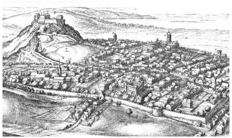
SHORT-DISTANCE CONSTRAINTS FOR HLBL



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Vetenskapsrådet

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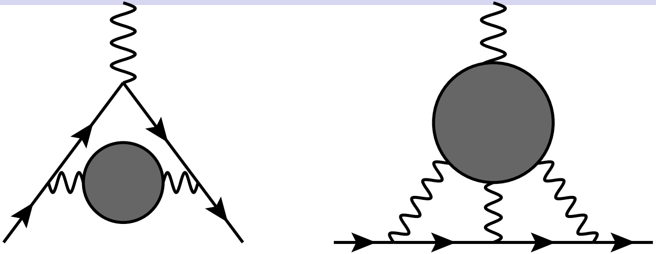
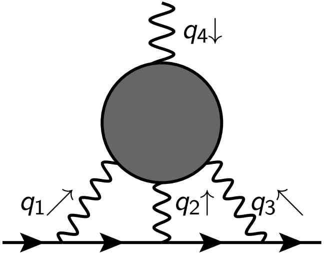
Hadronic contributions



LO-HVP HLbL

- Muon and photon lines, representative diagrams
- The blobs are hadronic contributions
- Higher order contributions of both types: known accurately enough
- $a_\mu^{HVP} = 6845(40)\ 10^{-11}$ (LO+NLO+NNLO)
- White paper numbers; HVP is not the subject of this talk but of others
- $a_\mu^{HLbL} = 92(18)\ 10^{-11}$ (LO+NLO)
- Some improvements since white paper, in particular a better lattice calculation, more work on short-distance and other parts

HLbL: the main object to calculate



- Muon line and photons: well known
- The blob: fill in with hadrons/QCD
- Trouble: low and high energy very mixed
- q_4 always at zero
- Double counting needs to be avoided: hadron exchanges versus quarks

Contributions

- Numbers from white paper
- "Long distance": under good control
  - Dispersive method: Berne group around G. Colangelo
  - π^0 (and η, η') pole: $93.8(4.0)\ 10^{-11}$
  - Pion and kaon box (pure): $-16.4(2)\ 10^{-11}$
  - $\pi\pi$-rescattering (include scalars below 1 GeV):$-8(1)\ 10^{-11}$
- Charm (beauty, top) loop: $3(1)\ 10^{-11}$
- "Short and medium distance"
  - Scalar ($\geq$ 1 GeV): $-1(3)\ 10^{-11}$
  - Axial vector: $6(6)\ 10^{-11}$
  - Short-distance: $15(10)\ 10^{-11}$
- Clearly the last item(s) needs improvement
- A guesstimate of the overlap went into this

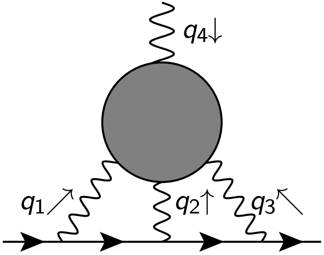
What are we up against?

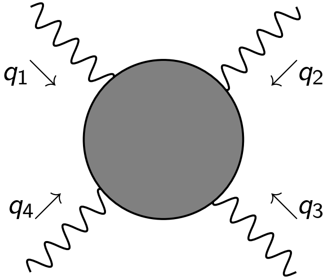
- Lots of resonances in light meson table from PDG (2022) 1-1.5 GeV

$\phi(1020)$	$h_1(1170)$	$b_1(1235)$	$a_1(1260)$	$f_2(1270)$
$f_1(1285)$	$\eta(1295)$	$\pi(1300)$	$a_2(1320)$	$f_0(1370)$
$\pi_1(1400)$	$\eta(1405)$	$h_1(1415)$	$f_1(1420)$	$\omega(1420)$
$a_0(1450)$	$\rho(1450)$	$\eta(1475)$	$f_0(1500)$	$\ldots$

- couplings to on-shell photons known for very few
- off-shell photons ($q_i^2 \neq 0$) even less
- Clearly we will need to go beyond data as it is now
- More data will always be useful as a constraint and we will still need improvement around 1 GeV

Definitions





$$= \Pi^{\mu\nu\lambda\sigma}(q_1, q_2, q_3)$$

- Actually we really need $\left.\dfrac{\delta\Pi^{\mu\nu\lambda\sigma}(q_1, q_2, q_3)}{\delta q_{4\rho}}\right|_{q_4=0}$
- Never purely short-distance: q_4 at zero
- $q_i^2 = -Q_i^2$

Definitions

$$\Pi^{\mu\nu\lambda\sigma} = -i \int d^4x d^4y d^4z e^{-i(q_1\cdot x+q_2\cdot y+q_3\cdot z)} \left\langle T\left(j^\mu(x) j^\nu(y) j^\lambda(z) j^\sigma(0)\right)\right\rangle$$

Use the Colangelo et al. conventions (mainly)

$$\Pi^{\mu\nu\lambda\sigma} = \sum_{i=1}^{54} T_i^{\mu\nu\lambda\sigma}\hat{\Pi}_i, \qquad \left.\frac{\delta\Pi^{\mu\nu\lambda\sigma}}{\delta q_{4\rho}}\right|_{q_4=0} = \sum_{i=1}^{54} \left.\frac{\delta T_i^{\mu\nu\lambda\sigma}}{\delta q_{4\rho}}\hat{\Pi}_i\right|_{q_4=0}$$

$$a_\mu = \frac{2\alpha^3}{3\pi^2}\int_0^\infty dQ_1 dQ_2 Q_1^3 Q_2^3 \int_{-1}^1 d\tau \sqrt{1-\tau^2}\sum_{i=1}^{12}\hat{T}_i\left(Q_1,Q_2,\tau\right)\overline{\Pi}_i\left(Q_1,Q_2,\tau\right)$$

$$Q_3^2 = Q_1^2 + Q_2^2 + 2Q_1Q_2\tau$$

- The 12 $\overline{\Pi}_i$ from $\hat{\Pi}_i$ for $i = 1, 4, 7, 17, 39, 54$
- The integral can be parametrized in many ways

Short-distance constraints

- There are very many different types of short-distance constraints (SDC)
- Those on hadronic properties
  - Couplings of hadrons to off-shell photons
  - Pure OPE (e.g. $\pi^0 \to \gamma^*\gamma^*$ at $Q_1^2 = Q_2^2$)
  - Brodsky-Lepage-Radyushkin-$\cdots$:
    - the overall power is very well predicted (counting rules)
    - the coefficient follows from the asymptotic wave functions and possible α_S corrections: larger uncertainty
  - Light-cone QCD sum rules
  - $\cdots$
- This type is mainly used in HLbL to put constraints on the form-factors in the individual contributions

Short-distance constraints

- On the full four-point function (4, 3 or 2 currents close)
- SD4: $\Pi^{\mu\nu\lambda\sigma}(q_1, q_2, q_3)$ all $Q_i \cdot Q_j$ large
  the standard OPE
- SD3: $\left.\frac{\delta\Pi^{\mu\nu\lambda\sigma}(q_1, q_2, q_3)}{\delta q_{4\rho}}\right|_{q_4=0}$ with $Q_1^2 \sim Q_2^2 \sim Q_3^2 \gg \Lambda_{QCD}^2$ JB,LL,NHT,ARS 19-21
- SD2: $\left.\frac{\delta\Pi^{\mu\nu\lambda\sigma}(q_1, q_2, q_3)}{\delta q_{4\rho}}\right|_{q_4=0}$ and $Q_1^2 \sim Q_2^2 \gg Q_3^2 (\gg \Lambda_{QCD}^2)$
  Melnikov-Vainshtein 03, JB, NHT, ARS 21-23
- ...
- Collaborators: Nils Hermansson-Truedsson, Laetitia Laub, Antonio Rodríguez-Sánchez
-
  - Phys.Lett. B798 (2019) 134994 [arxiv:1908.03331]: principle and next nonperturbative term
  - JHEP 10 (2020) 203 [arxiv:2008.13487]: proper description and up to NNLO nonperturbative terms
  - JHEP 04 (2021) 240 [arxiv:2101.09169]: perturbative correction

Implementation

Some general comments:

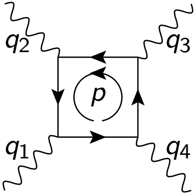
- Brodsky-Lepage constraints together with full n-point functions SDC often require an infinite number of resonances for obeying both
  JB,Gamiz,Lipartia,Prades 2003
- Have a model that fully implements SDC and then integrate everywhere
- Have a good description in the intermediate domain, use QCD expressions to do the short-distance part of the integration
- Varying the transitions can help with error estimates
- Make sure to avoid double counting: splitting the integration over different regions is one way to avoid this

Short-distance: first attempt

$$\Pi^{\mu\nu\lambda\sigma} = -i \int d^4x d^4y d^4z e^{-i(q_1\cdot x + q_2\cdot y + q_3\cdot z)} \left\langle T\left(j^\mu(x) j^\nu(y) j^\lambda(z) j^\sigma(0)\right)\right\rangle$$

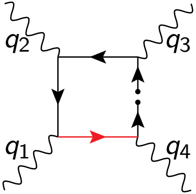
- Usual OPE: x, y, z all small (4 currents close)
- First term in the expansion is the massless quark loop
  no problem with $\partial/\partial q_4^\rho$ and $q_4 \to 0$

  p in loop $\Rightarrow$ no singular propagators:



- Next term problems: no loop momentum;

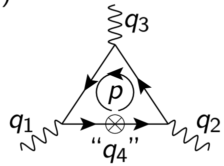
  $q_4 \to 0$ propagator diverges:



- Due to the symmetries: $1/q_4^2$ essentially unavoidable

Short-distance: correctly

- Similar problem in QCD sum rules for electromagnetic radii and magnetic moments
- Ioffe, Smilga, Balitsky, Yung, 1983
- For the q_4-leg use a constant background field and do the OPE in the presence of that constant background field
- Use radial gauge: $A_4^\lambda(w) = \frac{1}{2} w_\mu F^{\mu\lambda}$
  whole calculation is immediately with $q_4 = 0$.
- First term is exactly the massless quark loop
  (quark masses: next order)



- 3 quark currents close

Quark loop results: massless case

Known fully analyticallly

$$\hat{\Pi}_m^{QL0} = \frac{N_c\, e_q^4}{\pi^2} \sum_{i,j,k,n} \left[c_{i,j,k}^{(m,n)} + f_{i,j,k}^{(m,n)} F + g_{i,j,k}^{(m,n)} \log\left(\frac{Q_2^2}{Q_3^2}\right) + h_{i,j,k}^{(m,n)} \log\left(\frac{Q_1^2}{Q_2^2}\right) \right]$$
$$\times\, \lambda^{-n}\, Q_1^{2i}\, Q_2^{2j}\, Q_3^{2k}$$

$$\lambda = \left(Q_1^2 - Q_2^2 - Q_3^2\right)^2 - 4 Q_2^2 Q_3^2$$
$$F(Q_1^2, Q_2^2, Q_3^2) \equiv (4\pi)^2 i \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 (k-q_1)^2 (k-q_1-q_2)^2}$$

- The integral is known in terms of dilogarithm or Clausen's function
- Coefficients see JHEP 10 (2020) 203 [arxiv:2008.13487]

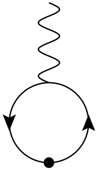
Short-distance: next term(s)

- Do the usual QCD sum rule expansion in terms of vacuum condensates
- Do this to first order in external field
- Operators for one quark flavour:

$$
\begin{aligned}
S_{1,\,\mu\nu} &\equiv e\, e_q F_{\mu\nu}\ , \\
S_{2,\,\mu\nu} &\equiv \bar{q}\sigma_{\mu\nu}q \\
S_{3,\,\mu\nu} &\equiv i\ \bar{q}\, G_{\mu\nu} q\ , \\
S_{4,\,\mu\nu} &\equiv i\, \bar{q}\, \bar{G}_{\mu\nu}\gamma_5 q\ , \\
S_{5,\,\mu\nu} &\equiv \bar{q}q\ e\, e_q F_{\mu\nu}\ , \\
S_{6,\,\mu\nu} &\equiv \frac{\alpha_s}{\pi}\, G_a^{\alpha\beta} G^a_{\alpha\beta}\ e\, e_q F_{\mu\nu}\ , \\
S_{7,\,\mu\nu} &\equiv \bar{q}(G_{\mu\lambda} D_\nu + D_\nu G_{\mu\lambda})\gamma^\lambda q - (\mu \leftrightarrow \nu)\ , \\
S_{\{8\},\,\mu\nu} &\equiv \alpha_s\, (\bar{q}\,\Gamma\, q\ \bar{q}\Gamma q)_{\mu\nu}\ .
\end{aligned}
$$

- Four quark operators: there are 12 independent ones
- Massless quark loop is the first operator

Short-distance: operator mixing

- These operators mix
- Alternatively: the perturbative values of these operators contain infrared divergences
- Mixing S_1 and S_2 is via 
- So an infrared well defined opertor is defined via

$$Q^0_{2,\mu\nu} = S_{2,\,\mu\nu} + S_{1,\,\mu\nu}\frac{N_c\, m_q}{4\pi^2}\left(-\frac{1}{\hat{\epsilon}} + \log(m_q^2)\right) + \cdots\,,$$

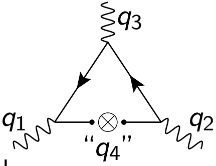
- Do this for all operators JHEP 10 (2020) 203 [arxiv:2008.13487]

Values of the condensates

- JHEP 10 (2020) 203 [arxiv:2008.13487]
- $\langle 0|Q_i^{\mu\nu}|\gamma(q_4)\rangle \equiv e e_q X_i \langle 0|F^{\mu\nu}|\gamma(q_4)\rangle$
- Get values for all X_i needed
- X_2: lattice QCD about 40 MeV (quark susceptibility) and VMD estimate Bali et al., arXiv:2004.08778
- $X_{3,4}$ VMD estimate
- X_5, X_6: from the usual quark and gluon condensates
- X_7: dimensional guess from the gluon condensate
- X_8: factorization and quark susceptibility

Results for the next terms

- Result derived from:



- $N_c = 3$ and one quark

$$\hat{\Pi}_1 = m_q X_q e_q^4 \frac{-4(Q_1^2 + Q_2^2 - Q_3^2)}{Q_1^2 Q_2^2 Q_3^4} \qquad \hat{\Pi}_7 = 0$$

$$\hat{\Pi}_4 = m_q X_q e_q^4 \frac{8}{Q_1^2 Q_2^2 Q_3^2} \qquad \hat{\Pi}_{17} = m_q X_q e_q^4 \frac{8}{Q_1^2 Q_2^2 Q_3^4}$$

$$\hat{\Pi}_{54} = m_q X_q e_q^4 \frac{-4(Q_1^2 - Q_2^2)}{Q_1^4 Q_2^4 Q_3^2} \qquad \hat{\Pi}_{39} = 0$$

- Formulas for all other operators are in JHEP 10 (2020) 203 [arxiv:2008.13487]

Short-distance: nonperturbative numerical results

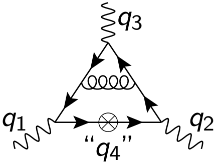
Order	Contribution	$Q_{\min} = 1\,\mathrm{GeV}$	$Q_{\min} = 2\,\mathrm{GeV}$
$1/Q^2_{\min}$	quark loop	$1.73 \cdot 10^{-10}$	$4.35 \cdot 10^{-11}$
$1/Q^4_{\min}$	quark loop, m_q^2	$-5.7 \cdot 10^{-14}$	$-3.6 \cdot 10^{-15}$
	$X_{2,m}$	$-1.2 \cdot 10^{-12}$	$-7.3 \cdot 10^{-14}$
$1/Q^6_{\min}$	X_{2,m^3}	$6.4 \cdot 10^{-15}$	$1.0 \cdot 10^{-16}$
	X_3	$-3.0 \cdot 10^{-14}$	$-4.7 \cdot 10^{-16}$
	X_4	$3.3 \cdot 10^{-14}$	$5.3 \cdot 10^{-16}$
	X_5	$-1.8 \cdot 10^{-13}$	$-2.8 \cdot 10^{-15}$
	X_6	$1.3 \cdot 10^{-13}$	$2.0 \cdot 10^{-15}$
	X_7	$9.2 \cdot 10^{-13}$	$1.5 \cdot 10^{-14}$
	$X_{8,1}$	$3.0 \cdot 10^{-13}$	$4.7 \cdot 10^{-15}$
	$X_{8,2}$	$-1.3 \cdot 10^{-13}$	$-2.0 \cdot 10^{-15}$

- $Q_1, Q_2, Q_3 \geq Q_{\min}$
- Suppression by small quark masses or small condensates
- Nonperturbative short-distance corrections are small

Short-distance: $1/Q^2_{\rm min}$

- Can we understand scaling with $Q_{\rm min}$?
- $a_\mu = \frac{2\alpha^3}{3\pi^2}\int_0^\infty dQ_1 dQ_2 Q_1^3 Q_2^3 \int_{-1}^1 d\tau\sqrt{1-\tau^2}\sum_{i=1,12}\hat{T}_i\overline{\Pi}_i$
- Do $Q_i \to \xi Q_i$
- overall factor goes as ξ^8
- Quark loop has no scale thus $\hat{\Pi}_i$ scale with their dimension
  $\hat{\Pi}_1, \hat{\Pi}_4 \sim \xi^{-4}, \qquad \hat{\Pi}_7, \hat{\Pi}_{17}, \hat{\Pi}_{39}, \hat{\Pi}_{54} \sim \xi^{-6}$
- $\Longrightarrow \overline{\Pi}_{1,\dots,4} \sim \xi^{-4} \qquad \overline{\Pi}_{5,\dots,12} \sim \xi^{-6}$
- Expand the T_i for $Q_i \gg m_\mu$: $T_1 \sim m_\mu^4$, $T_{i\neq 1} \sim m_\mu^2$
  $T_1 \sim \xi^{-8}$, $T_{2,3,4} \sim \xi^{-6}$, $T_{5,\dots,12} \sim \xi^{-4}$
- Put all together: quark loop scales as $a_\mu^{\rm SD\ ql} \sim \xi^{-2}$
- $m_q X_q$ adds an overall factor $\Longrightarrow a_\mu^{{\rm SD}X_q} \sim \xi^{-4}$
- And similarly for the others

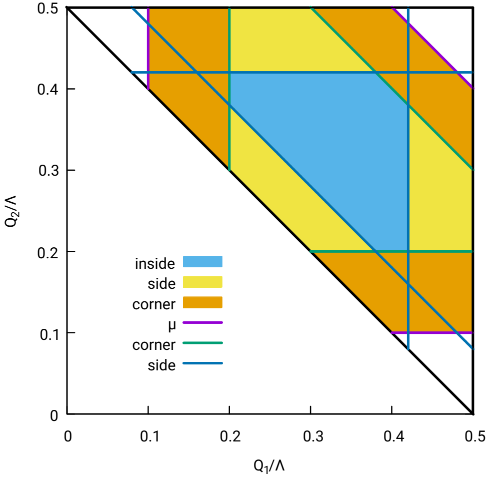
Perturbative corrections

- Representative diagram: 

- About -10%
- All integrals are known
- Infrared and UV divergences in individual diagrams
- Dimensional regularization: $d = 4 - 2\epsilon$
- All $1/\epsilon^3, 1/\epsilon^2, 1/\epsilon$ cancel
- Several independent calculations that agree
- Find some typos in integral papers (I hate signs)

Perturbative corrections

- Use method of master integrals: disadvantage: large numerical cancellations between integrals
- Especially near $\lambda = Q_1^4 + Q_2^4 + Q_3^4 - 2Q_1^2Q_2^2 - 2Q_2^2Q_3^2 - 2Q_3^2Q_1^2 = 0$



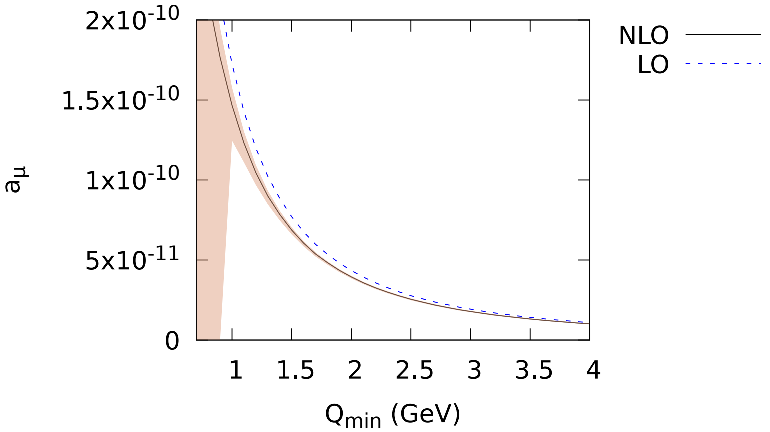
- $Q_1 + Q_2 + Q_3 = \Lambda$
- $Q_1, Q_2, Q_3 \geq \mu = Q_{\text{min}}$
- Need to expand on sides and corners
- Up to $1/\lambda^4$ occurs
- Analytical expressions for all regions available
- Simple for symmetric point and corners

Perturbative corrections: numerics

	Quark loop	Gluon corrections ($\frac{\alpha_s}{\pi}$ units)
$\bar{\Pi}_1$	0.0084	-0.0077
$\bar{\Pi}_2$	13.28	-12.30
$\bar{\Pi}_3$	0.78	-0.87
$\bar{\Pi}_4$	-2.25	0.62
$\bar{\Pi}_5$	0.00	0.20
$\bar{\Pi}_6$	2.34	-1.43
$\bar{\Pi}_7$	-0.097	0.056
$\bar{\Pi}_8$	0.035	0.41
$\bar{\Pi}_9$	0.623	-0.87
$\bar{\Pi}_{10}$	1.72	-1.61
$\bar{\Pi}_{11}$	0.696	-1.04
$\bar{\Pi}_{12}$	0.165	-0.16
Total	17.3	-17.0

- a_μ from integration from $Q_{\min} = 1\,\mathrm{GeV}$ in 10^{-11} units.
- Naive scaling to other Q_{min} applies (up to $\alpha_S(Q_{\min})$)
- $a_\mu^{\text{HLbL SD gluonic}} = -1.7\ 10^{-11}$
- $Q_{\min} = 1$ GeV, $\alpha_S = 0.33$
- Main uncertainty: how to handle α_S
- No sign that it is very large (about -10%)

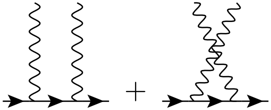
Perturbative corrections: numerics



- Uncertainty estimated by $\alpha_S(\mu)$ with $Q_{\text{min}}/\sqrt{2} \le \mu \le \sqrt{2}Q_{\text{min}}$
- Running $\alpha_S(M_Z)$ at 5 loops to $\alpha_S(m_\tau)$ or $\alpha_S(\mu)$

MV short-distance

- K. Melnikov, A. Vainshtein, Phys. Rev. **D70** (2004) 113006. [hep-ph/0312226]
- take $Q_1^2 \approx Q_2^2 \gg Q_3^2$: Leading term in OPE of two vector currents is proportional to axial current
- $\Pi^{\rho\nu\alpha\beta} \propto \frac{P^\rho}{Q_1^2} \langle 0 | T \left(J_A^\nu J_V^\alpha J_V^\beta \right) |0\rangle$ J_A comes from



- Coefficient of J_A has α_S and higher order OPE corrections
- AVV triangle anomaly: in particular nonrenormalization theorems
  - fully for longitudinal ($\overline{\Pi}_i, i = 1, 2, 3$)
  - perturbative for the others
- Recent discusions, implementations,. . . : M. Knecht, JHEP 08 (2020) 056 [2005.09929], P. Masjuan, P. Roig and P. Sanchez-Puertas, J. Phys. G **49** (2022) no.1, 015002 [2005.11761] Colangelo et al, JHEP 03 (2020) 101 [1910.13432], Eur.Phys.J.C 81 (2021) 8, 702 [2106.13222], Melnikov and Vainshtein, [1911.05874], L. Cappiello et al., Phys. Rev. D **102** (2020) no.1, 016009 [1912.02779], J. Leutgeb and A. Rebhan, Phys. Rev. D **104** (2021) 094017 [2108.12345] J. Lüdtke and M. Procura, Eur. Phys. J. C **80** (2020) no.12, 1108 [2006.00007]

MV Short-distance

- Only a proper prediction for $\hat{\Pi}_1$ Colangelo et al, JHEP 03 (2020) 101 [1910.13432], Eur.Phys.J.C 81 (2021) 8, 702 [2106.13222]
- $\overline{Q}_3 = Q_1 + Q_2$, $Q_3 \ll Q_1, Q_2$
- $\hat{\Pi}_1 = \frac{e_q^4}{\pi^2} \frac{-12}{Q_3^2 \overline{Q}_3^2} \left(1 - \frac{\alpha_S}{\pi}\right)$
- The quark loop and its gluonic correction reproduce this
- JB,NHT,ARS in progress: calculate the corrections: gluonic and OPE
- Next term in OPE has a number of features (from our corner expansions):
  - $\log \frac{Q_3^2}{\overline{Q}_3^2}$ show up already at $\alpha_S = 0$ (we have now understood how this comes about: operator mixing again)
  - For some of the terms the gluonic corrections dominate
- Should allow to resum some of the large corrections of the corners

Starting point

- We define:
$$\Pi^{\mu_1\mu_2} = \frac{i}{e^2}\int d^4x_1 \int d^4x_2 e^{-i(q_1x_1+q_2x_2)}\langle 0|T(J^{\mu_1}(x_1)J^{\mu_2}(x_2))|\gamma(q_3)\gamma(q_4)\rangle$$
- allows to get $\Pi^{\mu_1\mu_2} = \epsilon_{\mu_3}\epsilon_{\nu_4}\Pi^{\mu_1\mu_2\mu_3\nu_4}$.
- and $\partial/\partial q_{4,\mu_4}$ as well
- OPE on the two currents:

$$\Pi^{\mu_1\mu_2}_{\bar{q}q} \approx -\frac{e_q^2}{e^2}\frac{-\hat{q}_\alpha}{\hat{q}^2}\langle 0|\bar{q}(0)[\gamma^{\mu_1}\gamma^\alpha\gamma^{\mu_2} - \gamma^{\mu_2}\gamma^\alpha\gamma^{\mu_1}]q(0))|\gamma(q_3)\gamma(q_4)\rangle$$
$$-\frac{ie_q^2}{e^2\hat{q}^2}(g_{\mu_1\delta}g_{\mu_2\beta} + g_{\mu_2\delta}g_{\mu_1\beta} - g_{\mu_1\mu_2}g_{\delta\beta})\left(g^{\alpha\delta} - 2\frac{\hat{q}^\delta\hat{q}^\alpha}{\hat{q}^2}\right)\langle 0|\bar{q}(0)(\overrightarrow{D}^\alpha - \overleftarrow{D}^\alpha)\gamma^\beta q(0))|\gamma(q_3)\gamma(q_4)\rangle$$

- $\hat{q} = (q_1 - q_2)/2$ is large, q_3 is small.
- these are the $D = 3$ and $D = 4$ naive extrapolations
- Need more $D = 4$ terms: various combinations of $F_{\mu\nu}F_{\alpha\beta}$

What is needed

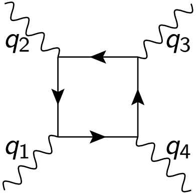
- $\lim_{q_4\to 0} \dfrac{\partial}{\partial q_{4,\mu 4}} \langle 0|O_i|\gamma(q_3)\gamma(q_4)\rangle$
- $D = 3$: $\hat{q}_\alpha \bar{q}(0)[\gamma^{\mu_1}\gamma^\alpha\gamma^{\mu_2} - \gamma^{\mu_2}\gamma^\alpha\gamma^{\mu_1}]q(0)$
- $D = 4$:
  - $O_1^{\alpha\beta} = \bar{q}(0)(\overrightarrow{D}^\alpha - \overleftarrow{D}^\alpha)\gamma^\beta q(0))\bar{q}(0)$
  - $O_2 = F^{\alpha\gamma}F^\beta{}_\gamma$
  - $O_3 = F^{\gamma\delta}F_{\gamma\delta}g^{\alpha\beta}$
  - Some more (see below)
- Starting expression at large scale (For $\Pi^{\mu_1\mu_2}$ take $\gamma(q_3)\gamma(q_4)$ matrix element):

$$
\begin{aligned}
O^{\mu_1\mu_2} =& \frac{8}{\hat{q}^2}F^{\mu_1\gamma}F^{\mu_2\delta}\hat{q}_\gamma\hat{q}_\delta - \frac{16}{3\hat{q}^6}\hat{q}^{\mu_1}\hat{q}^{\mu_2}F^{\alpha\gamma}F_{\alpha\delta}\hat{q}_\gamma\hat{q}^\delta \\
&+ \left(-\frac{32}{3} + \frac{16}{3}B(\hat{q}^2)\right)\left(\frac{1}{\hat{q}^2}F^{\mu_1\gamma}F^{\mu_2}{}_\alpha + \frac{1}{\hat{q}^4}F^{\mu_1\alpha}F_{\alpha\beta}\hat{q}^{\mu_2}\hat{q}^\beta + \frac{1}{\hat{q}^4}F^{\mu_2\alpha}F_{\alpha\beta}\hat{q}^{\mu_1}\hat{q}^\beta\right) \\
&+ \left(-\frac{8}{3} + \frac{8}{3}B(\hat{q}^2)\right)\left(\frac{2}{\hat{q}^4}F^{\alpha\gamma}F_{\alpha\delta}\hat{q}^\delta\hat{q}_\gamma g^{\mu_1\mu_2} + \frac{1}{\hat{q}^4}F^{\alpha\beta}F_{\alpha\beta}\hat{q}^{\mu_1}\hat{q}^{\mu_2} - \frac{1}{\hat{q}^2}F^{\alpha\beta}F_{\alpha\beta}g^{\mu_1\mu_2}\right)
\end{aligned}
$$

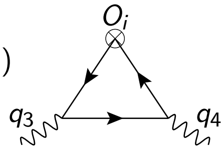
- with $B(\hat{q}^2) = \frac{2}{\bar{\epsilon}} + 2 - \log(\hat{q}^2)$

Next steps

- Comes from limit $q_3, q_4 \to 0$ using background gauge of:



- Now we have the operators at $\mu = \hat{q}$
- Use RGE to run down to $\mu = q_3$
- O_1 and $F_{\mu\nu}F_{\alpha\beta}$ mix via $(q_3, q_4 \to 0)$



- Take matrix elements from the $D = 3$ and $D = 4$ quark operators to e^2 and $F_{\mu\nu}F_{\alpha\beta}$ to e^0 (now not $q_3 \to 0$)
- Alternatively calculate the matrix elements at $\mu = \hat{q}$
- Infinities cancel as they should and logarithms become $\log(\hat{q}/q_3)$

Next steps

- We now have $\lim_{q_4\to 0}(\partial/\partial q_{4,\mu_4})\Pi^{\mu_1\mu_2\mu_3\nu_4}$ to two powers in $\hat{q}$
- Note gauge invariance for q_3 exact
- $q_4 \to 0$ gauge invariance is antisymmetry in $\nu_4\mu_4$
- BUT gauge invariance for q_1, q_2 only perturbatively in $\hat{q}$
- Consequence: be careful when using gauge equivalent expressions
- In particular when using projectors to get quantities without Lorentz indices (our intermediate $\widetilde{\Pi}$ or the $\hat{\Pi}_i$) need to use the projectors with lowest powers of $\hat{q}$ possible

Results

$\overline{Q}_3 = Q_1 + Q_2,\ \overline{Q}_2 = Q_1 + Q_3$

Corner with q_3 small:

$$\hat{\Pi}_1 = -\frac{4}{\pi^2\, Q_3^2\, \overline{Q}_3^2} + \mathcal{O}\left(\overline{Q}_3^{-4}\right)$$
$$\hat{\Pi}_4 = -\frac{16}{3\pi^2\, \overline{Q}_3^4} + \mathcal{O}\left(\overline{Q}_3^{-5}\right)$$
$$\hat{\Pi}_7 = \mathcal{O}\left(\overline{Q}_3^{-6}\right)$$
$$\hat{\Pi}_{17} = \frac{16}{3\pi^2\, Q_3^2\, \overline{Q}_3^4} + \mathcal{O}\left(\overline{Q}_3^{-5}\right)$$
$$\hat{\Pi}_{39} = \frac{16}{3\pi^2\, Q_3^2\, \overline{Q}_3^4} + \mathcal{O}\left(\overline{Q}_3^{-5}\right)$$
$$\hat{\Pi}_{54} = \mathcal{O}\left(\overline{Q}_3^{-5}\right)$$

Corner with q_2 small

$$\hat{\Pi}_1 = -\frac{16\left(5 + 6\log 2\,\frac{Q_2}{\overline{Q}_2}\right)}{9\pi^2\, \overline{Q}_2^4} + \mathcal{O}\left(\overline{Q}_2^{-5}\right)$$
$$\hat{\Pi}_4 = -\frac{4}{3\pi^2\, Q_2^2\, \overline{Q}_2^2} + \mathcal{O}\left(\overline{Q}_2^{-3}\right)$$
$$\hat{\Pi}_7 = -\frac{16}{3\pi^2\, Q_2^2\, \overline{Q}_2^4} + \mathcal{O}\left(\overline{Q}_2^{-5}\right)$$
$$\hat{\Pi}_{17} = \mathcal{O}\left(\overline{Q}_2^{-5}\right)$$
$$\hat{\Pi}_{39} = \frac{16}{3\pi^2\, Q_2^2\, \overline{Q}_2^4} + \mathcal{O}\left(\overline{Q}_2^{-5}\right)$$
$$\hat{\Pi}_{54} = -\frac{8}{3\pi^2\, Q_2^2\, \overline{Q}_2^4} + \mathcal{O}\left(\overline{Q}_2^{-5}\right)$$

Results

$$\overline{Q}_1 = Q_2 + Q_3$$

Corner with q_1 small

$$\hat{\Pi}_1 = -\frac{16\left(5 + 6\,\log 2\,\frac{Q_1}{\overline{Q}_1}\right)}{9\pi^2\,\overline{Q}_1^4} + \mathcal{O}\left(\overline{Q}_1^{-5}\right)$$

$$\hat{\Pi}_4 = -\frac{4}{3\pi^2\,Q_1^2\,\overline{Q}_1^2} + \mathcal{O}\left(\overline{Q}_1^{-3}\right)$$

$$\hat{\Pi}_7 = \mathcal{O}\left(\overline{Q}_1^{-4}\right)$$

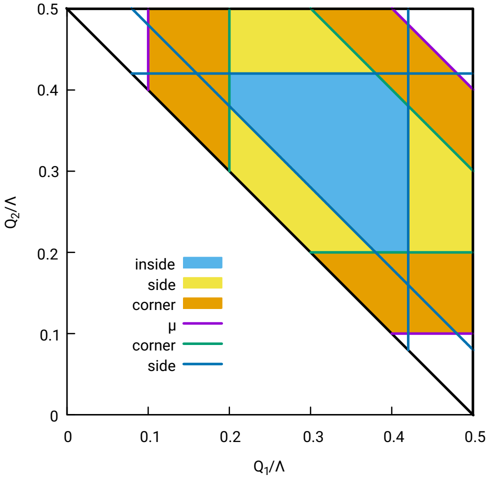
$$\hat{\Pi}_{17} = \mathcal{O}\left(\overline{Q}_1^{-5}\right)$$

$$\hat{\Pi}_{39} = \frac{16}{3\pi^2\,Q_1^2\,\overline{Q}_1^4} + \mathcal{O}\left(\overline{Q}_1^{-5}\right)$$

$$\hat{\Pi}_{54} = \frac{8}{3\pi^2\,Q_1^2\,\overline{Q}_1^4} + \mathcal{O}\left(\overline{Q}_1^{-5}\right)$$

- Agrees with quark loop expansion
- Gluon corrections are being worked on
- Can try nonperturbative matrix elements of the quark operators
- Higher orders can depend on $\delta_{23} = Q_2 - Q_3, \ldots$

Numerics



- Often asked question: how much of the result is from the MV limit (or the corners)
- Answer: depends on where we draw the border in $Q_3/(Q_1+Q_2)$,...
- Inside, sides and corner can be of similar size by choosing boundaries (done with quark loop expression expanded to higher orders)

Conclusions

- We have shown that the massless quark loop really is the first term of a proper OPE expansion for the HLbL
- We have shown how to properly go to higher orders
- We have calculated the next two terms in the OPE
    - NLO: suppressed by quark masses and a small X_q
    - NNLO: large number of induced condensates but all small
    - Numerically not relevant at the present precision
- Gluonic corrections about -10%
- We have worked out the NLO term in the corners (MV expansion), still some work in progress
- Why do this: matching of the sum over hadronic contributions to the expected short distance domain
- Finding the onset of the asymptotic domain

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