motivation

- 4.2σ discrepancy between $g-2$ experiments and WP: hint to BSM physics?
- **2.1σ tension** between R-ratio and BMWc lattice-QCD for HVP
- increases to **3.7σ for intermediate window**
- very recent results from ETMC, Mainz, RBC/UKQCD confirm BMWc intermediate window
- motivates **ongoing scrutiny** of R-ratio results

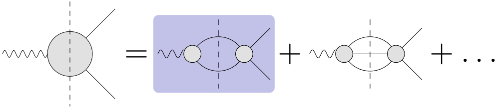
Two-pion contribution to HVP

- $\pi\pi$ contribution amounts to **more than 70%** of HVP contribution
- responsible for a similar fraction of HVP uncertainty
- can be expressed in terms of **pion vector form factor** $\Rightarrow$ constraints from analyticity and unitarity

  $\rightarrow$ Colangelo, Hoferichter, Stoffer, JHEP **02** (2019) 006

Dispersive representation of pion VFF

→ Colangelo, Hoferichter, Stoffer, JHEP **02** (2019) 006



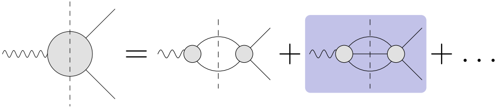
$$F_\pi^V(s) = \Omega_1^1(s) \times G_\omega(s) \times G_{\text{in}}^N(s)$$

- **Omnès function** with elastic $\pi\pi$-scattering P-wave phase shift $\delta_1^1(s)$ as input:

$$\Omega_1^1(s) = \exp\left\{ \frac{s}{\pi} \int_{4M_\pi^2}^{\infty} ds' \frac{\delta_1^1(s')}{s'(s'-s)} \right\}$$

Dispersive representation of pion VFF

→ Colangelo, Hoferichter, Stoffer, JHEP **02** (2019) 006



$$F_\pi^V(s) = \Omega_1^1(s) \times G_\omega(s) \times G_{\rm in}^N(s)$$

- isospin-breaking 3π intermediate state: negligible apart from **ω resonance** (ρ–ω interference effect)

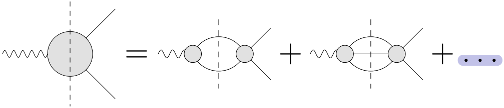
$$G_\omega(s) = 1 + \frac{s}{\pi}\int_{9M_\pi^2}^\infty ds' \frac{\mathrm{Im} g_\omega(s')}{s'(s'-s)} \left(\frac{1-\frac{9M_\pi^2}{s'}}{1-\frac{9M_\pi^2}{M_\omega^2}}\right)^4 ,$$

$$g_\omega(s) = 1 + \epsilon_\omega \frac{s}{(M_\omega - \frac{i}{2}\Gamma_\omega)^2 - s}$$

ϵ_ω: a priori a free **real** parameter

Dispersive representation of pion VFF

→ Colangelo, Hoferichter, Stoffer, JHEP **02** (2019) 006



$$F_\pi^V(s) = \Omega_1^1(s) \times G_\omega(s) \times G_{\text{in}}^N(s)$$

- heavier intermediate states: 4π (mainly $\pi^0\omega$), $\bar{K}K, \ldots$
- described in terms of a **conformal polynomial** with cut starting at $\pi^0\omega$ threshold

$$G_{\text{in}}^N(s) = 1 + \sum_{k=1}^{N} c_k(z^k(s) - z^k(0))$$

- correct P-wave threshold behavior imposed

Input and systematic uncertainties

- elastic $\pi\pi$-scattering P-wave phase shift $\delta_1^1(s)$ from Roy-equation analysis, including uncertainties

  $\rightarrow$ Ananthanarayan et al., 2001; Caprini et al., 2012

- high-energy continuation of phase shift above validity of Roy equations
- ω width
- systematics in conformal polynomial: order N, one mapping parameter

Free fit parameters

- value of the elastic $\pi\pi$-scattering P-wave **phase shift** δ_1^1 at two points (0.8 GeV and 1.15 GeV): number of free parameters dictated by Roy equations
- **ρ–ω mixing parameter** ϵ_ω
- **ω mass**
- **energy rescaling** for the experimental input, which allows for a calibration uncertainty
- $N-1$ coefficients in the **conformal polynomial**

VFF fit to the following data

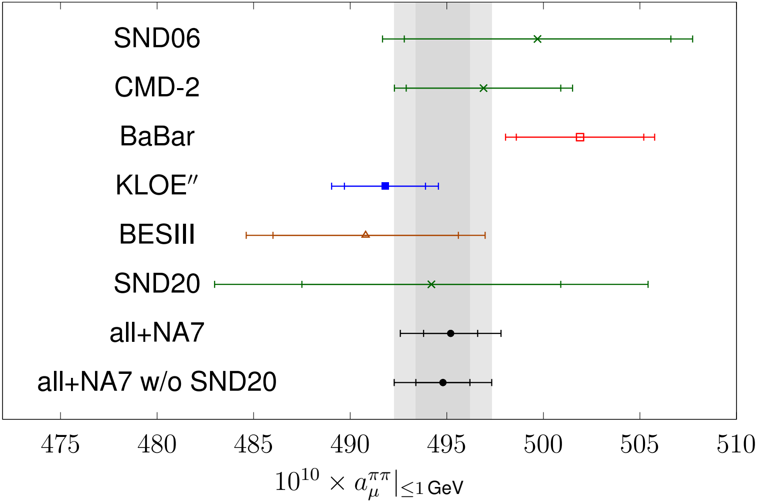
- time-like e^+e^- cross-section data
- space-like VFF data from NA7
- Eidelman–Łukaszuk bound on inelastic phase:

  $\rightarrow$ Eidelman, Łukaszuk, 2004
- iterative fit routine including full experimental covariance matrices and avoiding D'Agostini bias

  $\rightarrow$ D'Agostini, 1994; Ball et al. (NNPDF) 2010

Updated results for $a_\mu^{\mathrm{HVP},\pi\pi}$ below 1 GeV

→ Colangelo, Hoferichter, Kubis, Stoffer, arXiv:2208.08993 [hep-ph]



Updated results for $a_\mu^{\mathrm{HVP},\pi\pi}$ below 1 GeV

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	χ^2/dof	p-value	M_ω [MeV]	$10^3 \times \mathrm{Re}(\epsilon_\omega)$
SND06	1.40	5.3%	$781.49(32)(2)$	$2.03(5)(2)$
CMD-2	1.18	14%	$781.98(29)(1)$	$1.88(6)(2)$
BaBar	1.14	5.7%	$781.86(14)(1)$	$2.04(3)(2)$
KLOE	1.36	7.4×10^{-4}	$781.82(17)(4)$	$1.97(4)(2)$
KLOE″	1.20	3.1%	$781.81(16)(3)$	$1.98(4)(1)$
BESIII	1.12	25%	$782.18(51)(7)$	$2.01(19)(9)$
SND20	2.93	3.3×10^{-7}	$781.79(30)(6)$	$2.04(6)(3)$
all w/o SND20	1.23	3.0×10^{-5}	$781.69(9)(3)$	$2.02(2)(3)$

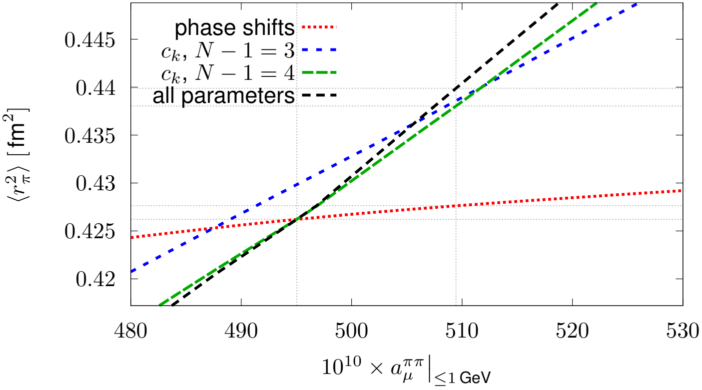
Modifying $a_\mu^{\pi\pi}|_{\leq 1\,\text{GeV}}$ to account for tension with lattice results

$\rightarrow$ Colangelo, Hoferichter, Stoffer, PLB **814** (2021) 136073

- either **large local changes** in the cross section in the ρ region
- or impact on **pion charge radius** and the space-like VFF $\Rightarrow$ chance for **independent lattice-QCD checks**

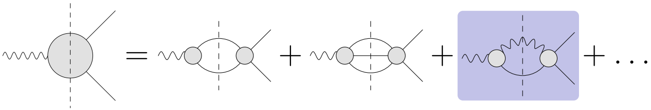
Modifying $a_\mu^{\pi\pi}|_{\leq 1\,\text{GeV}}$

correlations between $a_\mu^{\pi\pi}$ and $\langle r_\pi^2 \rangle$



Resonantly enhanced isospin-breaking effects

- with the given approximations, ϵ_ω is real by construction
- however, **additional radiative corrections** can be effectively mapped onto a phase in ϵ_ω
- additional channels in unitarity relation:



Resonantly enhanced isospin-breaking effects

- with the given approximations, ϵ_ω is real by construction
- however, **additional radiative corrections** can be effectively mapped onto a phase in ϵ_ω
- e.g., dominant $\pi^0\gamma$ channel can be implemented as

$$G_\omega(s) = 1 + \frac{s}{\pi}\int_{9M_\pi^2}^{\infty} ds' \frac{\mathrm{Re}\epsilon_\omega}{s'(s'-s)} \mathrm{Im}\left[\frac{s'}{(M_\omega - \frac{i}{2}\Gamma_\omega)^2 - s'}\right]\left(\frac{1-\frac{9M_\pi^2}{s'}}{1-\frac{9M_\pi^2}{M_\omega^2}}\right)^4$$

$$+ \frac{s}{\pi}\int_{M_{\pi^0}^2}^{\infty} ds' \frac{\mathrm{Im}\epsilon_\omega}{s'(s'-s)} \mathrm{Re}\left[\frac{s'}{(M_\omega - \frac{i}{2}\Gamma_\omega)^2 - s'}\right]\left(\frac{1-\frac{M_{\pi^0}^2}{s'}}{1-\frac{M_{\pi^0}^2}{M_\omega^2}}\right)^3$$

- resonance enhancement: **details of implementation irrelevant** (similar results with complex ϵ_ω in $g_\omega(s)$)

Effective phase in ρ–ω mixing parameter

- narrow-resonance estimate:

$$\mathrm{Im}\epsilon_\omega \simeq \frac{\sqrt{\Gamma[\omega \to \pi^0\gamma]\Gamma[\rho \to \pi^0\gamma]}}{3M_V}$$

- analogous relation for other intermediate states
- estimate leads to phases of $2.8^\circ(\pi^0\gamma)$, $0.4^\circ(\pi^+\pi^-\gamma)$, $0.2^\circ(\eta\gamma)$, $0.02^\circ(\pi^0\pi^0\gamma)$
  $\Rightarrow$ expect a phase $\arg(\epsilon_\omega) \approx 3.5(1.0)^\circ$

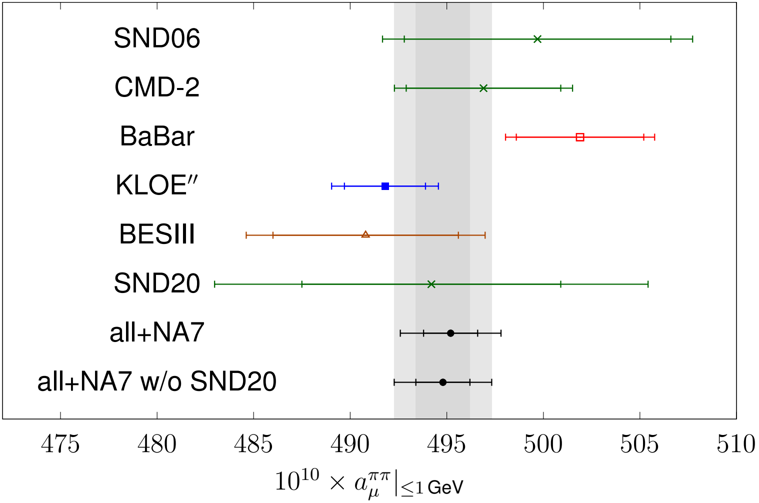
Including phase in ϵ_ω

$\rightarrow$ Colangelo, Hoferichter, Kubis, Stoffer, arXiv:2208.08993 [hep-ph]

	χ^2/dof	p-value	M_ω [MeV]	$10^3 \times \mathrm{Re}(\epsilon_\omega)$	$\arg(\epsilon_\omega)$ [°]
SND06	1.08	35%	782.11(32)(2)	1.98(4)(2)	8.5(2.3)(0.3)
CMD-2	1.01	45%	782.64(33)(4)	1.85(6)(4)	11.4(3.1)(1.0)
BaBar	1.14	5.5%	781.93(18)(4)	2.03(4)(1)	1.3(1.9)(0.7)
KLOE	1.27	6.7×10^{-3}	782.50(25)(6)	1.94(5)(2)	6.8(1.8)(0.5)
KLOE″	1.13	10%	782.42(23)(5)	1.95(4)(2)	6.1(1.7)(0.6)
BESIII	1.02	44%	783.05(60)(2)	1.99(19)(7)	17.6(6.9)(1.2)
SND20	1.87	4.1×10^{-3}	782.37(28)(6)	2.02(5)(2)	10.1(2.4)(1.4)
all w/o SND20	1.19	4.8×10^{-4}	782.09(12)(4)	1.97(2)(2)	4.5(9)(8)

No phase in ϵ_ω

→ Colangelo, Hoferichter, Kubis, Stoffer, arXiv:2208.08993 [hep-ph]



Including phase in ϵ_ω

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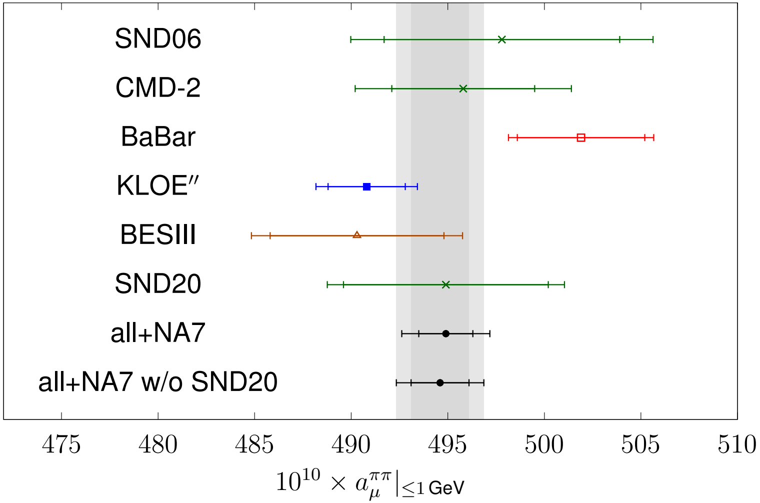


Extraction of IB contribution due to ρ–ω mixing

→ Colangelo, Hoferichter, Kubis, Stoffer, arXiv:2208.08993 [hep-ph]

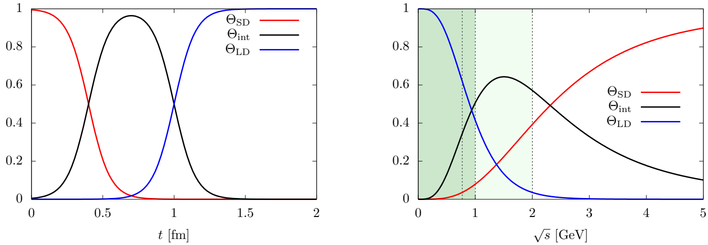
- extracted from full result vs. HVP integral with $\epsilon_\omega = 0$
- **similar size as FSR** contribution (sQED):

$\arg(\epsilon_\omega)$	$0°$	$4.5(1.2)°$
$10^{10} \times a_\mu^{\rho\text{–}\omega}$	$4.37(4)(7)$	$3.68(14)(10)$
$10^{10} \times a_\mu^{\pi\pi,\text{FSR}}$	$4.23(1)(2)$	$4.24(1)(2)$

- since we are considering 1-photon-irreducible HVP, entire effect should be assigned to $\mathcal{O}(m_u - m_d)$

  → thanks to Pablo Sanchez-Puertas for pointing this out

Some insights from the window quantities



- smooth window weight functions in Euclidean time

  → Blum et al. [RBC/UKQCD], PRL **121** (2018) 022003

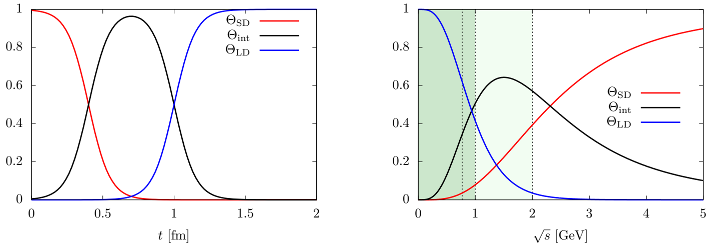
- total discrepancy:

$$a_\mu[\text{BMWc}] - a_\mu[\text{WP20}] = 14.4(6.8) \times 10^{-10}$$

- intermediate window: → Colangelo et al., PLB **833** (2022) 137313

$$a_\mu^{\text{int}}[\text{BMWc}] - a_\mu^{\text{int}}[e^+e^-] = 7.3(2.0) \times 10^{-10}$$

Some insights from the window quantities



- using form of weight functions:
  at least $\sim 40\%$ from **above 1 GeV**

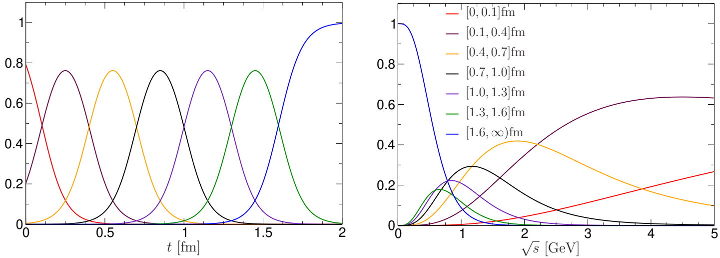
- assumptions:
  - rather uniform shifts in low-energy $\pi\pi$ region
  - no significant negative shifts

Data-driven evaluation of window quantities

→ Colangelo et al., PLB **833** (2022) 137313

- standard windows: $[0, 0.4]$ fm, $[0.4, 1.0]$ fm, $[1.0, \infty)$ fm with $\Delta = 0.15$ fm
- **additional windows**: cuts at $\{0.1, 0.4, 0.7, 1.0, 1.3, 1.6\}$ fm
- **data-driven evaluation** based on merging of KNT and CHHKS
- systematic effect due to BaBar vs. KLOE tension close to the WP estimate
- full covariance matrices for windows provided

Additional Euclidean-time windows

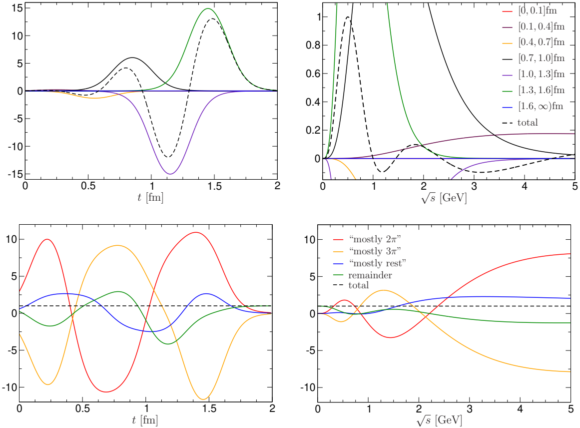


→ Colangelo et al., PLB **833** (2022) 137313

Localization in time-like region possible?

- better **localization in time-like region** could be achieved by taking linear combinations of Euclidean-time windows
- typically leads to **large cancellations** in Euclidean-time integral
- reflecting ill-posed inverse Laplace transform
- assessing usefulness requires knowledge of **full covariances**
- combinations dominated by **exclusive hadronic channels** suffer from similar problems

Localization in time-like region possible?



→ Colangelo et al., PLB **833** (2022) 137313

Conclusions

- reminder: unitarity/analyticity enable **independent checks** via pion VFF and $\langle r_\pi^2 \rangle$
- analysis of resonantly enhanced IB effects point at **systematic differences** between data sets
  - phase of mixing parameter
  - ω mass
- no good fit to SND20 data set possible

Conclusions

- **window quantities** and **analyticity constraints** point at an effect $\lesssim 8 \times 10^{-10}$ below 1 GeV, $\gtrsim 6 \times 10^{-10}$ above 1 GeV
- more detailed analysis might be possible with additional windows and knowledge of **correlations**

Backup

Tension with lattice QCD

→ Colangelo, Hoferichter, Stoffer, PLB **814** (2021) 136073

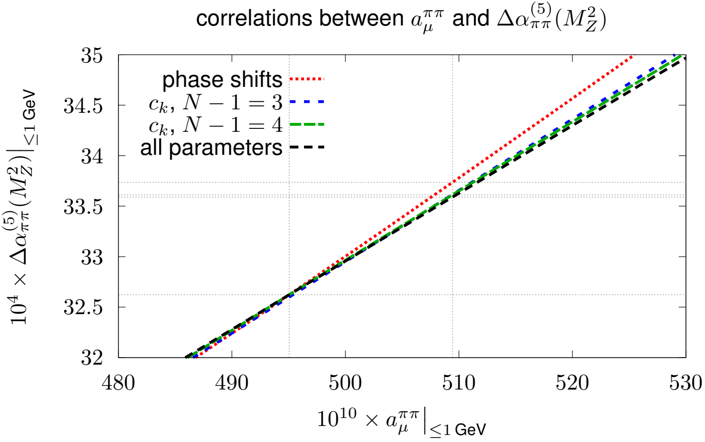
- force a different HVP contribution in VFF fits by including “lattice” datum with tiny uncertainty
- three different scenarios:
  - “low-energy” physics: $\pi\pi$ phase shifts
  - “high-energy” physics: inelastic effects, c_k
  - all parameters free
- study effects on pion charge radius, hadronic running of $\alpha_{\rm QED}^{\rm eff}$, phase shifts, cross sections

Modifying $a_\mu^{\pi\pi}|_{\leq 1\,\text{GeV}}$

$\rightarrow$ Colangelo, Hoferichter, Stoffer, PLB **814** (2021) 136073

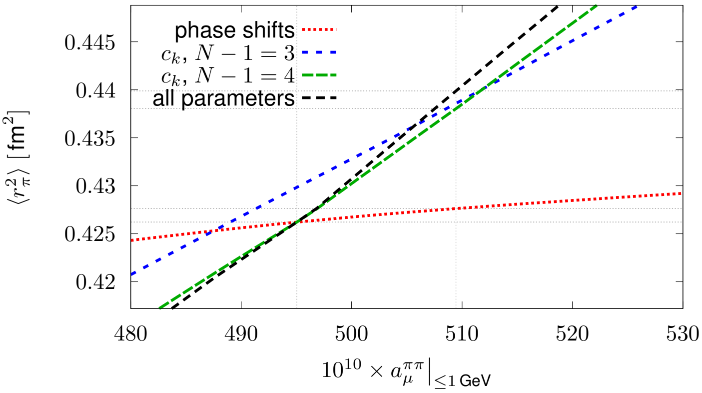
- "low-energy" scenario requires large local changes in the cross section in the ρ region
- "high-energy" scenario has an impact on **pion charge radius** and the space-like VFF $\Rightarrow$ chance for independent lattice-QCD checks

Modifying $a_\mu^{\pi\pi}|_{\leq 1\,\text{GeV}}$

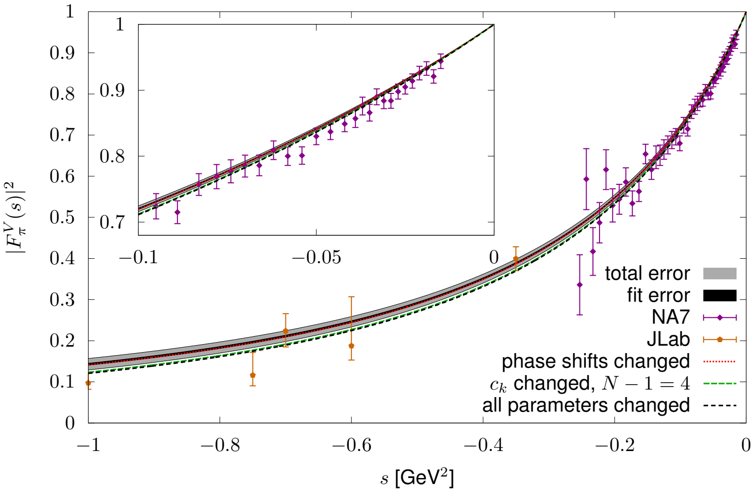


Modifying $a_\mu^{\pi\pi}|_{\leq 1\,\text{GeV}}$

correlations between $a_\mu^{\pi\pi}$ and $\langle r_\pi^2 \rangle$



Modifying $a_\mu^{\pi\pi}|_{\leq 1\,\text{GeV}}$



Modifying $a_\mu^{\pi\pi}|_{\leq 1\,\text{GeV}}$