Radiative corrections to decay amplitudes in lattice QCD

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- I am grateful for the invitation to “give an overview talk on the lattice approach taken by the Rome-Southampton collaboration, $\mathrm{QED_L}$”.
  - Personal note: This work began during my visit to Guido Martinelli in Trieste in 2013 when we began considering the cancellation of infrared divergences in a finite Euclidean volume.
  - Up until this time research has concentrated on including electromagnetic corrections to the spectrum where there are no infrared divergences.
- The story so far is reported in 7 papers and I will try to summarise some key points from each.

1. "QED Corrections to Hadronic Processes in Lattice QCD," arXiv:1502.00257
N. Carrasco, V. Lubicz, G. Martinelli, C.T.S, N. Tantalo, C. Tarantino and M. Testa.
• In this paper we develop the framework for the cancellation of infrared divergences.

2. "Finite-Volume QED Corrections to Decay Amplitudes in Lattice QCD,"
V. Lubicz, G. Martinelli, C.T.S, F. Sanfilippo, S. Simula and N. Tantalo. arXiv:1611.08497
• We demonstrate that the leading finite-volume corrections which are of $O(1/L)$ are universal, i.e. independent of the structure of the meson, and can be subtracted explicitly. The remaining FV corrections are of $O(1/L^2)$ are are structure-dependent.

3. "First lattice calculation of the QED corrections to leptonic decay rates,"
D. Giusti, V. Lubicz, G. Martinelli, C.T.S, F. Sanfilippo, S. Simula, N. Tantalo and C. Tarantino, arXiv:1711.06537

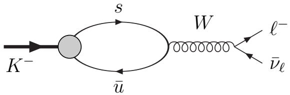
4. "Light-meson leptonic decay rates in lattice QCD+QED," arXiv:1904.08731
M. Di Carlo, D. Giusti, V. Lubicz, G. Martinelli, C.T.S., F. Sanfilippo, S. Simula and N. Tantalo.
• In these two papers we present numerical results for pion and kaon decays and discuss the scheme dependence of isospin breaking corrections.

5. "First lattice calculation of radiative leptonic decay rates of pseudoscalar mesons,"
A. Desiderio, R. Frezzotti, M. Garofalo, D. Giusti, M. Hansen, V. Lubicz, G. Martinelli, C.T.S., F. Sanfilippo, S. Simula and N. Tantalo. arXiv:2006.05358
   - Calculations of $P \to \ell\nu_\ell\,\gamma$ decay rates.

6. "Comparison of lattice QCD+QED predictions for radiative leptonic decays of light mesons with experimental data," arXiv:2012.02120
R. Frezzotti, M. Garofalo, V. Lubicz, G. Martinelli, C.T.S., F. Sanfilippo, S. Simula and N. Tantalo.
   - We compare our results with experimental measurements and find some tantalising "tensions".

7. "Virtual Photon Emission in Leptonic Decays of Charged Pseudoscalar Mesons,"
G. Gagliardi, V. Lubicz, G. Martinelli, F. Mazzetti, C.T.S., F. Sanfilippo, S. Simula and N. Tantalo. arXiv:2202.03833
   - Calculations of $P \to \ell\nu_\ell\,\ell'^+\ell'^-$ decay rates.

- As an example consider $K_{\ell 2}$ decays in QCD (without QED).
  (Similar discussion for π, $D_{(s)}$, $B_{(s)}$.)



- Without QED, the QCD effects in the leptonic decays $K \to \ell\nu_\ell$ are contained in a single number, the leptonic decay constant f_K:

$$\langle 0|A_\mu|K(p)\rangle = f_K p_\mu \qquad \text{and} \qquad \Gamma^{(0)} = \frac{G_F^2|V_{us}|^2 f_K^2}{8\pi} m_K^3 r_\ell^2 \left(1 - r_\ell^2\right)^2 ,$$

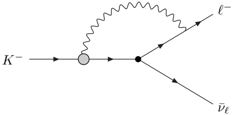
  where $r_\ell = m_\ell/m_K$.

- The measured value of Γ and the lattice computation of $f_K \Rightarrow |V_{us}|$, which is the quantity we wish to determine as precisely as possible.

- Lattice results for f_K are so precise that QED corrections must be included to make further progress, $f_K = 155.7(0.3)$ MeV. FLAG2021, arXiv:2111.09849

- Beyond $\sim$ 1% precision, radiative corrections must be included $\Rightarrow$ presence of infrared divergences.

  - f_K no longer contains all the QCD effects.



"QED Corrections to Hadronic Processes in Lattice QCD," arXiv:1502.00257

N. Carrasco, V. Lubicz, G. Martinelli, C.T.S, N. Tantalo, C. Tarantino and M. Testa.

- Aim is to calculate Γ including $O(\alpha_{\rm em})$ effects.
- Calculating electromagnetic corrections to decay amplitudes has an added major complication, not present in computations of the spectrum,

  the presence of infrared divergences

- This implies that when studying weak decays, such as e.g. $K^+ \to \ell^+\nu$, the physical observable must include soft photons in the final state

  F.Bloch and A.Nordsieck, PR 52 (1937) 54

$$\Gamma(K^+ \to \ell^+\nu_\ell(\gamma)) = \Gamma(K^+ \to \ell^+\nu_\ell) + \Gamma(K^+ \to \ell^+\nu_\ell\gamma) \equiv \Gamma_0 + \Gamma_1\,.$$

- The question is how best to combine this understanding with lattice calculations of non-perturbative hadronic effects (generic problem).

- Our proposal is to separate $\Gamma_0 + \Gamma_1$ into terms each of which is infrared convergent:

$$\begin{aligned}\Gamma(\Delta E_\gamma) &= \Gamma_0 + \Gamma_1(\Delta E_\gamma) = \Gamma_0 + \int_0^{2\Delta E_\gamma/m_P} dx_\gamma \, \frac{d\Gamma_1}{dx_\gamma} \\ &= \lim_{L\to\infty}\left[\Gamma_0(L) - \Gamma_0^{\mathrm{pt}}(L)\right] + \lim_{\mu_\gamma\to 0}\left[\Gamma_0^{\mathrm{pt}}(\mu_\gamma) + \Gamma_1^{\mathrm{pt}}(\Delta E_\gamma, \mu_\gamma)\right] \\ &\quad + \Gamma_1^{\mathrm{SD}}(\Delta E_\gamma) + \Gamma_1^{\mathrm{INT}}(\Delta E_\gamma)\,.\end{aligned}$$

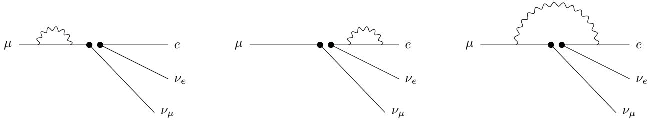
- $x_\gamma = 2E_\gamma/m_K$ in the kaon's rest frame.
- pt="point like", SD="Structure Dependent", INT="Interference" between pt and SD.

- The results for the widths are expressed in terms of G_F, the Fermi constant ($G_F = 1.16632(2)\times 10^{-5}\,\mathrm{GeV}^{-2}$). This is obtained from the muon lifetime:

$$\frac{1}{\tau_\mu} = \frac{G_F^2 m_\mu^5}{192\pi^3}\left[1-\frac{8m_e^2}{m_\mu^2}\right]\left[1+\frac{\alpha_{\rm em}}{2\pi}\left(\frac{25}{4}-\pi^2\right)\right].$$

S.M.Berman, PR 112 (1958) 267; T.Kinoshita and A.Sirlin, PR 113 (1959) 1652

- This expression can be viewed as the definition of G_F. Many EW corrections are absorbed into the definition of G_F; the explicit $O(\alpha_{\rm em})$ corrections come from the following diagrams in the effective theory:



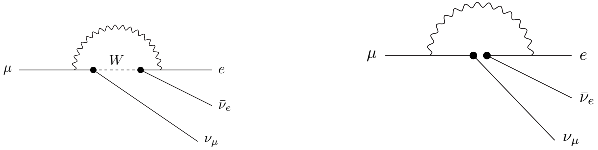
together with the diagrams with a real photon.

- These diagrams are evaluated in the W-regularisation in which the photon propagator is modified by:

A.Sirlin, PRD 22 (1980) 971

$$\frac{1}{k^2}\to\frac{M_W^2}{M_W^2-k^2}\,\frac{1}{k^2}. \qquad \left(\frac{1}{k^2}=\frac{1}{k^2-M_W^2}+\frac{M_W^2}{M_W^2-k^2}\,\frac{1}{k^2}\right)$$

- The $\gamma - W$ box diagram:



As an example providing some evidence & intuition that the W-regularization is useful consider the $\gamma - W$ box diagram.

- In the standard model (left-hand diagram) it contains both the γ and W propagators.
- In the effective theory this is preserved with the W-regularization where the photon propagator is proportional to

$$\frac{1}{k^2}\,\frac{1}{k^2 - M_W^2}$$

and the two diagrams are equal up to terms of $O(q^2/M_W^2)$, where q is the momentum of the e and ν_e.

- Most (but not all) of the EW corrections which are absorbed in G_F are common to leptonic and semileptonic decays $\Rightarrow$ factor in the amplitude of $(1 + 3\alpha_{\rm em}/4\pi(1+2\bar{Q})\log M_Z/M_W)$, where $\bar{Q} = \frac{1}{2}(Q_u + Q_s) = 1/6$.

  A.Sirlin, NP B196 (1982) 83; E.Braaten & C.S.Li, PRD 42 (1990) 3888

- We therefore need to calculate the kaon-decay diagrams in the effective theory with

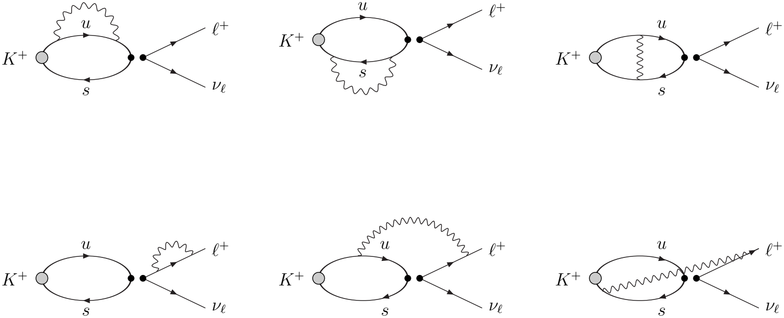
$$H_{\rm eff} = \frac{G_F}{\sqrt{2}} V_{us}^* \left(1 + \frac{\alpha_{\rm em}}{\pi} \log \frac{M_Z}{M_W}\right) (\bar{s}_L \gamma^\mu u_L)(\bar{\nu}_{\ell,L}\, \gamma_\mu\, \ell_L)$$

  in the W-regularization.

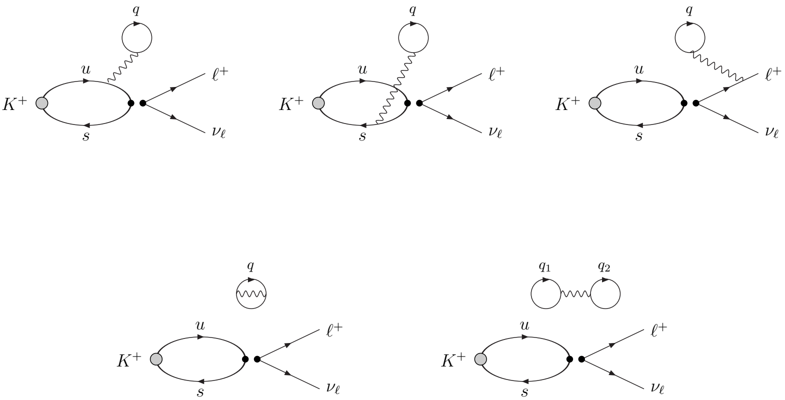
- Non-perturbative renormalisation at scales of $O(M_W)$ are not practicable at present so perturbative running is required from scales of a few GeV to M_W.
- Our current status was presented by Matteo Di Carlo at Lattice 2019 where the matching to W-regularisation was performed up to terms of $O(\alpha_{\rm em}\, \alpha_s(M_W))$.

  arXiv:1911.00938

Connected diagrams



Disconnected diagrams



- The total width, $\Gamma^{\rm pt}$ was calculated in 1958/9 using a Pauli-Villars regulator for the UV divergences and m_γ for the infrared divergences.

S.Berman, PR **112** (1958) 267, T.Kinoshita, PRL **2** (1959) 477

$$\begin{aligned}\Gamma^{\rm pt} \;&=\; \Gamma_0^{\rm tree}\times\Bigg\{1+\frac{\alpha}{4\pi}\,\bigg(3\log\left(\frac{m_K^2}{M_W^2}\right)-8\log(1-r_\ell^2)-\frac{3r_\ell^4}{(1-r_\ell^2)^2}\log(r_\ell^2)\\ &-8\frac{1+r_\ell^2}{1-r_\ell^2}\,\mathrm{Li}_2(1-r_\ell^2)+\frac{13-19r_\ell^2}{2(1-r_\ell^2)}+\frac{6-14r_\ell^2-4(1+r_\ell^2)\log(1-r_\ell^2)}{1-r_\ell^2}\;\log(r_\ell^2)\bigg)\Bigg\}\,.\end{aligned}$$

where $r_\ell=m_\ell/m_K$.

  - This is a very useful check on our perturbative calculation.

- Integrating E_γ up to ΔE we find:

$$
\begin{aligned}
\Gamma^{\rm pt}(\Delta E) \;=\; \Gamma_0^{\rm tree}\times\Bigg(1+\frac{\alpha}{4\pi}\Bigg\{ & 3\log\left(\frac{m_K^2}{M_W^2}\right)+\log\left(r_\ell^2\right)-4\log(r_E^2)+\frac{2-10r_\ell^2}{1-r_\ell^2}\log(r_\ell^2) \\
& -2\frac{1+r_\ell^2}{1-r_\ell^2}\log(r_E^2)\log(r_\ell^2)-4\frac{1+r_\ell^2}{1-r_\ell^2}\,\mathrm{Li}_2(1-r_\ell^2)-3 \\
& +\Bigg[\frac{3+r_E^2-6r_\ell^2+4r_E(-1+r_\ell^2)}{(1-r_\ell^2)^2}\log(1-r_E)+\frac{r_E(4-r_E-4r_\ell^2)}{(1-r_\ell^2)^2}\log(r_\ell^2) \\
& -\frac{r_E(-22+3r_E+28r_\ell^2)}{2(1-r_\ell^2)^2}-4\frac{1+r_\ell^2}{1-r_\ell^2}\,\mathrm{Li}_2(r_E)\Bigg]\Bigg\}\Bigg)\,,
\end{aligned}
$$

where $r_E = 2\Delta E/m_K$ and $r_\ell = m_\ell/m_K$.

- We believe that this is a new result which agrees with the Berman and Kinoshita calculations when $\Delta E = (\Delta E)^{\rm max}$.
- The total rate is readily obtained by setting r_E to its maximum value, namely $r_E = 1 - r_\ell^2$.

“Finite-Volume QED Corrections to Decay Amplitudes in Lattice QCD,”
V. Lubicz, G. Martinelli, C.T.S, F. Sanfilippo, S. Simula and N. Tantalo. arXiv:1611.08497

- The above discussion is general, but we implemented the framework in $\mathrm{QED_L}$ in which

$$A_\mu(\vec{k}=0,k_4)=0 \quad \text{for all } k_4\,.$$

M.Hayakawa and S.Uno, 0804.2044

  - Transfer matrix exists but locality is broken.
  - $L\to\infty$ limit should be taken first.

See talk by A.Patella

University of Southampton

- The evaluation of FV effects is based on the Poisson Summation Formula, e.g. in 1-D

$$\frac{1}{L}\sum_{n=-\infty}^{\infty} f(p_n^2) = \int_{-\infty}^{\infty}\frac{dp}{2\pi}f(p^2) + \sum_{n\neq 0}\int_{-\infty}^{\infty}\frac{dp}{2\pi}f(p^2)\,e^{inpL}\,.$$

- For decay constants, form-factors etc. the FV effects fall exponentially, typically $\propto \exp[-c\,m_\pi\,L]$.
- In the presence of a photon, if the integrand/summand $\to \frac{1}{(k^2)^{\frac{n}{2}}}$ as $k\to 0$ then we have the scaling law:

$$\xi' = \int\frac{dk_0}{(2\pi)}\left(\frac{1}{L^3}\sum_{\vec{k}\neq 0} - \int\frac{d^3k}{(2\pi)^3}\right)\frac{1}{(k^2)^{\frac{n}{2}}} = O\left(\frac{1}{L^{4-n}}\right)$$

- For the spectrum $n=3$ and the leading FV corrections are $O(1/L)$.
- For decay amplitudes $n=4$ and we have the form:

$$\Gamma_0^{\text{pt}}(L) = C_0(r_\ell) + \tilde{C}_0(r_\ell)\log(m_K L) + \frac{C_1(r_\ell)}{m_K L} + \dots\,,$$

where $r_\ell = m_\ell/m_K$ and m_ℓ is the mass of the final-state charged lepton.

$$\boxed{\Gamma_0+\Gamma_1(\Delta E)=\lim_{V\to\infty}(\Gamma_0-\Gamma_0^{\rm pt})+\lim_{V\to\infty}(\Gamma_0^{\rm pt}+\Gamma_1^{\rm pt}(\Delta E))+\Gamma_1^{\rm SD}(\Delta E)+\Gamma_1^{\rm INT}(\Delta E)}$$

- Here the finite-volume effects take the form:

$$\Gamma_0^{\rm pt}(L)=C_0(r_\ell)+\tilde{C}_0(r_\ell)\log\left(m_K L\right)+\frac{C_1(r_\ell)}{m_K L}+\dots,$$

  where $r_\ell=m_\ell/m_K$ and m_ℓ is the mass of the final-state charged lepton.

  The exhibited L-dependent terms are *universal*, i.e. independent of the structure of the meson!

- The leading structure-dependent FV effects in $\Gamma_0-\Gamma_0^{\rm pt}$ are of $O(1/L^2)$.
- For the spectrum the leading and next-to-leading finite-volume corrections are universal (independent of the hadron H):

$$m_H(L)=m_H\left[1-Q_H^2\alpha\left(\frac{\kappa}{m_H L}\left(1+\frac{2}{m_H L}\right)\right)+O\left(\frac{1}{(m_H L)^3}\right)\right],$$

  where $\kappa=1.41865$ is a universal constant and the structure-dependent terms start at $O(1/L^3)$.

S.Borsanyi et al., arXiv:1406.4088

- Writing

$$\Gamma_0^{\rm pt}(L) = \Gamma_0^{\rm tree}\,\left\{1 + 2\,\frac{\alpha}{4\pi}\,Y(L)\right\}\,,$$

and using the QED$_{\rm L}$ regulator of the zero mode we find

$$\begin{aligned}
Y(L) \quad = \quad & \left(1+r_\ell^2\right)\left[2(K_{31}+K_{32}) + \frac{\left(\gamma_E + \log\left[\frac{L^2 m_K^2}{4\pi}\right]\right)\log\left[r_\ell^2\right]}{(1-r_\ell^2)} + \frac{\log^2\left[r_\ell^2\right]}{2\,(1-r_\ell^2)}\right] + \\
& + \frac{(1-3\,r_\ell^2)\,\log\left[r_\ell^2\right]}{(1-r_\ell^2)} - \log\left[\frac{M_W^2}{m_K^2}\right] + \log[m_K^2 L^2] - \frac{1}{2}K_P + \frac{1}{12} + \\
& + \frac{1}{m_K L}\left(\frac{2r_\ell^2}{1-r_\ell^2}\,\left(K_{21}+K_{22} - 2\pi\left(\frac{1}{1+r_\ell^2} + \frac{1}{r_\ell}\right)\right) - \frac{\pi(1+r_\ell^2)}{(1-r_\ell^2)}\,(K_{11}+K_{12}-3)\right)\,,
\end{aligned}$$

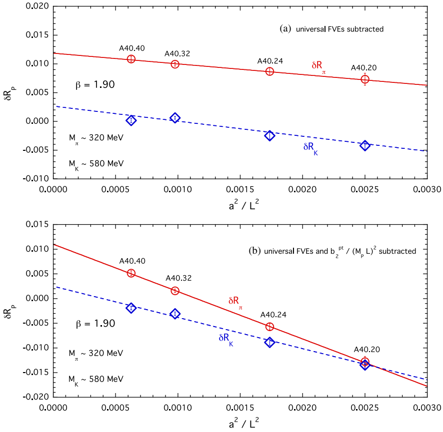
where $r_\ell = m_\ell/m_K$ and the K_{ij} are constants ($K_{21}+K_{22}$ and $K_{31}+K_{32}$ depends on the direction of $\vec{p}_\ell$).

See also the talk by Nils Hermasson-Truedsson

"First lattice calculation of the QED corrections to leptonic decay rates,"
D. Giusti, V. Lubicz, G. Martinelli, C.T.S, F. Sanfilippo, S. Simula, N. Tantalo and C. Tarantino,
arXiv:1711.06537

"Light-meson leptonic decay rates in lattice QCD+QED," arXiv:1904.08731
M. Di Carlo, D. Giusti, V. Lubicz, G. Martinelli, C.T.S., F. Sanfilippo, S. Simula and N. Tantalo.

- In QCD+QED what is meant by QCD and what is the QED correction is convention dependent.
  - QED shifts the quark and meson masses.
  - An advantage of lattice QCD is that directly measurable quantities, such as hadron masses, can be computed.
  - We advocate using such a set of hadronic quantities to determine the input quark masses and lattice spacing in defining QCD: *hadronic schemes*.
  - Mass counter-terms then have to be introduced to cancel the electromagnetic mass shifts and the lattice spacing is also shifted.
- The difference between the full, QCD+QED, result and that in QCD as defined above is then the "QED Correction".



- Finite-volume behaviour of 4-points, obtained at the same value of β and quark masses using ETMC twisted mass ensembles.
- The universal $1/L$ terms have been subtracted.
- The leading SD finite-volume terms appear to be of $O(1/L^2)$ as expected.
  - See however, the talk by M. Di Carlo.

- In our numerical calculations we used the physical quantity

$$\mathcal{F}_\pi^2 = \frac{\Gamma(\pi^\pm \to \ell^\pm \bar{\nu}_\ell(\gamma))}{\frac{G_F^2}{8\pi}|V_{ud}|^2 m_\ell^2 m_\pi \left(1 - \frac{m_\ell^2}{m_\pi^2}\right)} = [f_\pi^{(0)}]^2\,(1+\delta R_\pi)$$

to set the scale taking V_{ud} from super-allowed nuclear β-decays.

  - $|V_{ud}| = 0.97420(21)$ J.Hardy and I.S.Towner, CKM(2016) 028
  - This implies that we can no longer determine V_{ud} from this computation.

- Writing

$$\frac{\Gamma(K_{\mu 2})}{\Gamma(\pi_{\mu 2})} = \left|\frac{V_{us}}{V_{ud}}\frac{f_K^{(0)}}{f_\pi^{(0)}}\right|^2 \frac{m_\pi^3}{m_K^3}\left(\frac{m_K^2 - m_\mu^2}{m_\pi^2 - m_\mu^2}\right)^2 (1+\delta R_{K\pi})$$

where $m_{K,\pi}$ are the physical masses, using numerous twisted mass ensembles we find

$$\delta R_{K\pi} = -0.0126(14) \qquad \left[\delta R_\pi = +0.0153(19),\ \delta R_K = +0.0024(10)\right]$$

- $f_P^{(0)}$ are the decay constants obtained in iso-symmetric QCD with the renormalized $\overline{\mathrm{MS}}$ masses and coupling equal to those in the full QCD+QED theory extrapolated to infinite volume and to the continuum limit.
- Using ChPT $\delta R_\pi = +0.0176(21),\ \delta R_K = +0.0064(24)$ PDG(2018)

- We obtained

$$\left|\frac{V_{us}}{V_{ud}}\right| = 0.23135(46)\,.$$

- Taking $V_{ud} = 0.97420(21) \;\Rightarrow\; V_{us} = 0.22538(46)$ and with $|V_{ub}| = 0.00413(49)$,

$$|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 0.99988(46)\,.$$

- However, taking $|V_{ud}| = 0.97370(14)$ C.Y.Seng et al., arXiv:1807.10197

$$|V_{us}| = 0.22526(46),\; |V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 0.99885(34)\,.$$

- The latest PDG value is $V_{ud} = 0.97373(31)$ which is the average of the 15 most precise determinations and with a more conservative error. (Unitarity within a little more than 1σ.)

See also talk by M. Di Carlo

"First lattice calculation of radiative leptonic decay rates of pseudoscalar mesons,"
A. Desiderio, R. Frezzotti, M. Garofalo, D. Giusti, M. Hansen, V. Lubicz, G. Martinelli, C.T.S., F. Sanfilippo, S. Simula and N. Tantalo. arXiv:2006.05358

"Comparison of lattice QCD+QED predictions for radiative leptonic decays of light mesons with experimental data," arXiv:2012.02120
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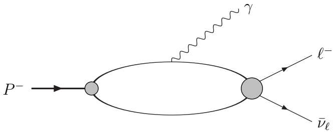
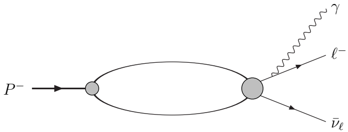




Non-perturbative contribution to $P \to \ell\bar\nu_\ell\gamma$ is encoded in:

$$
\begin{aligned}
H_W^{\alpha r}(k,\vec p) &= \epsilon_\mu^r(k)\, H_W^{\alpha\mu}(k,\vec p) = \epsilon_\mu^r(k) \int d^4y\, e^{ik\cdot y}\, \mathrm{T}\, \langle 0|\, j_W^\alpha(0) j_{\mathrm{em}}^\mu(y)|P(\vec p)\rangle \\
&= \epsilon_\mu^r(k) \left\{ \frac{H_1}{m_P}\left[k^2 g^{\mu\alpha} - k^\mu k^\alpha\right] + \frac{H_2}{m_P}\, \frac{\left[(p\cdot k - k^2)k^\mu - k^2(p-k)^\mu\right](p-k)^\alpha}{(p-k)^2 - m_P^2} \right.
\end{aligned}
$$

$$
\left. -i\frac{F_V}{m_P}\varepsilon^{\mu\alpha\gamma\beta}k_\gamma p_\beta + \frac{F_A}{m_P}\left[(p\cdot k - k^2)g^{\mu\alpha} - (p-k)^\mu k^\alpha\right] + f_P\left[g^{\mu\alpha} - \frac{(2p-k)^\mu (p-k)^\alpha}{(p-k)^2 - m_P^2}\right] \right\}
$$

- For decays into a real photon, $k^2 = 0$ and $\varepsilon \cdot k = 0$, only the decay constant f_p and the vector and axial form factors $F_V(x_\gamma)$ and $F_A(x_\gamma)$ are needed to specify the amplitude ($x_\gamma = 2p \cdot k/m_P^2$, $0 < x_\gamma < 1 - m_\ell^2/m_P^2$).
- In phenomenology $F^\pm \equiv F_V \pm F_A$ are more natural combinations.

- We have computed the form factors for π and K mesons for the full kinematic ranges and for D and D_s mesons for $x_\gamma \lesssim 0.4$. A.Desiderio et al, arXiV:2006.05358
    - The computations were performed on 11 ETMC $N_F = 2+1+1$ ensembles with $0.062\,\mathrm{fm} < a < 0.089\,\mathrm{fm}$, $227\,\mathrm{MeV} < m_\pi < 441\,\mathrm{MeV}$ and a range of volumes.
    - Computations are performed in the electroquenched approximation.
    - We are working towards computing the form factors for all x_γ for both light and heavy mesons (as well as improving the precision).
- Our data is fully consistent with a linear parametrisation of the form:

$$F^P_{A,V}(x_\gamma) = C^P_{A,V} + D^P_{A,V}\,x_\gamma\,,$$

and other parametrisations were also tried (and reported).

- For $D_{(s)}$ mesons the different parametrisations may diverge for x_γ where we don't yet have data.
- The values of the $C^P_{A,V}$ and $D^P_{A,V}$ are presented in the paper.

- $K \to e\nu_e\gamma$ KLOE, arXiv:0907.3594
  J-PARC E36, arXiv:2107.03583
- $K \to \mu\nu_\mu\gamma$ E787@BNL AGS, arXiv:hep-ex/0003019
  ISTRA+, arXiv:1005.3517, OKA, arXiv:1904.10078, both @ U-79, Protvino
- $\pi \to e\nu_e\gamma$ PIBETA, arXiv:0804.1815 @ πE1 beam line PSI

- NA62 will present the most precise results on $F^+(x_\gamma)$ from $K \to e\nu_e\gamma$ decays soon?
- The different experiments introduce cuts on E_γ, E_ℓ and $\cos\theta_{\ell\gamma}$, resulting in sensitivities to different form-factor(s).

Comparison with KLOE Data on $K \to e\nu_e\gamma$ decays

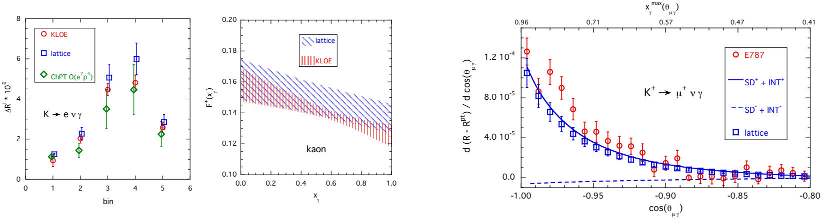
$$\Delta R^{\mathrm{exp},i} \equiv \frac{1}{\Gamma(K_{\mu 2(\gamma)})} \int_{E_\gamma^i}^{E_\gamma^{i+1}} \left[\frac{d\Gamma(K_{e2\gamma})}{dE_\gamma}\right]_{p_e>200\,\mathrm{MeV}}$$

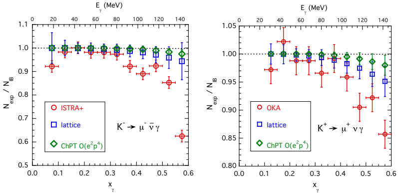
bin	E_γ MeV	$\Delta R^{\mathrm{exp},i} \times 10^6$	$\Delta R^{\mathrm{SD},i} \times 10^6$	$\Delta R^{\mathrm{th},i} \times 10^6$	exp / th	ChPT
1	10 - 50	$0.94 \pm 0.30 \pm 0.03$	0.26 ± 0.04	1.25 ± 0.04	0.75 ± 0.24	1.13 ± 0.03
2	50 - 100	$2.03 \pm 0.22 \pm 0.02$	2.26 ± 0.30	2.28 ± 0.30	0.89 ± 0.15	1.44 ± 0.36
3	100 - 150	$4.47 \pm 0.30 \pm 0.03$	5.06 ± 0.67	5.07 ± 0.67	0.88 ± 0.13	3.50 ± 0.96
4	150 - 200	$4.81 \pm 0.37 \pm 0.04$	6.00 ± 0.78	6.00 ± 0.78	0.80 ± 0.12	4.46 ± 1.25
5	200 - 250	$2.58 \pm 0.26 \pm 0.03$	2.85 ± 0.38	2.85 ± 0.38	0.91 ± 0.15	2.25 ± 0.63
1-5	10 - 250	$14.83 \pm 0.66 \pm 0.13$	16.43 ± 2.12	17.43 ± 2.12	0.85 ± 0.11	12.79 ± 3.24

- There is a universal cut of $p_e > 200\,$MeV on the above.
- KLOE is mainly sensitive to the form-factor $|F^+|$.
- We find good agreement between our theoretical predictions and the KLOE data.
- The J-PARC E36 experiment has recently presented their result for the rate integrated over the range

$$\frac{1}{\Gamma(K_{\mu 2(\gamma)})} \int_{10\,\mathrm{MeV}}^{250\,\mathrm{MeV}} \left[\frac{d\Gamma(K_{e2\gamma})}{dE_\gamma}\right]_{p_e>200\,\mathrm{MeV}} = (18.5 \pm 1.1 \pm 0.7)\,10^{-6}\,.$$

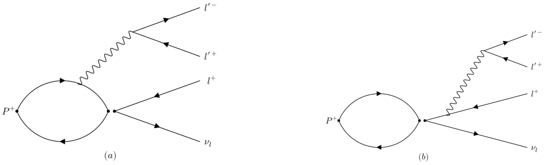
Comparison with Experiment





- Good agreement with KLOE.
- Significant tensions with $K \to \mu\nu_\mu\gamma$ experiments.

- We were also unable to find a phenomenological set of form-factors which would account for all the data.
- NA62 will soon have the most precise results for $K \to e\nu_e\gamma$ decay rates.
- Is it conceivable that we have LFU-violation here also?



- More recently we have begun to study decays into virtual photons $K \to \ell \bar{\nu}_\ell \, (\ell'^+ \ell'^-)$ for both $\ell = \ell'$ and $\ell \neq \ell'$ (4 cases in total).
- Now all four form factors ($F_{V,A}$ and $H_{1,2}$) contribute.
- To date we have only performed exploratory computations at unphysical quark masses, $m_\pi \simeq 320$ MeV and $m_K \simeq 530\,\mathrm{MeV}$ and so a comparison with experimental results is premature (but we couldn't resist doing one anyway).
  - A study with similar kinematics ($m_\pi \simeq 352$ MeV, $m_K \simeq 506$ MeV) has also been performed by a Tuo et al. X.-Y.Tuo et al. 2103.11331.
- The aim of this proof-of-principle exploratory calculation is to check that all four form factors can be determined with reasonable precision.
- The computations were performed on the A40.4 ensemble generated by the ETMC, with $N_f = 2+1+1$ dynamical quark flavours, $a = 0.0885(36)$ fm, $V = 32^3 \times 64$, $m_\pi \simeq 320$ MeV, $m_K \simeq 530$ MeV.

- Experimental results exist for three of the four channels, $K \to \mu\nu_\mu\, e^+e^-$, $K \to e\nu_e\, \mu^+\mu^-$ and $K \to e\nu_e\, e^+e^-$ from the E865 experiment at BNL. hep-ex/0204006, hep-ex/0505001

Channel	This work	Tuo et al.	ChPT	Experiment
$\mathrm{Br}[K \to \mu\nu_\mu\, e^+e^-]$	$8.26(13)\,10^{-8}$	$10.59(33)\,10^{-8}$	$8.25\,10^{-8}$	$7.93(33)\,10^{-8}$
$\mathrm{Br}[K \to e\nu_e\, \mu^+\mu^-]$	$0.762(49)\,10^{-8}$	$0.72(5)\,10^{-8}$	$0.62\,10^{-8}$	$1.72(45)\,10^{-8}$
$\mathrm{Br}[K \to e\nu_e\, e^+e^-]$	$1.95(11)\,10^{-8}$	$1.77(16)\,10^{-8}$	$1.75\,10^{-8}$	$2.91(23)\,10^{-8}$
$\mathrm{Br}[K \to \mu\nu_\mu\, \mu^+\mu^-]$	$1.178(35)\,10^{-8}$	$1.45(6)\,10^{-8}$	$1.10\,10^{-8}$	$-$

- Recall that both our results and those of Tuo et al. were obtained at unphysical u and d quark masses and at a single lattice spacing and single volume.The quoted errors do not include estimates of the corresponding systematic uncertainties.
- The results from Tuo et al. are from v2 of their paper posted in February 2022.
- Example: Analysis of the form factors from $K \to \mu\nu_\mu\, e^+e^-$ and $K \to e\nu_e\, e^+e^-$ decays gives $H_1(0,0) = 0.227(19)$. hep-ex/0204006
  From $K \to e\nu_e\, \mu^+\mu^-$ decays, $H_1(0,0) = 0.303(41)$. hep-ex/0505001
  We find $H_1(0,0) = 0.176(9)$ (at unphysical m_{ud}).

- We have developed and implemented a framework for computing QED corrections (and those from strong isospin breaking) to leptonic decay rates, opening a new precision regime for flavour physics.
    - Cancellation of infrared divergences and subtraction of $O(1/L)$ corrections included. Leading finite-volume corrections are $O(1/L^2)$.
    - Procedure consistent with definition of G_F at $O(\alpha_{\rm em})$.
    - Corrections are of $O(1\%)$ as expected.
- Future improvements and extensions:
    - Numerical application to leptonic decays of heavy mesons.
    - Improved matching of lattice operators to those in the W-regularization. (?)
    - Evaluation of disconnected diagrams.(?)
- See also the talk by Luchang Jin on the *Infinite-Volume Reconstruction* method in which the finite-volume corrections are exponentially small.
- Extension to other processes:
    - Leptonic decays are particularly simple in that there are no imaginary parts (in Minkowski space). This is generally not the case, leading to additional subtleties in relating FV sums to infinite-volume integrals.

Summary and Conclusions - Radiative Decays

- We have carried out a detailed study of $P \to \ell\nu_\ell\gamma$ decays for $P = \pi,\ K,\ D_{(s)}$ evaluating the form factors $F^{\pm} = F_V \pm F_A$.
  - Work is continuing to improve the precision as well as to extending the kinematical range for D_s decays beyond $x_\gamma = 0.4$.
- Experimental measurements are available for $K \to e\nu_e\gamma$, $K \to \mu\nu_\mu\gamma$ and $\pi \to e\nu_2\gamma$ decays.
- Our results agree well for $K \to e\nu_e\gamma$ decays but there exist significant tensions in some regions of phase space with the experimental data for $K \to \mu\nu_\mu\gamma$ and $\pi \to e\nu_2\gamma$ decays.
- (Setting aside our lattice results, we have been unable to find acceptable fits with phenomenological models for the form factors which would give acceptable fits to all the kaon decay data.)
- Exploratory calculation for $K \to \ell\nu_\ell\,\ell'^+\ell'^-$ performed with all four structure-dependent form factors evaluated.
  - This will be extended to physical kinematics.