A Complementary Dispersive Approach to Hadronic Light-by-light Scattering

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FWF Der Wissenschaftsfonds.

$\int dk\,\Pi$ Doktoratskolleg Particles and Interactions

Introduction and motivation

- **data-driven approach** to HLbL allowed for the first time to model-independently define individual contributions and calculate them with **small and reliable uncertainties** Colangelo, Hoferichter, Procura, Stoffer 2015, 2017
- error of data-driven HLbL $a_{\mu,\,\mathrm{phen}}^{\mathrm{HLbL}} = 92(19) \times 10^{-11}$ (White Paper) dominated by suppressed contributions from $1 - 2\,\mathrm{GeV}$ **resonances** (narrow-width approximation) and implementation of **short-distance constraints**
- 2 limitations for heavier resonances:
  - lack of (precise) **data input**
  - **conceptual difficulties** in the present dispersive framework
- will present novel formalism addressing these **conceptual difficulties**
- **combination** of both approaches should allow for **most precise** data-driven result

Outline

1. Dispersion relations in four-point kinematics
2. Dispersion relations in triangle kinematics
3. The sub-process $\pi\pi \to \gamma\pi\pi$
4. Conclusions and outlook

Dispersion relations in four-point kinematics

CHPS, JHEP 2015, Phys. Rev. Let. 2017, JHEP 2017

- generating set of Lorentz structures **free of kinematic singularities and zeros** Bardeen & Tung, Phys. Rev. 1968, Tarrach, Nuovo Cim. A 1975

$$\Pi^{\mu\nu\lambda\sigma} = \sum_{i=1}^{54} T_i^{\mu\nu\lambda\sigma} \Pi_i$$

- scalar functions Π_i depend on Mandelstam variables and q_i^2
- write dispersion relations for $\boldsymbol{\Pi_i}$ in s, t, u for **fixed** q_i^2 (4-point kinematics)
- BTT set $T_i^{\mu\nu\lambda\sigma}$ is **overcomplete**, which implies **ambiguities** in the scalar functions Π_i

Dispersion relations in four-point kinematics

Singly on-shell basis and sum rules

Colangelo et al., JHEP 2017

- sufficient to consider $q_4^2 = 0$
- in this limit a Lorentz **basis free of kinematic singularities** in s, t, u exists ($\check{\Pi}_i$)
- $\check{\Pi}_i$ have **different** mass dimensions
  $\rightarrow$ $\check{\Pi}_i$ with lower mass dimension fall off faster at high energies
  $\rightarrow$ implies **sum rules** of form $\int \mathrm{d}s' \, \mathrm{Im}\check{\Pi}_i(s') = 0$
- sum rules guarantee **basis independence** of a_μ^{HLbL}
- but: sum rules **only** fulfilled for (infinite) **sum** over intermediate states
  $\rightarrow$ individual contributions **basis dependent**
- **exception:** pseudoscalar poles and loops fulfill sum rules individually

Dispersion relations in four-point kinematics

Current limitation due to singularities in photon virtualities

- in addition: $\check{\Pi}_i$ have **singularities in q_i^2**
  $\rightarrow$ residues vanish due to **sum rules** for (infinite) **sum** over intermediate states
- poles lead to **non-convergent** master-formula integrals for **individual** contributions
  $\rightarrow$ must **subtract** poles using same prescription for all contributions
  $\rightarrow$ additional **ambiguity** for individual contributions
- in original basis this affects contributions with spin ≥ 1
- by basis change it can be **avoided** for axial-vector mesons

Colangelo, Hagelstein, Hoferichter, Laub, Stoffer, EPJC 2021

- **without additional** sum rules singularities are **unavoidable** for intermediate states with spin ≥ 2

How to overcome these difficulties?

$\rightarrow$ Dispersion relations at $q_4 = 0$ (triangle kinematics)

Dispersion relations in triangle kinematics

General idea and advantages

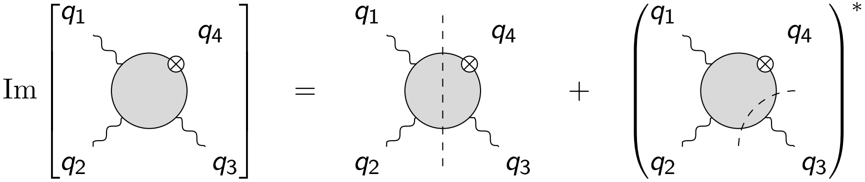
- has been highlighted that dispersion relations can also be written at $q_4 \to 0$ Colangelo, Hagelstein, Hoferichter, Laub, Stoffer, JHEP 2020
- at $q_4 \to 0$, all **redundancies disappear** and a Lorentz basis free of kinematic singularities ($\hat{\Pi}_i$) exists
- dispersion relations for $\hat{\Pi}_i$ **avoid ambiguities** coming from subtraction of spurious **poles**
- will allow to include D-wave $\pi\pi$ rescattering, tensor-meson poles, ...
- at $q_4 = 0$ **only** 3 independent kinematic variables: q_1^2, q_2^2, q_3^2

Dispersion relations in triangle kinematics

Addition of cuts

- suppressing additional arguments $\hat{\Pi}_{g_i}(q_3^2) = \lim_{s \to q_3^2} \check{\Pi}_i(s, q_3^2)$ with $s = (q_3 + q_4)^2$

$$\mathrm{Im}\,\hat{\Pi}_i(q_3^2) = \lim_{s \to q_3^2} \left[\mathrm{Im}_s \check{\Pi}_i(s, q_3^2 + i\epsilon) + \mathrm{Im}_3 \check{\Pi}_i(s + i\epsilon, q_3^2)^*\right]$$

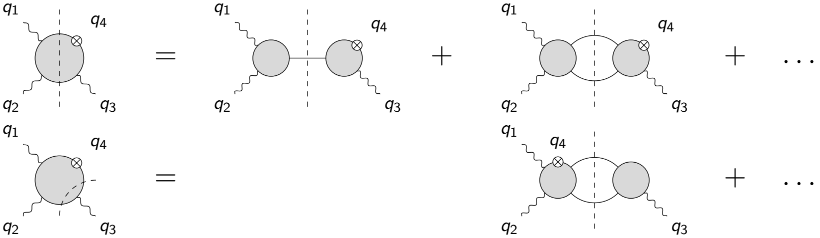


$\to$ s- and q_3^2-channel cuts have to be **added**

Dispersion relations in triangle kinematics

Topologies

- s- and q_3^2-channel cuts with 1- and 2-pion intermediate states



  → all sub-processes **except** for $\gamma^*\gamma^* \to \pi\pi\gamma$ well-known

  → **cancellation** of **soft singularities** in $\pi^+\pi^-$ intermediate states between s- and q_3^2-cuts **demonstrated**

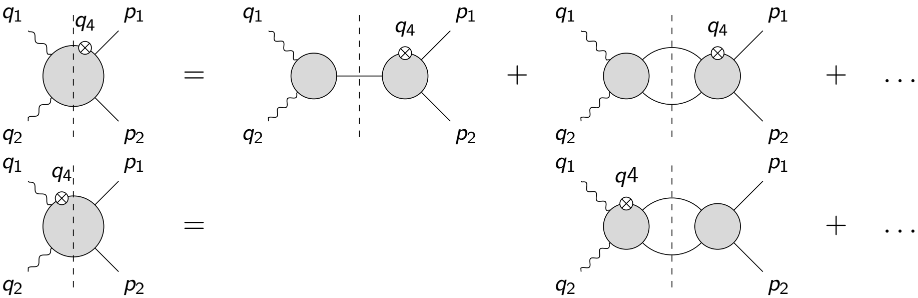
- q_3^2 cut with π^0-pole in sub-process gives momentum dependence of pion TFF Colangelo, Hagelstein, Hoferichter, Laub, Stoffer, JHEP 2020

  → reshuffling of contributions compared to 4-point kinematics

- s-channel **resonance contributions** given in terms of TFFs

The sub-process $\gamma^*\gamma^* \to \pi\pi\gamma$

- performed BTT decomposition and seen that **ambiguities disappear** in limit $q_4 \to 0$ using $\pi\pi$-crossing symmetry
- again 2 different (s-channel) cuts



$\to$ all sub-processes **except** for $\pi\pi \to \gamma\pi\pi$ well-known

The sub-process $\pi\pi \to \gamma\pi\pi$

Kinematics and Lorentz decomposition

- focus on sub-process $\pi\pi \to \gamma\pi\pi$ for the rest of the talk
  - shares **many** features with $\gamma^*\gamma^* \to \pi\pi\gamma$ and HLbL
  - Lorentz structure **much** simpler
- amplitude $\mathcal{M}(\pi^0\pi^0 \to \gamma\pi^+\pi^-) = \epsilon^*_\mu \mathcal{M}^\mu$
- charged channel $(\pi^+\pi^- \to \gamma\pi^+\pi^-)$ also needed, but related to mixed channel through **isospin symmetry** Kühn, Nucl. Phys. B 1999
- **BTT decomposition** $\mathcal{M}^\mu = \sum_{i=1}^6 \hat{T}_i^\mu \mathcal{M}_i$ leads to 3 Tarrach redundancies

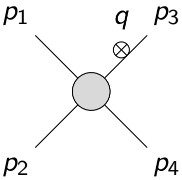
$$\hat{T}_1^\mu = p_2^\mu(p_3 \cdot q) - p_3^\mu(p_2 \cdot q)\,, \quad \hat{T}_4^\mu = q^\mu(p_1 \cdot q) - p_1^\mu q^2\,,$$
$$\hat{T}_2^\mu = p_3^\mu(p_1 \cdot q) - p_1^\mu(p_3 \cdot q)\,, \quad \hat{T}_5^\mu = q^\mu(p_2 \cdot q) - p_2^\mu q^2\,,$$
$$\hat{T}_3^\mu = p_1^\mu(p_2 \cdot q) - p_2^\mu(p_1 \cdot q)\,, \quad \hat{T}_6^\mu = q^\mu(p_3 \cdot q) - p_3^\mu q^2$$

- 5-particle process has 10 kinematic invariants, 5 fixed by on-shell conditions

The sub-process $\pi\pi \to \gamma\pi\pi$

Soft-photon limit and singularities

- in principle only $\lim_{q\to 0} \frac{\partial}{\partial q_\nu}\mathcal{M}^\mu$ needed
- **but**: limit **does not exist** due to



$\to$ **split** $\mathcal{M}^\mu = \mathcal{M}^\mu_{\text{sing}} + \mathcal{M}^\mu_{\text{reg}}$

- ambiguity to shift finite terms between $\mathcal{M}^\mu_{\text{sing}}$ and $\mathcal{M}^\mu_{\text{reg}}$
- for $\mathcal{M}^\mu_{\text{reg}}$ the limit can be performed and the problem reduces to **4-point kinematics** $\to$ Mandelstam variables
- Tarrach redundancies drop out in this limit (using crossing symmetry) $\to$ 2D basis exists
- **singularities cancel** when plugged into HLbL
- need **gauge-invariant** definition of $\mathcal{M}^\mu_{\text{sing}}$

The sub-process $\pi\pi \to \gamma\pi\pi$

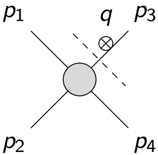
Definition of $\mathcal{M}^\mu_{\text{sing}}$

- Low's theorem relates terms of order q^{-1} and q^0 to $\pi\pi \to \pi\pi$
- but: higher-order singular pieces also needed (limit of $\frac{p_i \cdot q}{p_j \cdot q}$ depends on **direction** of q)
- suitable definition achieved from **unitarity** with a single-pion intermediate state
  $\to$ also only depends on $\pi\pi \to \pi\pi$ amplitude $\mathcal{T}$

$$\mathcal{M}^\mu_{\text{sing}} = F^V_\pi(q^2)\left(\frac{(2p_3+q)^\mu}{(p_3+q)^2 - m_\pi^2}\mathcal{T}(s,\tilde{t}-u) - \frac{(2p_4+q)^\mu}{(p_4+q)^2-m_\pi^2}\mathcal{T}(s,t-\tilde{u}) - 2(p_1-p_2)^\mu \Delta\mathcal{T}\right)$$

$$\Delta\mathcal{T} = \frac{\mathcal{T}(s,\tilde{t}-u) - \mathcal{T}(s,t-\tilde{u})}{\tilde{t}-u-t+\tilde{u}}$$

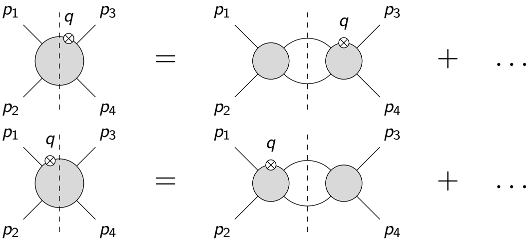
kinematic variables are defined in $2 \to 3$ kinematics



The sub-process $\pi\pi \to \gamma\pi\pi$

Unitarity relation and cancellation of singularities

- two-pion intermediate states in unitarity relations involve $\pi\pi \to \gamma\pi\pi$ as a **sub-process** (similarly in t- and u-channels)



$\to$ contains the **soft-singular** piece $\mathcal{M}^\mu_{\text{sing}}$

- checked that **sum of cuts** reproduces the singularities of $\text{Im}^{\pi\pi}\mathcal{M}^\mu_{\text{sing}}$ $\to$ **finite difference** is $\text{Im}^{\pi\pi}\mathcal{M}^\mu_{\text{reg}}$ and can be projected onto Lorentz basis in limit $q \to 0$ ($\bar{\mathcal{M}}_i$)

The sub-process $\pi\pi \to \gamma\pi\pi$

Threshold singularities

- $\mathcal{M}^\mu_{\text{reg}}$ is regular at $q \to 0$, but defined as **difference** of two singular functions $\mathcal{M}^\mu_{\text{reg}} = \mathcal{M}^\mu - \mathcal{M}^\mu_{\text{sing}}$
- first two terms in expansion around $q = 0$ **cancel** and the third term is the **result**
  $\to$ involves second derivatives of $\mathcal{M}^\mu$ and $\mathcal{M}^\mu_{\text{sing}}$
- $\mathcal{M}^\mu$ and $\mathcal{M}^\mu_{\text{sing}}$ can be expanded in **positive powers** of $\sqrt{t - 4m_\pi^2}$ at t-channel threshold
  $\to$ derivatives lead to **singularities** of form $(t - 4m_\pi^2)^{-3/2}$ and $(t - 4m_\pi^2)^{-1/2}$ in scalar functions $\bar{\mathcal{M}}_i$

The sub-process $\pi\pi \to \gamma\pi\pi$

Threshold singularities

- $(t - 4m_\pi^2)^{-3/2}$-**singularities** make standard dispersion relations for scalar functions invalid
  $\to$ write **instead** dispersion relations for

$$\check{\mathcal{M}}_i = \bar{\mathcal{M}}_i - \frac{c_i(s)}{(4m_\pi^2 - t)^{3/2}} \pm \frac{c_i(s)}{(4m_\pi^2 - u)^{3/2}}$$

  with $c_i(s)$ chosen such that the leading singularities cancel

- $c_i(s)$ have to be **real** for $\bar{\mathcal{M}}_i$ to be real between the cuts
  $\to$ fully determined from **unitarity**

$$c_1(s) = -\frac{1}{2}c_2(s) = -\frac{(a_0^2)^2}{1024\pi m_\pi}$$

- unitarity relations for $\check{\mathcal{M}}_i$ **follow** immediately from those for $\bar{\mathcal{M}}_i$

singularities do **not reduce** the **predictive power** of dispersion relations!

Conclusions

- the established dispersive formalism for HLbL has been **very successful** for most important contributions
- complementary dispersive approach **promises** to overcome roadblocks in inclusion of higher-spin intermediate states
- important steps towards this goal **already achieved**:
  - unitarity relations
  - cancellation of soft singularities
  - tensor decomposition for $\gamma^*\gamma^* \to \pi\pi\gamma$ (no redundancies at $q_4 = 0$!)
  - new dispersive formalism for $\pi\pi \to \gamma\pi\pi$

Outlook

- solution of dispersion relations will **complete** study of $\pi\pi \to \gamma\pi\pi$
- with $\pi\pi \to \gamma\pi\pi$ as input, **similar study** possible for $\gamma^*\gamma^* \to \pi\pi\gamma$
  - more complicated Lorentz structure (already derived)
  - but: **many similarities** concerning kinematics, cancellation of soft singularities and phase-space integrals expected
- this will allow for a treatment of all 1- and 2-particle intermediate states in HLbL including **angular momenta $\geq$ 2**
- study in detail **reshuffling** of contributions between the 2 dispersive approaches to HLbL
  - learn how to **combine** them to include as many contributions as possible

Thank you for your attention!

Backup

Master formula

- for a_μ^{HLbL} we need **two-loop integral** over

$$\lim_{q_4 \to 0} \frac{\partial}{\partial q_{4\rho}} \Pi^{\mu\nu\lambda\sigma}$$

- 35 linear combinations of the 54 structures **vanish** in this limit
- 5 of the 8 integrals can be performed in **full generality**
- due to **symmetry** only 12 linear combinations of scalar functions in the limit $q_4 \to 0$ enter the master formula

$$a_\mu^{\mathrm{HLbL}} = \frac{2\alpha^3}{3\pi^2} \int_0^\infty \mathrm{d}Q_1 \int_0^\infty \mathrm{d}Q_2 \int_{-1}^1 \mathrm{d}\tau \sqrt{1-\tau^2} Q_1^3 Q_2^3 \sum_{i=1}^{12} T_i(Q_1, Q_2, \tau) \bar{\Pi}_i(Q_1, Q_2, \tau)$$

- $Q_i = \sqrt{-q_i^2}$, $\tau = \tau(Q_1, Q_2, Q_3)$
- kernel functions T_i **known** analytically

Addition of cuts

- suppress additional arguments and use simplified notation $\mathrm{Im}\hat{\Pi}_i(q_3^2) = \lim\limits_{s\to q_3^2} \check{\Pi}_i(s, q_3^2)$ with $s = (q_3 + q_4)^2$

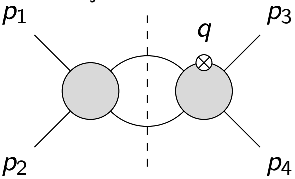
$$
\begin{aligned}
\mathrm{Im}\,\hat{\Pi}_i(s) &= \lim_{q_3^2\to s} \frac{\check{\Pi}_i(s+i\epsilon, q_3^2+i\epsilon) - \check{\Pi}_i(s-i\epsilon, q_3^2-i\epsilon)}{2i} \\
&= \lim_{q_3^2\to s} \Bigg[\frac{\check{\Pi}_i(s+i\epsilon, q_3^2+i\epsilon) - \check{\Pi}_i(s-i\epsilon, q_3^2+i\epsilon)}{2i} \\
&\qquad + \frac{\check{\Pi}_i(s-i\epsilon, q_3^2+i\epsilon) - \check{\Pi}_i(s-i\epsilon, q_3^2-i\epsilon)}{2i}\Bigg] \\
&= \lim_{q_3^2\to s} \Bigg[\frac{\check{\Pi}_i(s+i\epsilon, q_3^2+i\epsilon) - \check{\Pi}_i(s-i\epsilon, q_3^2+i\epsilon)}{2i} \\
&\qquad + \left(\frac{\check{\Pi}_i(s+i\epsilon, q_3^2+i\epsilon) - \check{\Pi}_i(s+i\epsilon, q_3^2-i\epsilon)}{2i}\right)^*\Bigg] \\
&=: \lim_{q_3^2\to s} \left[\mathrm{Im}_s\check{\Pi}_i(s, q_3^2+i\epsilon) + \mathrm{Im}_3\check{\Pi}_i(s+i\epsilon, q_3^2)^*\right]
\end{aligned}
$$

Isospin symmetry and mixing of *s*- and *t*-channel amplitudes

- in the isospin limit charged and mixed channels are related through

$$\mathcal{M}_{\pi^+\pi^-\to\gamma\pi^+\pi^-}(p_1,p_2,p_3,p_4) = -\mathcal{M}_\mu(p_3,p_4,p_2,p_1) - \mathcal{M}_\mu(p_3,p_1,p_2,p_4) - \mathcal{M}_\mu(p_2,p_4,p_3,p_1) - \mathcal{M}_\mu(p_2,p_1,p_3,p_4)$$

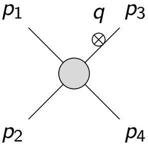
- the charged channel appears in a sub-process of the *s*-channel unitarity relation for the mixed channel



- this makes the *s*-channel imaginary part depend on the *t*-channel amplitude

Low's theorem

- **Low's theorem**: terms of order $\boldsymbol{q^{-1}}$ obtainable from



  with scalar QED vertex for soft photon,
  terms of order $\boldsymbol{q^0}$ fixed by imposing Ward identity Adler & Dothan, Phys. Rev. 1966
  $\rightarrow$ terms up to order q^0 given in terms of $\pi\pi \rightarrow \pi\pi$

- but: $\lim_{q\rightarrow 0} \frac{\partial}{\partial q_\nu}(\mathcal{M}^\mu - \mathcal{M}^\mu_{\text{sing}})$ still does **not exist** due to terms like $q^\mu \frac{p_i \cdot q}{p_j \cdot q}$ (of order q^1, but limit depends on **direction** of q)

Dispersive definition of $\mathcal{M}^{\mu}_{\text{sing}}$

$$\mathcal{M}^{\mu}_{\text{sing}} = F^V_\pi(q^2)\left(\frac{(2p_3+q)^\mu}{(p_3+q)^2-m_\pi^2}\mathcal{T}(s,\tilde{t}-u) - \frac{(2p_4+q)^\mu}{(p_4+q)^2-m_\pi^2}\mathcal{T}(s,t-\tilde{u}) - 2(p_1-p_2)^\mu\Delta\mathcal{T}\right)$$

$$\Delta\mathcal{T} = \frac{\mathcal{T}(s,\tilde{t}-u)-\mathcal{T}(s,t-\tilde{u})}{\tilde{t}-u-t+\tilde{u}}$$

- variables defined as

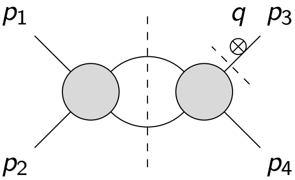
$$s=(p_1+p_2)^2\,,\qquad t=(p_1+p_3)^2\,,\qquad u=(p_1+p_4)^2\,,$$
$$\tilde{s}=(p_3+p_4)^2\,,\qquad \tilde{t}=(p_2+p_4)^2\,,\qquad \tilde{u}=(p_2+p_3)^2$$

- obtained from dispersion relation in $w_3=(p_3+q)^2$ with s, $t'-u'=\frac{1}{2}(t+\tilde{t}-u-\tilde{u})$, q^2, w_1 and w_2 fixed

- w_4-singularity appears as pole in w_3

Cancellation of soft singularities

- for $\pi^0\pi^0$ intermediate state, the cancellation takes place between the **different orders of cuts** in



- in the case of the $\pi^+\pi^-$ intermediate state, there is **in addition** a cancellation between the two cuts