$\eta^{(\prime)}$ Pole Contribution with Dispersive Methods

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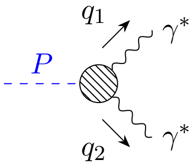
η and η' transition form factors

- Pseudoscalar ($P = \pi^0,\, \eta,\, \eta'$) transition form factors defined by

$$i \int \mathrm{d}^4x\, e^{iq_1\cdot x} \langle 0 \big| T\{j_\mu(x) j_\nu(0)\} \big| P(q_1+q_2) \rangle = \epsilon_{\mu\nu\alpha\beta} q_1^\alpha q_2^\beta F_{P\gamma^*\gamma^*}(q_1^2, q_2^2)$$



- Normalization related to di-photon decays governed by chiral anomaly:

$$\Gamma(P\to\gamma\gamma) = \frac{\pi\alpha_{\mathrm{em}}^2 M_P^3}{4} \big| F_{P\gamma^*\gamma^*}(0,0) \big|^2$$

- For pion: low-energy theorem predicts its value

Bell, Jackiw 1969; Adler 1969; Bardeen 1969

- For η and η': complicated by η–η' mixing

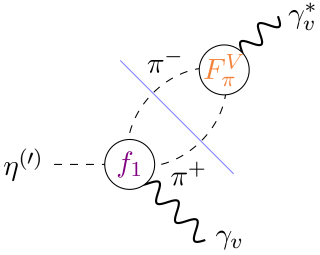
Feldmann, Kroll, Stech 1998–2000;

Escribano, Gonzàlez-Solís, Masjuan, Sánchez-Puertas 2016

Transition form factor $\eta^{(\prime)} \to \gamma\gamma^*$

Isospin decomposition: $F^{\rm disp}_{\eta^{(\prime)}\gamma^*\gamma^*}(q_1^2, q_2^2) = F^{\eta^{(\prime)}}_{vv}(q_1^2, q_2^2) + F^{\eta^{(\prime)}}_{ss}(q_1^2, q_2^2)$

Reconstruction from the lowest-lying hadronic states:



Isovector part:

- largest contribution: $\pi^+\pi^-$ intermediate state
- Dispersively combine data on $\eta^{(\prime)} \to \pi^+\pi^-\gamma$ and $e^+e^- \to \pi^+\pi^-$

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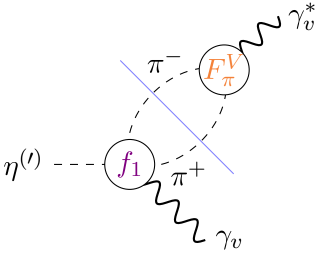

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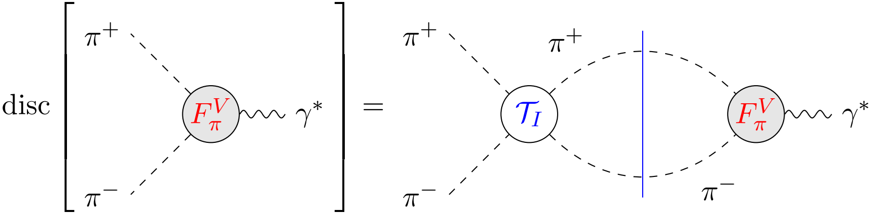
Isoscalar part:

- Dominated by narrow resonances:
  ω & ϕ Hanhart et al. 2013
- Employ VMD and fix couplings by exp. det. decay widths for
  - $\omega \to \eta\gamma$
  - $\eta' \to \omega\gamma$
  - $\phi \to \eta^{(\prime)}\gamma$
  - $\omega,\ \phi \to e^+e^-$

η: strong cancellation between ω and ϕ

η': isoscalar contribution more significant than for η (*e.g.* in norm $\sim 20\,\%$)

Pion vector form factor



- solution of discontinuity equation: Omnès 1958

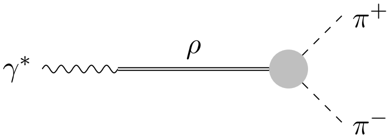
$$F_\pi^V(s) = R(s)\Omega(s)\ , \quad \Omega(s) = \exp\left(\frac{s}{\pi}\int_{4M_\pi^2}^{\infty}\frac{\delta_1^1(\omega)}{\omega(\omega-s)}\mathrm{d}\omega\right)$$

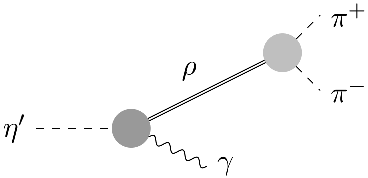
- δ_1^1: $I=1$ $\pi\pi$ P-wave phase shift, $R(s)=(1+\alpha_\pi s)$

$$F_\pi^{V,e^+e^-}(s) = \left(1+\epsilon_{\rho\omega}\frac{s}{M_\omega^2 - s - iM_\omega\Gamma_\omega}\right)F_\pi^V(s)$$

From $\eta' \to \pi^+\pi^-\gamma$ to $\eta' \to \gamma\gamma^*$

- Include isospin-breaking ρ–ω-mixing effect
  - Effect enhanced by presence of resonance propagator
  - present in both $\eta' \to \pi^+\pi^-\gamma$ and $e^+e^- \to \pi^+\pi^-$
  - cannot consider 2π channel in isolation
- Employ multichannel formalism (based on Hanhart 2012) to include both the 2π and 3π discontinuities
- Inclusion in both the isovector and isoscalar channels of TFF
  - double discontinuity (in 2π and 3π) of TFF vanishes





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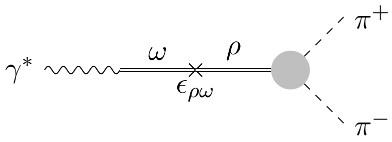
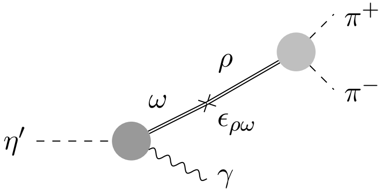
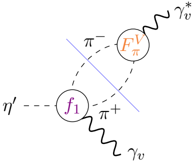




$\eta' \to \pi^+\pi^-\gamma$ amplitude

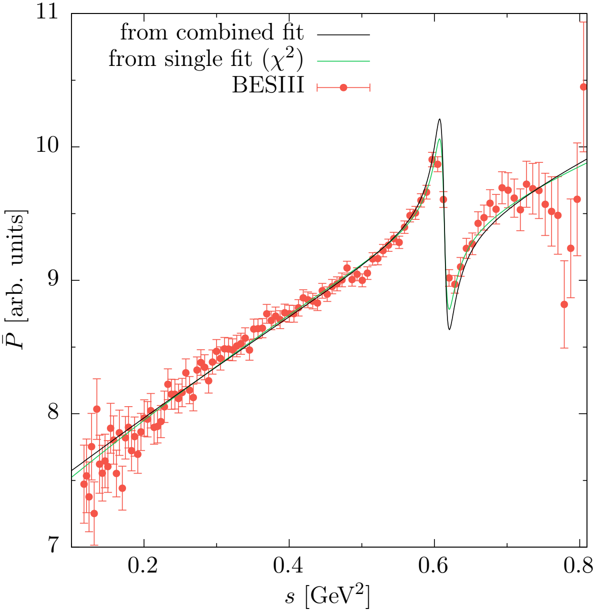
- parameterized by F_π^V due to universality of $\pi\pi$ final-state interactions
- ρ–ω-mixing strength proportional to the one found in $e^+e^- \to \pi^+\pi^-$



$$\frac{\mathrm{d}\Gamma(\eta' \to \pi^+\pi^-\gamma)}{\mathrm{d}s} = 16\pi\alpha_{\text{em}}\Gamma_0|F_\pi^V(s)|^2\left|P(s) + \frac{g_{\eta'\omega\gamma}\epsilon_{\rho\omega}}{g_{\omega\gamma}}\frac{1}{M_\omega^2 - s - iM_\omega\Gamma_\omega}\right|^2,$$

- where $s = M_{\pi\pi}^2$, Γ_0: phase space, and $P(s) = A(1 + \alpha s + \beta s^2)$
  Stollenwerk et al. 2012; Kubis, Plenter 2015
- Fit parameters: (among others) $\epsilon_{\rho\omega}$ and 3 in quad. polynomial $P(s)$

Combined fit to $\eta' \to \pi^+\pi^-\gamma$ and $e^+e^- \to \pi^+\pi^-$



- $\bar{P}$: $\eta' \to \pi^+\pi^-\gamma$ spectrum div. by Omnès function and phase space
- $\eta' \to \pi^+\pi^-\gamma$ data
  - BESIII
- pion vector form factor data
  - energy scan experiments: SND, CMD-2
  - radiative return: BaBar, KLOE

Singly-virtual η' TFF representation – I

- representation of the singly-virtual TFF from multichannel formalism
  - isovector contribution

$$F_{\eta'\gamma^*\gamma^*}(s,0) = F_{\eta'\gamma\gamma} + \frac{s}{48\pi^2}\int_{4M_\pi^2}^{\infty} \mathrm{d}s' \frac{\sigma_\pi^3(s')P(s')|F_\pi^V(s')|^2}{s'-s-i\epsilon}$$

Singly-virtual η' TFF representation – I

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  - isoscalar contribution

$$F_{\eta'\gamma^*\gamma^*}(s,0) = F_{\eta'\gamma\gamma}$$

$$+ \frac{s}{48\pi^2}\int_{4M_\pi^2}^{\infty} \mathrm{d}s' \frac{\sigma_\pi^3(s')P(s')|F_\pi^V(s')|^2}{s'-s-i\epsilon}$$

$$+ \frac{F_{\eta'\gamma\gamma} w_{\eta'\omega\gamma} s}{M_\omega^2 - s - iM_\omega\Gamma_\omega}$$

$$+ \frac{F_{\eta'\gamma\gamma} w_{\eta'\phi\gamma} s}{M_\phi^2 - s - iM_\phi\Gamma_\phi}$$

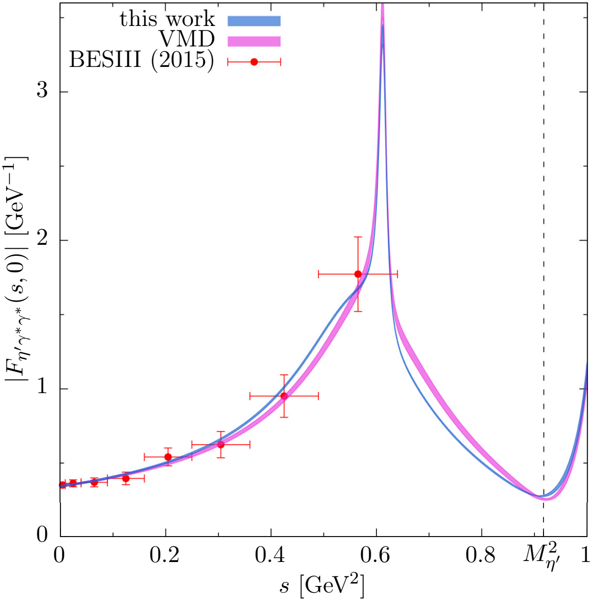
Singly-virtual η' TFF representation – I

- representation of the singly-virtual TFF from multichannel formalism
  - isovector contribution
  - isoscalar contribution
  - isospin-breaking ρ–ω-mixing effect

$$
\begin{aligned}
F_{\eta'\gamma^*\gamma^*}(s,0) = F_{\eta'\gamma\gamma} &+ \left[1 + \frac{\epsilon_{\rho\omega} s}{M_\omega^2 - s - iM_\omega\Gamma_\omega}\right] \\
&\times \frac{s}{48\pi^2}\int_{4M_\pi^2}^{\infty} \mathrm{d}s' \frac{\sigma_\pi^3(s')P(s')|F_\pi^V(s')|^2}{s'-s-i\epsilon} \\
&+ \frac{F_{\eta'\gamma\gamma} w_{\eta'\omega\gamma} s}{M_\omega^2 - s - iM_\omega\Gamma_\omega}\left[1 + \frac{\epsilon_{\rho\omega} s}{48\pi^2 g_{\omega\gamma}^2}\int_{4M_\pi^2}^{\infty} \mathrm{d}s' \frac{\sigma_\pi^3(s')|F_\pi^V(s')|^2}{s'(s'-s-i\epsilon)}\right] \\
&+ \frac{F_{\eta'\gamma\gamma} w_{\eta'\phi\gamma} s}{M_\phi^2 - s - iM_\phi\Gamma_\phi}
\end{aligned}
$$

Singly-virtual η' TFF representation – II

[SH, Hanhart, Hoferichter, Kubis 2022]



Singly-virtual η' TFF representation – II

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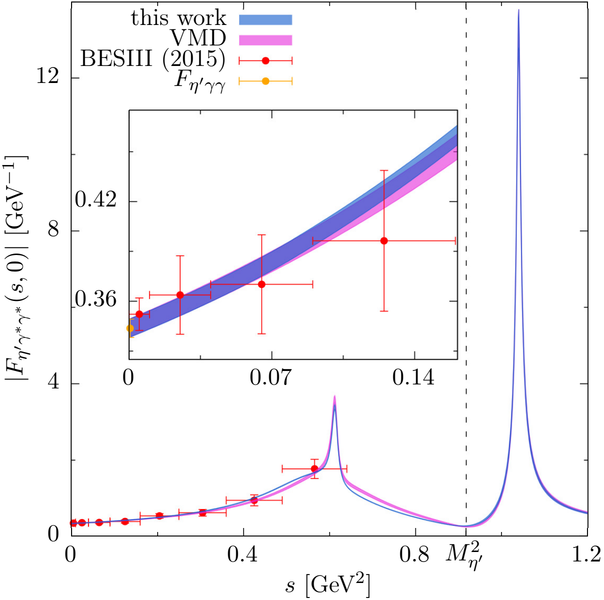


Factorization breaking in the η and η' TFFs

- Simplest approach: Application of VMD form factor in the low-energy regime

$$F_{\eta^{(\prime)}\gamma^*\gamma^*}(-Q_1^2, -Q_2^2) \propto \frac{1}{Q_1^2 + M_V^2} \times \frac{1}{Q_2^2 + M_V^2}$$

- For high energies $(Q_1^2, Q_2^2 \to \infty)$ pQCD predicts Walsh, Zerwas 1972

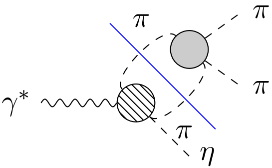
$$F_{\eta^{(\prime)}\gamma^*\gamma^*}(-Q_1^2, -Q_2^2) \propto \frac{1}{Q_1^2 + Q_2^2}$$

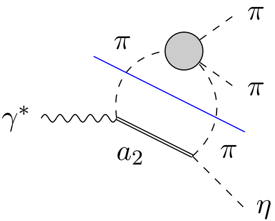
- No factorization in the singly-virtual TFFs present
- Model-independent description of intermediate energy regime with factorization breaking of paramount importance for control over uncertainties
- Recent exp. study (BaBar 2018) showed for $Q_1^2 = Q_2^2 \in [6.5,\, 45]\text{GeV}^2$ VMD factorization is breaking down
- factorization-breaking present also in Canterbury approximant approach to TFF

Escribano, Gonzàlez-Solís, Masjuan, Sánchez-Puertas 2013–2016

Towards a doubly-virtual TFF representation

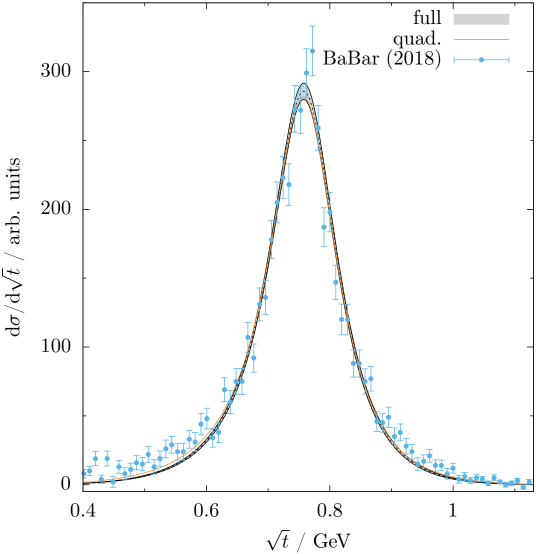
- Pin down doubly-virtual η TFF in specific kinematic regime ($q_1^2 \ll 1\,\mathrm{GeV}^2$, $q_2^2 > 1\,\mathrm{GeV}^2$)
  - analyze data for $e^+e^- \to \pi^+\pi^-\eta$ BaBar 2007+2018





- Left-hand cut contribution modeled by exchange of $a_2(1320)$ tensor meson Kubis, Plenter 2015
  - natural factorization-breaking mechanism, leads to the solution of an inhomogeneous Omnès problem
- test factorization hypothesis by comparing to simplified models

Cross section $e^+e^- \to \pi^+\pi^-\eta$



- differential spectra $\mathrm{d}\sigma/\mathrm{d}\sqrt{t}$ integrated over large kinematic range
- $\pi\pi$ spectrum: tensions below the $\rho(770)$ peak
- excited $I = 1$ vector resonances ρ' and ρ'' included via Breit–Wigner parameterization

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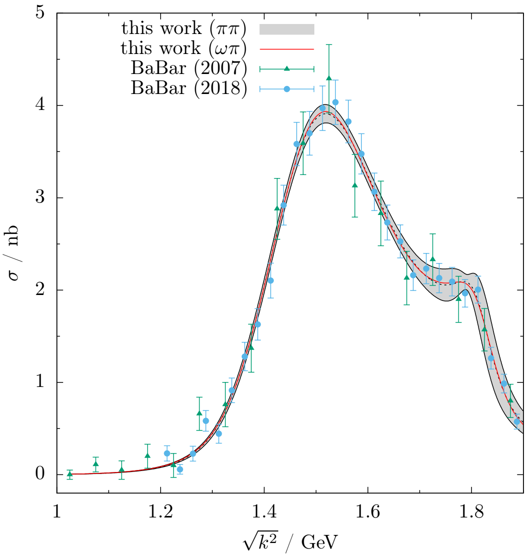

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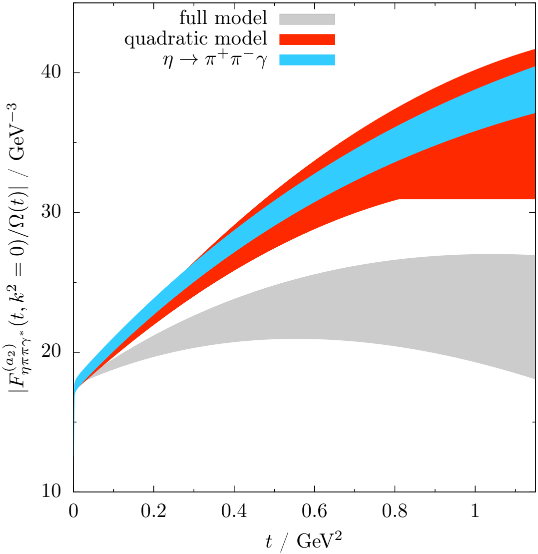
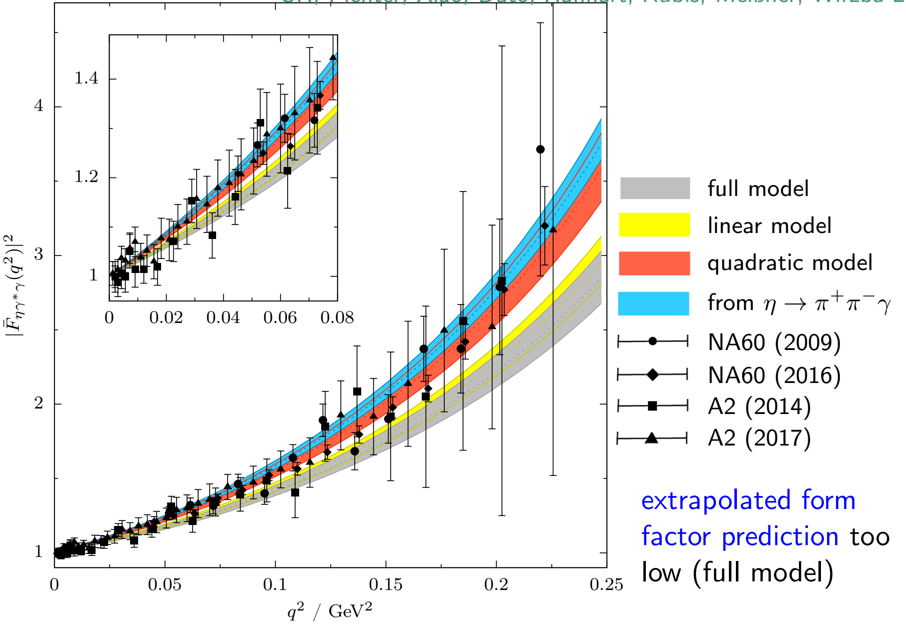
Model comparison



- subtractions fixed from k^2-integrated $\pi\pi$ spectra — compatible with $\eta \to \pi^+\pi^-\gamma$?
  - yes in naïve, factorizing, quad. model
  - no with motivated a_2 model

Prediction of singly-virtual TFF

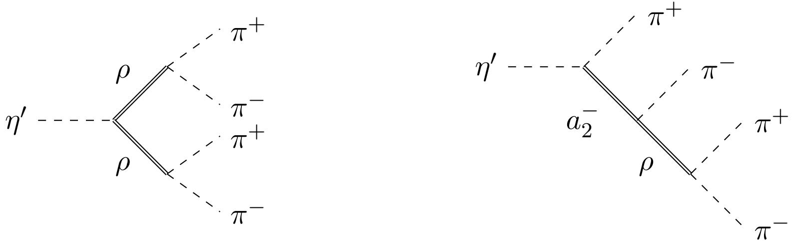
SH, Plenter, Xiao, Dato, Hanhart, Kubis, Meißner, Wirzba 2021



extrapolated form factor prediction too low (full model)

Formalism for doubly-virtual representations

- Start from $\eta' \to 2(\pi^+\pi^-)$ amplitude
  - describe decay via two rho resonance by hidden local symmetry (HLS) model Guo, Kubis, Wirzba 2012
  - left-hand-cut contribution due to a_2 exchange by phenomenological Lagrangian models



Final-state interaction

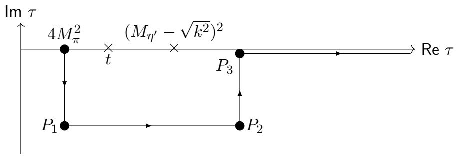
- in HLS amplitude: introduce pair-wise pion rescattering by replacing ρ propagators by Omnès functions
- in a_2 exchange amplitude $\Rightarrow$ inhomogenous Omnès problem

Inhomogeneous Omnès problem in $\eta' \to 2(\pi^+\pi^-)$

- Solution (P-wave) expressed in twice subtracted dispersion integral

$$\tilde{\mathcal{F}}(t,k^2) = \left[P(t) + \frac{t^2}{\pi}\int_{4M_\pi^2}^{\infty} \frac{\mathrm{d}\tau}{\tau^2}\frac{\hat{G}(\tau,k^2)\sin\delta_1^1(\tau)}{(\tau - t - i\epsilon)|\Omega(\tau)|}\right]\Omega(t)$$

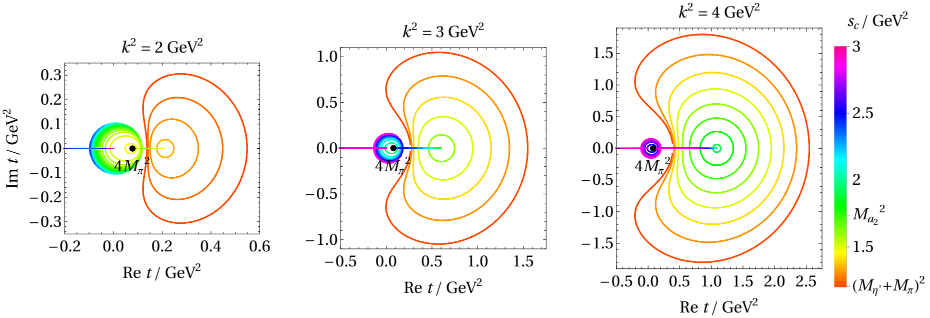
- Inhomogeneity $\hat{G}$ known for phenomenological model, but challenges direct evaluation due to singularity structure



- deform path of integration into complex plane (inspired by ideas of Gasser, Rusetsky 2018)

Path deformation into the complex plane

- express $\pi\pi$ P-wave phase shift through inverse amplitude method (IAM) amplitude Niehus, Hoferichter, Kubis 2019
  - fit low-energy constants to solution of Roy equations
    Caprini, Colangelo, Leutwyler 2012
- analytical continuation of inhomogeneity $\hat{G}$ for complex arguments
- identification critical regions Gasser, Rusetsky 2018



Doubly-virtual representations

- successive application of dispersion relations enables description of $\eta^{(\prime)} \to \gamma^*\gamma^*$
  - doubly-virtual representation of isovector part of TFF:

$$F^{(I=1)}_{\eta^{(\prime)}\gamma^*\gamma^*}(q_1^2, q_2^2) = \frac{-1}{2304\pi^4} \int_{4M_\pi^2}^{\infty} \mathrm{d}x\mathrm{d}y \, \frac{xy\sigma_\pi^3(x)\sigma_\pi^3(y)F_\pi^{V*}(x)F_\pi^{V*}(y)\left[f_1(x,y)\Omega(y) + f_1(y,x)\Omega(x)\right]}{(x - q_1^2 - i\epsilon)(y - q_2^2 - i\epsilon)}$$

- f_1 from solution of inhomogeneous Omnès problem
  - encodes non-factorization in TFF
  - subtraction constants fixed by fit to $\eta^{(\prime)} \to \pi^+\pi^-\gamma$ data

KLOE 2013, BESIII 2018

- unsubtracted dispersion relation (in q_i^2) necessary for desired (singly-virtual) space-like asymptotics

$$\lim_{Q^2\to\infty} Q^2 F_{\eta^{(\prime)}\gamma^*\gamma}(-Q^2, 0) = \text{const.}$$

- Bose symmetry respected $F^{(I=1)}_{\eta^{(\prime)}\gamma^*\gamma^*}(q_1^2, q_2^2) = F^{(I=1)}_{\eta^{(\prime)}\gamma^*\gamma^*}(q_2^2, q_1^2)$

Conclusions and outlook

Isospin breaking in the η' TFF [SH, Hanhart, Hoferichter, Kubis 2022]

- dispersive reconstruction of form factor via $\eta' \to \pi^+\pi^-\gamma$ and $e^+e^- \to \pi^+\pi^-$
- multichannel formalism: consistent implementation of ρ–ω-mixing effect, tension seen between reactions
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Towards doubly-virtual TFFs: Starting from $e^+e^- \to \pi^+\pi^-\eta$
[SH, Plenter, Xiao, Dato, Hanhart, Kubis, Meißner, Wirzba 2021]

- $\pi\pi$ spectra in $e^+e^- \to \pi^+\pi^-\eta$ [BaBar] vs. $\eta \to \pi^+\pi^-\gamma$ [KLOE]
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Towards doubly-virtual TFFs: Starting from $\eta' \to 2(\pi^+\pi^-)$

- Doubly-virtual representations of η and η' TFFs aimed towards $(g-2)_\mu$ application (including factorization-breaking effects)
- Numerical evaluation work in progress