Pseudoscalar pole contributions to HLbL from lattice QCD with physical quark masses

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Flavor Singlet Project for ETMC
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Project overview

Goal

Computing $\mathcal{F}_{P \to \gamma^* \gamma^*}$, $P = \pi_0, \eta, \eta'$ to determine the corresponding contributions to HLbL in the muon $g-2$.

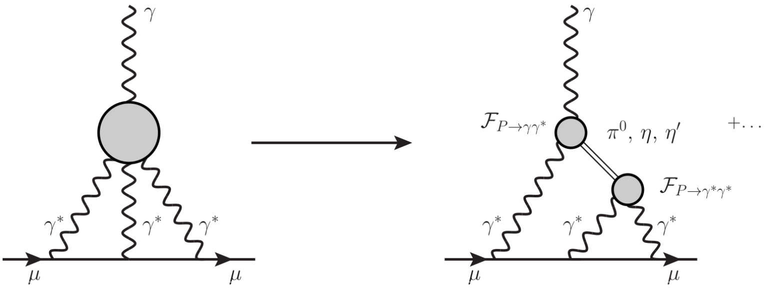
Using

- Twisted mass clover improved lattice QCD at maximal twist
- Four dynamical flavors ($N_f = 2 + 1 + 1$) [C. Alexandrou et al., arXiv:2104.06747]
- Analysis on

ensemble	$L^3 \cdot T/a^4$	m_π [MeV]	a [fm]	$a \cdot L_x$ [fm]	$m_\pi \cdot L_x$
cB072.64	$64^3 \cdot 128$	140.2	0.080	5.09	3.62
cC06.80	$80^3 \cdot 160$	136.7	0.068	5.46	3.78
cD054.96	$96^3 \cdot 192$	140.8	0.057	5.46	3.90

- Ensembles at the physical point (m_s also physical to within a few percent)

Hadronic light-by-light scattering



- Governed by rank-four hadronic vacuum polarization tensor.
- Numerically dominant role played by the π^0 pole, followed by η and η' poles.
- Nonperturbative information is encapsulated in the pole masses and the transition form factors $\mathcal{F}_{P\gamma^*\gamma^*}$.

Following [M. Knecht, A. Nyffeler, Phys. Rev. D65, 073034 (2002)], the transition form factors in Minkowski space-time are defined via the matrix element

$$M_{\mu\nu}(p,q_1) = i\int d^4x\, e^{iq_1x}\,\langle 0|T\{j_\mu(x)j_\nu(0)\}|P(p)\rangle$$
$$= \varepsilon_{\mu\nu\alpha\beta}\,q_1^\alpha q_2^\beta \mathcal{F}_{P\gamma^*\gamma^*}(q_1^2,q_2^2).$$

Defining

$$\tilde{A}_{\mu\nu}(\tau) = \langle 0|T\{j_\mu(\vec{q}_1,\tau)j_\nu(\vec{p}-\vec{q}_1,0)\}|P(p)\rangle\,,$$

the matrix element in Euclidean space-time is recovered by integration:

$$M^E_{\mu\nu} = \int_{-\infty}^{\infty} d\tau\, e^{\omega_1\tau}\tilde{A}_{\mu\nu}(\tau), \quad i^{n_0}M^E_{\mu\nu}(p,q_1) = M_{\mu\nu}(p,q_1).$$

On the lattice, starting from the amplitude

$$C_{\mu\nu}(\tau,t_P) = a^6\sum_{\vec{x},\vec{z}}\langle j_\mu(\vec{x},t_i)j_\nu(\vec{0},t_f)P^\dagger(\vec{z},t_0)e^{i\vec{p}\vec{z}}e^{-i\vec{x}\vec{q}_1}\rangle,$$

one constructs

$$\tilde{A}_{\mu\nu}(\tau) = \frac{2E_P}{Z_P}\lim_{t_P\to\infty} e^{E_P(t_f-t_0)}C_{\mu\nu}(\tau,t_P).$$

Kinematics

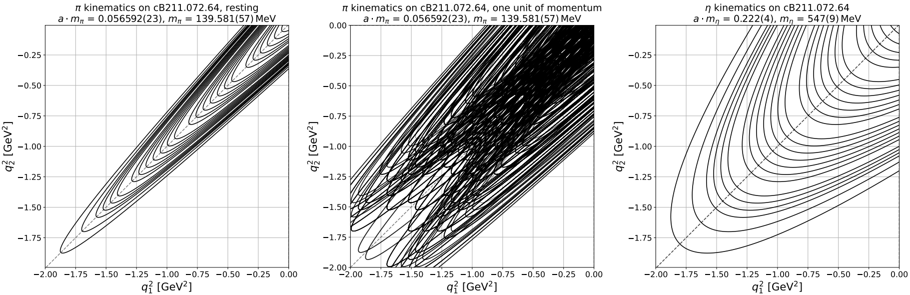
Free parameter ω_1, finite volume momenta q_i, p.

More coverage for larger m_P and $\vec{p} \neq \vec{0}$.

z-expansion fit for full coverage of (q_1^2, q_2^2) plane.

For $\vec{p} = \vec{0}$:

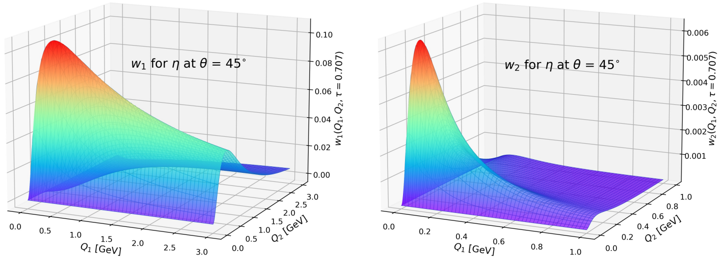
- $q_1^2 = \omega_1^2 - \vec{q}_1^2$,
  $q_2^2 = (m_P - \omega_1)^2 - \vec{q}_1^2$
- $-\sqrt{m_V^2 + \vec{q}_1^2} + m_P < \omega_1 < \sqrt{m_V^2 + \vec{q}_1^2}$
- $\tilde{A}(\tau) = i m_P^{-1} \varepsilon_{ijk} \frac{\vec{q}_1^i}{\vec{q}_1^2} \tilde{A}_{jk}(\tau)$



Pseudoscalar pole contribution

3d integral representation [A. Nyffeler, Phys. Rev. D94, 053006 (2016) and refs. therein]

$$a_\mu^{P-\text{pole}} = \left(\frac{\alpha}{\pi}\right)^3 \int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^{+1} d\tau \Big[w_1(Q_1, Q_2, \tau)\mathcal{F}_{P\gamma^*\gamma^*}(-Q_1^2, -(Q_1+Q_2)^2)\mathcal{F}_{P\gamma^*\gamma}(-Q_2^2, 0) + w_2(Q_1, Q_2, \tau)\mathcal{F}_{P\gamma^*\gamma^*}(-Q_1^2, -Q_2^2)\mathcal{F}_{P\gamma^*\gamma}(-(Q_1+Q_2)^2, 0)\Big]$$



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Current operators and isospin combinations

- Decompose current into definite isospin combination

$$j_\mu^l(x) = \frac{2}{3}\bar{u}\gamma_\mu u(x) - \frac{1}{3}\bar{d}\gamma_\mu d(x) = \frac{1}{6}j_\mu^{(0,0)} + \frac{1}{2}j_\mu^{(1,0)}.$$

- For the amplitude, this yields

$$C_{\mu\nu} = \frac{1}{6}\left\langle \pi_0\, j_\mu^{(1,0)}\, j_\nu^{(0,0)} \right\rangle .$$

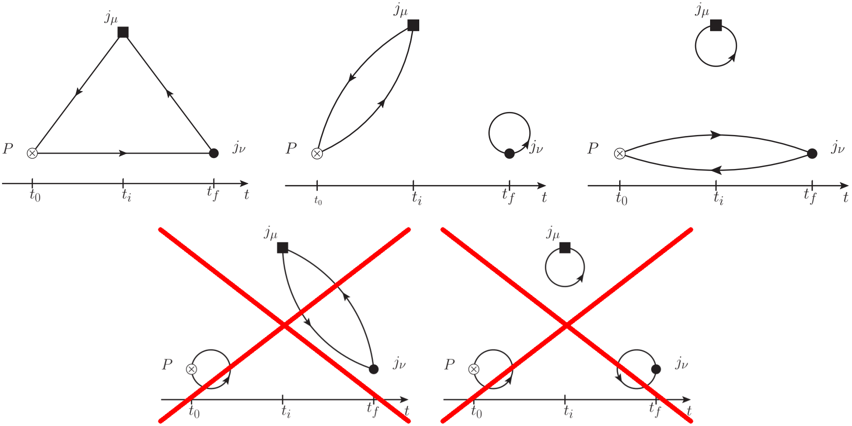
- Isospin symmetry allows the rotation

$$\left\langle \pi_0 j_\nu^{(1,0)} j_\mu^{(0,0)} \right\rangle \to \left\langle \pi_- j_\nu^{(1,+)} j_\mu^{(0,0)} \right\rangle + \left\langle \pi_+ j_\nu^{(1,-)} j_\mu^{(0,0)} \right\rangle ,$$

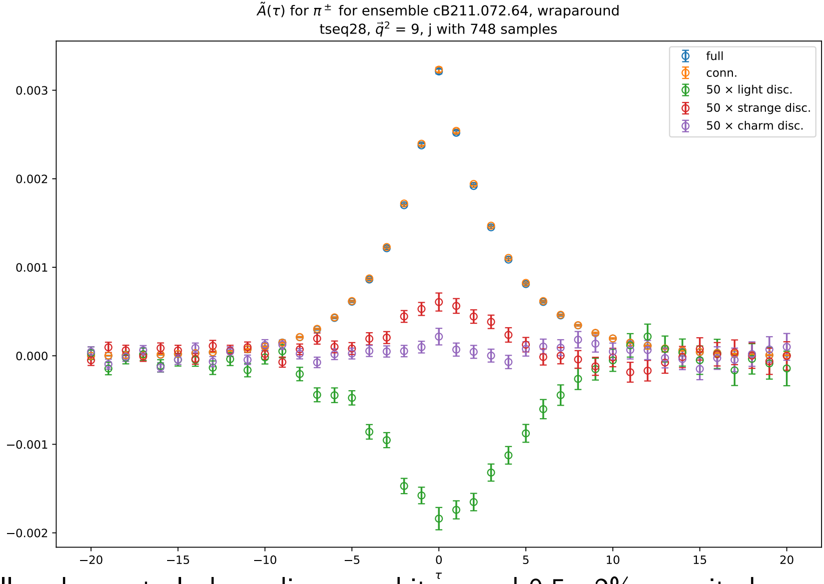
removing two disconnected Wick contractions. They differ only by an $\mathcal{O}(a^2)$ lattice artefact.

Considered diagrams π

The amplitude $C_{\mu\nu}$ only contains connected and vector current disconnected Wick contractions after rotating to the charged pion.

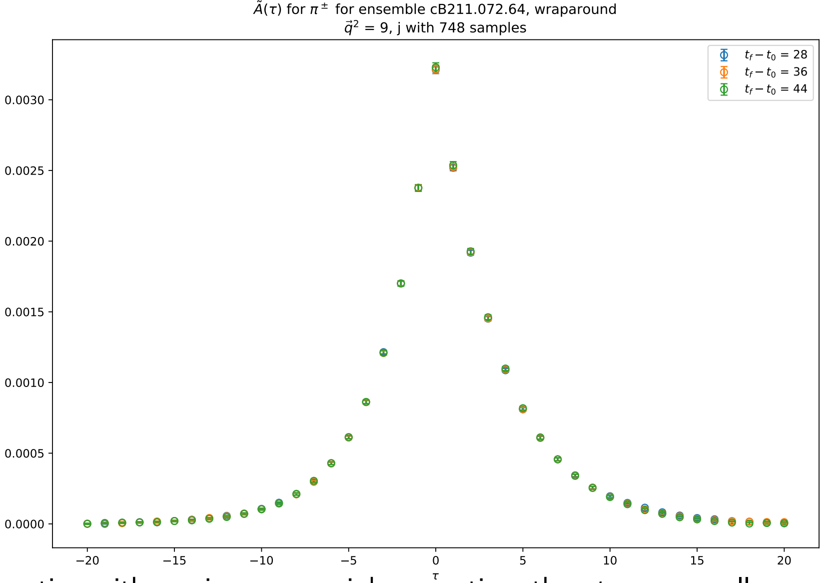


Vector current disconnected



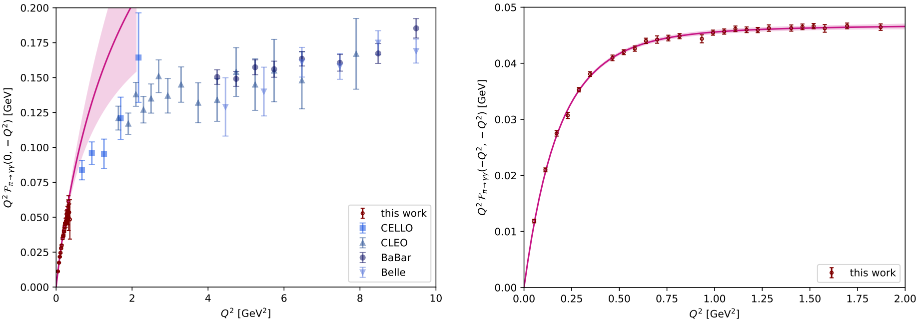
Well under control, depending on orbit around 0.5 - 2% magnitude compared to connected.

Source-sink separations



Saturation with varying source-sink separation, thus $t_P \to \infty$ well approximated.

Comparison to experimental data

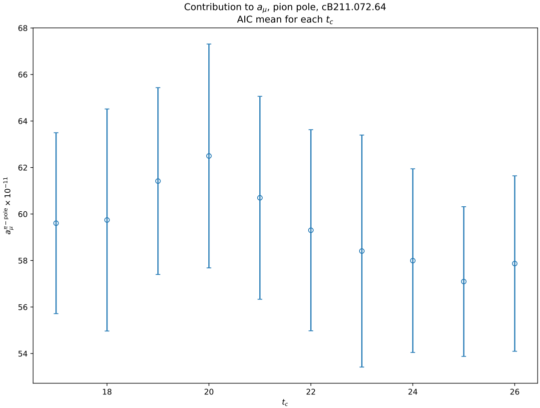


TFFs from data most precise for diagonal and close to diagonal (space-like) kinematics for the pion.

[H. J. Behrend et al. (CELLO), Z. Phys. C49, 401 (1991)] [J. Gronberg et al. (CLEO), Phys. Rev. D57, 33 (1998)] [B. Aubert et al. (BABAR), Phys. Rev. D80, 052002 (2009)] [P. del Amo Sanchez et al. (BABAR), Phys. Rev. D84, 052001 (2011)] [S. Uehara et al. (Belle), Phys. Rev. D86, 092007 (2012)]

Pion pole contribution to a_μ

- Scan over range of parameters: Fitwindows, t_c, sampling of momentum plane, order of z-expansion fit, ...
- Average and systematic errors obtained from AIC [Borsanyi et al., Science 347, 1452 (2015)], inclusive error budgets for choices of paramters.

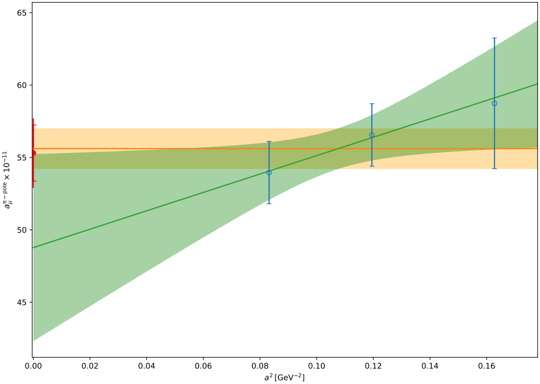


Pion pole contribution to a_μ

Preliminary continuum result

$$a_\mu^{\pi-\text{pole}} = 55.3(2.0)_{stat}(1.5)_{sys}[2.5]_{tot} \times 10^{-11}.$$

- Constant fit underestimates error, linear fit overestimates error.
- Take weighted average and variation between fits as systematic error [Alexandrou et al. (ETMC), 2104.13408].



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Current operators and isospin combinations

- Also consider strange contributions in the electromagnetic current, i.e.

$$j_\mu(x) = \underbrace{\frac{2}{3}\bar{u}\gamma_\mu u(x) - \frac{1}{3}\bar{d}\gamma_\mu d(x)}_{=j^l_\mu(x)} - \underbrace{\frac{1}{3}\bar{s}\gamma_\mu s(x)}_{=j^s_\mu(x)} = \frac{1}{6}j^{l,(0,0)}_\mu + \frac{1}{2}j^{l,(1,0)}_\mu + j^s_\mu .$$

- Decomposition into definite isospin for the light contribution yields

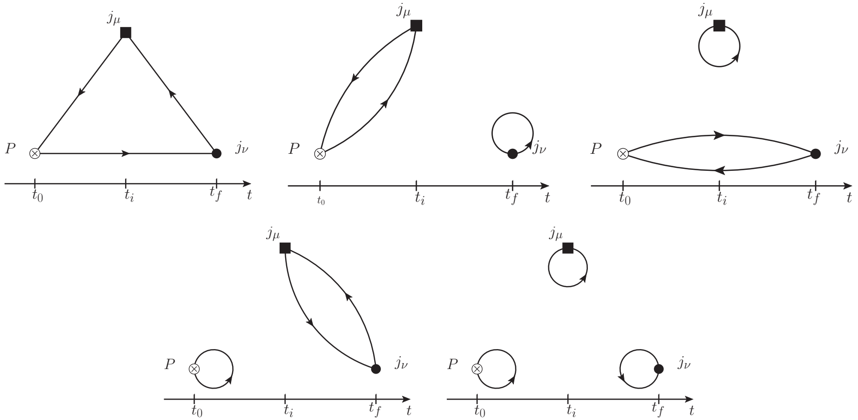
$$C^{l-current}_{\mu\nu} = \frac{1}{4}\left\langle \eta\, j^{l,(0,0)}_\mu\, j^{l,(0,0)}_\nu \right\rangle + \frac{1}{36}\left\langle \eta\, j^{l,(1,0)}_\mu\, j^{l,(1,0)}_\nu \right\rangle$$

- Osterwalder-Seiler is used for introducing the strange quark. [R. Frezzotti, G. C. Rossi, JHEP 10 (2004) 070]
- We project onto the correct ground state for the η. Mixing would be needed for η'.

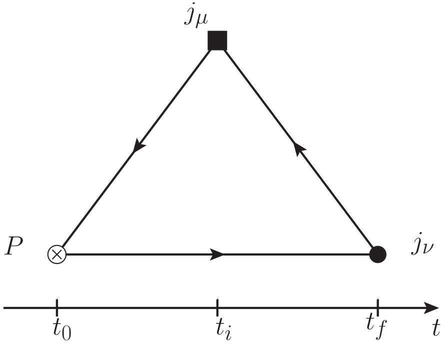
$$\eta \approx \eta_8 \propto \bar{u}u + \bar{d}d - 2\bar{s}s$$
$$\eta' \approx \eta_1 \propto \bar{u}u + \bar{d}d + \bar{s}s$$

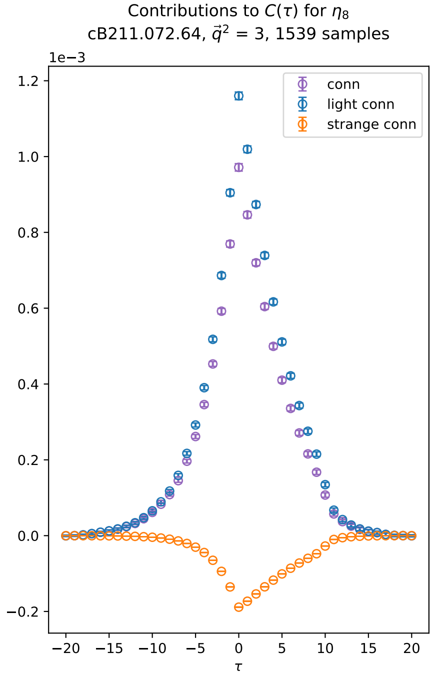
Considered diagrams η

The amplitude $C_{\mu\nu}$ contains connected, vector current disconnected, pseudoscalar disconnected and doubly disconnected Wick contractions.

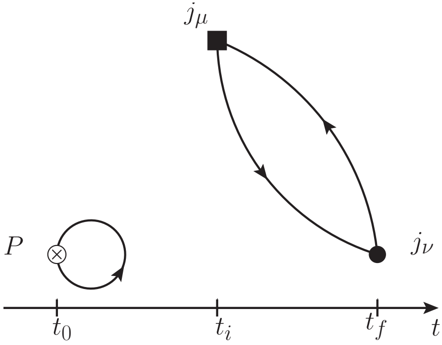


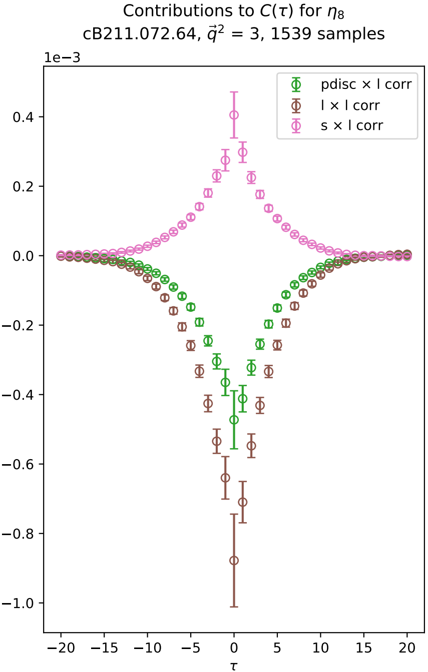
Considered diagrams η



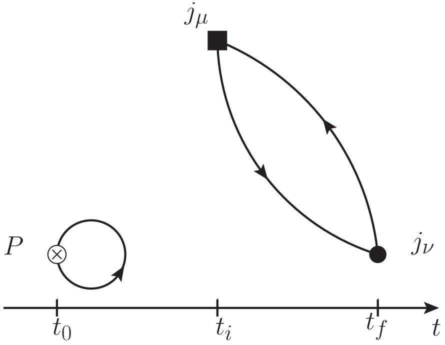


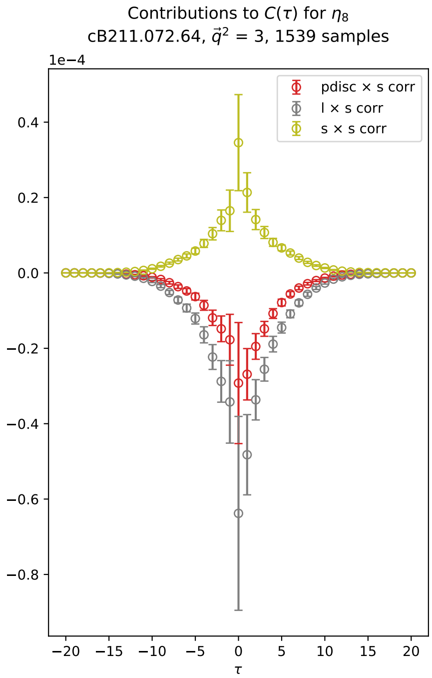
Considered diagrams η



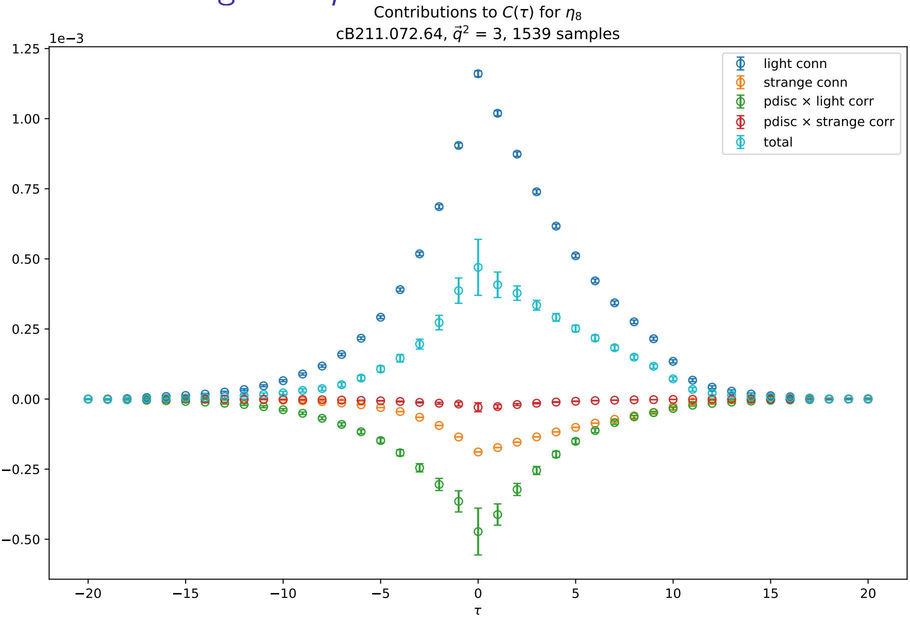


Considered diagrams η





Considered diagrams η



Tail fits

We need

$$\mathcal{F}(q_1^2, q_2^2) = \int_{-t_c}^{t_c} d\tau\, \tilde{A}^{(latt.)}(\tau) e^{\omega_1 \tau} + \int_{\pm t_c}^{\pm\infty} d\tau\, \tilde{A}^{(fit)}(\tau) e^{\omega_1 \tau}.$$

Following [Gerardin et al., Phys. Rev. D94, 074507 (2016) and refs. therein], consider

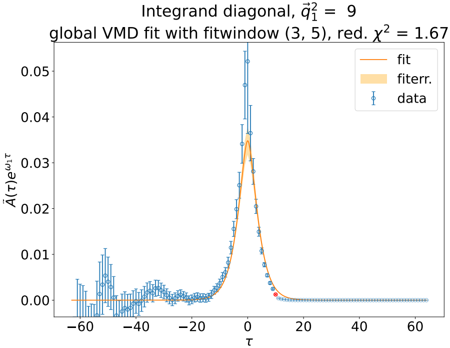
- Vector meson dominance (VMD) model:

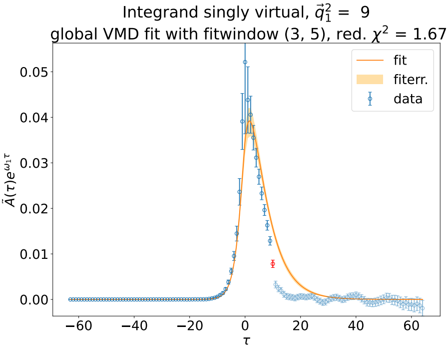
$$\mathcal{F}^{VMD}_{P\gamma^*\gamma^*}(q_1^2, q_2^2) = \frac{\alpha M_V^4}{(M_V^2 - q_1^2)(M_V^2 - q_2^2)} \to \tilde{A}^{(fit,VMD)}(\tau).$$

- Lowest meson dominance (LMD) model:

$$\mathcal{F}^{LMD}_{P\gamma^*\gamma^*}(q_1^2, q_2^2) = \frac{\alpha M_V^4 + \beta(q_1^2 + q_2^2)}{(M_V^2 - q_1^2)(M_V^2 - q_2^2)} \to \tilde{A}^{(fit,LMD)}(\tau).$$

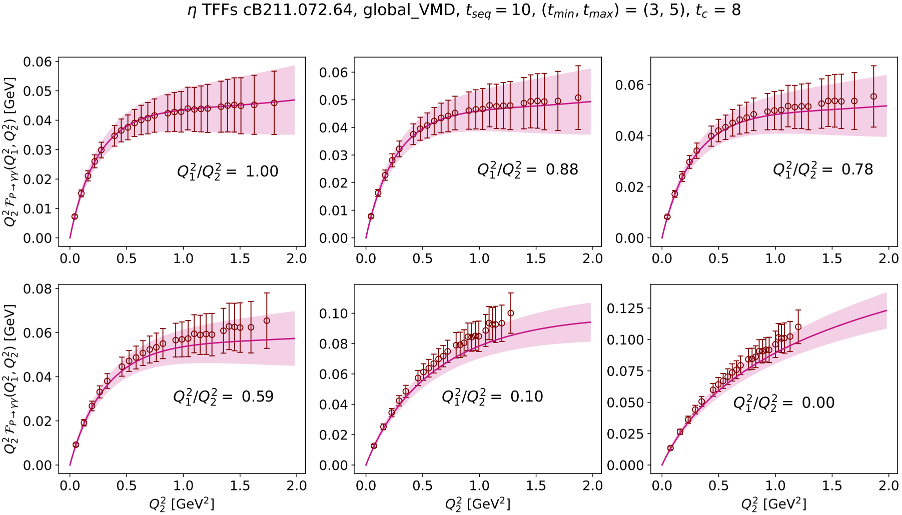
Integrands





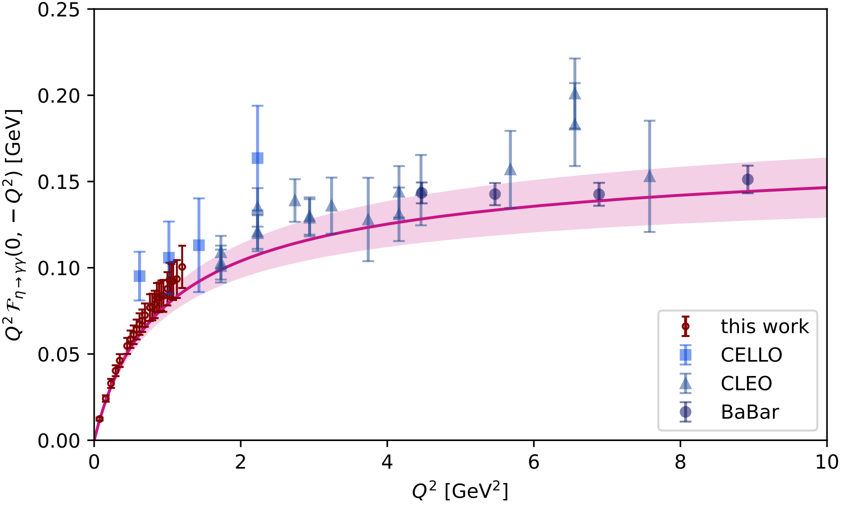
- Diagonal kinematics: $q_1^2 = q_2^2 \Rightarrow \omega_1 = m_\eta/2$.
- Singly virtual kinematics: $q_1^2 = 0 \Rightarrow \omega_1 = |\vec{q}_1|$.
- Global fit to model, i.e. simultaneous for all orbits $1 \leq \vec{q}^2(2\pi/L_x)^2 \leq 32$.

Example TFFs



z-expansion fit: [Gerardin et al., Phys. Rev. D100, 034520 (2019) and refs. therein]

Comparison to experimental data



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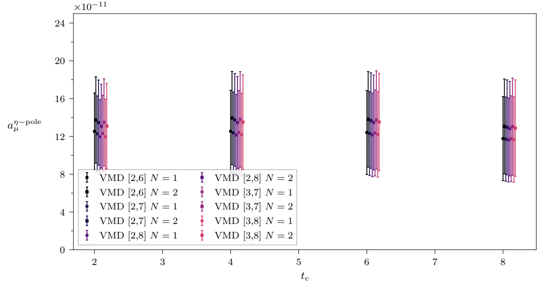
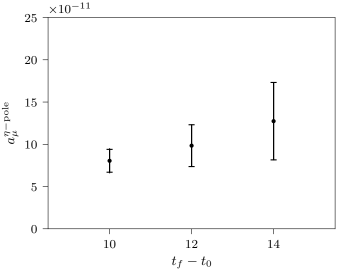
Eta pole contribution to a_μ





AIC model averaging gives a preliminary

$$a_\mu^{\eta-\text{pole}} = 12.7(4.6)_{stat}(0.7)_{sys}[4.6]_{tot} \times 10^{-11}.$$

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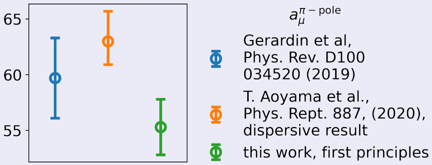
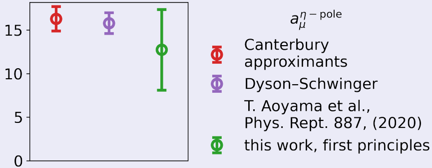
Conclusion & Outlook

Summary

Our setup allows the determination of $a_\mu^{\pi-\text{pole}}$ and a first determination of $a_\mu^{\eta-\text{pole}}$ from lattice QCD directly at the physical point. We get promising preliminary results, compatible with other calculations, e.g.





Next steps

- More precise characterization of the systematic error for η, wrap up analysis for π.
- Analysis of η on the other physical point ensembles, continuum limit estimation.

BACKUP - Decay width

At leading order in α_e, one gets for the pseudoscalar decay width

$$\Gamma(P \to \gamma\gamma) = \frac{\pi\alpha_e^2 m_P^3}{4}\mathcal{F}^2_{P\to\gamma\gamma}(0,0).$$

The current experimental results are

- $\Gamma(\pi^0 \to \gamma\gamma) = 7.802(117)\,\text{eV}$ [I. Larin et al.(2020), Science, 368 (6490) : 506-509]
- $\Gamma(\eta \to \gamma\gamma) = 0.516(19)\,\text{keV}$ [P.A. Zyla et al. (PDG), Prog.Theor.Exp.Phys.2020, 083 C01 (2020)]
  ($\Gamma(\eta \to \gamma\gamma) = 0.324(46)\,\text{keV}$ [A. Browman et al., Phys. Rev. Lett. 32, 1067 (1974)])

We find systematically smaller values:

	cB211.072.64	cC211.06.80	cD211.054.96	cont. lim
$\Gamma(\pi^0 \to \gamma\gamma)$ [eV]	6.01(44)(12)[45]	6.98(35)(11)[37]	7.64(49)(32)[58]	6.91(0.35)(0.54)[0.65]
$\Gamma(\eta \to \gamma\gamma)$ [keV]	0.312(76)(16)[78]			

BACKUP - Ensembles

ensemble	$L^3 \cdot T/a^4$	β	c_{SW}	M_π [MeV]	a [fm]	$a\mu_\ell$	$a\mu_\sigma$	$a\mu_\delta$	κ_{crit}
cA53.24	$24^3 \cdot 48$	1.726	1.74	350	0.0927	0.0053	0.1408	0.1521	0.1400645
cA40.24	$24^3 \cdot 48$	1.726	1.74	301	0.0927	0.0040	0.1408	0.1521	0.1400645
cA30.32	$32^3 \cdot 64$	1.726	1.74	260	0.0927	0.0030	0.1408	0.1521	0.1400645
cA12.48	$48^3 \cdot 96$	1.726	1.74	165	0.0927	0.0012	0.1408	0.1521	0.1400645
cB25.48	$48^3 \cdot 96$	1.778	1.69	255	0.0800	0.0025	0.1247	0.1315	0.1394267
cB140.64	$64^3 \cdot 128$	1.778	1.69	190	0.0800	0.00140	0.1247	0.1315	0.1394267
cB072.64	$64^3 \cdot 128$	1.778	1.69	135	0.0800	0.00072	0.1247	0.1315	0.1394265
cC06.80	$80^3 \cdot 160$	1.836	1.645	135	0.069	0.0006	0.1060	0.1135	0.1387510

Table: Parameter values for the gauge configurations available through (and under production by) ETMC with $N_f = 2+1+1$ Wilson clover twisted mass quark flavours. For each ensemble we provide the volume, the gauge coupling β, the clover coefficient c_{SW}, the pion mass M_π and the lattice spacing a in physical units, the bare twisted mass values $a\mu_\ell$, $a\mu_\sigma$, $a\mu_\delta$, and the hopping parameter κ_{crit}.

BACKUP - Clover improved tmLQCD action [C. Alexandrou et al., Phys. Rev. D98, 054518 (2018)]

$$S = S_g + S_{tm}^l + S_{tm}^h$$

Iwasaki improved gauge action for S_g:

$$S_g = \frac{\beta}{3} \sum_x \left(b_0 \sum_{\substack{\mu,\nu=1 \\ 1 \leq \mu < \nu}}^{4} \left\{ 1 - \mathrm{Re}\,\mathrm{Tr}\left(U^{1\times1}_{x,\mu,\nu}\right) \right\} + b_1 \sum_{\substack{\mu,\nu=1 \\ \mu \neq \nu}}^{4} \left\{ 1 - \mathrm{Re}\,\mathrm{Tr}\left(U^{1\times2}_{x,\mu,\nu}\right) \right\} \right)$$

Light up and down doublet:

$$S_{tm}^l = \sum_x \bar{\chi}_l(x) \left[D_W(U) + \frac{i}{4} c_{SW} \sigma^{\mu\nu} \mathcal{F}^{\mu\nu}(U) + m_l + i\mu_l \tau^3 \gamma^5 \right] \chi_l(x)$$

Heavy quark action, non-degenerate strange and charm quarks*:

$$S_{tm}^h = \sum_x \bar{\chi}_h(x) \left[D_W(U) + \frac{i}{4} c_{SW} \sigma^{\mu\nu} \mathcal{F}^{\mu\nu}(U) + m_h - \mu_\delta \tau_1 + i\mu_\sigma \tau^3 \gamma^5 \right] \chi_h(x)$$

*In practise, we use $m_h = m_l$, this constitutes an additional $\mathcal{O}(a^2)$ lattice artefact which is small for the strange quark and practically vanishes for the charm quark.

BACKUP - Extraction of the form factors

We follow the conventions from [A. Gérardin, H. B. Meyer and A. Nyffeler, Phys. Rev. D94, 074507 (2016)]

- The relevant hadronic quantity is the rank-four hadronic vacuum polarization tensor

$$\Pi_{\mu\nu\lambda\rho}(q_1, q_2, q_3) = \int d^4x_1 d^4x_2 d^4x_3 \Big[e^{i(q_1 x_1 + q_2 x_2 + q_3 x_3)} \cdot \langle 0 | T\{ j_\mu(x_1) j_\nu(x_2) j_\lambda(x_3) j_\rho(0) \} | 0 \rangle \Big]$$

- Each pseudoscalar pole $P \in \{\pi_0, \eta, \eta'\}$ contributes to the amplitude via one-particle-reducible single pseudoscalar exchanges

$$\Pi^{(P)}_{\mu\nu\lambda\rho}(q_1, q_2, q_3) = \Big[i \frac{\mathcal{F}_{P\gamma^*\gamma^*}(q_1^2, q_2^2) \mathcal{F}_{P\gamma^*\gamma^*}(q_3^2, (q_1 + q_2 + q_3)^2)}{(q_1 + q_2)^2 + M_P^2} \cdot \epsilon_{\mu\nu\alpha\beta} q_1^\alpha q_2^\beta \epsilon_{\lambda\rho\gamma\delta} q_3^\gamma (q_1 + q_2)^\delta \Big] + (\text{crosses})$$

BACKUP - z-expansion

Extract $\mathcal{F}_{\pi_0\gamma^*\gamma^*}$ over the whole kinematical range using the modified z-expansion [Gerardin et al., Phys. Rev. D100, 034520 (2019) and refs. therein]:

$$P(Q_1^2,Q_2^2)\mathcal{F}_{\pi_0\gamma^*\gamma^*}(Q_1^2,Q_2^2) = \sum_{m,n=0}^{N} c_{nm}\left(z_1^n-(-1)^{N+n+1}\frac{n}{N+1}z_1^{N+1}\right)\left(z_2^m-(-1)^{N+m+1}\frac{m}{N+1}z_2^{N+1}\right)$$

- $z_k = z_k(Q_k^2)$, $P(Q_1^2,Q_2^2)$ a polynomial in four-momenta
- Sampling in momentum plane by fixing Q_2^2/Q_1^2 using continuous free parameter ω.
- Correlated order $N \in 1, 2, 3$ fits.

BACKUP - z-expansion - detailed

Extract $\mathcal{F}_{\pi_0\gamma^*\gamma^*}$ over the whole kinematical range using the modified z-expansion [Gerardin et al., Phys. Rev. D100, 034520 (2019)]:

- $z_k = \frac{\sqrt{t_c+Q_k^2}-\sqrt{t_c-t_0}}{\sqrt{t_c+Q_k^2}+\sqrt{t_c-t_0}}$, $k \in \{1,2\}$
- $P(Q_1^2, Q_2^2) = 1 + \frac{Q_1^2+Q_2^2}{M_V^2}$

- Map branch cut starting at $t_c = 4m_\pi^2$ onto unit circle $|z_k| = 1$.
- Free parameter t_0, optimal choice $t_0 = t_c(1 - \sqrt{1 + Q_{max}^2/t_c})$.

$$P(Q_1^2,Q_2^2)\mathcal{F}_{\pi_0\gamma^*\gamma^*}(Q_1^2, Q_2^2) = \sum_{m,n=0}^{N} c_{nm}\left(z_1^n - (-1)^{N+n+1}\frac{n}{N+1}z_1^{N+1}\right)\left(z_2^m - (-1)^{N+m+1}\frac{m}{N+1}z_2^{N+1}\right)$$

- Sampling in momentum plane by fixing Q_2^2/Q_1^2 using continuous free parameter ω.
- Correlated order $N \in 1, 2, 3$ fits.

BACKUP - Projection on η

By the following argument, we still extract $a_\mu^{\eta-\text{pole}}$ easily: Note that η is the ground state in the η/η' system and η_8 the ground state in the η_8/η_1 system, and that η/η' are linear combinations of η_8/η_1. Explicitely,

$$\eta = \cos\theta_P \cdot \eta_8 + \sin\theta_P \cdot \eta_1. \tag{1}$$

For asymptotically large Euclidean time the ground state is uniquely the state of lowest mass, which survives. Thus

$$\begin{aligned}\langle 0|\mathcal{O}(t)|\cos\theta_P \cdot \eta_8 + \sin\theta_P \cdot \eta_1\rangle \overset{t\to\infty}{=} &\cos\theta_P \langle 0|\mathcal{O}(0)|\eta_8\rangle \cdot e^{-m_{\eta_8} t} \\ &+ \sin\theta_P \langle 0|\mathcal{O}(0)|\eta_1\rangle \cdot e^{-m_{\eta_1} t}\end{aligned} \tag{2}$$

So with ground state projection, after dividing out the overlap, we directly obtain $\tilde{A}$ for η (of course $\langle 0|\mathcal{O}(0)|\eta_8\rangle$ and $\langle 0|\mathcal{O}(0)|\eta\rangle$ differ, but $\langle 0|\mathcal{O}(0)|\eta_8\rangle$ vanishes in the division).