Isospin Breaking corrections in τ-decays for $(g-2)_\mu$

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Fifth plenary workshop of the Muon g-2 theory initiave
Edinburgh, Scotland, September, 2022

Window fever

Hadronic Vacuum Polarization (HVP) contribution to a_μ

Time-momentum representation [Bernecker, Meyer, '11]

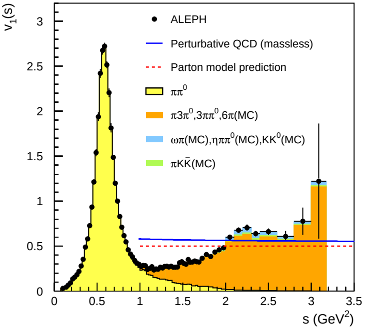
$$G^\gamma(t) = \frac{1}{3}\sum_k \int d\boldsymbol{x}\ \langle j_k^\gamma(x) j_k^\gamma(0)\rangle \quad \rightarrow \quad a_\mu = 4\alpha^2 \sum_t w_t G^\gamma(t)$$

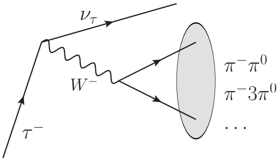
Windows in Euclidean time [RBC/UKQCD '18]

$$a_\mu^W = 4\alpha^2 \sum_t w_t\, G^\gamma(t)\, [\Theta(t, t_0, \Delta) - \Theta(t, t_1, \Delta)]$$

$$t_0 = 0.4 \text{ fm} \quad t_1 = 1.0 \text{ fm} \quad \Delta = 0.15 \text{ fm}$$

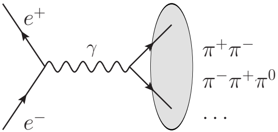
Motivations for τ





$V-A$ current

Final states $I=1$ charged



EM current

Final states $I=0,1$ neutral

τ data can improve $a_\mu[\pi\pi]$

$\rightarrow$ 72% of total Hadronic LO

$\rightarrow$ competitive precision on a_μ^W

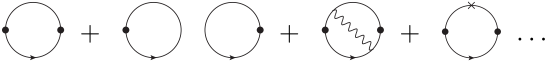
Contribution to a_μ

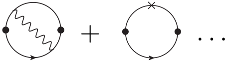
Time-momentum representation [Bernecker, Meyer, '11]

$$G^\gamma(t) = \frac{1}{3}\sum_k \int d\boldsymbol{x}\ \langle j_k^\gamma(x) j_k^\gamma(0)\rangle \quad \rightarrow \quad a_\mu = 4\alpha^2 \sum_t w_t G^\gamma(t)$$

Isospin decomposition of u, d current

$$j_\mu^\gamma = \frac{i}{6}\left(\bar{u}\gamma_\mu u + \bar{d}\gamma_\mu d\right) + \frac{i}{2}\left(\bar{u}\gamma_\mu u - \bar{d}\gamma_\mu d\right) = j_\mu^{(0)} + j_\mu^{(1)}$$

$G_{00}^\gamma \leftarrow \langle j_k^{(0)}(x) j_k^{(0)}(0)\rangle =$ 

$G_{01}^\gamma \leftarrow \langle j_k^{(0)}(x) j_k^{(1)}(0)\rangle =$ 

$G_{11}^\gamma \leftarrow \langle j_k^{(1)}(x) j_k^{(1)}(0)\rangle =$ 

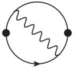
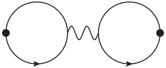
Decompose $a_\mu = a_\mu^{(0,0)} + a_\mu^{(0,1)} + a_\mu^{(1,1)}$

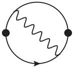
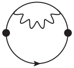
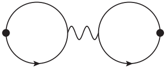
Neutral vs Charged

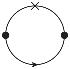
$$\tfrac{i}{2}\left(\bar{u}\gamma_\mu u - \bar{d}\gamma_\mu d\right), \begin{bmatrix} I=1 \\ I_3=0 \end{bmatrix} \quad \rightarrow \quad j_\mu^{(1,-)} = \tfrac{i}{\sqrt{2}}\left(\bar{u}\gamma_\mu d\right), \begin{bmatrix} I=1 \\ I_3=-1 \end{bmatrix}$$

Isospin 1 charged correlator $G_{11}^W = \frac{1}{3}\sum_k \int d\boldsymbol{x}\, \langle j_k^{(1,+)}(x) j_k^{(1,-)}(0)\rangle$

$\delta G_{11} \equiv G_{11}^\gamma - G_{11}^W$ [MB et al.' Latt18]

$$= Z_V^4 (4\pi\alpha) \frac{(Q_\mathrm{u} - Q_\mathrm{d})^4}{4} \Big[$$  $+$  $\Big]$

$$G_{01}^\gamma = Z_V^4 \frac{(Q_u^2 - Q_d^2)^2}{2} (4\pi\alpha) \Big[$$  $+ 2\times$  $+$  $+ \dots \Big]$

$$+ Z_V^2 \frac{Q_u^2 - Q_d^2}{2} (m_u - m_d) \Big[2\times$$  $+ \dots \Big]$

$\dots$ = subleading diagrams

Intermezzo

Given definition of isosymmetric world [RBC/UKQCD '18, BMWc '20]

comparision of isosymmetric windows from LQCD well-defined

in continuum, infinite volume

comparison of isospin breaking shift also well-defined

A possible new probe for LQCD+QED calculations

$\delta G_{11} \equiv G_{11}^{\gamma} - G_{11}^{W}$

observable defined in QCD+QED $\rightarrow$ no scheme ambiguity

allows for testing a smaller combination of diagrams

windows of δG_{11} provide additional angle

Strategy

$G^\gamma_{00} + G^\gamma_{01} + G^\gamma_{11} =$	$+G^\gamma_{00}$	LQCD+QED	future
	$+G^\gamma_{01}$	LQCD+QED	this talk
	$+G^W_{11}$	τ-data	this talk
	$+\delta G_{11}$	LQCD+QED	this talk

Restriction to $\pi\pi$ channel

assume $G^\gamma_{00} \simeq 0$ and G^γ_{01} dominated by IB effects of $\pi\pi$ channel

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$$v_-(s) = \frac{m_\tau^2}{6|V_{ud}|^2} \frac{\mathcal{B}_{\pi\pi^0}}{\mathcal{B}_e} \frac{1}{N_{\pi\pi^0}} \frac{dN_{\pi\pi^0}}{ds} \left(1 - \frac{s}{m_\tau^2}\right)^{-1} \left(1 + \frac{2s}{m_\tau^2}\right)^{-1} \frac{1}{S_{\rm EW}}$$

0. $S_{\rm EW}$ electro-weak radiative correct. [Marciano, Sirlin '88][Braaten, Li '90]

1. Laplace transform to Euclidean time

Long distance QED

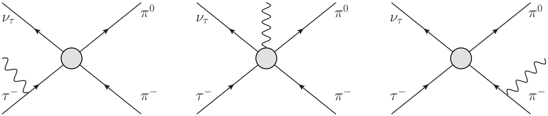
At low energies relevant degrees of freedom are mesons

Chiral Perturbation Theory [Cirigliano et al. '01, '02]

Meson dominance model [Flores-Talpa et al. '06, '07]

Corrections casted in one function $v_-(s) \to v_-(s) G_{\mathrm{EM}}(s)$

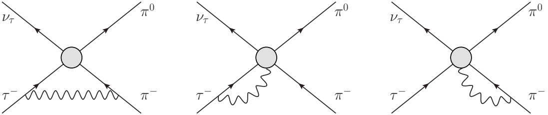
Real photon corrections



Real + virtual

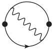
$\to$ IR divergences cancel

Virtual photon corrections (τ and π self-energy)

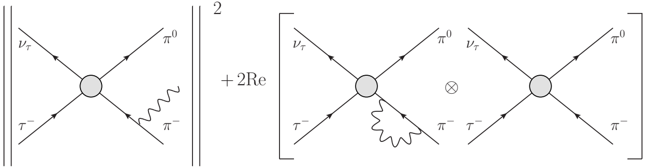


Matching to Lattice Q[C,E]D

Re-evaluation of $G_{\rm EM} \to G^{\pi}_{\rm EM}$

lattice contains $\pi^0\pi^-\gamma$ states $\to$ 

$G^{\pi}_{\rm EM}$ = "remove" infrared safe sub-components of rate from $G_{\rm EM}$



$G^{\pi}_{\rm EM}$ preliminary results in the Leading Low approximation

- keep terms $O(1/k^2)$ (k photon momentum)
  - subleading terms $O(1/k)$ [in collab. with Cirigliano]
- full $G^{\pi}_{\rm EM}$ shift as systematic error

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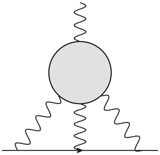
Synergy



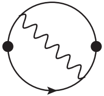
from QCD we need a 4-point function $f(x, y, z, t)$:

known kernel with details of photons and muon line

1 pair of point sources (x, y), sum over z, t exact at sink

stochastic sampling over (x, y) (based on $|x - y|$)

Successfull strategy: x10 error reduction [RBC '16]



from QCD we need a 4-point function $f(x, y, z, t)$:

$(g-2)_\mu$ kernel + photon propagator

Similar problem $\rightarrow$ re-use HLbL point sources!

The RBC & UKQCD collaborations

UC Berkeley/LBNL
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BNL and BNL/RBRC
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Peter Boyle (Edinburgh)
Taku Izubuchi
Chulwoo Jung
Christopher Kelly
Meifeng Lin
Nobuyuki Matsumoto
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Amarjit Soni
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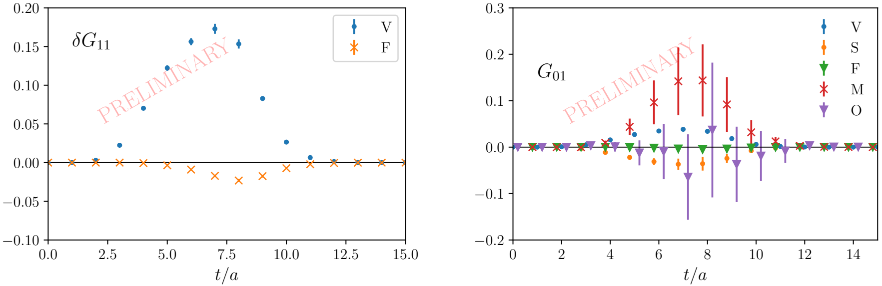
Nikolai Husung
Rajnandini Mukherjee
Callum Radley-Scott
Chris Sachrajda

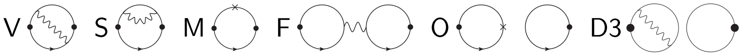
Stony Brook University
Jun-Sik Yoo
Sergey Syritsyn (RBRC)

Results - Preliminary

Preliminary from 48I ensemble

- phys. pions, $a^{-1} \simeq 1.73$ GeV, 17 configs
- cross-checks of code, data, analysis still missing





Results - Preliminary

✓ statistical errors: point source sampling based on HLbL data
plans to improve SIB-valence (see backup)

✗ systematic errors: this talk very conservative, plans to improve

1. finite vol. $\to$ repeat calculation on $6\ \mathrm{fm}$ box (see backup)
2. discretization errors $\to$ repeat calculation on finer 64I
3. add QED-sea and SIB-sea effects

$$\Delta a_\mu^W[\tau] = 4\alpha^2 \int w_t \left[G_{01}^{\gamma}(t) + \delta G_{11}(t)\right] \left[\Theta(t,t_0,\Delta) - \Theta(t,t_1,\Delta)\right]$$

$\Delta a_\mu^W \times 10^{10} = +2.1(1.3)|_{G^{\pi}_{\mathrm{EM}}+\mathrm{LQCD}}$ [PRELIMINARY]

Note: S_{EW} shift treated separately

Isospin Corrections

Restriction to $e^+e^- \to \pi^+\pi^-$ and $\tau^- \to \pi^-\pi^0\,\nu_\tau$

$$v_0(s) = \frac{s}{4\pi\alpha^2}\sigma_{\pi^+\pi^-(\gamma)}(s)$$

$$v_-(s) = \frac{m_\tau^2}{6|V_{ud}|^2}\frac{\mathcal{B}_{\pi\pi^0}}{\mathcal{B}_e}\frac{1}{N_{\pi\pi^0}}\frac{dN_{\pi\pi^0}}{ds}\left(1-\frac{s}{m_\tau^2}\right)^{-1}\left(1+\frac{2s}{m_\tau^2}\right)^{-1}\frac{1}{S_{\mathrm{EW}}}$$

Isospin correction $v_0 = R_{\mathrm{IB}} v_-$ $\quad R_{\mathrm{IB}} = \frac{\mathrm{FSR}}{G_{\mathrm{EM}}}\frac{\beta_0^3|F_\pi^0|^2}{\beta_-^3|F_\pi^-|^2}$ [Alemani et al. '98]

0. S_{EW} electro-weak radiative correct. [Marciano, Sirlin '88][Braaten, Li '90]

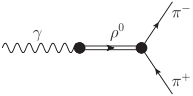
1. Final State Radiation of $\pi^+\pi^-$ system [Schwinger '89][Drees, Hikasa '90]

2. G_{EM} (long distance) radiative corrections in τ decays
- Chiral Resonance Theory [Cirigliano et al. '01, '02]
- Meson Dominance [Flores-Talpa et al. '06, '07]

3. Phase Space ($\beta_{0,-}$) due to ($m_{\pi^\pm} - m_{\pi^0}$)

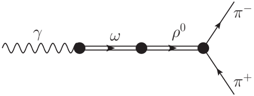
Pion form factors

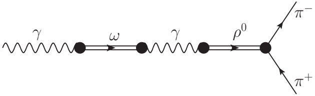
$$F_\pi^0(s) \propto \frac{m_\rho^2}{D_\rho(s)} + (\rho', \rho'')$$



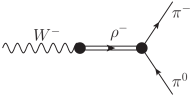
[Gounaris, Sakurai '68]
[Kühn, Santamaria '90]

$$\times \left[1 + \delta_{\rho\omega} \frac{s}{D_\omega(s)}\right]$$





$$F_\pi^-(s) \propto \frac{m_{\rho^-}^2}{D_{\rho^-}(s)} + (\rho', \rho'')$$



$$\Delta a_\mu^W \times 10^{10} = +0.7|_{G_{\rm EM}+R_{\rm IB}}$$

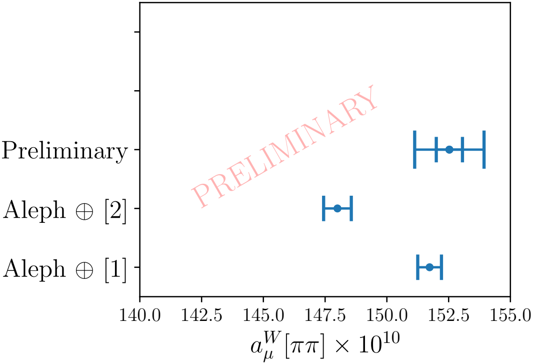
[Davier et al. '09] expected to agree well with this estimate

$$\Delta a_\mu^W \times 10^{10} = -2.8|_{G_{\rm EM}+R_{\rm IB}+\rho\gamma}$$

Window fever - τ

my PRELIMINARY analysis of exp. + latt. data

- only exp. errs, no attempt at estimating sys. errs for [1] and [2]
- LQCD syst. errs require further investigation/improvements



Isospin-breaking:

[1]: w/o $\rho\gamma$ mixing

[2]: w/ $\rho\gamma$ mixing

What is $\rho\gamma$? too much to say, too little time to explain everything...

Window fever - τ

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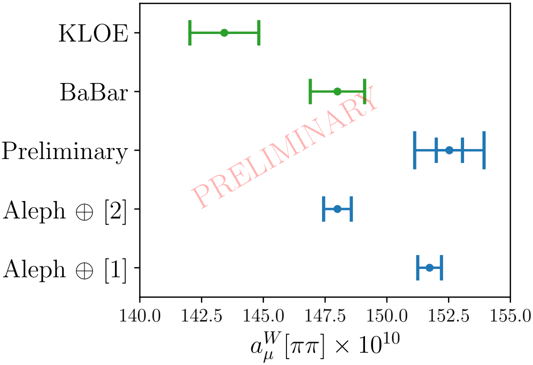

Isospin-breaking:

[1]: w/o $\rho\gamma$ mixing

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Preliminary Conclusions

Windows very powerful quantities: intermediate window a_μ^W
hadronic τ-decays can shed light on tension lattice vs e^+e^-

IB effects from Lattice (preliminary) $\oplus$ τ-data: hints to

1. shift $\approx +7 \times 10^{-10}$ in $\pi\pi$ channel w.r.t. e^+e^-
   points towards agreement of τ w/ LQCD+QED of a_μ^W
2. qualitative agreement w/ pheno estimates [Davier et al. '09]
3. disagreement w/ $\rho\gamma$ mixing [Jegerlehner et al. '11]
   does not mean it is not there, a lot to unpack here

Note: largest shift from short distance $S_{\mathrm{EW}} \simeq +3.4(0.1) \times 10^{-10}$

Outlook

Complete calculation on 48I

- cross-checks of lattice data, cross-check of $G^{\pi}_{\rm EM}$
- analyse QED-sea & SIB-qed diagrams (building blocks on disk)

Remove restriction to $\pi\pi$ channel

- analyze full spectral density from experiment
- include G^{γ}_{00} from LQCD, ie disconneted diagram

Compare against data-driven approaches

Complete calculation of $G_{\rm EM}$ beyond Leading Low

- structure-depent model [in collab. w/ Cirigliano]
- using dispersive methods [Ruiz de Elvira's talk]
- ideas for full lattice calculation of $G_{\rm EM}$ under develop.

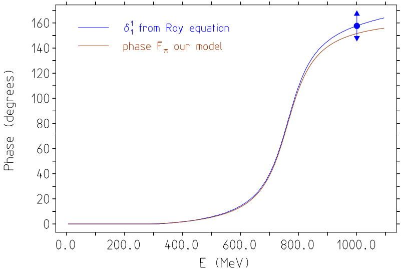
Extend τ-data analysis w/ all experimets [Goltermann et al.][Davier et al.]

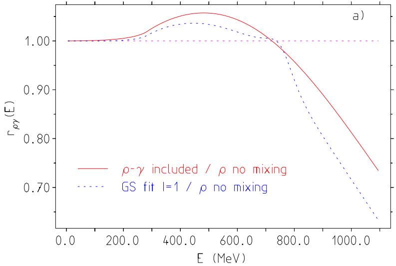
Thanks for your attention

Backup slides

$\rho\gamma$ MIXING

VMD model with gauge-invariance [Kroll, Lee, Zumino '67]
at 1-loop s-dependent mass matrix [Jegerlehner, Szafron '11]





30% correction at 1 GeV, δ_1^1 in good agreement $E < 800$ MeV
$\rightarrow$ perhaps restrict the $\rho\gamma$ below 800 MeV?

From $(g-2)$ White Paper: " .. an increasing effect above the ρ peak that appears uncomfortably large."

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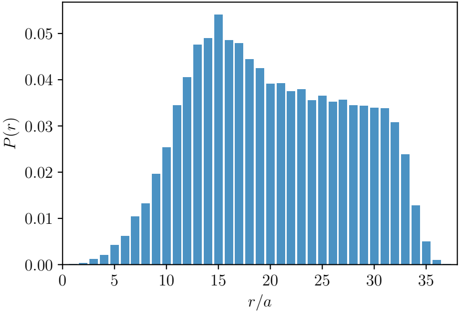
Sampling strategy

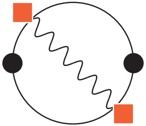
Propagators on disk from HLbL project [Phys.Rev.Lett. 118 (2017)]

$$\tilde{V}_\Gamma(x_0, z_0, r) = \sum_{\boldsymbol{x},\boldsymbol{z}} \mathrm{tr}\left[\Gamma D^{-1}(x,0)\gamma_\nu D^{-1}(0,z)\Gamma D^{-1}(z,r)\gamma^\nu D^{-1}(r,x)\right]$$

$$V_\Gamma(|x_0 - z_0|) = \sum_r \Delta(r)\tilde{V}_\Gamma(x_0, z_0, r)$$

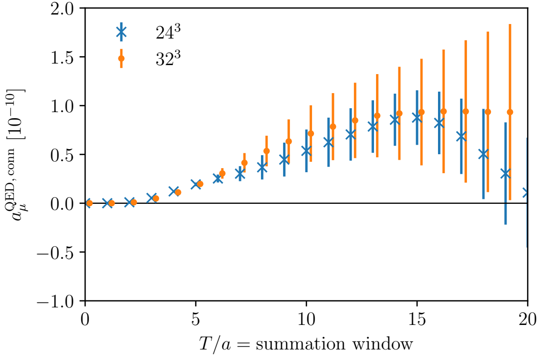
$O(10^3)$ points $\rightarrow$ $O(10^6)$ pairs





contract photon offline
$\rightarrow$ study QED_L vs QED_∞

Finite volume errors



$$a_\mu^{\mathrm{QED,conn}} = V + 2S$$

FV study at coarse
$a^{-1} \sim 1$ GeV

Finite volume errors

empirical observation: diagrams may have largish FV errors

cancellation of FV effects in physical combinations

similar observation in ChPT, e.g. [Bijnens, Portelli '19]

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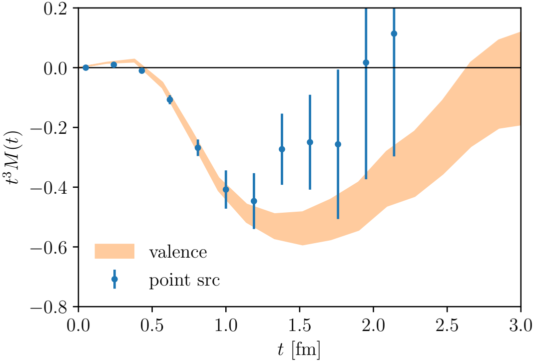
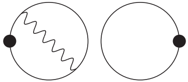
Strong isospin breaking

Accurate determination from multiple valence calculations

- independent determination from point sources only 8k / 1M
- on-going check if full 1M can be competitive

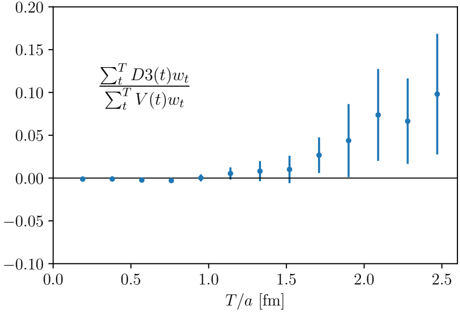


QED valence disconnected



Preliminary (run2)
Point sources at exchanged photon vertices

Coarse lattice $a \simeq 0.2$ fm



Observe suppression relative to V

matches target accuracy

not yet explored full statistics (running)