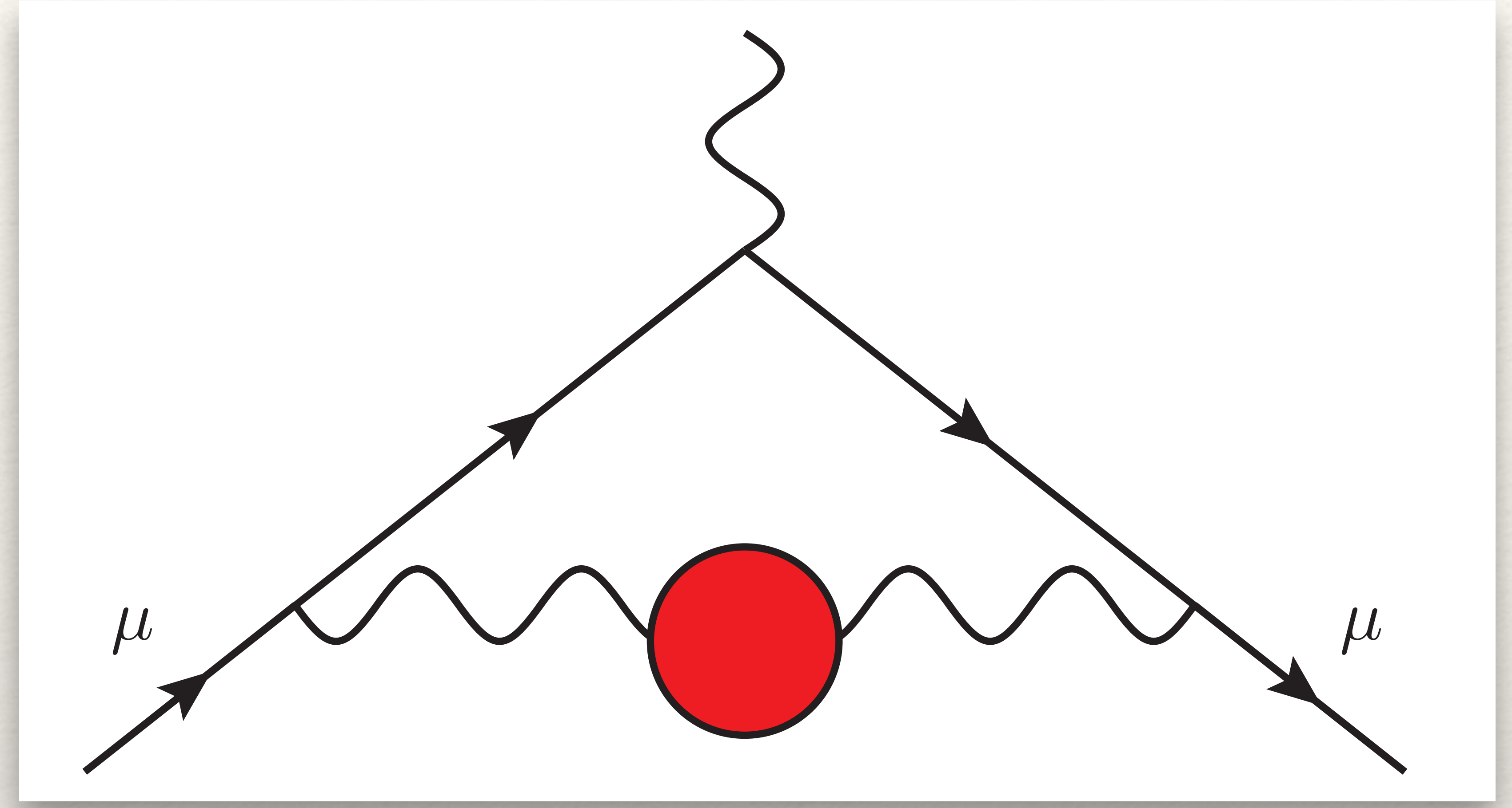
Fifth Plenary Workshop of the Muon g-2 Theory Initiative
Edinburgh
Sept 5-9 2022

NLO and NNLO HVP contributions to the muon $g-2$ in the space-like region

Elisa Balzani
Università degli Studi di Padova
INFN Padova

Work in Collaboration with S. Laporta and M. Passera, arXiv: 2112.05704

The hadronic LO contribution: time-like method



$$a_\mu^{HVP}(LO) = \frac{\alpha}{\pi^2}\int_{s_0^2}^{\infty}\frac{ds}{s}K^{(2)}(s/m^2)\mathrm{Im}\Pi_\mathrm{h}(\mathrm{s})$$

$$\mathrm{Im}\Pi_\mathrm{h}(\mathrm{s}) = \frac{\alpha}{3}\mathrm{R(s)} \qquad \mathrm{R(s)} = \frac{\sigma(\mathrm{e^+e^-} \to \mathrm{hadrons})}{4\pi\alpha^2/(3\mathrm{s})}$$

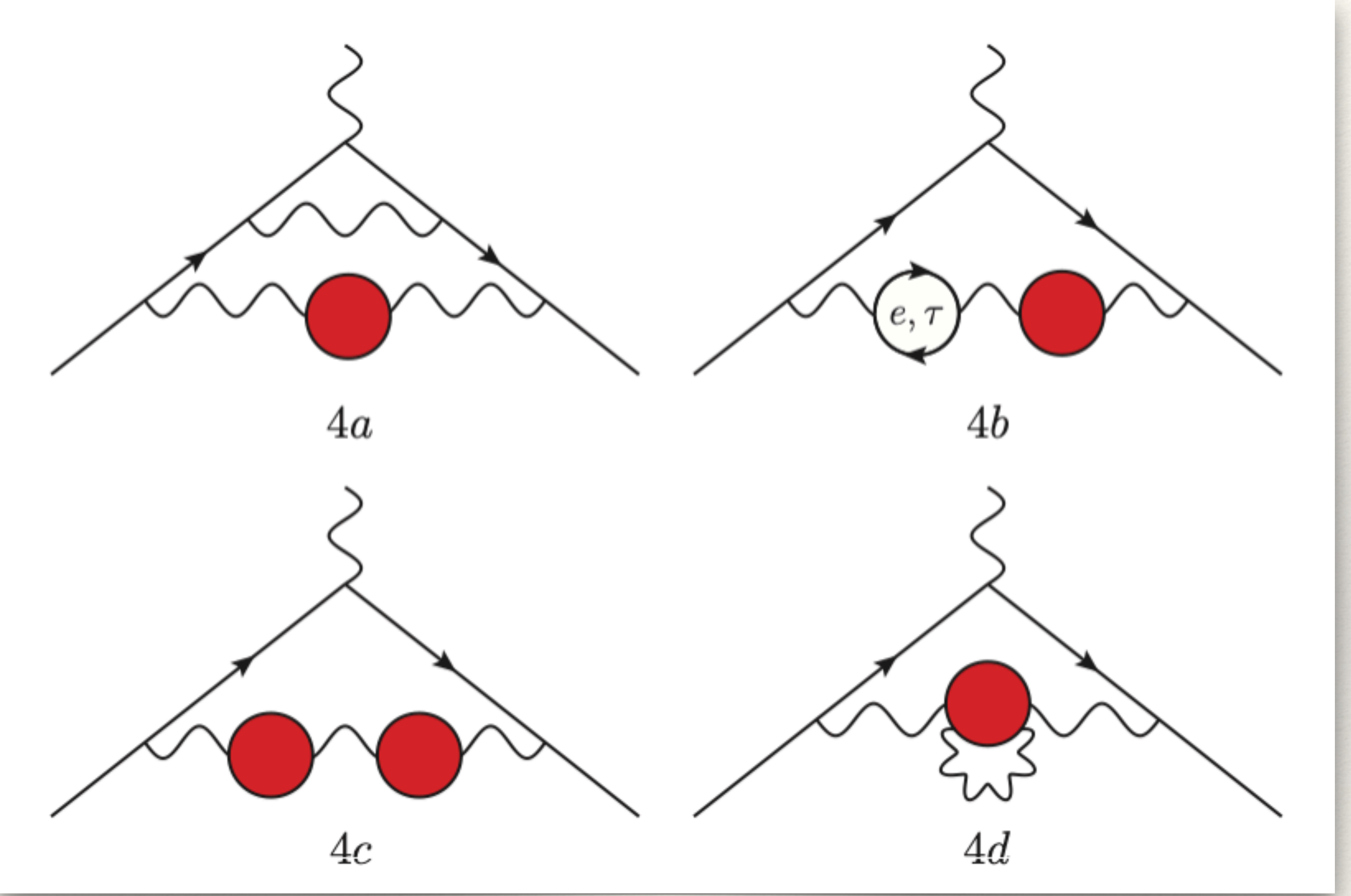
$$K^{(2)}(s/m^2) = \int_0^1 dx\frac{x^2(1-x)}{x^2+(1-x)(s/m^2)}$$

Defining $z = \frac{q^2}{m^2}$ & $y(z) = \frac{z-\sqrt{z(z-4)}}{z+\sqrt{z(z-4)}}$:

$$K^{(2)}(z) = \frac{1}{2} - z + \left(\frac{z^2}{2} - z\right)\ln z + \frac{\ln y(z)}{\sqrt{z(z-4)}}\left(z - 2z^2 + \frac{z^3}{2}\right)$$

The hadronic NLO VP contribution

- $O(\alpha^3)$ contributions containing HVP insertion

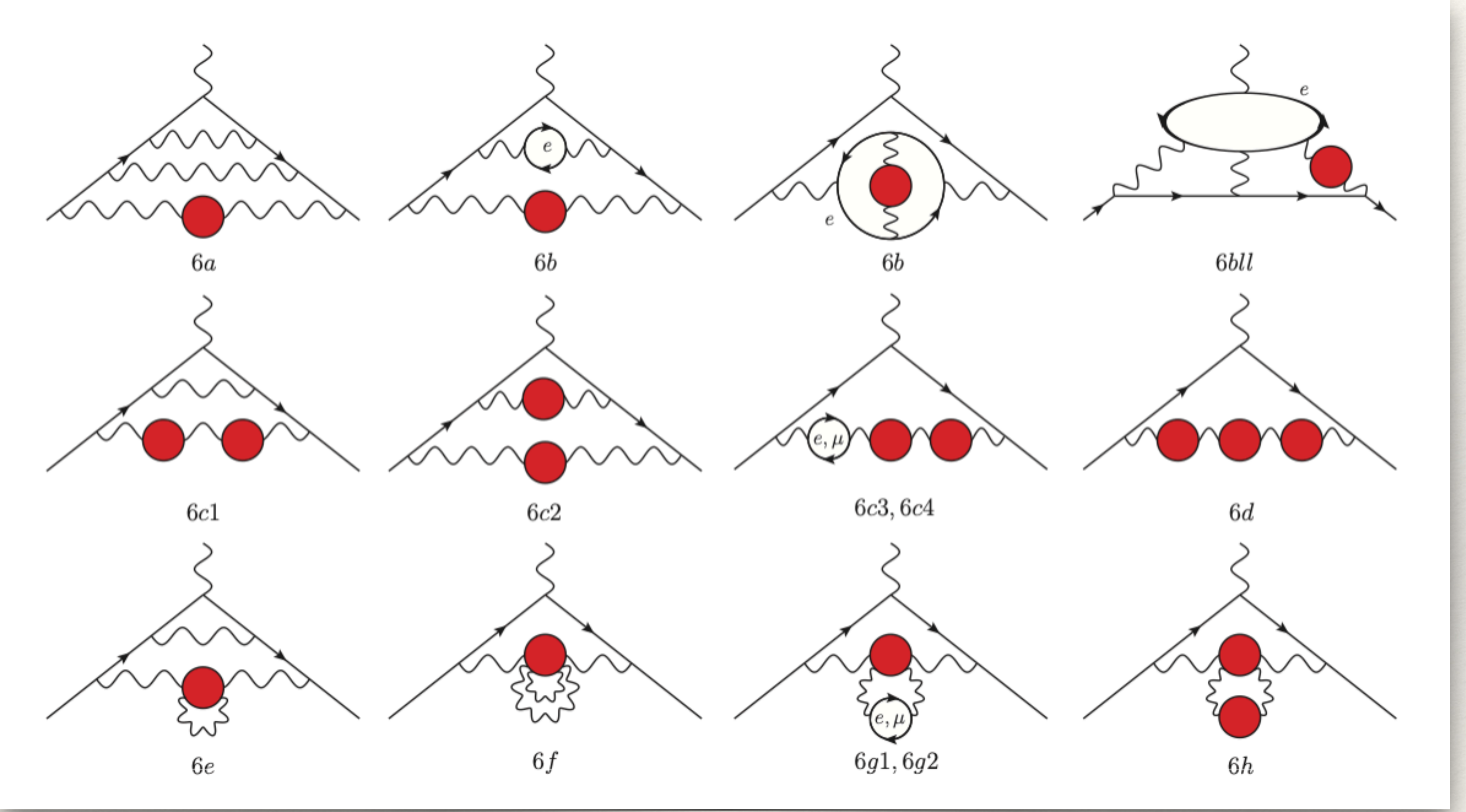


$$a_\mu^{HVP}(NLO) = -98.3(7) \times 10^{-11}$$

Krause '96; Keshevarzi, Nomura, Teubner 2019; WP20

The hadronic NNLO VP contribution

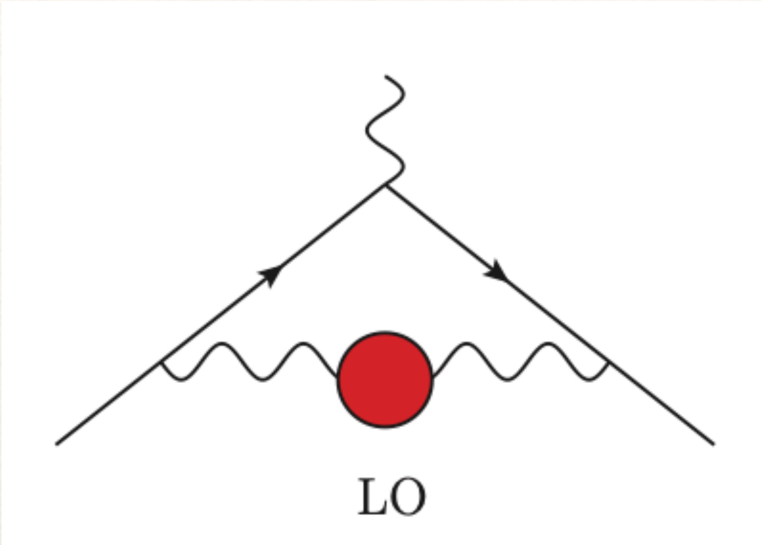
- $O(\alpha^4)$ contributions containing HVP insertion



$$a_\mu^{HVP}(NNLO) = 12.4(1) \times 10^{-11}$$

Kurz, Liu, Marquard, Steinhauser 2014

Space-like method: LO HVP contribution



$$K^{(2)}(z) = \frac{1}{\pi}\int_{-\infty}^{0} dz' \frac{\mathrm{Im}K^{(2)}(z')}{z'-z}, \quad z > 0 \qquad \frac{\Pi(q^2)}{q^2} = \frac{1}{\pi}\int \frac{ds}{s}\frac{\mathrm{Im}\Pi(s)}{s-q^2}, \quad q^2 < 0$$

$$a_\mu^{HVP}(LO) = -\frac{\alpha}{\pi^2}\int_{-\infty}^{0}\frac{dt}{t}\Pi_h(t)\mathrm{Im}K^{(2)}(t/m^2)$$

With the imaginary part:

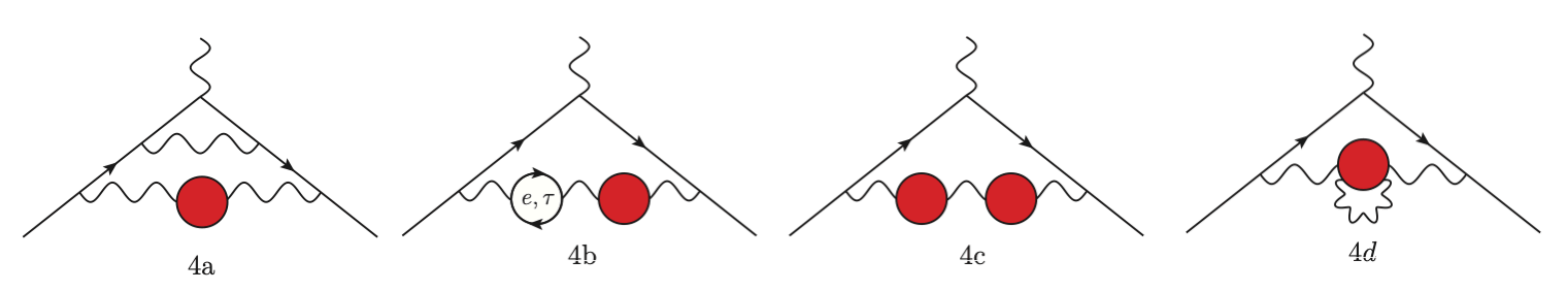
$$\mathrm{Im}K^{(2)}(z+i\epsilon) = \pi\theta(-z)\left[\frac{z^2}{2} - z + \frac{z - 2z^2 + z^3/2}{\sqrt{z(z-4)}}\right] = \pi\theta(-z)F^{(2)}(1/y(z)), \qquad F^{(2)}(u) = \frac{u+1}{u-1}u^2$$

Changing the variable in the dispersive integral $t \to y \to x$: $t(x) = \frac{m^2x^2}{1-x^2}$, $\quad x = 1 + y \qquad \Delta\alpha_h(t) = -\Pi_h(t)$

$$a_\mu^{HVP}(LO) = \frac{\alpha}{\pi}\int_0^1 dx\, \kappa^{(2)}(x)\Delta\alpha_h(t(x)) \qquad \kappa^{(2)}(x) = 1 - x$$

Lautrup, Peterman, de Rafael 1972

Space-like method: NLO HVP contribution



$$a_\mu^{(4a)} = \frac{\alpha^2}{\pi^3}\int_{s_0}^{\infty}\frac{ds}{s}2K^{(4)}(s/m^2)\mathrm{Im}\Pi_h(s)$$

$K^{(4)}$ from Barbieri Remiddi 1975

$$a_\mu^{(4a)} = -\frac{\alpha^2}{\pi^3}\int_{-\infty}^{0}\frac{dt}{t}\Pi_h(t)\mathrm{ImK}^{(4)}(t/m^2)$$

$$\mathrm{ImK}^{(4)}(z+i\epsilon) = \pi\theta(-z)F^{(4)}(1/y(z))$$

$$F^{(4)}(u) = R_1(u) + R_2(u)\ln(-u) + R_3(u)\ln(1+u) + R_4(u)\ln(1-u) + R_5(u)\left[4\,\mathrm{Li}_2(u) + 2\,\mathrm{Li}_2(-u) + \ln(-u)\ln\left((1-u)^2(1+u)\right)\right]$$

$$R_1 = \frac{23u^6 - 37u^5 + 124u^4 - 86u^3 - 57u^2 + 99u + 78}{72(u-1)^2u(u+1)},$$

$$R_2 = \frac{12u^8 - 11u^7 - 78u^6 + 21u^5 + 4u^4 - 15u^3 + 13u + 6}{12(u-1)^3u(u+1)^2},$$

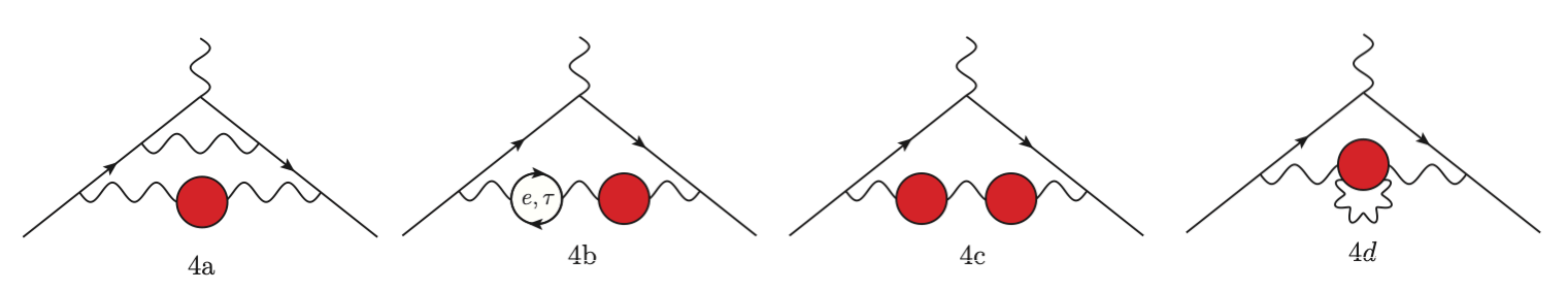
$$R_3 = \frac{(u+1)\left(-u^3 + 7u^2 + 8u + 6\right)}{12u^2},$$

$$R_4 = \frac{-7u^4 - 8u^3 + 8u + 7}{12u^2},$$

$$R_5 = -\frac{3u^4 + 5u^3 + 7u^2 + 5u + 3}{6u^2}.$$

Obtained independently by Nesterenko, arXiv: 2112.05009

Space-like method: NLO hadronic vacuum polarization contribution



$$a_\mu^{(4a)} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^1 dx\ \kappa^{(4)}(x) \Delta\alpha_h(t(x)),$$

$$\kappa^{(4)} = \frac{2(2-x)}{x(x-1)} F^{(4)}(x-1)$$

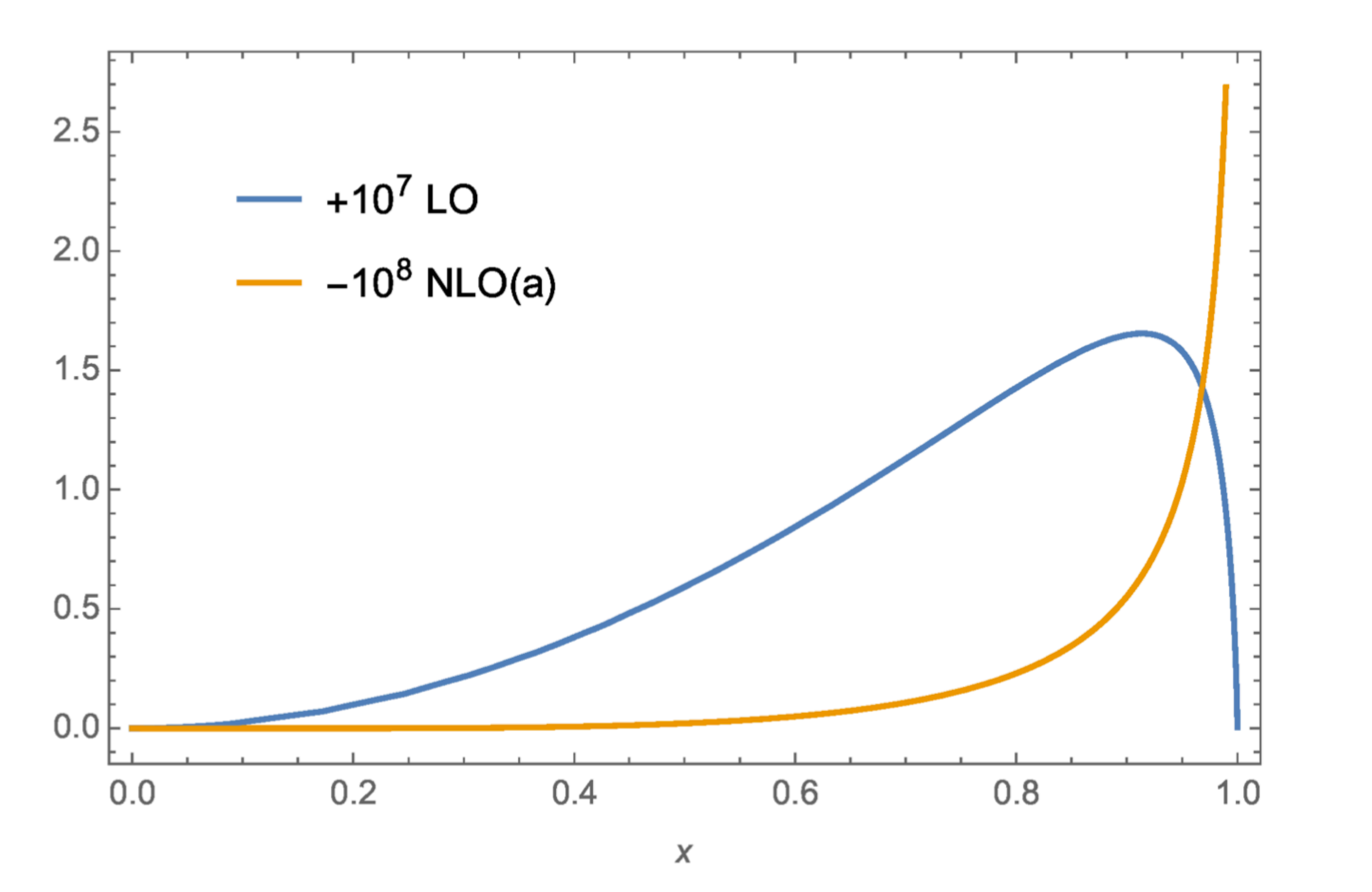
$$a_\mu^{(4b)} = \left(\frac{\alpha}{\pi}\right) \int_0^1 dx\ \kappa^{(2)}(x) \Delta\alpha_h(t(x)) \times 2 \left[\Delta\alpha_e^{(2)}(t(x)) + \Delta\alpha_\tau^{(2)}(t(x))\right]$$

$$a_\mu^{(4c)} = \left(\frac{\alpha}{\pi}\right) \int_0^1 dx\ \kappa^{(2)}(x) [\Delta\alpha_h(t(x))]^2$$

Chakraborty at al. (arXiv: 1806.08190) provided an approximated expression for the space-like kernel function. They added a $O(10\%)$ uncertainty to their final result.

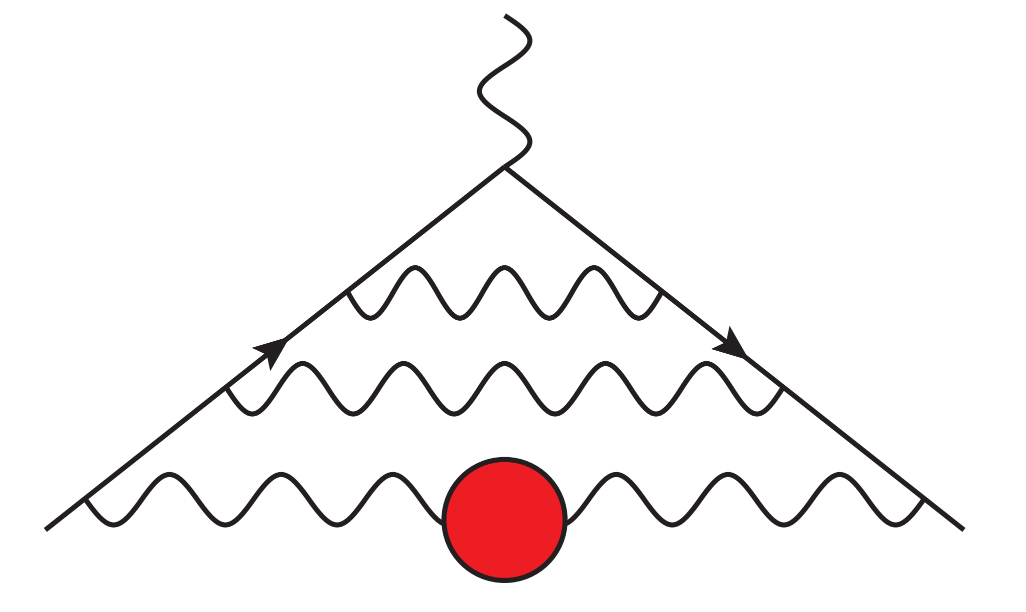
This uncertainty can be eliminated using our exact formula $\kappa^{(4)}(x)$.

Space-like method: NLO hadronic vacuum polarization contribution



- The LO integrand has a peak at $x \sim 0.914$
- The NLO integrand has an integrable logarithmic singularity at $x \to 1$

Space-like method: NNLO HVP contribution



$$a_\mu^{(6a)} = \frac{\alpha^3}{\pi^4}\int_{s_0}^{\infty}\frac{ds}{s}K^{(6a)}(s/m^2)\mathrm{Im}\Pi_\mathrm{h}(s)$$

Approximated series expansion of $K^{(6a)}(z)$ in the parameter $r = m^2/s$ contains powers r^n of degree $n = 1,2,3,4$, multiplied by constants, $\ln r$, $\ln^2 r$, $\ln^3 r$ terms.

$$K^{(6a)}(s/m^2) = \int_0^1 d\xi \left[\frac{P^{(6a)}(\xi)}{1+r\xi} + \frac{L^{(6a)}(\xi)}{\xi + r}\right]$$

Exploiting generating integral representation to fit all the term.

Groote, Körner, Pivovarov, arXiv:0111206v2

$$P^{(6a)}(\xi) = p_0^{(6a)} + p_1^{(6a)}\xi + p_2^{(6a)}\xi^2 + p_3^{(6a)}\xi^3$$

$$G^{(6a)}(\xi) = g_0^{(6a)} + g_1^{(6a)}\xi + g_2^{(6a)}\xi^2 + g_3^{(6a)}\xi^3$$

$$H^{(6a)}(\xi) = h_0^{(6a)} + h_1^{(6a)}\xi + h_2^{(6a)}\xi^2 + h_3^{(6a)}\xi^3$$

$$J^{(6a)}(\xi) = j_0^{(6a)} + j_1^{(6a)}\xi + j_2^{(6a)}\xi^2 + j_3^{(6a)}\xi^3$$

$$L^{(6a)}(\xi) = G^{(6a)}(\xi) + H^{(6a)}(\xi)\ln\xi + j^{(6a)}(\xi)\ln^2\xi$$

Space-like method: NNLO HVP contribution

$$a_\mu^{(6a)} = \left(\frac{\alpha}{\pi}\right)^3 \int_0^1 dx\ \bar{\kappa}^{(6a)}(x)\Delta\alpha_h(t(x))$$

For $0 < x < x_\mu = (\sqrt{5}-1)/2 = 0.618...$

$$\bar{\kappa}^{(6a)} = \frac{2-x}{x(1-x)} P^{(6a)}\left(\frac{x^2}{1-x}\right)$$

For $x_\mu < x < 1$

$$\bar{\kappa}^{(6a)} = \frac{2-x}{x^3} L^{(6a)}\left(\frac{1-x}{x^2}\right)$$

(6a)

$$j_0 = 0;\quad j_1 = -\frac{3793}{864};\quad j_2 = \frac{35087}{21600};\quad j_3 = \frac{1592093}{43200};$$

$$h_0 = -\frac{359}{36};\quad h_1 = \frac{122293}{5184};\quad h_2 = -\frac{43879427}{648000};\quad h_3 = \frac{14388407}{48000};$$

$$g_0 = \frac{1301}{144} - \frac{19\pi^2}{9};$$

$$g_1 = \frac{441277}{10368} + \pi^2\left(-\frac{355}{648} + \ln 4\right) + \frac{25\ \zeta(3)}{2};$$

$$g_2 = -\frac{5051645167}{38880000} + \pi^2\left(\frac{221411}{32400} - 18\ln 2\right) - \frac{3919\ \zeta(3)}{60};$$

$$g_3 = \frac{14588342017}{38880000} + \pi^2\left(-\frac{2479681}{64800} + 112\ln 2\right) + \frac{3113\ \zeta(3)}{10};$$

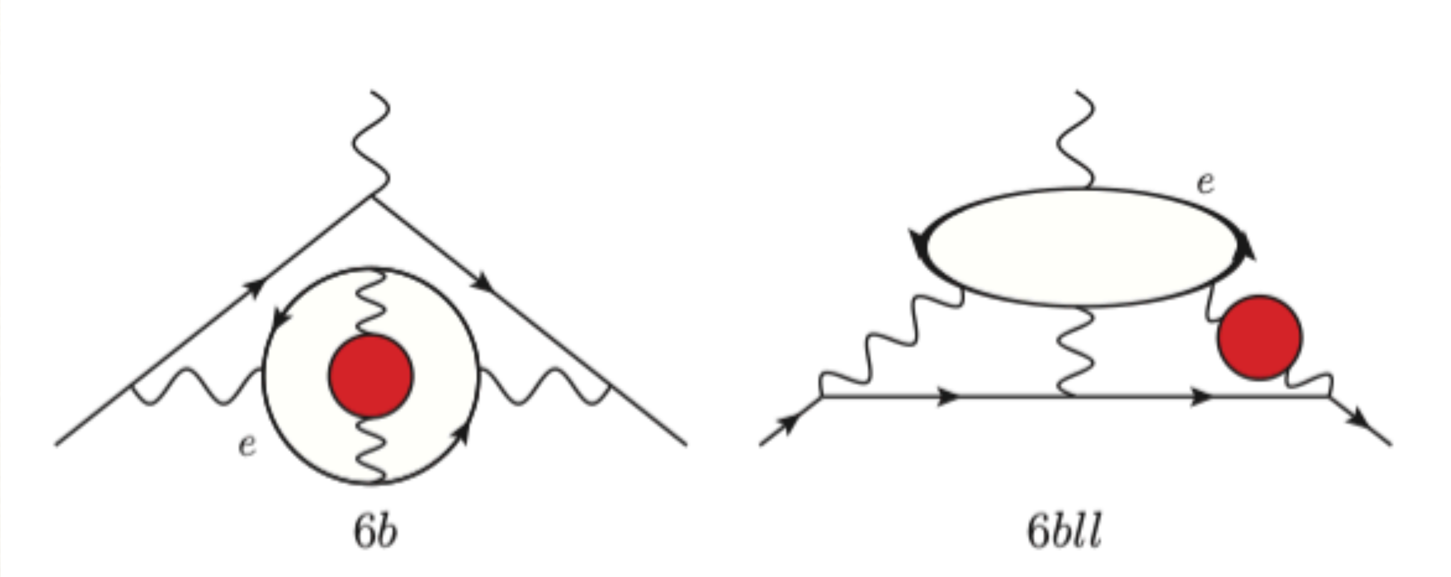
$$p_0 = -\frac{1808080780513}{14580000} + \frac{41851\pi^4}{15} + \frac{8432\ln^4 2}{3} + 67456\ a_4 + \frac{2085448\ \zeta(3)}{15} + \\ +\pi^2\left(-\frac{11944163099}{194400} + \frac{272}{3}(180 - 31\ln 2)\ln 2 + \frac{115072\ \zeta(3)}{3}\right) - \frac{575360\ \zeta(5)}{3};$$

$$p_1 = \frac{134017456919}{96000} - \frac{4481182\pi^4}{135} - \frac{98420\ln^4 2}{3} - 787360\ a_4 + 2255200\ \zeta(5) + \\ +\pi^2\left(\frac{23549054249}{32400} - 201122\ln 2 + \frac{98420\ln^2 2}{3} - 451040\ \zeta(3)\right) - \frac{57189259\ \zeta(3)}{36};$$

$$p_2 = -\frac{13069081405453}{3888000} + \frac{330073\pi^4}{4} + 80790\ln^4 2 + 1938960\ a_4 + \frac{77371609\ \zeta(3)}{20} + \\ +\pi^2\left(-\frac{729995599}{405} + 6(85313 - 13465\ln 2)\ln 2 + 1114360\ \zeta(3)\right) - 5571800\ \zeta(5);$$

$$p_3 = \frac{1274611832039}{583200} - \frac{986377\pi^4}{18} - 53340\ln^4 2 - 1280160\ a_4 + \frac{11057200\ \zeta(5)}{3} + \\ +\pi^2\left(\frac{5809659289}{4860} + 420\ln 2(-823 + 127\ln 2) - \frac{2211440\ \zeta(3)}{3}\right) - \frac{22833188\ \zeta(3)}{9};$$

Space-like method: NNLO HVP contribution



(6b)

j	h
$j_0 = 0;$	$h_0 = \frac{65}{54};$
$j_1 = \frac{11}{27};$	$h_1 = -\frac{3559}{1296} + \rho^2 + \frac{5}{18}\ln\rho;$
$j_2 = \frac{41}{120};$	$h_2 = \frac{3917}{432} - \frac{82\rho^2}{3} + \frac{61}{10}\ln\rho;$
$j_3 = -\frac{507}{40};$	$h_3 = -\frac{4109}{80} + \frac{2211\rho^2}{10} - \frac{1763}{30}\ln\rho;$

$$g_0 = \frac{1}{108}\left(259 - 72\rho^2 + 276\ln\rho\right);$$
$$g_1 = -\frac{9215}{1296} + \frac{65\pi^2}{162} - \frac{3\pi^2\rho}{4} + \frac{49\rho^2}{36} + \left(-\frac{301}{54} + 8\rho^2\right)\ln\rho + \frac{4}{3}\ln^2\rho + 2\,\zeta(3);$$
$$g_2 = \frac{501971}{40500} - \frac{113\pi^2}{36} + \frac{270\pi^2\rho}{36} - \frac{8417\rho^2}{180} + \left(\frac{3479}{900} - 44\rho^2\right)\ln\rho - 8\ln^2\rho - 12\,\zeta(3);$$
$$g_3 = -\frac{2523823}{324000} + \frac{625\pi^2}{36} - 49\pi^2\rho + \frac{84946\rho^2}{225} + \left(\frac{987}{50} + 200\rho^2\right)\ln\rho + \frac{112}{3}\ln^2\rho + 56\,\zeta(3);$$

$$p_0 = -\frac{95519053063}{486000} - 7275\pi^2\rho + \left(-\frac{587150693}{5400} + \frac{75272\rho^2}{3} + \frac{120800\pi^2}{9}\right)\ln\rho + \left(\frac{1135508}{9} + 96\rho^2\right)\zeta(3) + $$
$$+4720\ln^2\rho + \frac{1067115409\rho^2}{5400} + \pi^2\left(\frac{24382331}{810} - \frac{285184}{9}\ln 2\right) - 32\pi^2\rho^2\left(687 + \ln 4\right);$$
$$p_1 = \frac{279489728279}{121500} + \frac{179283\pi^2\rho}{2} + \left(\frac{2280933773}{1800} - 309540\rho^2 - \frac{1419328\pi^2}{9}\right)\ln\rho - \frac{10}{3}\left(446023 + 216\rho^2\right)\zeta(3) + $$
$$-\frac{174712}{3}\ln^2\rho - \frac{174350167\rho^2}{75} + \pi^2\left(-\frac{143574463}{405} + \frac{3352256\ln 2}{9}\right) + \frac{16}{3}\pi^2\rho^2\left(48481 + 90\ln 2\right);$$
$$p_2 = -\frac{229560199193}{40500} - \frac{912495\pi^2\rho}{4} + \left(-\frac{1867939691}{600} + 788488\rho^2 + \frac{1168336\pi^2}{3}\right)\ln\rho + \left(\frac{11034553}{3} + 1440\rho^2\right)\zeta(3) + $$
$$+148348\ln^2\rho + \frac{258653648\rho^2}{45} + \frac{4}{135}\pi^2\left(29597029 - 31048560\ln 2\right) - \frac{320}{3}\pi^2\rho^2\left(5989 + \ln 512\right);$$
$$p_3 = \frac{72762177677}{19440} + 154035\pi^2\rho - \frac{7}{108}\left(-31650719 + 3973440\pi^2 + 8220240\rho^2\right)\ln\rho - \frac{280}{9}\left(78283 + 27\rho^2\right)\zeta(3) + $$
$$-100240\ln^2\rho - \frac{513692207\rho^2}{135} + \frac{35}{162}\pi^2\left(-2687659 + 2816064\ln 2\right) + \frac{140}{3}\pi^2\rho^2\left(9055 + \ln 4096\right);$$

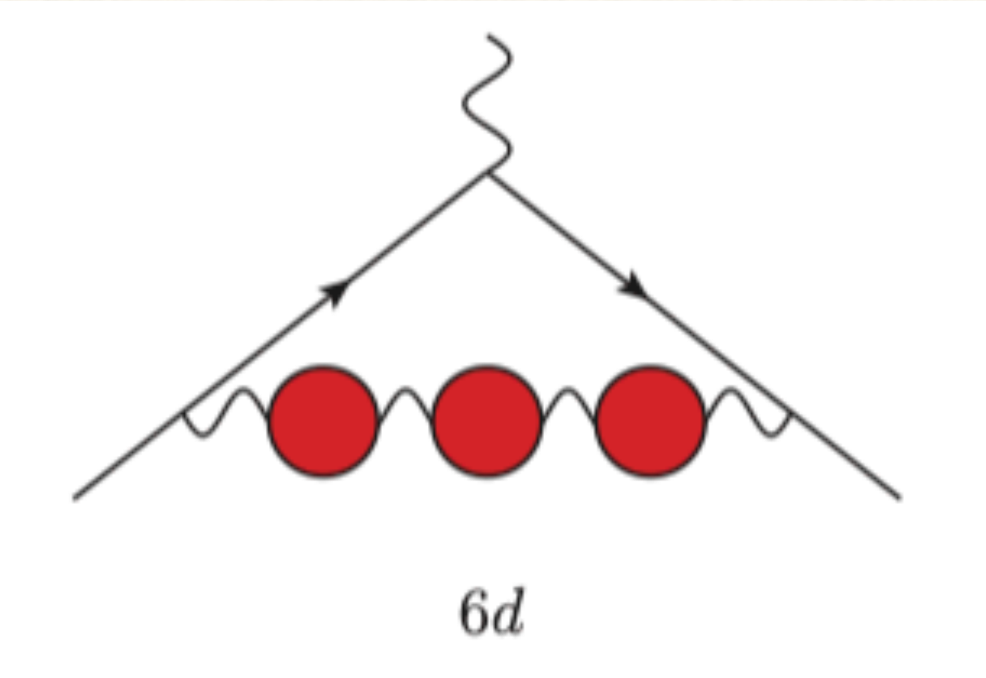
(6bll)

j	h
$j_0 = 0;$	$h_0 = -\frac{9}{2};$
$j_1 = \frac{4}{27} - \frac{9\rho^2}{2};$	$h_1 = \frac{59}{9} - \frac{275\rho^2}{36} - 18\rho^2\ln\rho;$
$j_2 = -\frac{41}{48} + \frac{2201\rho^2}{216};$	$h_2 = -\frac{485}{32} + \frac{1351\rho^2}{48} + \frac{659\rho^2}{18}\ln\rho;$
$j_3 = \frac{3037}{900} - \frac{5909\rho^2}{216};$	$h_3 = \frac{282617}{6750} - \frac{10481\rho^2}{108} - \frac{851\rho^2}{9}\ln\rho;$

$$g_0 = \frac{43}{8} - 4\pi^2\rho + 15\rho^2 + \pi^2\rho^2 - 18\rho^2\ln\rho + 6\rho^2\ln^2\rho;$$
$$g_1 = -\frac{73}{81} + \frac{8\pi^2}{81} + \frac{40\pi^2\rho}{9} + \frac{2437\rho^2}{108} + \frac{17\pi^2\rho^2}{9} + \frac{607\rho^2}{18}\ln\rho - \frac{20\rho^2}{3}\ln^2\rho + \frac{2}{3}\zeta(3) + 2\rho^2\zeta(3);$$
$$g_2 = -\frac{385}{162} - \frac{41\pi^2}{72} - \frac{28\pi^2\rho}{3} - \frac{89873\rho^2}{5184} - \frac{997\pi^2\rho^2}{324} - \frac{1961\rho^2}{72}\ln\rho + 14\rho^2\ln^2\rho - \frac{5}{2}\zeta(3) - \frac{16\rho^2}{3}\zeta(3);$$
$$g_3 = \frac{2691761}{202500} + \frac{3037\pi^2}{1350} + 24\pi^2\rho + \frac{655429\rho^2}{97200} + \frac{2359\pi^2\rho^2}{324} + \frac{6943\rho^2}{360}\ln\rho - 36\rho^2\ln^2\rho + \frac{42}{5}\zeta(3) + 15\rho^2\zeta(3);$$

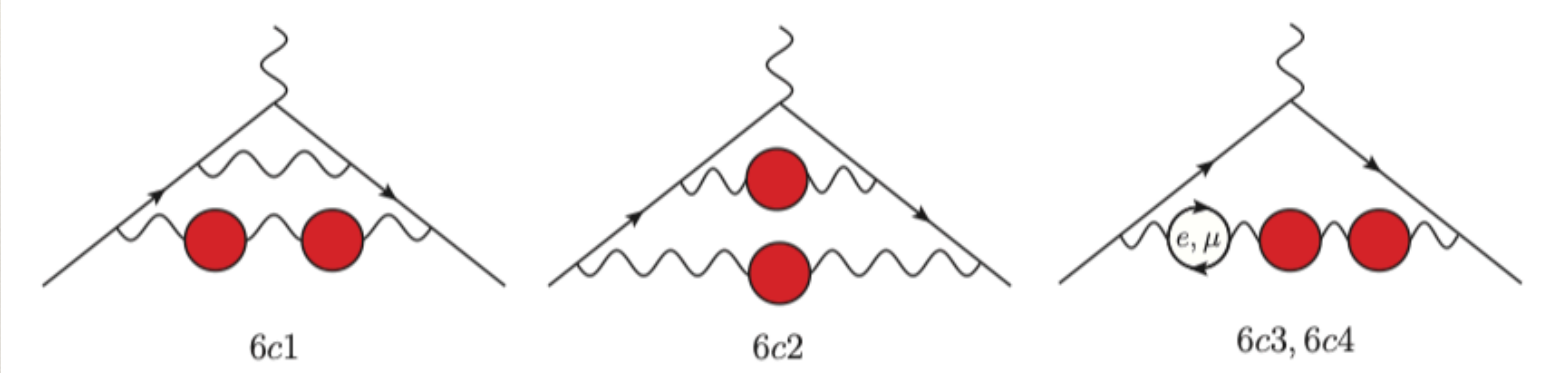
$$p_0 = -\frac{343277101}{45000} - \frac{33156604927\rho^2}{583200} + \pi^2\left(-\frac{615427}{4050} + \frac{6776\rho}{3} + \frac{763121\rho^2}{972}\right) - \frac{4\pi^4}{135}\left(7817 + 3212\rho^2\right) + $$
$$+\left(-\frac{7290521}{3240} + \frac{49622\pi^2}{27} - \frac{128\pi^4}{9}\right)\rho^2\ln\rho + \left(-3388 - \frac{80\pi^2}{3}\right)\rho^2\ln^2\rho + $$
$$+\left(25642 + \frac{1515724\rho^2}{27} - 128\pi^2\rho^2 - 160\rho^2\ln\rho\right)\zeta(3) - \frac{1280}{3}\rho^2\zeta(5);$$
$$p_1 = \frac{89280434843}{972000} + \frac{248834878697\rho^2}{388800} - \frac{1}{324}\pi^2\left(-533001 + 9110736\rho + 3110417\rho^2\right) + \frac{2}{135}\pi^4\left(180247 + 73530\rho^2\right) + $$
$$+\left(\frac{11101973}{1080} - \frac{193400\pi^2}{9} + \frac{320\pi^4}{3}\right)\rho^2\ln\rho + \frac{2}{3}\left(63269 + 300\pi^2\right)\rho^2\ln^2\rho + $$
$$+\frac{1}{45}\left(-13410977 + 100\left(-292301 + 432\pi^2\right)\rho^2 + 54000\rho^2\ln\rho\right)\zeta(3) + 3200\rho^2\zeta(5);$$
$$p_2 = -\frac{6209532853}{27000} - \frac{29997466847\rho^2}{19440} + \pi^2\left(-\frac{114521}{30} + 71840\rho + \frac{1970140\rho^2}{81}\right) - \frac{4}{9}\pi^4\left(14685 + 6032\rho^2\right) + $$
$$-\frac{1}{54}\left(190613 - 2847360\pi^2 + 11520\pi^4\right)\rho^2\ln\rho - 80\left(1347 + 5\pi^2\right)\rho^2\ln^2\rho + $$
$$-\frac{10}{9}\left(-658509 + \left(-1431463 + 1728\pi^2\right)\rho^2 + 2160\rho^2\ln\rho\right)\zeta(3) - 6400\rho^2\zeta(5);$$
$$p_3 = \frac{49726331179}{324000} + \frac{7324831423\rho^2}{7290} + \pi^2\left(\frac{3897971}{1620} - \frac{145880\rho}{3} - \frac{3977785\rho^2}{243}\right) + \frac{14}{27}\pi^4\left(8269 + 3419\rho^2\right) + $$
$$+\frac{7}{81}\left(-81551 - 401520\pi^2 + 1440\pi^4\right)\rho^2\ln\rho + \frac{140}{3}\left(1563 + 5\pi^2\right)\rho^2\ln^2\rho + $$
$$+\frac{35}{27}\left(-371889 + 16\left(-50437 + 54\pi^2\right)\rho^2 + 1080\rho^2\ln\rho\right)\zeta(3) + \frac{11200}{3}\rho^2\zeta(5);$$

Space-like method: NNLO HVP contribution



$$a_\mu^{(6d)} = \frac{\alpha}{\pi^4}\int_{s_0}^{\infty}\frac{ds}{s}\frac{ds'}{s'}\frac{ds''}{s''}K^{(6d)}(s,s',s'') \times \mathrm{Im}\Pi_\mathrm{h}(\mathrm{s})\mathrm{Im}\Pi_\mathrm{h}(\mathrm{s}')\mathrm{Im}\Pi_\mathrm{h}(\mathrm{s}'')$$

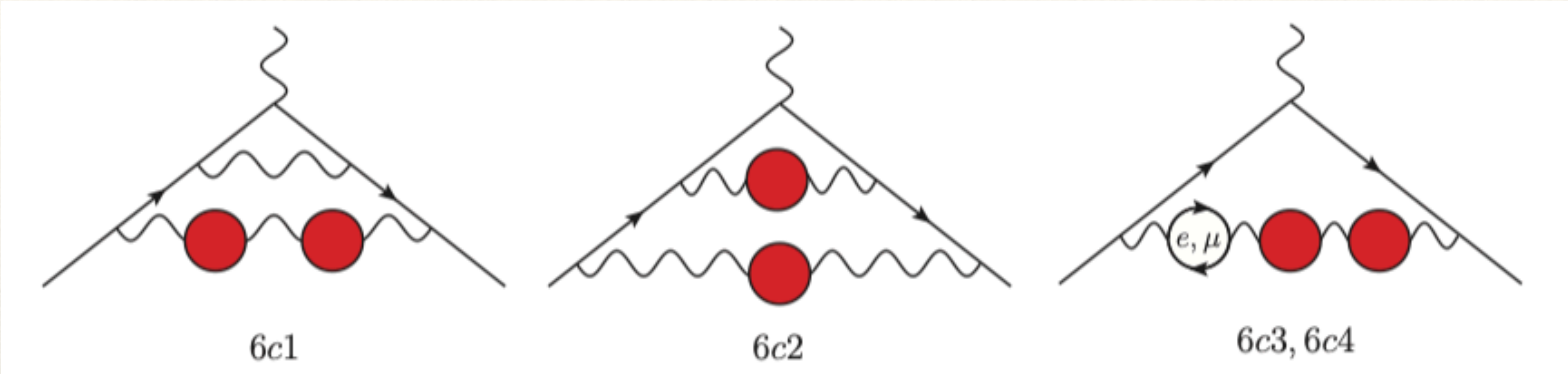
$$\rightarrow \quad \boxed{a_\mu^{(6d)} = \left(\frac{\alpha}{\pi}\right)\int_0^1 dx\ \kappa^{(2)}(x)[\Delta\alpha_h(t(x))]^3}$$



$$a_\mu^{(6c)} = \frac{\alpha^2}{\pi^4}\int_{s_0}^{\infty}\frac{ds}{s}\frac{ds'}{s'}K^{(6c)}(s/m^2, s'/m^2) \times \mathrm{Im}\Pi_\mathrm{h}(\mathrm{s})\mathrm{Im}\Pi_\mathrm{h}(\mathrm{s}')$$

($6c$) diagrams contain two HVP insertions → $a_\mu^{(6c)}$ requires two dispersive integrations

Space-like method: NNLO hadronic vacuum polarization contribution



We start from the time-like asymptotic expansions:

- $K^{(6c)}(s/m^2, s'/m^2)$ in $s' \approx s \gg m^2$
- $K^{(6c)}(s/m^2, s'/m^2)$ in $s' \gg s \gg m^2$

$$a_\mu^{(6c1)} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^1 dx\ \lambda^{(4)}(x)[\Delta\alpha_h(t(x))]^2$$

$$\lambda^{(4)}(x) = \kappa^{(4)}(x) - \frac{2\pi}{\alpha}\kappa^{(2)}(x)\Delta\alpha_\mu^{(2)}(t(x))$$

$$a_\mu^{(6c3)} = \frac{\alpha}{\pi}\int_0^1 dx\ \kappa^{(2)}(x)[3\Delta\alpha_h(t(x))^2\Delta\alpha_e^{(2)}(t(x))]$$

$$a_\mu^{(6c4)} = \frac{\alpha}{\pi}\int_0^1 dx\ \kappa^{(2)}(x)[3\Delta\alpha_h(t(x))^2\Delta\alpha_\mu^{(2)}(t(x))]$$

Two steps:

1. Computed the approximated time-like kernel $K^{(6c2)}(s/m^2, s'/m^2)$
2. Matched the LO of the approximated time-like kernels $K^{(6c2)}(s/m^2, s'/m^2)$, with those of a series expansion of a two dimensional generating integral representation

Space-like method: NNLO hadronic vacuum polarization contribution

$$a_\mu^{(6c2)} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^1 dx \int_0^1 dx' \; \bar{\kappa}^{(6c2)} \times \Delta\alpha_h(t(x))\Delta\alpha_h(t(x'))$$

For $x_\mu < \{x, x'\} < 1$

$$\bar{\kappa}^{(6c2)}(x, x') = \frac{2-x}{x^3}\frac{2-x'}{x'^3} G^{(6c2)}\left(\frac{1-x}{x^2}, \frac{1-x'}{x'^2}\right)$$

$$G^{(6c2)}\left(\xi, \xi'\right) = \frac{1}{4\left(32\pi^2 - 315\right)} \times$$

$$\times \left[\left(1855 - 188\pi^2\right)\frac{\min\left(\xi, \xi'\right)}{\max\left(\xi, \xi'\right)^2} + \left(988\pi^2 - 9765\right)\frac{\min\left(\xi, \xi'\right)^2}{\max\left(\xi, \xi'\right)^3} + 24\left(435 - 44\pi^2\right)\frac{\min\left(\xi, \xi'\right)^3}{\max\left(\xi, \xi'\right)^4}\right]$$

Conclusions

- We provide simple analytic expressions to calculate the HVP contributions to the muon $g-2$ in the space-like region up to NNLO.

- These results will allow to extend MUonE's original proposal to determine the LO contribution to $a_\mu^{(HVP)}$ to NNLO. They can also be employed in lattice QCD computations.

- An existing approximation for the NLO kernel induced large uncertainties. These can be eliminated using our exact expression $\kappa^{(4)}(x)$.

- For NNLO a combination of exact and approximated one- and two-dimensional kernels is provided. The uncertainty due to the kernel approximations is estimated to be less than $O(10^{-12})$.

- Calculation of higher-order HVP corrections require a precise treatment of the QED radiative corrections to the HVP function. MUonE will naturally include their leading effects in the space-like approach.

- These results allow to compare time-like and space-like calculations of $a_\mu^{(HVP)}$ at NNLO accuracy.

Thank you for your attention!