Update on the hadronic light-by-light scattering contribution to the muon $g-2$ (RBC/UKQCD)

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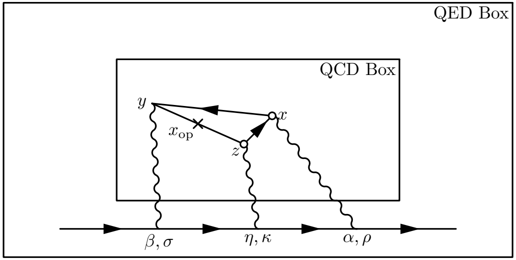
Fifth Plenary Workshop of the Muon g-2 Theory Initiative, Edinburgh



- Mainz group pioneered infinite volume QED approach (QED_∞): (semi-)analytically calculate the QED kernel. J. Green *et al*. 2015 (PRL 115, 22, 222003)
- RBC-UKQCD collaboration used same volume for QED and QCD boxes (QED_L scheme). T. Blum *et al*. 2015 (PRL 114, 1, 012001)
- Mainz 2021: $L_{\mathrm{QED}} = \infty$, m_π: $200 \sim 422$ MeV, Wilson fermions E.H. Chao *et al*. 2021 (EPJC 81, 7, 651)
- RBC-UKQCD 2019: $L_{\mathrm{QED}} = L_{\mathrm{QCD}}$: $4.67 \sim 6.22$ fm, m_π: $135 \sim 144$ MeV, (Möbius) Domain wall fermions T. Blum *et al*. 2020 (PRL 124, 13, 132002)

Outline

1. **HLbL from finite volume QED box (QED$_L$)**

   RBC-UKQCD 2019 T. Blum *et al*. 2020 (PRL 124, 13, 132002)

2. HLbL from infinite volume QED box (QED$_\infty$)

   Work in progress.

PHYSICAL REVIEW LETTERS **124**, 132002 (2020)

Editors' Suggestion Featured in Physics

Hadronic Light-by-Light Scattering Contribution to the Muon Anomalous Magnetic Moment from Lattice QCD

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We report the first result for the hadronic light-by-light scattering contribution to the muon anomalous magnetic moment with all errors systematically controlled. Several ensembles using $2+1$ flavors of physical mass Möbius domain-wall fermions, generated by the RBC and UKQCD collaborations, are employed to take the continuum and infinite volume limits of finite volume lattice QED + QCD. We find $a_\mu^{\mathrm{HLbL}} = 7.87(3.06)_{\mathrm{stat}}(1.77)_{\mathrm{sys}} \times 10^{-10}$. Our value is consistent with previous model results and leaves little room for this notoriously difficult hadronic contribution to explain the difference between the standard model and the BNL experiment.

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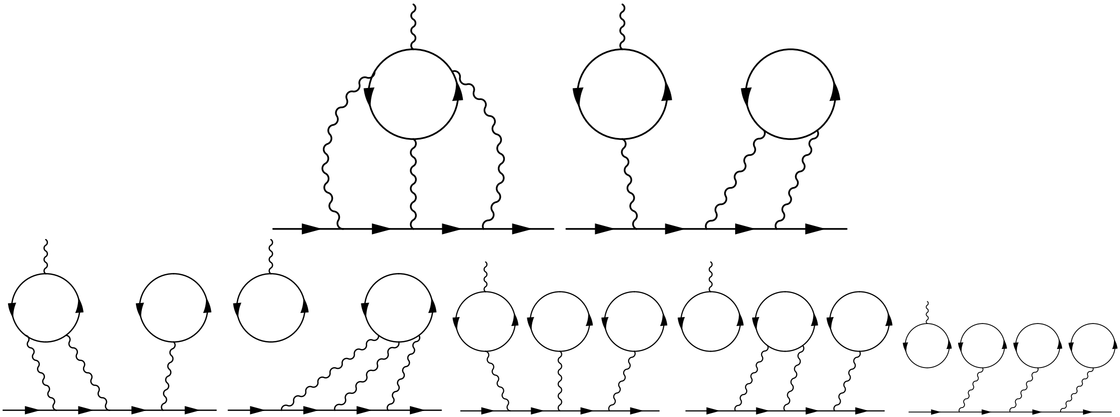
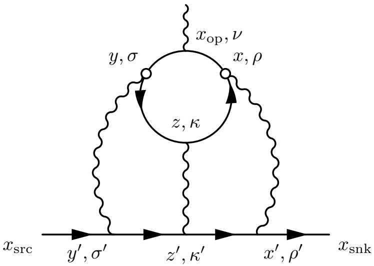


- There are additional distinct permutations of photons not shown.
- Diagrams in the second row are suppressed by flavor SU(3) symmetry and are numerically very small.

Exact photon and the moment method



- Point sources at x, y (randomly sample)
- Importance sampling: focus on small $|x - y|$. (complete for $|x - y| \leq 5a$)
- Translational symmetry: $x_{\rm ref} = (x + y)/2$

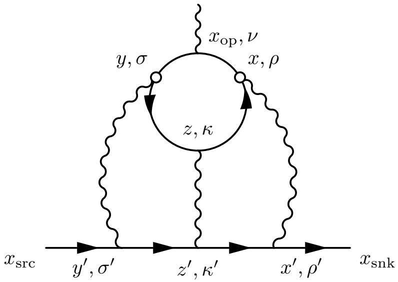
$$\langle \mu(\vec{p}')|J_\nu(0)|\mu(\vec{p})\rangle \quad = \quad -e\bar{u}(\vec{p}')\left(F_1(q^2)\gamma_\nu + i\frac{F_2(q^2)}{4m}[\gamma_\nu, \gamma_\rho]q_\rho\right)u(\vec{p}),$$

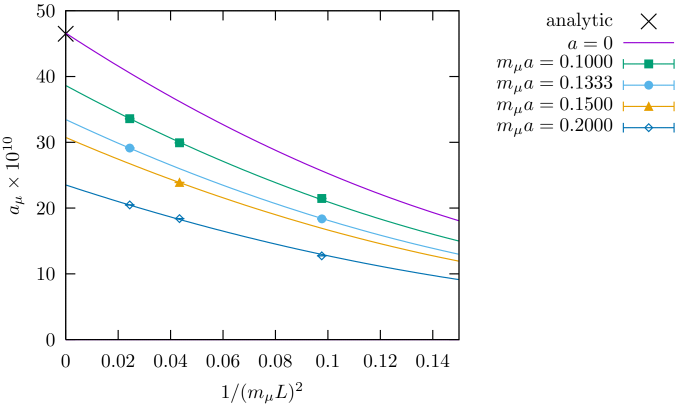
$$\frac{a_\mu}{m_\mu}\bar{u}_{s'}(\vec{0})\frac{\Sigma}{2}u_s(\vec{0}) = \sum_{r=x-y}\sum_{z}\sum_{x_{\rm op}}\frac{1}{2}(\vec{x}_{\rm op} - \vec{x}_{\rm ref}) \times \bar{u}_{s'}(\vec{0})i\vec{\mathcal{F}}^C(\vec{0}; x, y, z, x_{\rm op})u_s(\vec{0})$$

- WI exact on each configuration (other 2 diagrams not shown)
  - q "exact" and allows "moment method" $\rightarrow q = 0$ directly
- Reorder summation $|x - y| \leq \min(|y - z|, |x - z|)$ and $\times 3$

T. Blum *et al*. 2016 (PRD 93, 1, 014503)

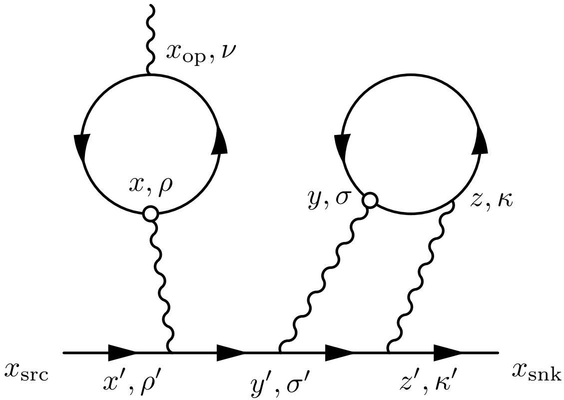
- We test our setup by computing **muon leptonic light by light** contribution to muon $g-2$.





$$F_2(a,L) = F_2\left(1 - \frac{c_1}{(m_\mu L)^2} + \frac{c_1'}{(m_\mu L)^4}\right)(1 - c_2 a^2 + c_2' a^4) \;\rightarrow\; F_2 = 46.6(2) \times 10^{-10} \quad (19)$$

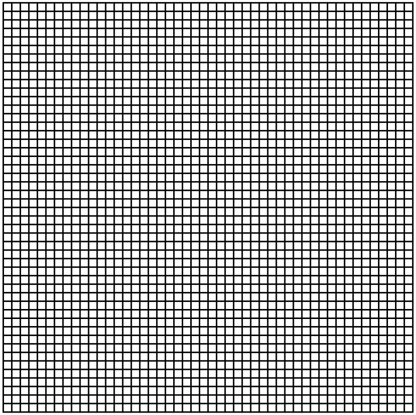
- **Pure QED computation**. Muon leptonic light by light contribution to muon $g-2$. Phys.Rev. D93 (2016) 1, 014503. arXiv:1510.07100.
- Analytic results: $0.371 \times (\alpha/\pi)^3 = 46.5 \times 10^{-10}$.
- $\mathcal{O}(1/L^2)$ finite volume effect, because the photons are emitted from a conserved loop.



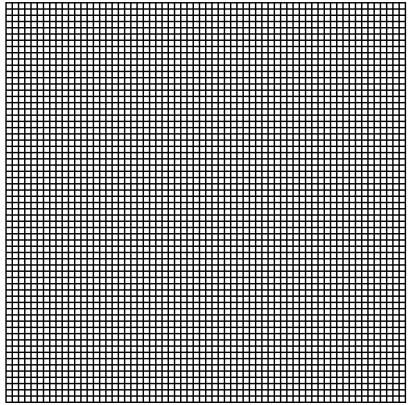
- Point x is used as the reference point for the moment method.
- We can use two point source photons at x and y, which are chosen randomly. The points $x_{\rm op}$ and z are summed over exactly on lattice.
- Only point source quark propagators are needed. We compute M point source propagators and all M^2 combinations of them are used to perform the stochastic sum over $r = x - y$.

T. Blum *et al*. 2017 (PRL 118, 2, 022005)

48I



64I

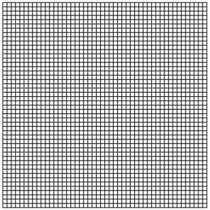


- Möbius domain wall fermion action (chiral symmetry, no $\mathcal{O}(a)$ lattice artifacts)
- Iwasaki gauge action
- $M_\pi = 135$ MeV *, $L = 5.5$ fm box, $1/a_{48\text{I}} = 1.73$ GeV, $1/a_{64\text{I}} = 2.359$ GeV
  RBC-UKQCD 2016 (PRD 93, 074505)

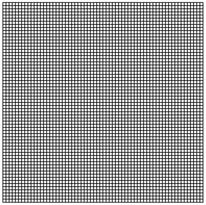
*: Valence pion mass. Slightly different from the 139 MeV unitary pion mass used in the ensemble generation.

T. Blum *et al*. 2020 (PRL 124, 13, 132002)

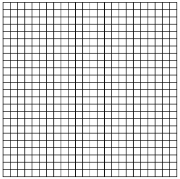
48I



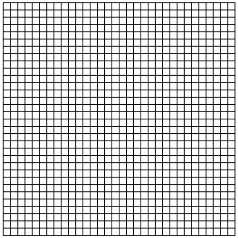
64I



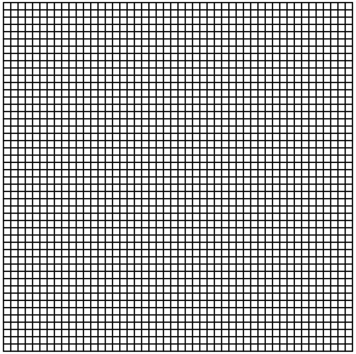
24D



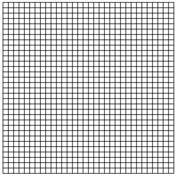
32D



48D



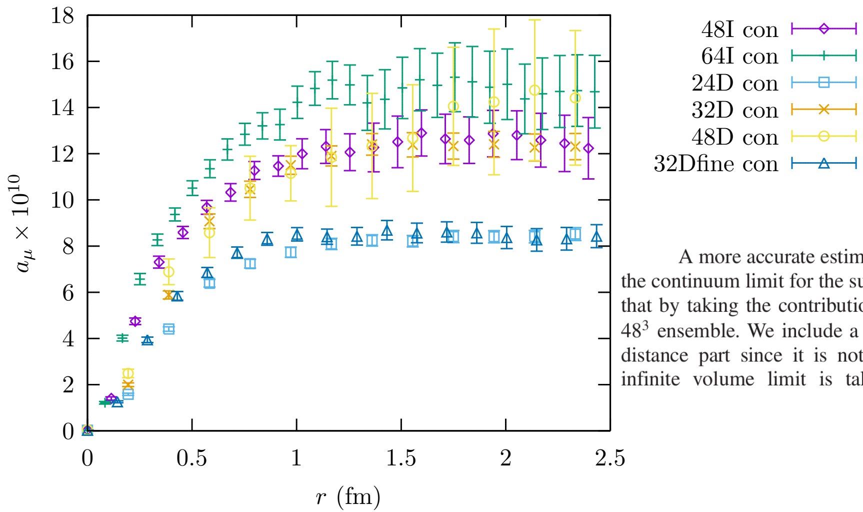
32Dfine



- 24D, 32D, 48D for FV effects,
- $M_\pi \approx 140$ MeV and $1/a = 1.015$

T. Blum *et al*. 2020 (PRL 124, 13, 132002)

$$\frac{a_\mu}{m_\mu}\bar{u}_{s'}(\vec{0})\frac{\Sigma}{2}u_s(\vec{0}) = \sum_{r=x-y}\sum_{z}\sum_{x_{\rm op}}\frac{1}{2}(\vec{x}_{\rm op}-\vec{x}_{\rm ref})\times\bar{u}_{s'}(\vec{0})i\vec{\mathcal{F}}^C(\vec{0};x,y,z,x_{\rm op})u_s(\vec{0})$$

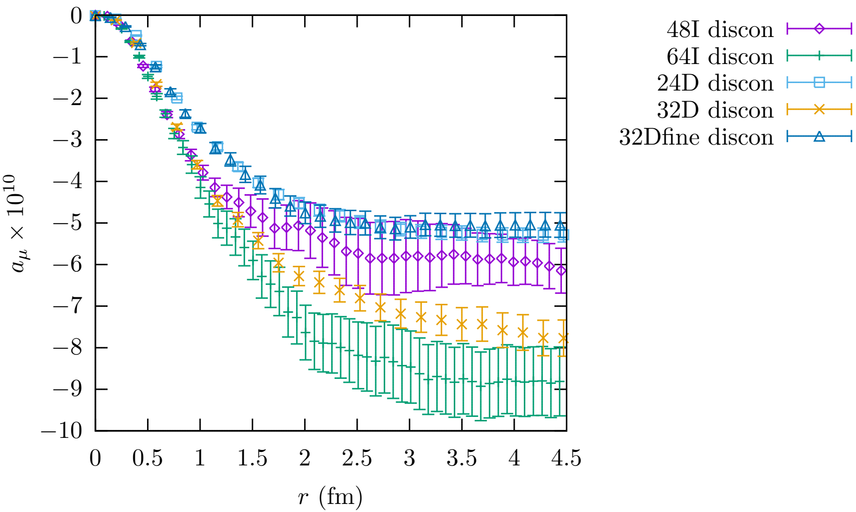


A more accurate estimate can be obtained by taking the continuum limit for the sum up to $r = 1$ fm, and above that by taking the contribution from the relatively precise 48^3 ensemble. We include a systematic error on this long distance part since it is not extrapolated to $a = 0$. The infinite volume limit is taken as before.

Partial sum. Full sum is the right most data point

T. Blum *et al*. 2020 (PRL 124, 13, 132002)

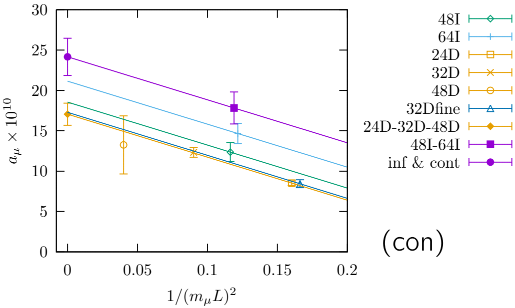
$$\frac{a_\mu}{m_\mu}\bar{u}_{s'}(\vec{0})\frac{\Sigma}{2}u_s(\vec{0}) = \sum_{r=x-y}\sum_{z}\sum_{x_{\rm op}}\frac{1}{2}(\vec{x}_{\rm op}-\vec{x}_{\rm ref})\times\bar{u}_{s'}(\vec{0})i\vec{\mathcal{F}}^C(\vec{0};x,y,z,x_{\rm op})u_s(\vec{0})$$

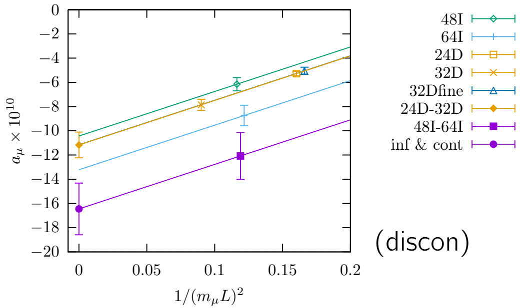


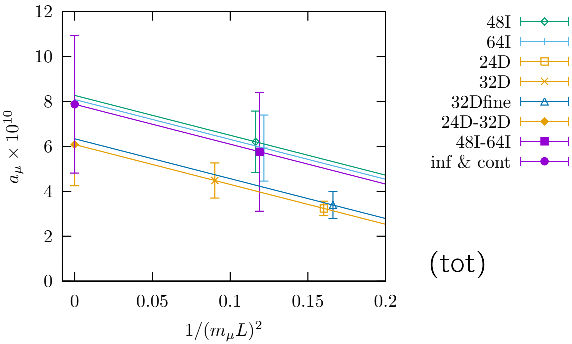
Partial sum. Full sum is the right most data point

T. Blum *et al*. 2020 (PRL 124, 13, 132002)

$$a_\mu(L, a^{\mathrm{I}}, a^{\mathrm{D}}) = a_\mu\left(1 - \frac{b_2}{(m_\mu L)^2} - c_1^{\mathrm{I}}(a^{\mathrm{I}}\ \mathrm{GeV})^2 - c_1^{\mathrm{D}}(a^{\mathrm{D}}\ \mathrm{GeV})^2 + c_2^{\mathrm{D}}(a^{\mathrm{D}}\ \mathrm{GeV})^4\right)$$







T. Blum *et al*. 2020 (PRL 124, 13, 132002)

	con	discon	tot
a_μ	24.16(2.30)	-16.45(2.13)	7.87(3.06)
sys hybrid $\mathcal{O}(a^2)$	0.20(0.45)	0	0.20(0.45)
sys $\mathcal{O}(1/L^3)$	2.34(0.41)	1.72(0.32)	0.83(0.56)
sys $\mathcal{O}(a^4)$	0.88(0.31)	0.71(0.28)	0.95(0.92)
sys $\mathcal{O}(a^2 \log(a^2))$	0.23(0.08)	0.25(0.09)	0.02(0.11)
sys $\mathcal{O}(a^2/L)$	4.43(1.38)	3.49(1.37)	1.08(1.57)
sys strange con	0.30	0	0.30
sys sub-discon	0	0.50	0.50
sys all	5.11(1.32)	3.99(1.29)	1.77(1.13)

- Systematic errors are estimated by altering the form of the extrapolation.
- Same method used for estimating the systematic errors of individual and total contributions.
- Systematic error cancels somewhat between the connected and disconnected diagrams.

T. Blum *et al*. 2020 (PRL 124, 13, 132002)

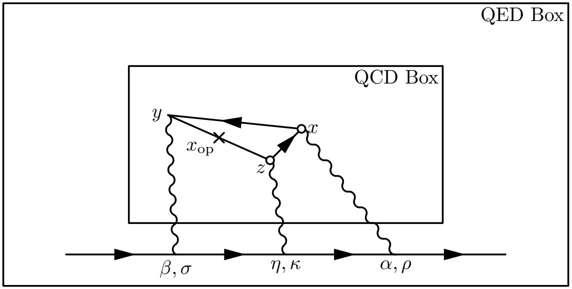
HLbL: RBC-UKQCD 2019 - Computation @ MIRA under ALCC allocations 14 / 26

1. HLbL from finite volume QED box (QED_L)

   RBC-UKQCD 2019 T. Blum *et al*. 2020 (PRL 124, 13, 132002)

2. **HLbL from infinite volume QED box (QED_∞)**

   Work in progress.



$$
\begin{aligned}
i^3\mathcal{G}_{\rho,\sigma,\kappa}(x,y,z) &= \mathfrak{G}_{\rho,\sigma,\kappa}(x,y,z) + \mathfrak{G}_{\sigma,\kappa,\rho}(y,z,x) + \mathfrak{G}_{\kappa,\rho,\sigma}(z,x,y) \\
&\quad + \mathfrak{G}_{\kappa,\sigma,\rho}(z,y,x) + \mathfrak{G}_{\rho,\kappa,\sigma}(x,z,y) + \mathfrak{G}_{\sigma,\rho,\kappa}(y,x,z), \\
\mathfrak{G}_{\sigma,\kappa,\rho}(y,z,x) &= \lim_{t_{\rm src}\to-\infty,\, t_{\rm snk}\to\infty} e^{m_\mu (t_{\rm snk}-t_{\rm src})} \int_{\alpha,\beta,\eta} G(x,\alpha)G(y,\beta)G(z,\eta) \\
&\quad \times \int_{\vec{x}_{\rm snk},\vec{x}_{\rm src}} S_\mu\left(x_{\rm snk},\beta\right) i\gamma_\sigma S_\mu(\beta,\eta) i\gamma_\kappa S_\mu(\eta,\alpha) i\gamma_\rho S_\mu\left(\alpha, x_{\rm src}\right),
\end{aligned}
$$

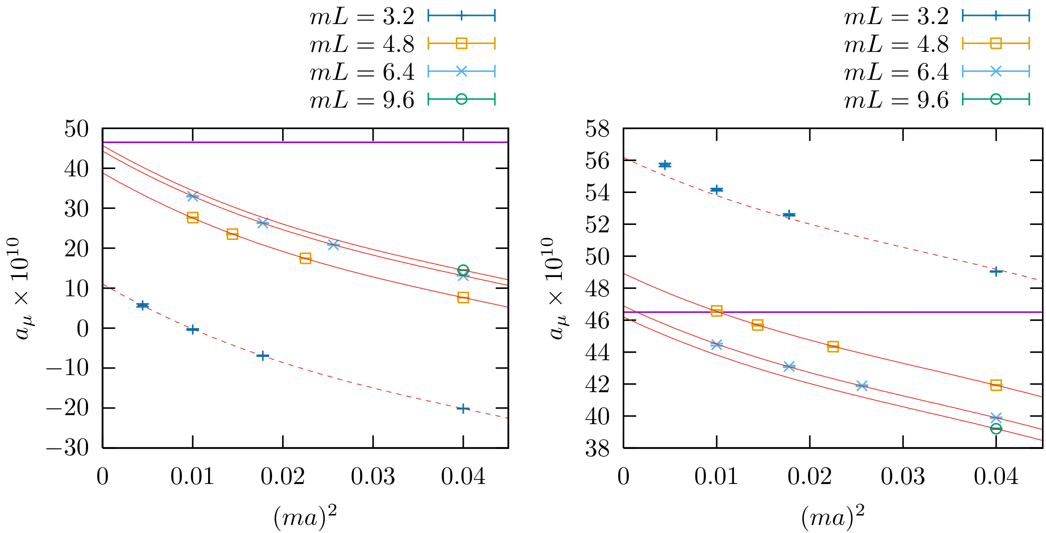
Subtraction to (1) remove infrared divergence; (2) reduce discretization and finite volume effects.

$$
\begin{aligned}
\mathfrak{G}^{(1)}_{\sigma,\kappa,\rho}(y,z,x) &= \frac{1}{2}\mathfrak{G}_{\sigma,\kappa,\rho}(y,z,x) + \frac{1}{2}[\mathfrak{G}_{\rho,\kappa,\sigma}(x,z,y)]^\dagger, \\
\mathfrak{G}^{(2)}_{\sigma,\kappa,\rho}(y,z,x) &= \mathfrak{G}^{(1)}_{\sigma,\kappa,\rho}(y,z,x) - \mathfrak{G}^{(1)}_{\sigma,\kappa,\rho}(z,z,x) - \mathfrak{G}^{(1)}_{\sigma,\kappa,\rho}(y,z,z).
\end{aligned}
$$

T. Blum *et al*. 2017. (PRD 96 3, 034515)

- Code and hardware: Peter Boyle's Grid and Christoph Lehner's GPT are now used on SUMMIT's NVIDIA V100 GPUs. Contraction code ported to GPU.
  Previously used Peter Boyle's BFM on MIRA's IBM BG/Q system.
- Measurements: 113 48I configurations, 2048 pt src propagators on each
  Previously 34 configurations, each with 1024 pt src propagators
  6.65 times increase of the number of propagators.
- Connected diagram ($x_{\text{ref}} = z$)
  - Increase (x, y) pairs to $\sim 57,000$ per configuration (8$\times$ increase per propagator)
  - $\sim 53\times$ statistics, or $7\times$ error reduction in total
- Disconnected diagram:
  - Use all possible combinations of point pairs (2$\times$ increase per propagator)
  - More efficient adaptive sampling scheme for the third vertex z. Sampling probability determined based on $\sum_{\mu,\nu} |\Pi_{\mu,\nu}(z-y) - \langle \Pi_{\mu,\nu}(z-y) \rangle|^2$ instead of $|z-y|$. Effectively $7\times$ statistics, or $\sim 10\times$ error reduction in total

- Compare the two $\mathfrak{G}_{\rho,\sigma,\kappa}(x, y, z)$ in **pure QED computation**.

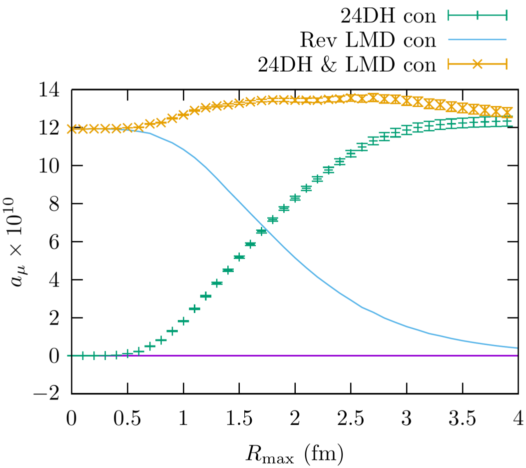


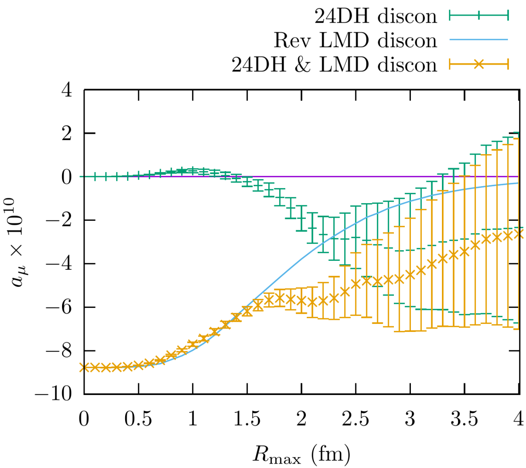
Original QED kernel

Subtracted QED kernel

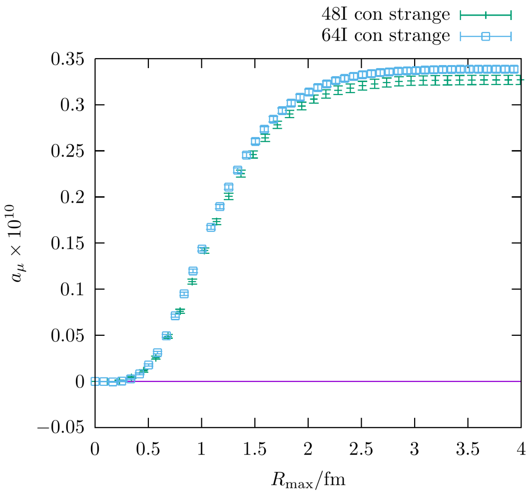
- Notice the vertical scales in the two plots are different.

T. Blum *et al*. 2017. (PRD 96 3, 034515)



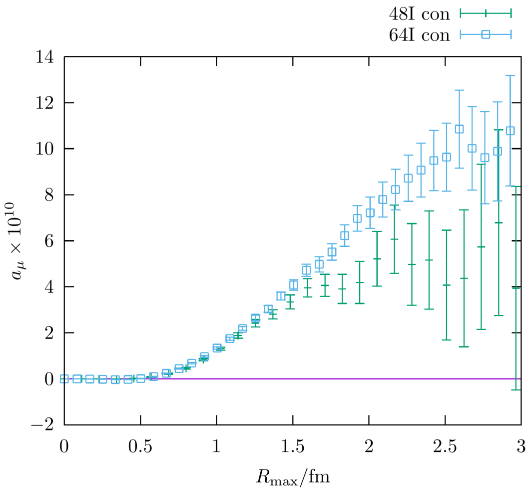


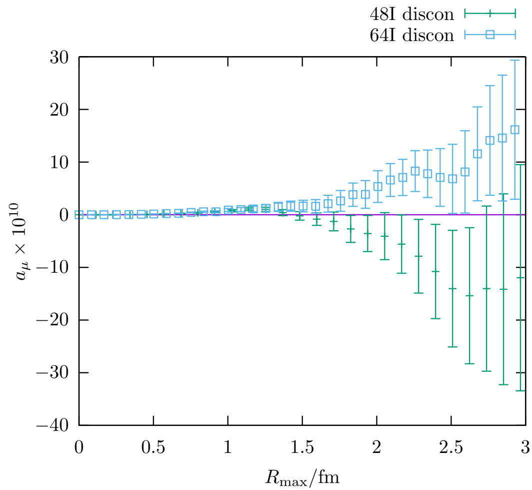
- 24DH: . $R_{\max} = \max(|x-y|, |x-z|, |y-z|)$. Partial sum up to $R_{\max}$
- Long distance part computed with the LMD model (reverse partial sum).
  (lattice calculation is in progress)
- At $R_{\max} = 2.0$ fm, the combination gives: $a_\mu^{\text{tot}} = 7.46(62)_{\text{stat}} \times 10^{-10}$
  where $a_\mu^{\text{con}} = 13.44(10)_{\text{stat}} \times 10^{-10}$, $a_\mu^{\text{discon}} = -5.70(58)_{\text{stat}} \times 10^{-10}$



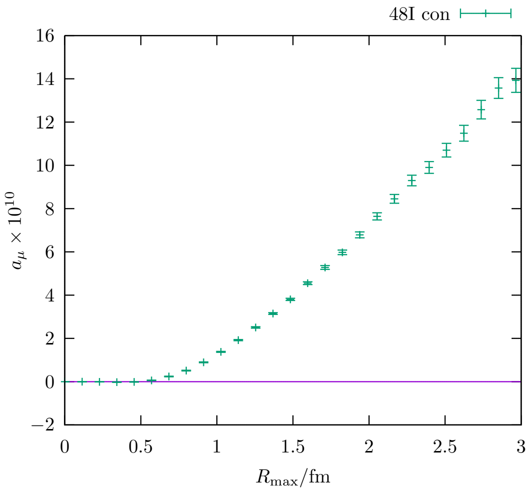
- 48I: $a_\mu^{\text{con-strange}} = 0.319(5)_{\text{stat}} \times 10^{-10}$.
- 64I: $a_\mu^{\text{con-strange}} = 0.338(3)_{\text{stat}} \times 10^{-10}$.
- Continuum limit:
  $a_\mu^{\text{con-strange}} = 0.361(7)_{\text{stat}} \times 10^{-10}$.

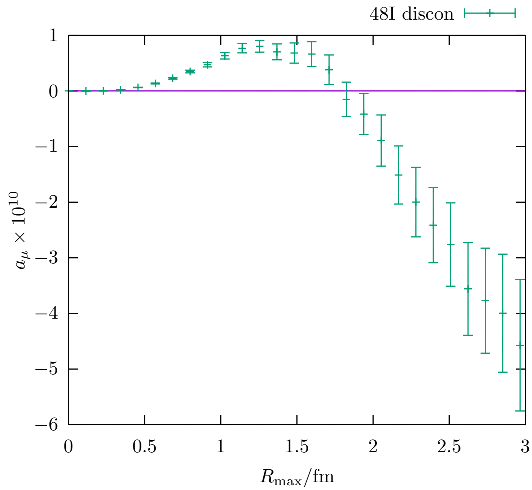
- 48I: $a = 0.114$ fm. 64I: $a = 0.084$ fm.
- $R_{\max} = \max(|x-y|, |x-z|, |y-z|)$. Partial sum up to $R_{\max}$.





- 48I: $a = 0.114$ fm. 64I: $a = 0.084$ fm.
- $R_{\text{max}} = \max(|x-y|, |x-z|, |y-z|)$. Partial sum up to R_{max}.





- 48I: $a = 0.114$ fm, only the light quark contributions.
- $R_{\rm max} = \max(|x-y|, |x-z|, |y-z|)$. Partial sum up to $R_{\rm max}$.
- Much more statistics for 48I ensemble with last year's ALCC allocation on SUMMIT.

For the four-point-function, when its two ends, x and y, are far separated, but x' is close to x and y' is close to y, the four-point-function is dominated by π^0 exchange.

Both the connected and the disconnected diagram will contribute in these region. We can find a connection between the connnected diagram and the disconnected diagram by first investigating the η correlation function.

$$\langle \bar{u}\gamma_5 u(x)(\bar{u}\gamma_5 u + \bar{d}\gamma_5 d)(y)\rangle \sim e^{-m_\eta |x-y|} \tag{24}$$

$$\langle \bar{u}\gamma_5 u(x)(\bar{u}\gamma_5 u - \bar{d}\gamma_5 d)(y)\rangle + 2\langle \bar{u}\gamma_5 u(x)\bar{d}\gamma_5 d(y)\rangle \sim e^{-m_\eta |x-y|} \tag{25}$$

That is

$$\langle \bar{u}\gamma_5 u(x)\bar{d}\gamma_5 d(y)\rangle = -\frac{1}{2}\langle \bar{u}\gamma_5 u(x)(\bar{u}\gamma_5 u - \bar{d}\gamma_5 d)(y)\rangle + \mathcal{O}\big(e^{-m_\eta |x-y|}\big) \tag{26}$$

Above is a relation between disconnected diagram π^0 exchange (left hand side) and connected diagram π^0 exchange (right hand side).

The nearby two current operater can be viewed as an interpolating operator for π^0, just like $\bar{u}\gamma_5 u$ or $\bar{d}\gamma_5 d$ with appropriate charge factors.

Multiplied by appropriate charge factors:

$$\text{Connected contribution} \quad \left[\left(\frac{2}{3}\right)^4 + \left(-\frac{1}{3}\right)^4\right] = \frac{17}{81} \tag{27}$$

$$\text{Disconnected contribution} \quad \left[\left(\frac{2}{3}\right)^2 + \left(-\frac{1}{3}\right)^2\right]^2 \left(-\frac{1}{2}\right) = \frac{25}{81}\left(-\frac{1}{2}\right) \tag{28}$$

$$\text{Connected} : \text{Disconnected} = 34 : -25 \tag{29}$$

Different approach by J. Bijnens and J. Relefors: **JHEP 1609 (2016) 113**.

- The ratio between connected and disconnected diagram is exactly $-34/25$ at long distance. True before taking the continuum or infinite volume limit.
- We use the same (subtracted) weighting function for both the connected diagram and the disconnected diagram.

$$a_\mu = (a_\mu^{\text{discon}} + \frac{25}{34} a_\mu^{\text{con}}) + \frac{9}{34} a_\mu^{\text{con}}$$

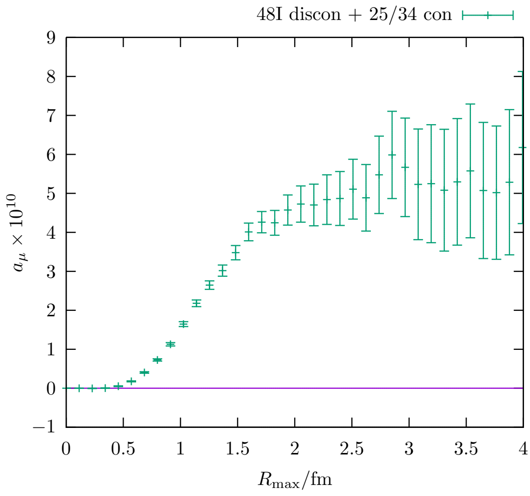
- At $R_{\text{max}} = 2.0$ fm:

$$a_\mu^{\text{discon}} + \frac{25}{34} a_\mu^{\text{con}} = 4.65(42)_{\text{stat}} \times 10^{-10}$$

- At $R_{\text{max}} = 4.0$ fm:

$$a_\mu^{\text{con}} = 18.61(1.22)_{\text{stat}} \times 10^{-10}$$

- Combine: $a_\mu = 9.58(52)_{\text{stat}} \times 10^{-10}$



PRELIMINARY: Pending finite volume corrections, continuum limit, strange & charm contributions, LD contribution


G
F
OAK RIDGE
National Laboratory
IBM
summit

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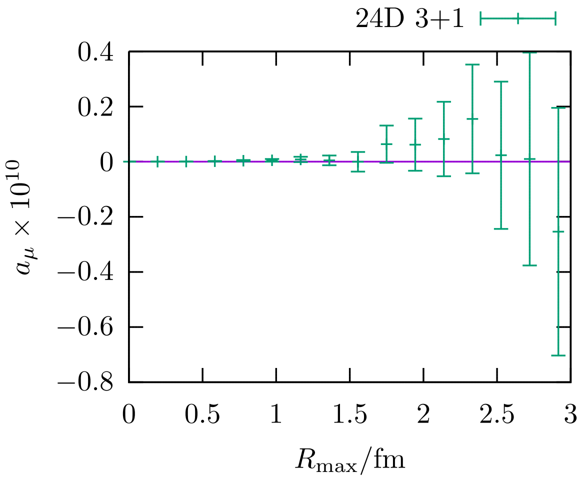
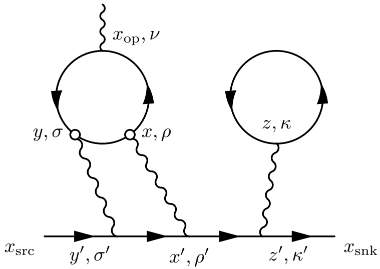
Computations done on

- Mira at ALCF (ALCC)
- Summit at OLCF (ALCC)
- Oakforest-PACS at Tokyo University (HPCI Research Project)
- Clusters at BNL SDCC

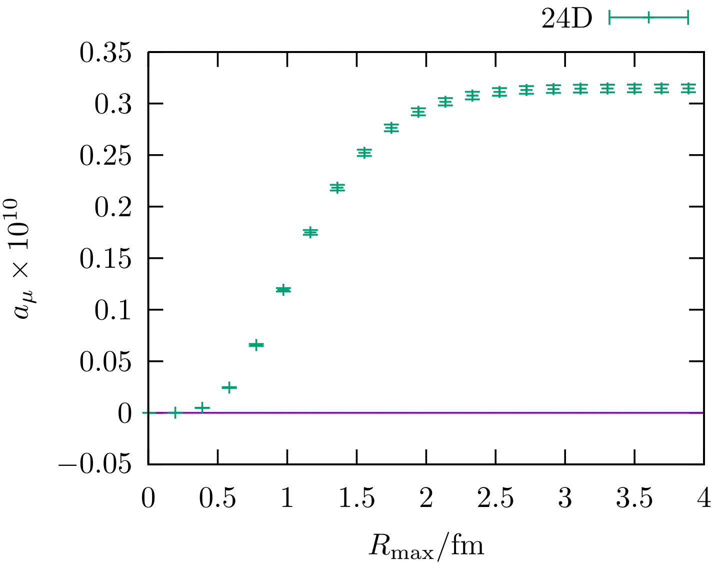
Backup slides

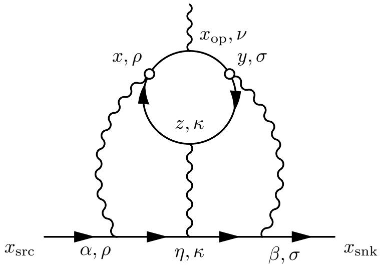




- 24D: $24^3 \times 64$
  $L = 4.8$ fm
- $a^{-1} = 1.015$ GeV
  $M_\pi = 142$ MeV
  $M_K = 512$ MeV

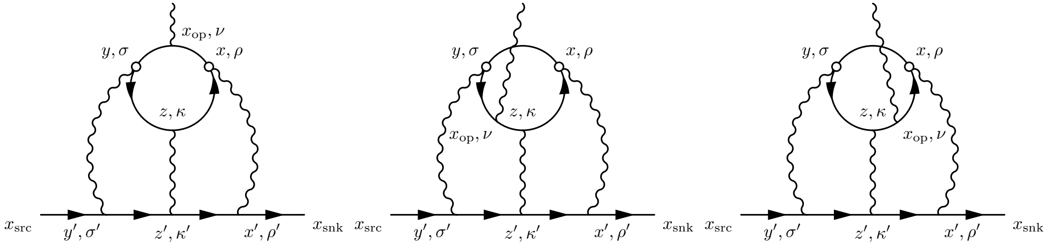
- Partial sum up to R_{max}
  $R_{\mathrm{max}} = \max(|x-y|, |x-z|, |y-z|)$
- The tadpole part comes from C. Lehner et al. 2016 (PRL 116, 232002)
- Systematic error (subdiscon): 0.5×10^{-10}





- 24D: $24^3 \times 64$
  $L = 4.8$ fm
- $a^{-1} = 1.015$ GeV
  $M_\pi = 142$ MeV
  $M_K = 512$ MeV

- Partial sum up to $R_{\max}$
  $R_{\max} = \max(|x-y|, |x-z|, |y-z|)$
- Systematic error (strange con): 0.3×10^{-10}



- The three internal vertex attached to the quark loop are equivalent (all permutations are included).
- We can pick the closer two points as the point sources x, y.

$$\sum_{x,y,z} \to \sum_{x,y,z} \begin{cases} 3 & \text{if } |x-y| < |x-z| \text{ and } |x-y| < |y-z| \\ 3/2 & \text{if } |x-y| = |x-z| < |y-z| \\ 3/2 & \text{if } |x-y| = |y-z| < |x-z| \\ 1 & \text{if } |x-y| = |y-z| = |x-z| \\ 0 & \text{others} \end{cases}$$

Split the a_μ^{con} into two parts:

$$a_\mu^{\text{con}} = a_\mu^{\text{con,short}} + a_\mu^{\text{con,long}}$$

- $a_\mu^{\text{con,short}} = a_\mu^{\text{con}}(r \leq 1\text{fm})$:
  most of the contribution, small statistical error.

- $a_\mu^{\text{con,long}} = a_\mu^{\text{con}}(r > 1\text{fm})$:
  small contribution, large statistical error.

Perform continuum extrapolation for short and long parts separately.

- $a_\mu^{\text{con,short}}$: conventional a^2 fitting.

- $a_\mu^{\text{con,long}}$: simply use 48I value.
  Conservatively estimate the relative $\mathcal{O}(a^2)$ error: it may be as large as for $a_\mu^{\text{con,short}}$ from 48I.

$$a_\mu(L, a^{\mathrm{I}}, a^{\mathrm{D}}) = a_\mu\Big(1 - \frac{b_2}{(m_\mu L)^2} - c_1^{\mathrm{I}}(a^{\mathrm{I}}\ \mathrm{GeV})^2 - c_1^{\mathrm{D}}(a^{\mathrm{D}}\ \mathrm{GeV})^2 + c_2^{\mathrm{D}}(a^{\mathrm{D}}\ \mathrm{GeV})^4\Big)$$

$\mathcal{O}(1/L^3)$

$$a_\mu(L, a^{\mathrm{I}}, a^{\mathrm{D}}) = a_\mu\Big(1 - \frac{b_2}{(m_\mu L)^2} + \frac{b_2}{(m_\mu L)^3} - c_1^{\mathrm{I}}(a^{\mathrm{I}}\ \mathrm{GeV})^2 - c_1^{\mathrm{D}}(a^{\mathrm{D}}\ \mathrm{GeV})^2 + c_2^{\mathrm{D}}(a^{\mathrm{D}}\ \mathrm{GeV})^4\Big)$$

$\mathcal{O}(a^2 \log(a^2))$

$$a_\mu(L, a^{\mathrm{I}}, a^{\mathrm{D}}) = a_\mu\Bigg(1 - \frac{b_2}{(m_\mu L)^2} - \Big(c_1^{\mathrm{I}}(a^{\mathrm{I}}\ \mathrm{GeV})^2 + c_1^{\mathrm{D}}(a^{\mathrm{D}}\ \mathrm{GeV})^2 - c_2^{\mathrm{D}}(a^{\mathrm{D}}\ \mathrm{GeV})^4\Big) \times\Big(1 - \frac{\alpha_S}{\pi}\log\big((a\ \mathrm{GeV})^2\big)\Big)\Bigg)$$

Sys error from difference of fits

$$a_\mu(L, a^{\mathrm{I}}, a^{\mathrm{D}}) = a_\mu\Big(1 - \frac{b_2}{(m_\mu L)^2} - c_1^{\mathrm{I}}(a^{\mathrm{I}}\ \mathrm{GeV})^2 - c_1^{\mathrm{D}}(a^{\mathrm{D}}\ \mathrm{GeV})^2 + c_2^{\mathrm{D}}(a^{\mathrm{D}}\ \mathrm{GeV})^4\Big)$$

$\mathcal{O}(a^4)$ (maximum of the following two)

$$a_\mu(L, a^{\mathrm{I}}, a^{\mathrm{D}}) = a_\mu\Big(1 - \frac{b_2}{(m_\mu L)^2} - c_1^{\mathrm{I}}(a^{\mathrm{I}}\ \mathrm{GeV})^2 - c_1^{\mathrm{D}}(a^{\mathrm{D}}\ \mathrm{GeV})^2 + c_2(a\ \mathrm{GeV})^4\Big)$$

$$a_\mu(L, a^{\mathrm{I}}, a^{\mathrm{D}}) = a_\mu\Big(1 - \frac{b_2}{(m_\mu L)^2} - c_1(a\ \mathrm{GeV})^2 + c_2^{\mathrm{I}}(a^{\mathrm{I}}\ \mathrm{GeV})^4 + c_2^{\mathrm{D}}(a^{\mathrm{D}}\ \mathrm{GeV})^4\Big)$$

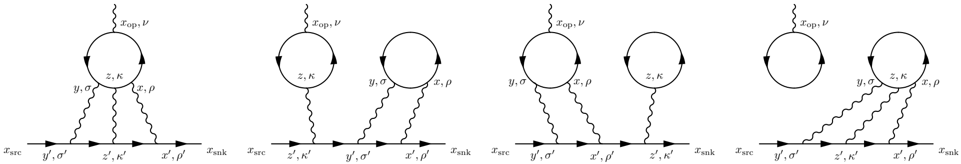
$$a_\mu(L, a^{\mathrm{I}}, a^{\mathrm{D}}) = a_\mu\Big(1 - \frac{b_2}{(m_\mu L)^2} - c_1^{\mathrm{I}}(a^{\mathrm{I}}\ \mathrm{GeV})^2 - c_1^{\mathrm{D}}(a^{\mathrm{D}}\ \mathrm{GeV})^2 + c_2^{\mathrm{D}}(a^{\mathrm{D}}\ \mathrm{GeV})^4\Big)$$

$\mathcal{O}(a^2/L)$ (maximum of the following two)

$$a_\mu(L, a^{\mathrm{I}}, a^{\mathrm{D}}) = a_\mu\left(1 - \frac{b_2}{(m_\mu L)^2} - \Big(c_1^{\mathrm{I}}(a^{\mathrm{I}}\ \mathrm{GeV})^2 + c_1^{\mathrm{D}}(a^{\mathrm{D}}\ \mathrm{GeV})^2 - c_2^{\mathrm{D}}(a^{\mathrm{D}}\ \mathrm{GeV})^4\Big)\Big(1 - \frac{1}{m_\mu L}\Big)\right)$$

$$a_\mu(L, a^{\mathrm{I}}, a^{\mathrm{D}}) = a_\mu\Big(1 - \frac{b_2}{(m_\mu L)^2}\Big) \times \Big(1 - c_1^{\mathrm{I}}(a^{\mathrm{I}}\ \mathrm{GeV})^2 - c_1^{\mathrm{D}}(a^{\mathrm{D}}\ \mathrm{GeV})^2 + c_2^{\mathrm{D}}(a^{\mathrm{D}}\ \mathrm{GeV})^4\Big)$$

When the 4 points of the 4-point function are all far separated:

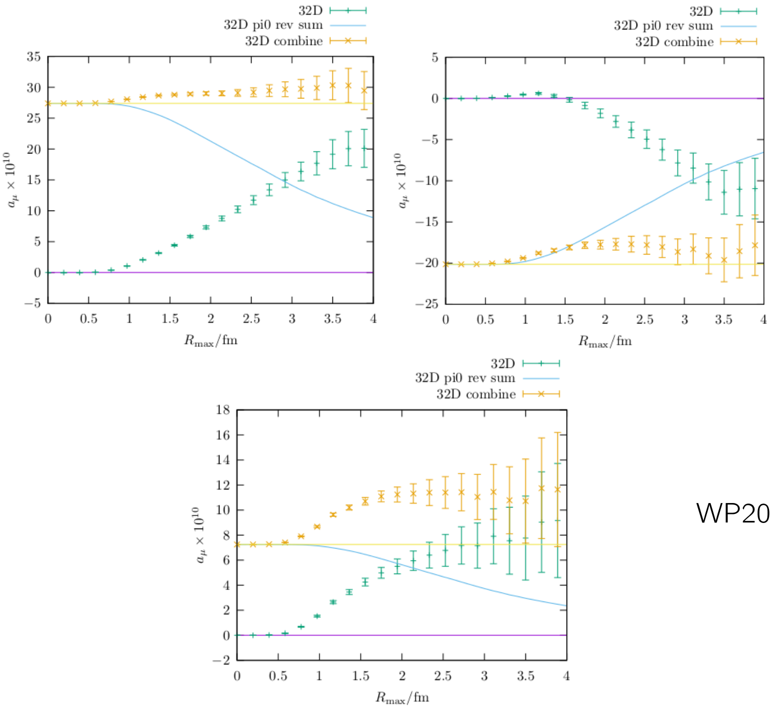


$$
\begin{aligned}
(e_u^4 + e_d^4) C \quad & \text{Connected diagram contribution} \\
(e_u^2 + e_d^2)^2 D \quad & \text{Leading order disconnected diagram contribution} \\
(e_u + e_d)(e_u^3 + e_d^3) D' \quad & \text{Next leading order disconnected diagram contribution}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{M} &\approx (e_u^4 + e_d^4) C + (e_u^2 + e_d^2)^2 D + (e_u + e_d)(e_u^3 + e_d^3) D' \\
&\propto (e_u - e_d)^4
\end{aligned} \tag{65}
$$

$$
C : D : D' = -2 : -3 : 4 \tag{66}
$$

$$
\text{Connected} : \text{LO-disconnected} : \text{NLO-disconnected} = -34 : -75 : 28 \tag{67}
$$



WP2020

Figure 91: Combined lattice and pion-pole contributions to the HLbL scattering part of the muon anomaly. Partial sums for the hadronic contributions, connected (top-left), leading disconnected (top-right), and total (bottom), computed with QED$_\infty$. $a^{-1} = 1$ GeV, $L = 6.4$ fm, and $M_\pi = 142$ MeV. Lines denote the π^0-pole contribution computed from the LMD model and are summed right-to-left.