Sum rules for the hadronic vacuum polarization

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Motivation

- A new set of sum rules allowing for comparison between spectral functions from experimental data and lattice correlators, or between spectral functions from different data sets; may help shed light on discrepancies such as lattice-exp. data and KLOE-BaBar

- Similar to "window method," but start with a window on the spectral function instead of a window defined in Euclidean time; narrow region in $\sqrt{s}$ instead of in t (See also Colangelo *et al*. '22)

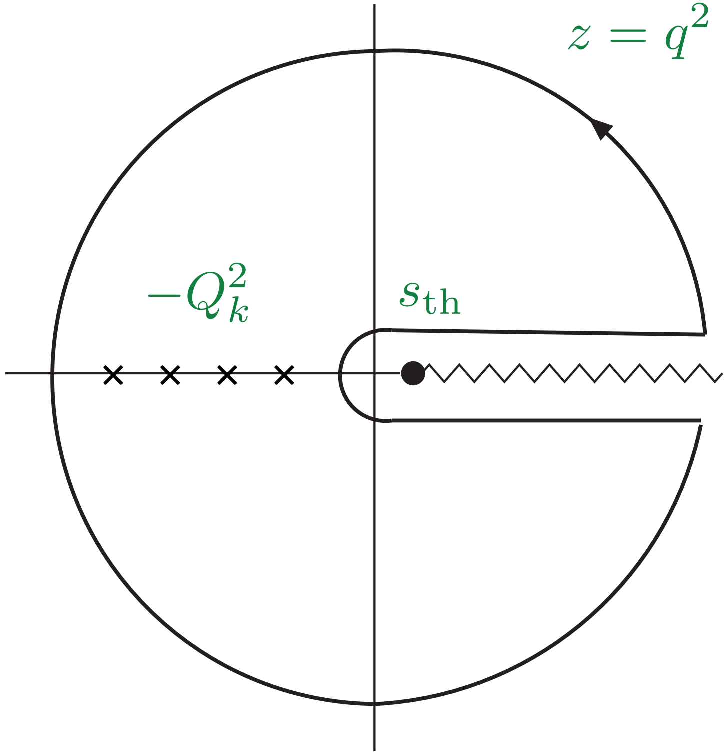
- Applications:

➢ Comparisons between R-ratio data and lattice data

➢ Potentially useful for reconsidering hadronic τ-decay data: more precise τ spectral function (Boito *et al*. '20) available; progress with lattice computation of isospin-breaking (IB) effects (see Mattia Bruno's talk)

Rational-weight sum rules

Consider a class of rational weights $W_{m,n}(s) = \dfrac{(s - s_{\rm th})^m}{\prod_{k=1}^{n}(s + Q_k^2)}$, Q_k^2 Euclidean, fixed



$z = q^2$ plane

Taking radius to infinity leads to sum rule

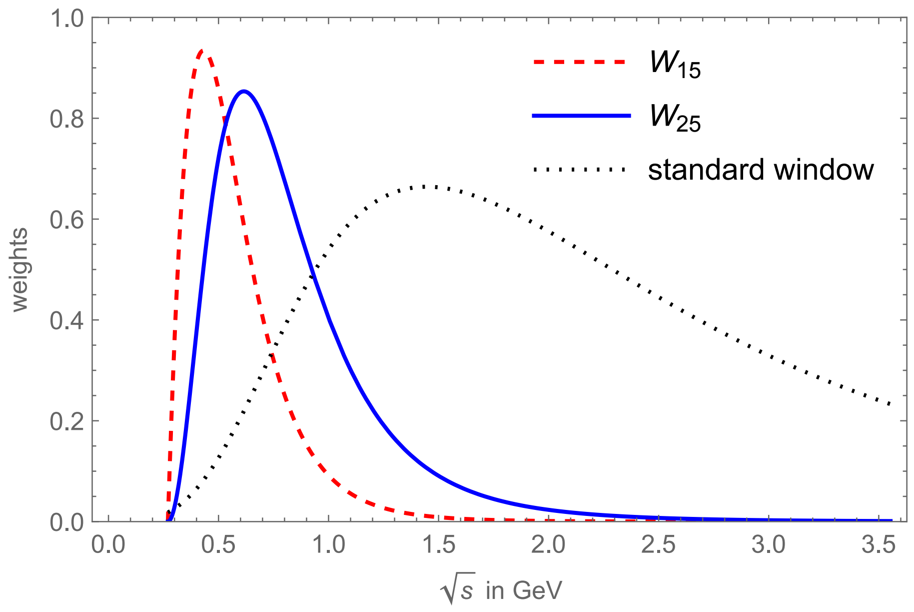
$$\frac{1}{2\pi i}\oint_C dz\, W_{m,n}(z)\,\hat{\Pi}(-z) = (-1)^m \sum_{k=1}^{n} \frac{(Q_k^2 + s_{\rm th})^m}{\prod_{\ell\neq k}(Q_\ell^2 - Q_k^2)}\hat{\Pi}(Q_k^2)$$
$$= \int_{s_{\rm th}}^{\infty} ds\, W_{m,n}(s)\,\rho(s)$$

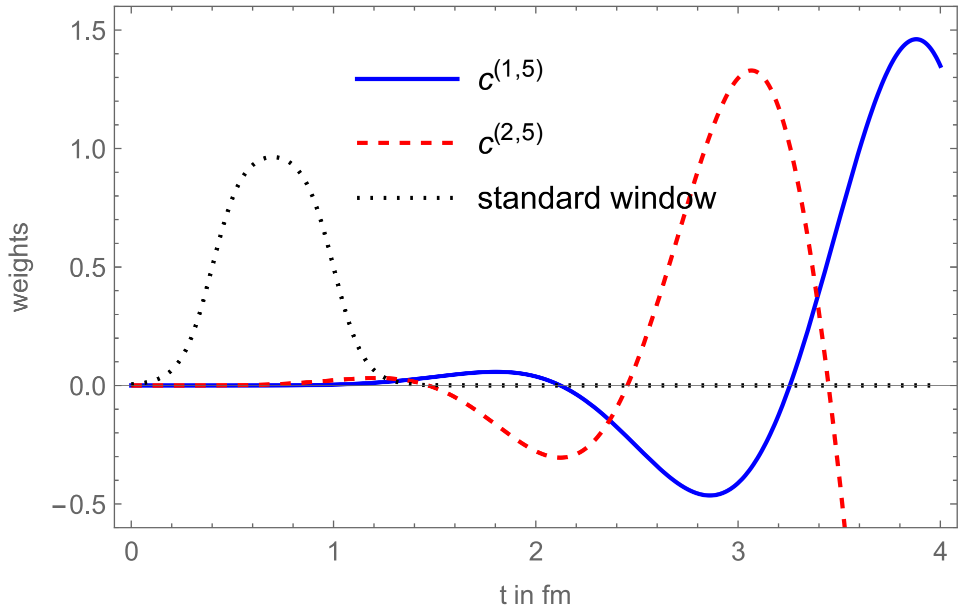
Relation between spectral weight and Euclidean quantity

This can be recast in terms of $C(t) = \frac{1}{3}\sum_{\vec{x},i}\langle j_i^{\rm EM}(\vec{x},t) j_i^{\rm EM}(0)\rangle = -\int \frac{dQ}{2\pi}\, e^{iQt}\, Q^2\, \hat{\Pi}(Q^2)$ as

$$\int_{s_{\rm th}}^{\infty} ds\, W_{m,n}(s)\,\rho(s) = \int_0^{\infty} dt\, \underbrace{(-1)^m \sum_{k=1}^{n} \frac{(Q_k^2+s_{\rm th})^m}{\prod_{\ell\neq k}(Q_\ell^2-Q_k^2)} \left(\frac{4\sin^2(Q_k t/2)}{Q_k^2} - t^2\right)}_{c^{(m,n)}} C(t)$$

Examples: choose $Q_k^2 = 0.25,\ 0.325,\ 0.4,\ 0.475,\ 0.55\ {\rm GeV}^2$ and $n = 1,\ 2$:





Note: arbitrary vertical scales!

Another type of sum rule: exponential weights

Because $$C(t) = \int_{E_{\rm th}}^{\infty} dE\, E^2 e^{-Et}\, \rho(E)\ , \qquad t > 0$$

choosing $$w_n(E) = \sum_{j=1}^{n} x_j\, E^2 e^{-Et_j}\ , \qquad t_j > 0$$

leads to new sum rule $$\int_{E_{\rm th}}^{\infty} dE\, w_n(E)\, \rho(E) = \sum_{j=1}^{n} x_j C(t_j)$$

Why is this interesting?

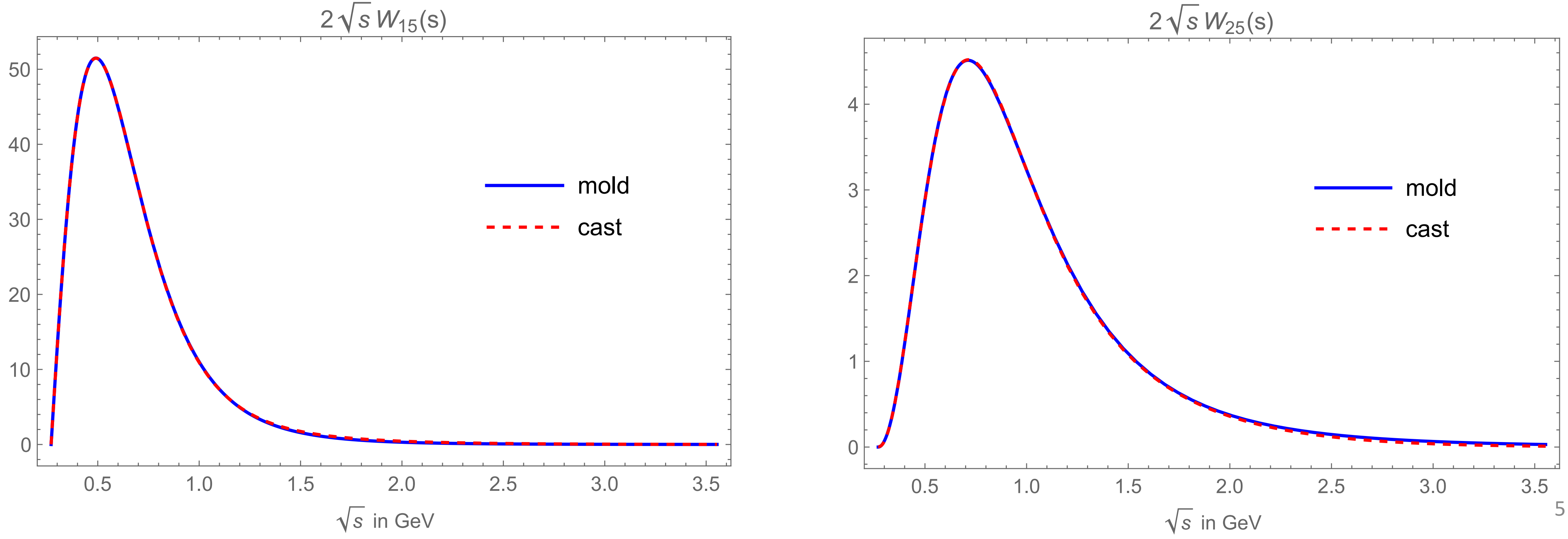
Choose a physically interesting weight $2EW(s = E^2)$ (the "mold"), and replace by approximation $w_n(E)$ (the "cast") (Hansen, Lupo and Tantalo '19)

- Get to choose values of t_j ; pick values where $C(t_j)$ has small errors
- Sum rule is exact for the cast $\Rightarrow$ throw away the mold!

For a given weight $W(s = E^2)$, minimize $\int_{E_{\rm th}}^{\infty} dE \left| w_n(E; \{t_j\}, \{x_j\})/E^2 - 2W(E^2)/E \right|^2$

Given the times $t_n > \cdots > t_1 > 0$ this yields the coefficients x_j , determining the exp. weights $w_n(E)$

Examples: choose $t_j = 3,\ 6,\ 9,\ 12,\ 15\ \mathrm{GeV}^{-1} \approx\ 0.6,\ 1.2,\ 1.8,\ 2.4,\ 3\ \mathrm{fm}$:



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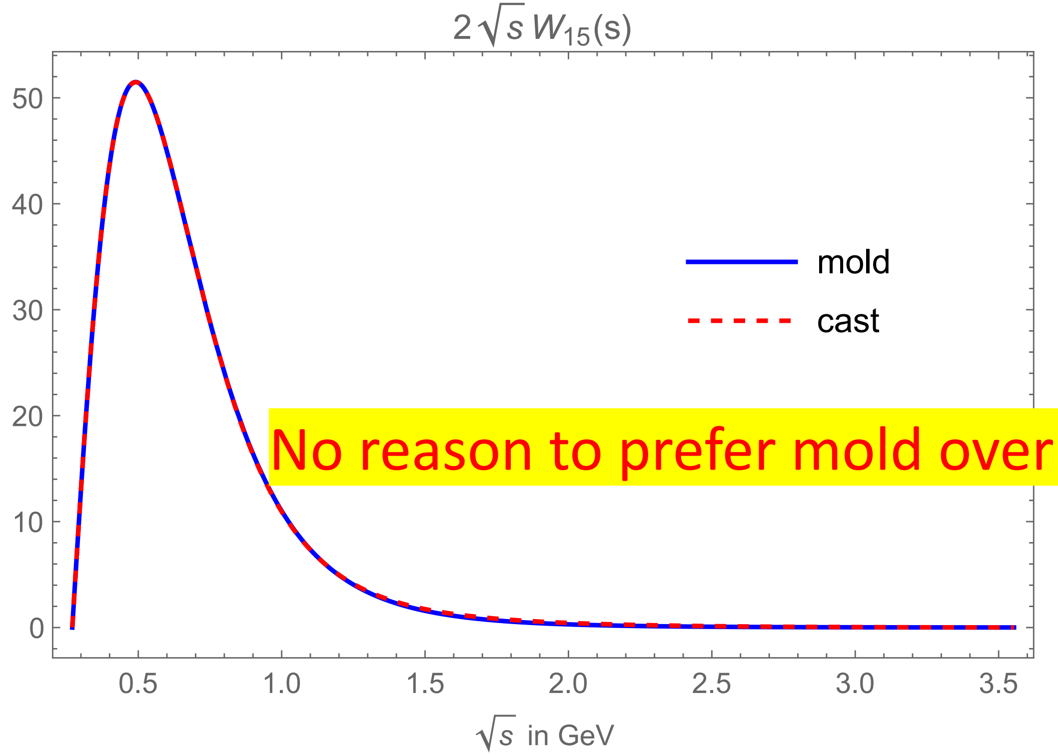

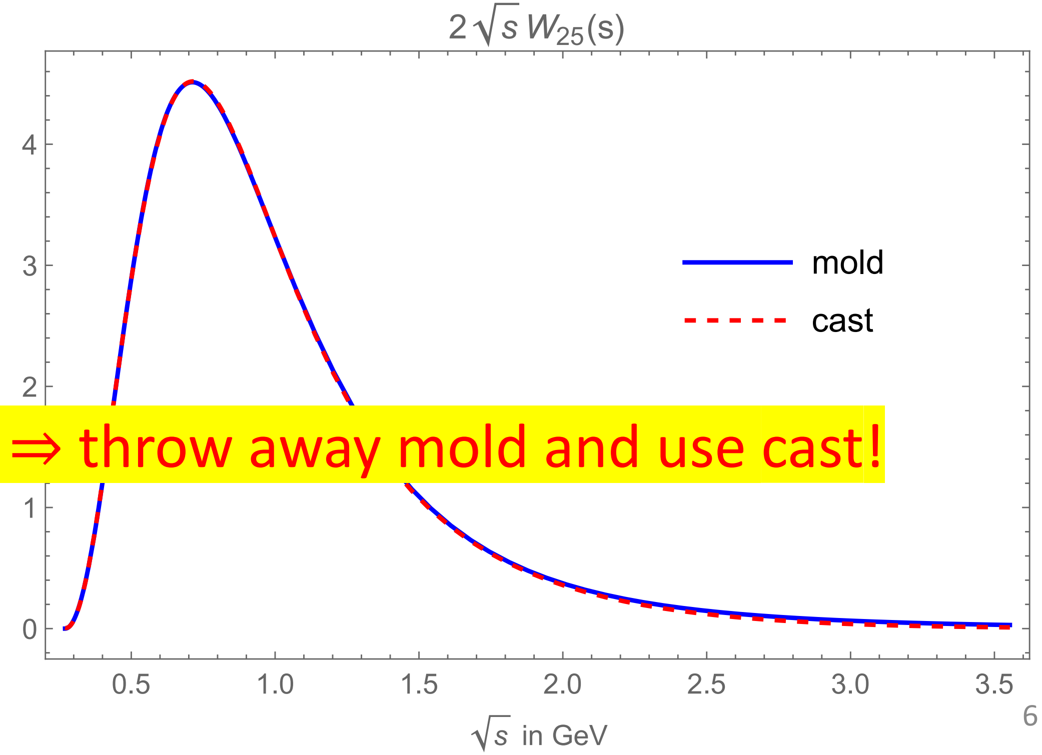

No reason to prefer mold over cast ⇒ throw away mold and use cast!

Comparison between R-ratio and lattice: rational-weight sum rules

- R-ratio data (Keshavarzi, Nomura and Teubner ’19)
- Lattice: from ABGP ‘22 *light-quark connected part only*; compare only errors!

  lattice spacings $a = 0.06,\ 0.09,\ 0.12,\ 0.15\ \mathrm{fm}$, extrapolate to continuum limit

  (pion-mass mistunings and finite-volume effects smaller than statistical errors)

	R-ratio	rel. error	lattice	rel. error
W_{15}	0.4756(16)	0.3%	0.468(26)	5.6%
W_{25}	0.08912(34)	0.4%	0.0838(33)	3.9%

- Do NOT compare central values! Lattice errors order of magnitude larger

Comparison between R-ratio and lattice: exponential-weight sum rules

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	R-ratio	rel. error	lattice	rel. error
W'_{15}	0.4785(16)	0.3%	0.496(17)	3.4%
W'_{25}	0.08857(34)	0.4%	0.0798(18)	2.3%

(W' is exponential-weight version of *W*)

- Do NOT compare central values! Lattice errors order of magnitude larger
- Lattice errors almost factor 2 smaller! (No attempt yet to optimize choice of weights)

How useful are these numbers?

Compare with BaBar-KLOE discrepancy

Assume that above s = 0.95 GeV^2 there is no discrepancy (no KLOE data above 0.95 GeV^2): allows us to to compute the spectral weights with W'_{15} and W'_{25}:

$$\Delta_{15}(\mathrm{BaBar} - \mathrm{KLOE}) = 0.0091(39)$$
$$\Delta_{25}(\mathrm{BaBar} - \mathrm{KLOE}) = 0.00153(52)$$

Lattice accuracy: 0.017 for W'_{15} ; 0.0018 for W'_{25}

$\Longrightarrow$ Need factor ~4 improvement for W'_{15} ; factor ~2 improvement for W'_{25} (W'_{25} better "designed" than W'_{15})

Comparison between R-ratio and hadronic τ-decay data

- In an isospin-conserving world, one can simply replace any exclusive I = 1 spectral distribution by the corresponding distribution obtained from non-strange vector hadronic τ decays (for example, $\pi^-\pi^+$ by $\pi^-\pi^0$ and the sum of $2\pi^+ 2\pi^- + \pi^+\pi^- 2\pi^0$ by $2\pi^-\pi^+\pi^0 + \pi^- 3\pi^0$, etc.)

Why is this again interesting?

- A new $2\pi + 4\pi$ τ-based spectral function exists which combines ALEPH and OPAL data, can be improved by BELLE II data (Boito *et al*. '20)

- Isospin breaking (IB) is a problem, but the lattice can help out! (Mattia Bruno's talk)

Issues: exclusive vs. inclusive

Lattice can compute (1st order) IB correction: $\Pi_{\rm IB}(Q^2) \equiv \frac{2}{\sqrt{3}}\Pi_{38}(Q^2) + \left[\Pi_{33}(Q^2) - \frac{1}{2}\Pi_{ud;V}(Q^2)\right]$

- However, this is inclusive!

  Only 10% of $a_\mu^{\rm HVP}$ from above $\sqrt{s}$ = 1.937 GeV, assuming IB ~ 1% leads to 0.1% uncertainty

  Below $\sqrt{s}$ = 1.937 GeV only 2π, 3π, 4π, $K\overline{K}$ relevant at the 1-2% IB level

  $K\overline{K}$ dominated by I = 0 (Babar '18, Boito *et al.* '22), 2π and 4π are I = 1, 3π is I = 0 (not quite)

  Thus $$\Delta_{mn} \equiv \int_{s_{\rm th}}^\infty ds\, W_{mn}(s)\rho^{e^+e^-}_{2\pi+4\pi}(s) - \int_{s_{\rm th}}^\infty ds\, W_{mn}(s)\rho^{\tau}_{2\pi+4\pi}(s)$$
  $$= (-1)^m \sum_{k=1}^n \frac{(Q_k^2+s_{\rm th})^m}{\prod_{\ell\neq k}(Q_\ell^2-Q_k^2)}\Pi_{\rm IB}(Q_k^2)$$

- This is Euclidean! Problem, because on the τ side, we are restricted to $s < m_\tau^2$

  However, bulk of 2π, 4π for the weights we consider from s below m_τ^2

➢ For practical purposes, IB in 2π+ 4π can be considered inclusive!

Issues: 3π overcounting; missed 4π contribution

- The lattice IB correction contains a 3π contribution due to ρ-ω interference, but since we only replace $2\pi + 4\pi$ by τ-based data, this leads to overcounting the 3π contribution

  BaBar '20 analysis of $e^+e^- \to \rho \to 3\pi$ allows for quantitative estimate of overcount: amounts to -0.54(54) × 10^{-10} for a_μ^{HVP} -- estimates for sum rules follow

- $s > m_\tau^2$ contributions: negligible for 2π

  small for 4π if we bound IB by 2% in this (non-resonant) region

Results

- Define $I^W(s_0) = \int_{s_{\rm th}}^{s_0} ds\, W(s)\, \rho(s)$ then we obtain from the R-ratio:

	$I_W^{2\pi+4\pi}$	$I_W^{2\pi+4\pi}(m_\tau^2)$	$\mathrm{IB}_{3\pi\ \mathrm{overcount}}$	$\mathrm{IB}_{s>m_\tau^2}$
W_{15}	0.4008 ± 0.0014	0.3995 ± 0.0014	-0.00041(41)	< 0.00026
W_{25}	0.06620 ± 0.00029	0.06459 ± 0.00027	-0.00012(12)	< 0.000033

Note: IB corrections much smaller than statistical errors

- Can compute these spectral weights also from the $2\pi + 4\pi$ τ-based spectral function;

  We find for the differences:
$$I_{\rm EM}^{W_{15}}(m_\tau^2) - I_\tau^{W_{15}}(m_\tau^2) = -0.0108(26)$$
$$I_{\rm EM}^{W_{25}}(m_\tau^2) - I_\tau^{W_{25}}(m_\tau^2) = -0.00233(35)$$

- Sets the stage for the required lattice computation of the IB part

Remarks

- New set of sum rules potentially useful for comparing spectral and lattice data, as well as electroproduction and hadronic τ-decay data

- To do: more precise lattice correlators
  search for optimal sum-rule weights

- τ-decays: need lattice computation of IB part of the hadronic vacuum polarization
  if we use both the 2π and 4π data, an inclusive lattice computation is sufficient!

- Potential improvement of τ-decay data from BELLE II – only 2π and 4π need to be analyzed!