

The hadronic running of the electromagnetic coupling from lattice QCD

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based on JHEP08(2022)220 [arXiv:2203.08676]

with

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Edinburgh

the running of the electromagnetic coupling

the QED coupling $\alpha = g^2/(4\pi)$ runs with energy

- in the Thomson limit ($q^2 \rightarrow 0$), the fine-structure constant is known at 0.23 ppb
 $\alpha^{-1} = \alpha(0)^{-1} = 137.035\,999\,139(31)$

[PDG 2018, CODATA 2014]

- at the Z pole, $\hat{\alpha}^{(5)}(M_Z)^{-1} = 127.955(10)$ (in the $\overline{\text{MS}}$ scheme)

$$\alpha(q^2) = \frac{\alpha}{1 - \Delta\alpha(q^2)} \quad \Delta\alpha_{\text{had}}(q^2) = 4\pi\alpha \text{Re } \bar{\Pi}(q^2), \quad \bar{\Pi}(q^2) = \Pi(q^2) - \Pi(0)$$

main uncertainty in the Z -pole value: the **hadronic contribution** to the running, proportional to the subtracted **hadronic vacuum polarization** (HVP) function $\bar{\Pi}(q^2)$

- extracted from the exp. R -ratio data via dispersive integral (data-driven method) [Erler 1999; Davier *et al.* 2017; PDG 2018]

$$\text{Re } \bar{\Pi}(q^2) = \frac{q^2}{12\pi} P \int_{m_\pi^2}^{\infty} \frac{R(s)}{s(s - q^2)} ds, \quad \text{Im } \bar{\Pi}(q^2) = \frac{R(q^2)}{12\pi}, \quad R(s) = \frac{\sigma_{e^+e^- \rightarrow \text{hadrons}}(s)}{4\pi\alpha^2/(3s)}$$

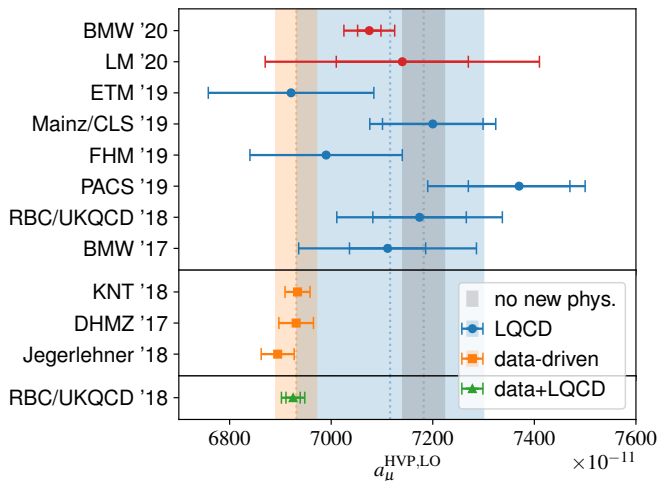
$$\Delta\alpha_{\text{had}}^{(3)}(4 \text{ GeV}^2) = 58.71(50) \times 10^{-4}, \quad \Delta\alpha_{\text{had}}^{(5)}(M_Z^2) = 0.027\,64(7), \quad (\text{on-shell scheme})$$

- or computed **on the lattice**

[Burger *et al.* 2015; Francis *et al.* Lattice 2015; Borsanyi *et al.* 2018; MC *et al.* Lattice 2019; MC *et al.* 2022]

$$\bar{\Pi}(-Q^2) = -\frac{1}{3} \int d^4x e^{iQ \cdot x} \langle j_\mu^\gamma(x) j_\mu^\gamma(0) \rangle$$

the leading-order HVP contribution to a_μ



- extracted from the exp. R -ratio data via dispersive integral (data-driven method)

$$a_\mu^{\text{HVP,LO}} = \left(\frac{\alpha m_\mu}{3\pi} \right)^2 \int_{m_\pi^2}^{\infty} \frac{\hat{K}(s)}{s^2} R(s) ds$$

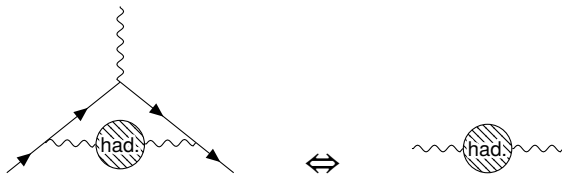
$$\hat{K}(4m_\pi^2) \approx 0.63, \quad \lim_{s \rightarrow \infty} \hat{K}(s) = 1$$

- or computed on the lattice

$$a_\mu^{\text{HVP,LO}} = 4\alpha^2 \int_0^\infty dQ^2 f(Q^2) \bar{\Pi}(-Q^2)$$

with $f(Q^2) \geq 0$ a known QED kernel

motivation – the HVP connection



if the lattice QCD confirms the **larger value** of $a_\mu^{\text{HVP,LO}}$

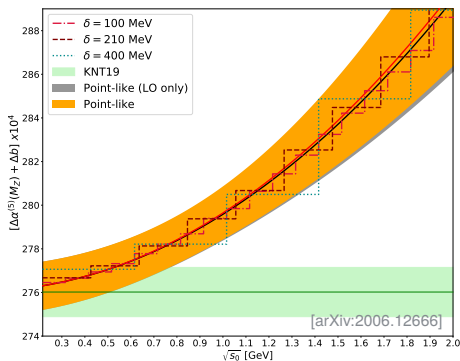
$$a_\mu^{\text{HVP,LO}} = \left(\frac{\alpha m_\mu}{3\pi} \right)^2 \int_{m_\pi^2}^{\infty} ds \frac{\hat{K}(s)}{s^2} R(s), \quad \bar{\Pi}(-Q^2) = \frac{Q^2}{12\pi^2} \int_{m_\pi^2}^{\infty} ds \frac{R(s)}{s(s+Q^2)}$$

with $0.63 \lesssim \hat{K}(s) < 1 \Rightarrow R(s)$ is **larger for some s** , $\bar{\Pi}(-Q^2)$ is also larger!

\Rightarrow depending on the energy bin of the increase, also $\Delta\alpha_{\text{had}}(M_Z^2)$ is affected

[Passera, Marciano, Sirlin 2008; Crivellin, Hoferichter, Manzari, Montull 2020; Keshavarzi, Passera, Marciano, Sirlin 2020; Colangelo, Hoferichter, Stoffer 2021]

motivation – global Standard Model fits



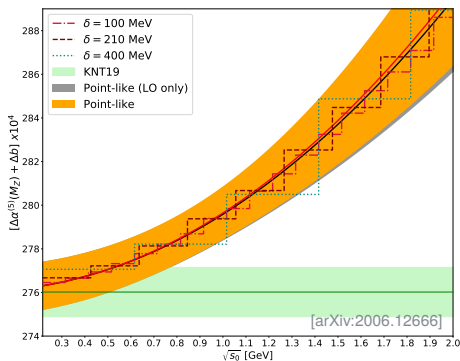
if the hadronic cross-section increases [Keshavarzi, Passera, Marciano, Sirlin 2020]

- at higher \sqrt{s} :
the increase in $\Delta\alpha^{(5)}(M_Z)$ is in **tension with global SM fits**
 $\Rightarrow M_W, \sin^2 \theta_{\text{eff}}^{\text{lep}}, M_H$ exclude shifts for $\sqrt{s} > 0.7 \text{ GeV}$ at 95 % C.L.
- below 0.7 GeV (ρ -resonance region):
no significant change in $\Delta\alpha_{\text{had}}$, no tension in global SM fits,
a 9 % increase of the integrated cross-section would solve the $(g - 2)_\mu$ discrepancy
 \Rightarrow **tension with the experimental hadronic cross-section data!**

can we use lattice computations to say something more about this?

yes, by studying the running with energy of the electromagnetic coupling!

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motivation – t -channel scattering

the leading hadronic contribution to $(g - 2)_\mu$ from the running of α

[Lautrup, Peterman, de Rafael 1972]

$$a_\mu^{\text{HVP,LO}} = \frac{\alpha}{\pi} \int_0^1 dx (1-x) \Delta\alpha_{\text{had}}(-Q^2), \quad Q^2 = \frac{x^2 m_\mu^2}{1-x},$$

with the integrand peaked at $x \approx 0.914$, $Q^2 \approx 0.108 \text{ GeV}^2$.

[Carloni Calame *et al.* 2015]

upcoming **MUonE experiment** @ CERN: measure the energy dependence of α at **space-like** Q^2

[Abbiendi *et al.* 2017]

- independent determination of $a_\mu^{\text{HVP,LO}}$
- kinematic range $0 < x < 0.932$, corresponding to $Q^2 \lesssim 0.14 \text{ GeV}^2$
- $0.932 < x < 1$ or $Q^2 \gtrsim 0.14 \text{ GeV}^2$ accounts for 13 % of $a_\mu^{\text{HVP,LO}}$
- test run in 2023!

⇒ **lattice input** for the intermediate region $Q^2 = 0.14 - 4 \text{ GeV}^2$

the time-momentum representation (TMR) method

uses the zero-momentum-projected Euclidean-time correlator

$$G(t) = -\frac{1}{3} \int d^3x \sum_{k=1}^3 \langle j_k(x) j_k(0) \rangle,$$

and known kernel functions $K(t, Q^2)$ and $w(t)$

[Bernecker, Meyer 2011; Francis *et al.* 2013]

$$\begin{aligned} \bar{\Pi}(-Q^2) &= \int_0^\infty dt K(t, Q^2) G(t), & K(t, Q^2) &= t^2 - \frac{4}{Q^2} \sin^2\left(\frac{Qt}{2}\right), \\ a_\mu^{\text{HVP,LO}} &= \int_0^\infty dt w(t) G(t), & w(t) &= 4\pi^2 \int_0^\infty dQ^2 f(Q^2) K(t, Q^2) \end{aligned}$$

w.r.t. the traditional approach

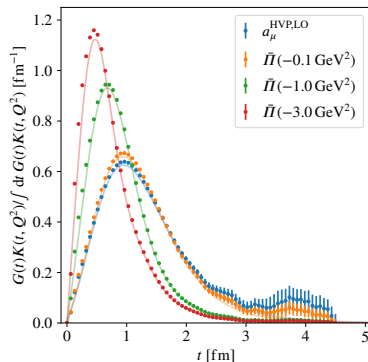
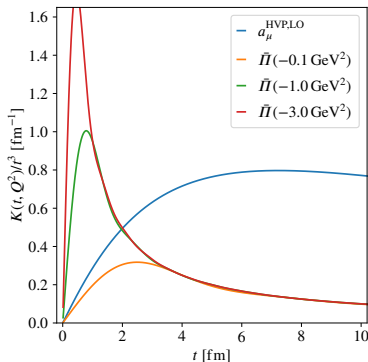
- same statistical power
- better understanding of the systematics
 - finite-size effects correction
 - (improved) bounding method
- in principle, $\bar{\Pi}(-Q^2)$ can be computed for **any** Q^2

similar alternative approach: time moments

[Chakraborty 2014]

the TMR method – the kernel

- the $a_\mu^{\text{HVP,LO}}$ kernel is very long range
- the $\bar{\Pi}(-Q^2)$ kernel has a shorter range depending on Q^2



$$\bar{\Pi}(-Q^2): K(t, Q^2) = t^2 - \frac{4}{Q^2} \sin^2\left(\frac{Qt}{2}\right),$$

$$a_\mu^{\text{HVP,LO}}: w(t) = 4\pi^2 \int_0^\infty dQ^2 f(Q^2) K(t, Q^2)$$

systematic effects

controlling the **tail of the correlator** \Rightarrow main source of statistical uncertainty

- the bounding method is used
- crucial to control the statistical error on $a_\mu^{\text{HVP,LO}}$, less critical for $\bar{\Pi}(-Q^2)$

[Lehner LGT2016; Gérardin, MC *et al.* 2019]

simulations of finite-size lattices \Rightarrow correction of **finite-size effects**

- computing FSE on the zero-momentum correlator $G(t)$
- with $m_\pi L \approx 4$ ($L \approx 6$ fm at the physical point), about 1 % shift fully under control

[Lüscher 1991; Lellouch, Lüscher 2000; Meyer 2011]

extrapolation to the continuum limit, extra- or interpolation to physical meson masses

- ensembles around physical meson masses are used by almost all collaborations
- continuum-limit extrapolation with $\sim a^2$ and $\sim a^3$ effects at $Q^2 \gtrsim 2.5 \text{ GeV}^2$

QED and strong isospin breaking corrections

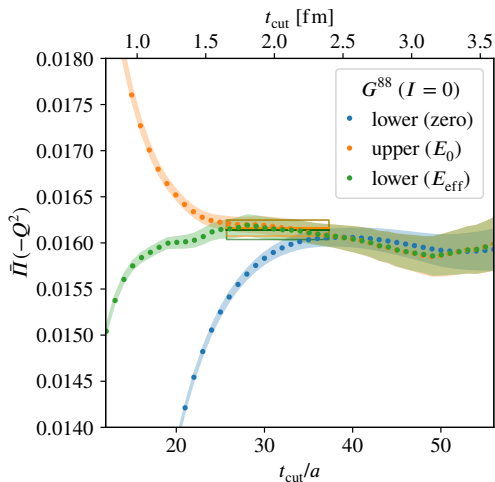
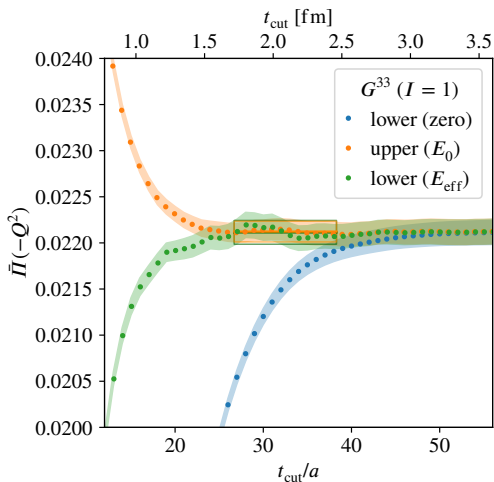
- only available on a few ensembles, work in progress \Rightarrow included as systematics
- ≈ 0.3 %, small contribution to the total error

[Risch, Wittig Lattice 2019; Lattice 2021]

scale setting systematics

- a 1 % uncertainty on the scale is a ≈ 2 % systematic error on $a_\mu^{\text{HVP,LO}}$, ≈ 1 % on $\bar{\Pi}(-1 \text{ GeV}^2)$ (Q^2 dependent)
 \Rightarrow a per-mille level scale determination is needed

bounding method – example



$$0 \leq G(t_{\text{cut}})e^{-E_{\text{eff}}(t_{\text{cut}})(t-t_{\text{cut}})} \leq G(t) \leq G(t_{\text{cut}})e^{-E_0(t-t_{\text{cut}})}, \quad \text{for } t \geq t_{\text{cut}},$$

with $aE_{\text{eff}}(t) = \log(G(t)/G(t+a))$ and E_0 ground state in the channel

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correction of finite-size effects

added to the $I = 1$ correlator $G^{33}(t)$, with **two different strategies** and $t_i = (m_\pi L/4)^2/m_\pi$ [Gérardin, MC *et al.* 2019]

$t < t_i$: HP method with $\vec{n}^2 \leq 3$, with the size of the $\vec{n}^2 = 3$ level included as a systematic error

$t > t_i$: average of MLL-GS and HP methods, with the half-difference included as an extra systematic error

\Rightarrow a model is used **only** for the small correction $G^{33}(t, L) - G^{33}(t, \infty)$

explicit check of FSE with two pair of ensembles at different volume and otherwise identical simulation parameters

- H105 and N101 at $m_\pi \approx 280$ MeV
- H200 and N202 at the SU(3)-symmetric point

we observe a **good agreement** between the MLL-GS and HP methods, especially for $t \gtrsim 2$ fm,
with the two methods relying on very different input \Rightarrow **robustness of the evaluation of finite size effects**

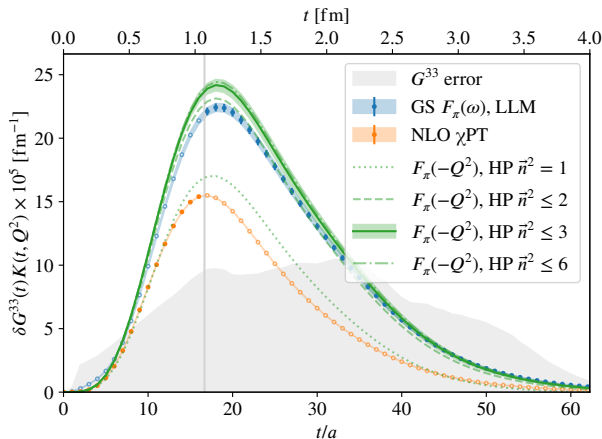
correction of finite-size effects – Hansen-Patella (HP) method

series expansion in $\exp\{-|\vec{n}|m_\pi L\}$, $\vec{n}^2 = 1, 2, 3, 6, \dots$

[Hansen, Patella 2019; 2020]

- implementation by K. Miura
- neglects $\exp\{-1.93m_\pi L\}$ contributions
- fast convergence at short and medium distances
- input: forward Compton amplitude, $F_\pi(Q^2)$ monopole ansatz with $M^2 = 0.517(23) \text{ GeV}^2 + 0.647(30)m_\pi^2$

[Brommel et al. 2007]



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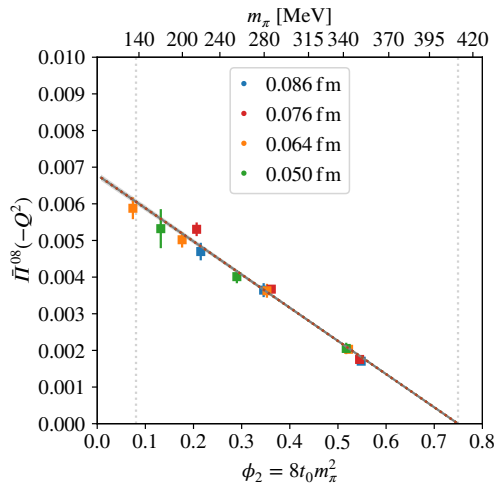
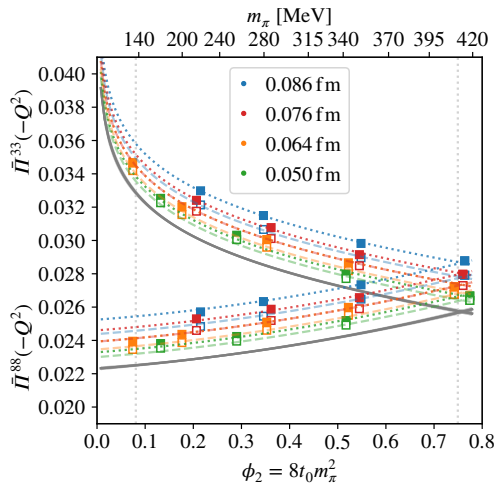
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extrapolation results

at $Q^2 = 1.0 \text{ GeV}^2$



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definition of the isosymmetric QCD world

we set the scale with

[Bruno, Korzec, Schaefer 2015]

$$\sqrt{8t_0} = 0.415(4)(2) \text{ fm}$$

and we define the isospin symmetric point as

[discussion at the 4th Muon $g - 2$ workshop at KEK (virtual), 2021]

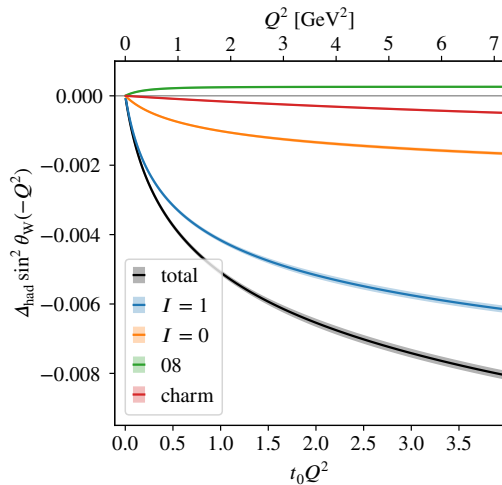
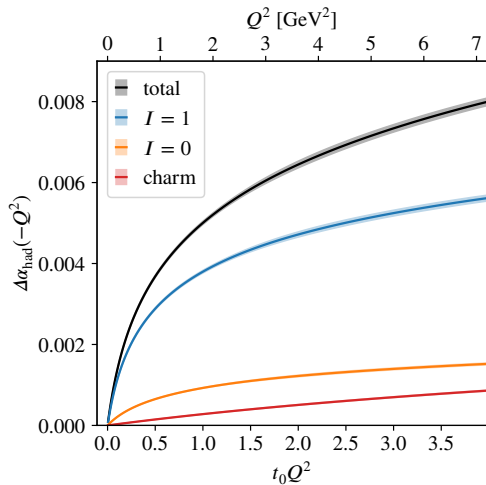
$$m_\pi = m_{\pi^0} = 134.9768 \text{ MeV}$$
$$m_K^2 - \frac{m_\pi^2}{2} = \frac{m_{K^\pm}^2 + m_{K^0}^2 - m_{\pi^\pm}^2}{2} \Rightarrow m_K = 495.011 \text{ MeV}$$

and the valence charm quark mass is tuned to reproduce the physical D_s meson mass

[Gérardin *et al.* 2019]

we also publish the derivatives w.r.t. ϕ_2 and $\phi_4 \Rightarrow m_\pi, m_K$

running with energy – results



running with energy – rational approximation

the running is obtained varying Q^2 in the TMR kernel \Rightarrow each Q^2 choice is a different fit
we present the results

- tabulated for selected values of Q^2 up to 7 GeV^2
- with a rational approximation that interpolates to other values of Q^2

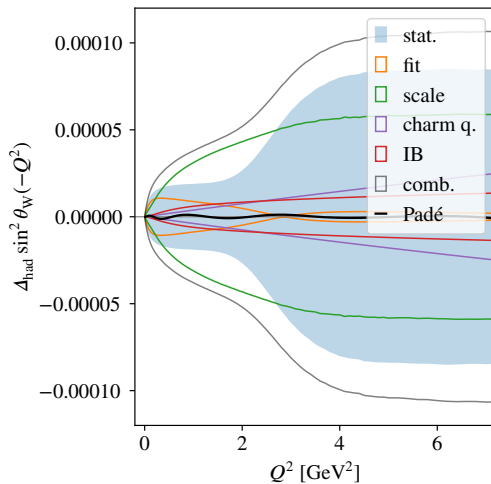
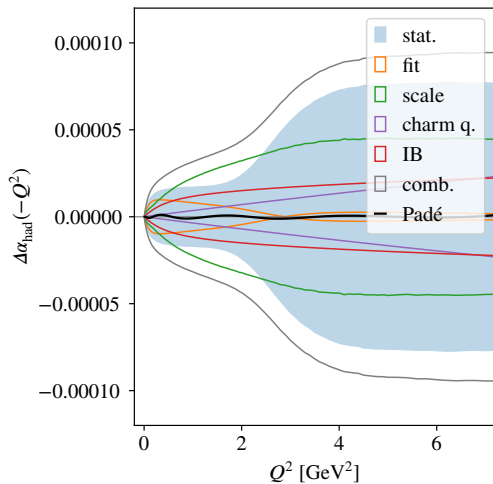
$$\bar{\Pi}(-Q^2) \approx R_M^N(Q^2) = \frac{\sum_{j=0}^M a_j Q^{2j}}{1 + \sum_{k=1}^M b_k Q^{2k}}$$

with $M = N = 3$ and $a_0 = 0$, by solving the over-constrained system via a least-squares fit

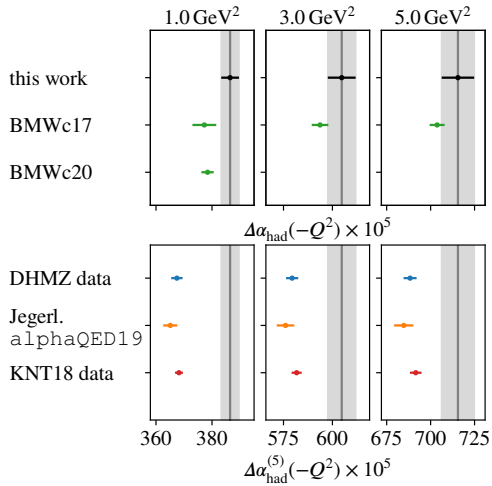
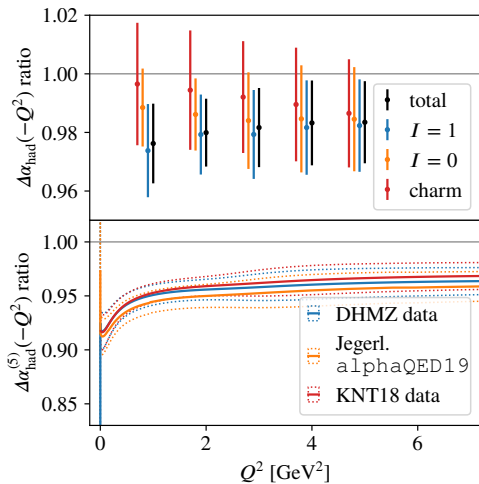
$$\sum_i \frac{1}{\delta \bar{\Pi}(-Q_i^2)} \left[\sum_{j=0}^M a_j Q_i^{2j} - \left(1 + \sum_{k=1}^M b_k Q_i^{2k} \right) \bar{\Pi}(-Q_i^2) \right] = 0$$

the rational approximations are provided also for the derivatives w.r.t. the meson masses

running with energy – summary of systematics



running with energy – comparison



running to M_Z

we use the Euclidean split technique (or Adler function approach)

[Chetyrkin *et al.* 1996; Eidelman *et al.* 1999, Jegerlehner 2008]

$$\Delta\alpha_{\text{had}}^{(5)}(M_Z^2) = \Delta\alpha_{\text{had}}^{(5)}(-Q_0^2) + \left[\Delta\alpha_{\text{had}}^{(5)}(-M_Z^2) - \Delta\alpha_{\text{had}}^{(5)}(-Q_0^2) \right] + \left[\Delta\alpha_{\text{had}}^{(5)}(M_Z^2) - \Delta\alpha_{\text{had}}^{(5)}(-M_Z^2) \right]_{\text{pQCD}}$$

- $\Delta\alpha_{\text{had}}^{(5)}(-Q_0^2)$ with Q_0^2 between 3 and 7 GeV² is evaluated on the lattice

$$\Delta\alpha_{\text{had}}^{(5)}(-5 \text{ GeV}^2) = 0.007 16(9)$$

- $[\Delta\alpha_{\text{had}}^{(5)}(-M_Z^2) - \Delta\alpha_{\text{had}}^{(5)}(-Q_0^2)]$ from either pQCD or R -ratio data (KNT18)

$$\left[\Delta\alpha_{\text{had}}^{(5)}(-M_Z^2) - \Delta\alpha_{\text{had}}^{(5)}(-5 \text{ GeV}^2) \right] = 0.020 53(11) \quad \text{or} \quad 0.020 66(9)$$

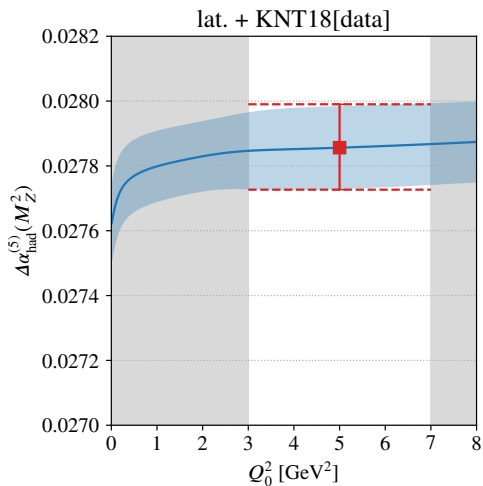
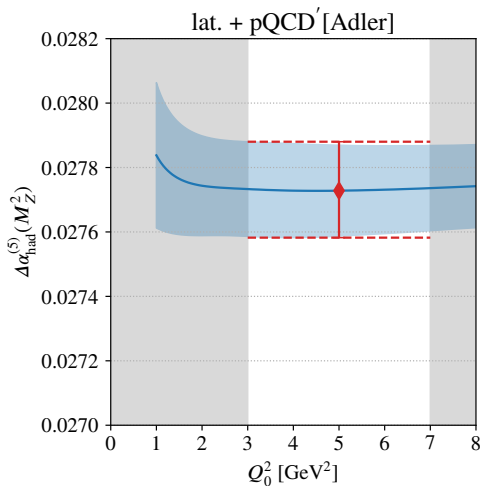
- $[\Delta\alpha_{\text{had}}^{(5)}(M_Z^2) - \Delta\alpha_{\text{had}}^{(5)}(-M_Z^2)]_{\text{pQCD}} = 0.000 045(2)$ has a negligible error

[Jegerlehner 1986, 2020]

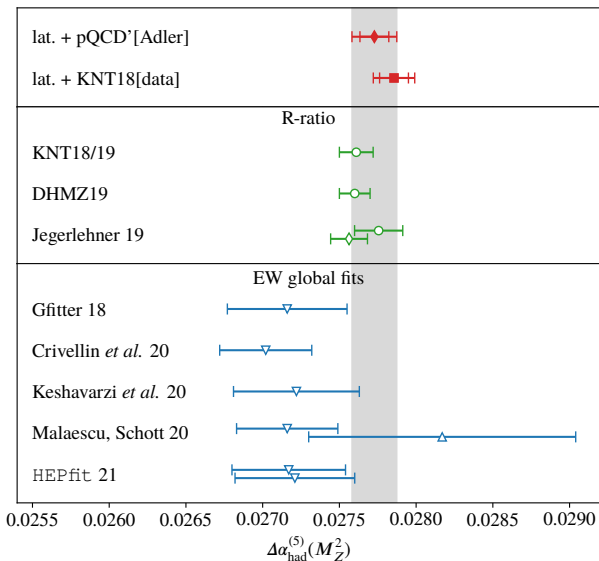
using pQCD \Rightarrow result independent from R -ratio input

$$\Delta\alpha_{\text{had}}^{(5)}(M_Z^2) = 0.027 73(15)$$

running to M_Z – results



running to M_Z – results and comparison



conclusions

we computed on the lattice the **HVP contribution to the running of α**

- with sub-percent statistical errors
- bounding method helps at small Q^2 • correction for finite-size effects is essential

$$\Delta\alpha_{\text{had}}^{(5)}(-Q^2) = 0.007\,16(9) \quad \text{at } Q^2 = 5 \text{ GeV}^2$$

- approaching the precision of the data-driven estimate • **up to 3.5σ tension with the data-driven estimate**
- consistent with the lattice results for the intermediate $a_\mu^{\text{HVP,LO}}$ window
- precision limited by current scale setting on CLS ensembles
- full correction for isospin breaking effects still missing

[work in progress: Risch, Wittig Lattice 2019; Lattice 2021]

and the running up to M_Z

$$\Delta\alpha_{\text{had}}^{(5)}(M_Z^2) = 0.027\,73(15)$$

- employing the Euclidean split technique and pQCD for the running at large $Q^2 \Rightarrow$ no R -ratio data dependency
- lattice contributes to $\approx 25\%$ of the value and up to 50% of the error
- **the result agrees with ones based on the R -ratio** within the uncertainties

the SM can accommodate a larger value for a_μ^{HVP} without contradicting electroweak precision data

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we computed on the lattice the **HVP contribution to the running of α**

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outlook

implement changes already used for the muon $g - 2$ window result

[talk by Kuberski]

- mostly the same dataset
- two extra ensembles at the **finer lattice spacing** ($a \approx 0.039$ fm)
- alternative set of renormalization and improvement coefficients
- variety of fits combined with the Akaike Information Criterion (AIC)

[Heitger, Joswig 2021; Fritsch 2018]

and add full isospin breaking corrections, improve the scale setting

computing higher Q^2 (up to M_Z^2) directly on the lattice?

using the **discrete Adler function** $\Delta_2(Q^2) = \Pi(-Q^2) - \Pi(-Q^2/4)$

[MC, Harris, Meyer, Toniato, Török 2021; Harris Lattice 2021]

- naively, $\Lambda \ll |Q| \ll a^{-1}$
- thermal effects are $\sim (\pi T/Q)^4 \Rightarrow$ using **finite-temperature ensembles** at $T = Q/8\pi$
- such that only $T \ll |Q| \ll a^{-1}$ needs to be satisfied
- different from step scaling with a finite-volume scheme \Rightarrow the volume is parametrically large, *e.g.* $L \approx 4/T$
- similar proposal from BMWc

[Frech Lattice 2021; Lattice 2022]

thanks
for your attention!



questions?

backup slides

the running of the electroweak mixing angle

the electroweak mixing (Weinberg) angle θ_W parametrizes the mixing between the $SU(2)_L$ and $U(1)_Y$ sectors of the Standard Model. At tree level,

$$\sin^2 \hat{\theta}_W = \frac{g'^2}{g^2 + g'^2}, \quad \text{or} \quad \sin^2 \theta_W = 1 - \frac{M_W^2}{M_Z^2},$$

where g and g' are the $SU(2)_L$ and $U(1)_Y$ coupling respectively

- Z vector coupling $v_f = T_f - 2Q_f \sin^2 \theta_f^{\text{eff}}$
- weak charge of the proton $Q_W(p) \sim 1 - 4 \sin^2 \theta_W(0)$

like the couplings, the mixing angle is renormalization scheme and **energy dependent**

$$\sin^2 \theta_W(Q) = \sin^2 \theta_W(0) [1 + \Delta \sin^2 \theta_W(Q^2)],$$

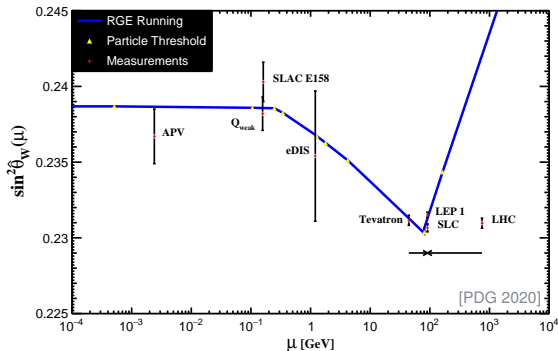
and the leading **hadronic contribution** to the running

[Jegerlehner 1986; 2011]

$$\Delta_{\text{had}} \sin^2 \theta_W(Q^2) = -\frac{4\pi\alpha}{\sin^2 \theta_W} \bar{\Pi}^{Z\gamma}(Q^2), \quad \bar{\Pi}^{Z\gamma}(Q^2) = \Pi^{Z\gamma}(Q^2) - \Pi^{Z\gamma}(0),$$

is proportional to the subtracted **$Z\gamma$ -mixing HVP function**

the running of the electroweak mixing angle



experimental values measured at colliders enter global SM fits

[PDG 2022]

$$\sin^2 \hat{\theta}_W(M_Z) = 0.231\,22(4)$$

upcoming experiments at low Q^2

- P2 @ MESA, Mainz [Becker *et al.* 2018]
0.13 % target precision at $Q^2 = 0.005 \text{ GeV}^2$
- MOLLER @ JLab [Benesch *et al.* 2014]

the running to the Thomson limit is affected by **non-perturbative QCD physics** that

- can be extracted from hadronic cross-section data

[Erler, Ferro-Hernández 2017]

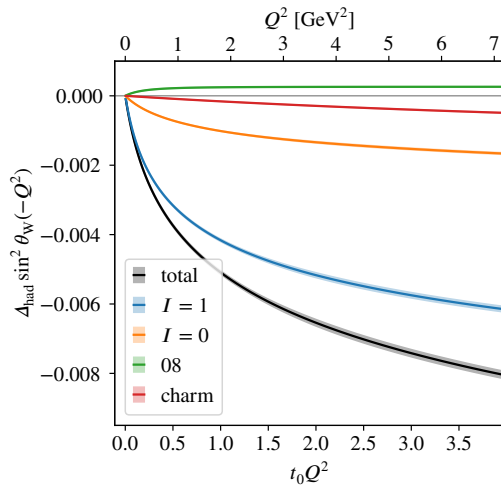
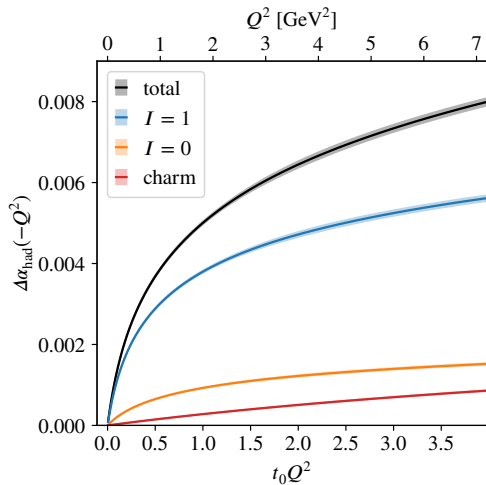
$$\sin^2 \hat{\theta}_W(0) = 0.238\,68(5)(2), \quad (\overline{\text{MS}} \text{ scheme})$$

with additional input for **flavor separation**

- or can be computed **on the lattice**
⇒ lattice easily provides flavour separation

[Burger *et al.* 2015; Francis *et al.* 2015; Gülpers *et al.* 2015]

the running of the electroweak mixing angle – results



the running of the electroweak mixing angle – conclusions

we present a result for the $Z\gamma$ -mixing HVP contribution to the running of $\sin^2 \theta_W$

$$\bar{\Pi}^{08}(-Q^2) = \frac{0.021\,7(11)x + 0.015\,1(12)x^2}{1 + 2.93(8)x + 2.15(12)x^2}, \quad x = \frac{Q^2}{\text{GeV}^2}$$

that has a finite limit for large Q^2

$$\bar{\Pi}^{08}(-Q^2) = 0.007\,04(17) \quad \text{for } Q^2 \gtrsim 7 \text{ GeV}^2$$

- using flavor separation on the lattice
- most precise determination to date

ensembles

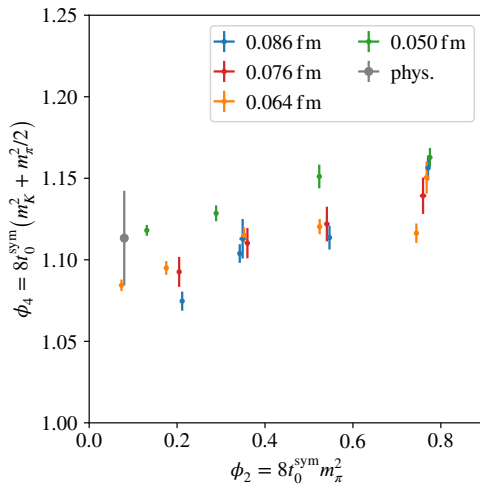
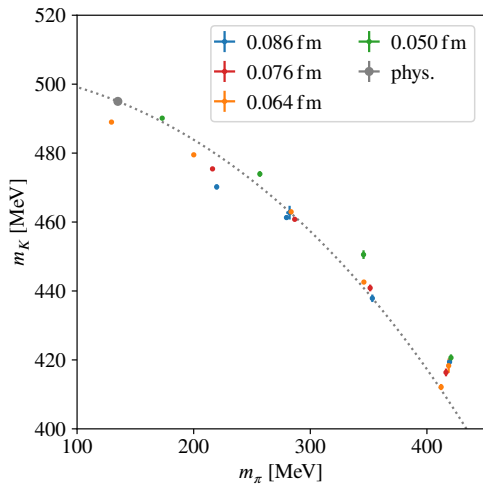
from the CLS initiative

[Bruno *et al.* 2015, Bruno, Korzec, Schaefer 2017]

tree-level Lüscher-Weisz gauge action, non-perturbatively $\mathcal{O}(a)$ -improved Wilson fermions,
open BCs in time, except B450, N451, D450, and E250 that have periodic BCs in time,

	T/a	L/a	t_0^{sym}/a^2	a [fm]	L [fm]	m_π, m_K [MeV]	$m_\pi L$	#cnfg (con., dis.)		
H101	96	32	2.860	0.086	2.8	415	5.8	2 000 -		
H102	96	32			2.8	355 440	5.0	1 900 1 900		
H105	96	32			2.8	280 460	3.9	1 000 1 000		
N101	128	48			4.1	280 460	5.8	1 155 1 155		
C101	96	48			4.1	220 470	4.6	2 000 2 000		
B450	64	32	3.659	0.076	2.4	415	5.1	1 600 -		
S400	128	32			2.4	350 440	4.3	1 720 1 720		
N451	128	48			3.7	285 460	5.3	1 000 1 000		
D450	128	64			4.9	215 475	5.3	500 500		
H200	96	32	5.164	0.064	2.1	420	4.4	1 980 -		
N202	128	48			3.1	410	6.4	875 -		
N203	128	48			3.1	345 440	5.4	1 500 1 500		
N200	128	48			3.1	285 465	4.4	1 695 1 695		
D200	128	64			4.1	200 480	4.2	2 000 1 000		
E250	192	96			6.2	130 490	4.1	485 485		
N300	128	48			8.595	0.050	2.4	420	5.1	1 680 -
N302	128	48					2.4	345 460	4.2	2 190 2 190
J303	192	64	3.2	260 475			4.2	1 040 1 040		
E300	192	96	4.8	175 490			4.3	600 600		

ensemble landscape



lattice correlators

on $N_f = 2 + 1$ ensembles from the CLS initiative

[Bruno *et al.* 2015, Bruno, Korzec, Schaefer 2017]

with $SU(3)_F$ notation, in the isospin-symmetric limit (light quark ℓ : either u or d):

$$I = 1 \text{ contribution:} \quad G_{\mu\nu}^{33}(x) = \frac{1}{2} C_{\mu\nu}^{\ell,\ell}(x),$$

$$I = 0 \text{ contribution:} \quad G_{\mu\nu}^{88}(x) = \frac{1}{6} \left[C_{\mu\nu}^{\ell,\ell}(x) + 2C_{\mu\nu}^{s,s}(x) + 2D_{\mu\nu}^{\ell-s,\ell-s}(x) \right],$$

$$Z\text{-}\gamma \text{ mixing:} \quad G_{\mu\nu}^{08}(x) = \frac{1}{2\sqrt{3}} \left[C_{\mu\nu}^{\ell,\ell}(x) - C_{\mu\nu}^{s,s}(x) + D_{\mu\nu}^{2\ell+s,\ell-s}(x) \right],$$

where the **connected** and **disconnected** Wick's contractions are

$$C_{\mu\nu}^{f_1, f_2} = - \left\langle \gamma_\mu \begin{array}{c} \xrightarrow{f_1} \\ \xleftarrow{f_2} \end{array} \gamma_\nu \right\rangle, \quad D_{\mu\nu}^{f_1, f_2} = \left\langle \begin{array}{c} \gamma_\mu \begin{array}{c} \circlearrowleft^{f_1} \\ \circlearrowright^{f_2} \end{array} \end{array} \right\rangle$$

and the relevant correlators are given by

(note: $G_{\text{con}}^\ell = 2G^{33}$ and $G_{\text{con}}^s = 3G_{\text{con}}^{88} - G^{33}$)

$$G^{\gamma\gamma} = G^{33} + \frac{1}{3}G^{88} + \frac{4}{9}C^{c,c},$$

$$G^{Z\gamma} = \left(\frac{1}{2} - \sin^2 \theta_W \right) G^{\gamma\gamma} - \frac{1}{6\sqrt{3}}G^{08} + \frac{4}{9} \left(\frac{3}{8} - \sin^2 \theta_W \right) C^{c,c}.$$

renormalization and $\mathcal{O}(a)$ improvement

for the local current

[Bhattacharya *et al.* 2006, [...], Gérardin, Harris, Meyer 2018]

$$V_{\mu,R}^3 = Z_V(1 + 3\bar{b}_V am_q^{av} + b_V am_{q,\ell}) V_{\mu}^{3,I} = Z_3 V_{\mu}^{3,I},$$

$$\begin{pmatrix} V_{\mu}^8 \\ V_{\mu}^0 \end{pmatrix}_R = Z_V \begin{pmatrix} 1 + 3\bar{b}_V am_q^{av} + b_V \frac{a(m_{q,\ell} + 2m_{q,s})}{3} & \left(\frac{b_V}{3} + f_V\right) \frac{2a(m_{q,\ell} - m_{q,s})}{\sqrt{3}} \\ r_V d_V \frac{a(m_{q,\ell} - m_{q,s})}{\sqrt{3}} & r_V 1 + (3\bar{d}_V + d_V) am_q^{av} \end{pmatrix} \begin{pmatrix} V_{\mu}^8 \\ V_{\mu}^0 \end{pmatrix}^I = \begin{pmatrix} Z_8 & Z_{80} \\ Z_{08} & Z_0 \end{pmatrix} \begin{pmatrix} V_{\mu}^8 \\ V_{\mu}^0 \end{pmatrix}^I$$

where

$$V_{\mu}^{a,I} = V_{\mu}^a + ac_V \partial_0 T_{0\mu}^a, \quad V_{\mu}^{0,I} = V_{\mu}^0 + a\bar{c}_V \partial_0 T_{0\mu}^0.$$

while for the conserved current

$$V_{\mu,R}^a = V_{\mu}^a + ac_V^{cs} \partial_0 T_{0\mu}^a, \quad V_{\mu,R}^0 = V_{\mu}^0 + a\bar{c}_V^{cs} \partial_0 T_{0\mu}^0.$$

⇒ we use only the conserved vector current for the flavor-singlet component, and we set

$$f_V = 0, \quad \bar{c}_V = c_V \quad \bar{c}_V^{cs} = c_V^{cs}.$$

bounding method

$$G(t) = \sum_{n=0}^{\infty} \frac{Z_n^2}{2E_n} e^{-E_n t}$$

for a correlator with positive spectral decomposition, and $t > t_c$

$$0 \leq G(t_c) e^{-E_{\text{eff}}(t_c)(t-t_c)} \leq G(t) \leq G(t_c) e^{-E_0(t-t_c)},$$

where $E_{\text{eff}}(t) = -(1/a) \log G(t+a)/G(t)$ is the effective mass

and E_0 is the ground state in the given channel, depending on the volume L^3 and on m_π

• for $I = 1$, $E_0 = m_\rho$ or $E_{2\pi}$, • for $I = 0$, $E_0 = m_\omega \approx m_\rho$ or $E_{3\pi}$

improved bounding method:

[Lehner LGT2016; Gérardin, MC *et al.* 2019]

if E_0, \dots, E_N and Z_0, \dots, Z_{N-1} are available, one can bound the subtracted correlator

$$\tilde{G}(t) = G(t) - \sum_{n=0}^{N-1} \frac{Z_n^2}{2E_n} e^{-E_n t},$$

that approaches zero faster \Rightarrow dedicated spectroscopy effort

extrapolation to the physical point

a combined fit of $\bar{\Pi}^{33}$, $\bar{\Pi}^{88}$ and $\bar{\Pi}^{08}$, with two discretization each (one discr. for $\bar{\Pi}^{08}$) is used

$$\bar{\Pi}^{33,X}(a^2/t_0^{\text{sym}}, \phi_2, \phi_4) = \bar{\Pi}^{\text{sym}} + \delta_2^X a^2/t_0^{\text{sym}} + \gamma_1^{33}(\phi_2 - \phi_2^{\text{sym}}) + \gamma_{\log}^{33} \log \phi_2/\phi_2^{\text{sym}} + \eta_1(\phi_4 - \phi_4^{\text{sym}}),$$

$$\bar{\Pi}^{88,X}(a^2/t_0^{\text{sym}}, \phi_2, \phi_4) = \bar{\Pi}^{\text{sym}} + \delta_2^X a^2/t_0^{\text{sym}} + \gamma_1^{88}(\phi_2 - \phi_2^{\text{sym}}) + \gamma_2^{88}(\phi_2 - \phi_2^{\text{sym}})^2 + \eta_1(\phi_4 - \phi_4^{\text{sym}}),$$

$$\bar{\Pi}^{08,\text{CL}}(a^2/t_0^{\text{sym}}, \phi_2, \phi_4) = \lambda_1(\phi_4 - 3/2\phi_2),$$

where $X = \text{CL}$ or LL , $\phi_2 = 8t_0 m_\pi^2$, $\phi_4 = 8t_0(m_K^2 + m_\pi^2/2)$.

- we add also a $\delta_3^X a^3/(t_0^{\text{sym}})^{3/2} \Rightarrow$ better fit at large $Q^2 \Rightarrow$ smooth transition around $Q^2 = 2.5 \text{ GeV}^2$
- $\sim a^2 \log a$ term?
 \Rightarrow assuming free theory coefficient, up to 0.4 % downward shift, within the statistical error
- extrapolation of the charm contribution done separately

[MC, Harris, Meyer, Toniato, Török 2021]