Massive QED on the lattice

Michele Della Morte

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UNIVERSITY OF SOUTHERN DENMARK

In collaboration with: K. Clark, Z. Hall, B. Hörz, A. Nicholson, A. Shindler, J. T. Tsang, A. Walker Loud and H. Yan

Plan of the talk

Isospin symmetry

The formal N_f flavor QCD Lagrangian

$$L_{QCD}^{N_f} = \sum_{i=1}^{N_f} \overline{\psi}_i(\mathrm{i}(\gamma_\mu D^\mu) - m)\psi_i - \frac{1}{4} G^a_{\mu\nu} G_a^{\mu\nu}$$

in the case of degenerate up and down quarks, is invariant under SU(2) rotations in the (u-d) flavor space.

Isospin breaking (IB) has two sources

$m_u \neq m_d$ (strong IB)
$Q_u \neq Q_d$ (EM IB)

The separation makes sense classically. Renormalization effects induce a mass gap, even with bare degenerate masses ($\rightarrow$ scheme dependence).

IB is responsible for the neutron-proton mass splitting, whose value played an important role in nucleosynthesis and the evolution of stars [BMW, Science 347 (2015)].

More motivations

The 2021 FLAG review [arXiv:2111.09849] gives

$$f_\pi = 130.2(8)\ \mathrm{MeV}\ , \qquad f_K = 155.7(7)\ \mathrm{MeV} \quad [N_f = 2+1]$$
$$f_D = 212.0(7)\ \mathrm{MeV}\ , \qquad f_{D_s} = 249.9(5)\ \mathrm{MeV} \quad [N_f = 2+1+1]$$

obtained in the isospin limit. EM corrections can be included following [Phys.Rev. D91 (2015) no.7, 074506 and Phys.Rev.D 103 (2021) 1, 014502 (Rome-Soton)]

These hadronic parameters are relevant for the extraction of CKM elements from purely leptonic decays. In that game the error is dominated by experiments, as opposed to the semileptonic case. [arXiv:1811.06364 (Rome-Soton)]

Periodic boundary conditions (PBC)

$$\psi(x + L_\mu \hat{\mu}) = \psi(x) \,, \quad A_\mu(x + L_\nu \hat{\nu}) = A_\mu(x)$$

The Lagrangian with one fermion of charge 1 (and $e = 1$) invariant for

$$\begin{array}{rcl} A_\mu(x) & \to & A_\mu(x) + \partial_\mu \Lambda(x) \\ \psi(x) & \to & e^{i\Lambda(x)}\psi(x) \\ \overline{\psi}(x) & \to & \overline{\psi}(x) e^{-i\Lambda(x)} \end{array}$$

$\Lambda(x)$ does not need to be periodic

$$\Lambda(x + L_\mu \hat{\mu}) = \Lambda(x) + 2\pi r_\mu$$

The quantization in r_μ follows from the periodicity of the fermions. In general

$$\Lambda(x) = \Lambda^0(x) + 2\pi \left(\frac{r}{L}\right)_\mu x_\mu$$

with $\Lambda^0(x)$ periodic.

Let us consider the "large gauge transformations" defined by $\Lambda^0 = 0$

$$A_\mu(x) \to A_\mu(x) + 2\pi \frac{r_\mu}{L_\mu} \,, \quad \psi(x) \to \psi(x) e^{i2\pi \left(\frac{r}{L}\right)_\mu x_\mu}$$

they act as a finite volume shift symmetry on the gauge fields.

Considering now the correlator $\langle \psi(T/4, \underline{0}) \overline{\psi}(0, \underline{0}) \rangle$, it is clear that it vanishes as a consequence of invariance under large gauge transformations (choose $r_0$mod(4)=2).

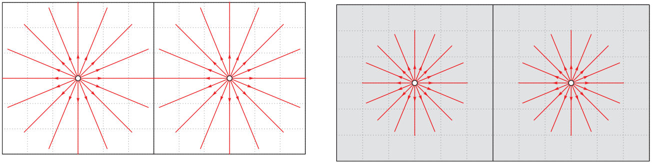
OK, let's gauge away the shift symmetry and require the 0-mode of A_μ to vanish

$$\int d^4x A_\mu(x) = 0$$

that is a non-local constraint, which cannot be imposed through a local gauge-fixing ! Not a derivative one at least .... We like those because gauge-independence of physical quantities is manifest.

Another way to look at the problem

Electric field of a point charge cannot be made periodic and continuous



$$Q = \int d^3x \rho(x) = \int d^3x \partial_i E_i(x) = 0$$

Introduce uniform, time-independent background current c_μ then

$$\int d^3x \rho(x) + \int d^3x c_0 = 0 \, ,$$

which allows to have a net charge.

Promoting c_μ to a field, the Lagrangian density is modified by a term

$$A_\mu(x) \int d^4y \, c_\mu(y)$$

whose EoM is $\int d^4x A_\mu(x) = 0$. When enforcing this on each conf (not just on average) one obtains the QED_{TL} prescription used first in [Duncan et al.,Phys.Rev.Lett. 76 (1996)]. It is

- non-local
- without a Transfer matrix

An Hamiltonian formulation can be recovered adopting the QED_L prescription [Hayakawa and Uno, Prog.Theor.Phys. 120 (2008)], requiring

$$\int d^3x A_\mu(t, \underline{x}) = 0$$

(Imagine coupling a uniform but time-dependent current, as for charged particles propagators).

Both prescriptions

- Introduce some degree of non-locality (issues with renormalization ? O(a) improvement ? Mixing of IR and UV ?)
- Remove modes, which in the electroquenched approximation, would be un-constrained and cause algorithmic problems (wild fluctuations)

QED_L is to be preferred as it has a Transfer matrix. The 'quenched' modes should not play a role in the infinite-vol dynamics (fields vanish at infinity), so it is a matter of finite volume effects (see for example [Davoudi et al., arXiv:1810.05923] for studies in PT and numerically for scalar-QED).

Another natural approach:

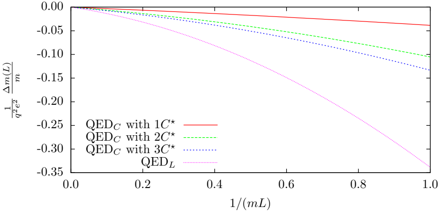
the quantization of the shift symmetry was due to BC for fermions. How about changing it to: [Lucini et al., JHEP 1602 (2016) 076] (C^* BC)

$$A_\mu(x + L_\nu \hat{\nu}) = -A_\mu(x) = A^C_\mu(x)$$
$$\psi(x + L_\nu \hat{\nu}) = \psi^C(x) = C^\dagger \overline{\psi}^T(x)$$
$$\overline{\psi}(x + L_\nu \hat{\nu}) = -\psi(x)^T C \quad \text{with} \quad C^\dagger \gamma_\mu C = -\gamma_\mu^T$$

Completely local, no zero-modes allowed, however at the price of violations of flavor and charge conservation (by boundary effects).

Also, SU(3) dynamical configurations need to be generated again.

It is useful to look at finite volume corrections, e.g. to point-like particles at O(α) ($1/L$ and $1/L^2$ universal) [Lucini et al., JHEP 1602 (2016) 076]



A PT-inspired approach [RM123, JHEP 1204 (2012) 124, Phys.Rev. D87 (2013) no.11, 114505]

Simpler in the case of strong IB:

$$
\begin{aligned}
\mathcal{L} &= \mathcal{L}_{kin} + \mathcal{L}_m \\
&= \mathcal{L}_{kin} + \frac{m_u + m_d}{2}(\bar{u}u + \bar{d}d) - \frac{m_d - m_u}{2}(\bar{u}u - \bar{d}d) \\
&= \mathcal{L}_{kin} + m_{ud}\,\bar{q}q - \Delta m_{ud}\,\bar{q}\tau^3 q \\
&= \mathcal{L}_0 - \Delta m_{ud}\,\hat{\mathcal{L}}\,,
\end{aligned}
$$

$$
\begin{aligned}
\langle \mathcal{O} \rangle &\simeq \frac{\int D\phi\ \mathcal{O}\,(1 + \Delta m_{ud}\,\hat{S})\,e^{-S_0}}{\int D\phi\ \ (1 + \Delta m_{ud}\,\hat{S})\,e^{-S_0}} = \frac{\langle \mathcal{O} \rangle_0 + \Delta m_{ud}\,\langle \mathcal{O}\hat{S}\rangle_0}{1 + \Delta m_{ud}\,\langle \hat{S}\rangle_0} \\
&= \langle \mathcal{O} \rangle_0 + \Delta m_{ud}\,\langle \mathcal{O}\hat{S}\rangle_0\,,
\end{aligned}
$$

Similarly, for QED corrections, one inserts $J_\mu(x)$ (and possible lattice tadpoles) over 4dim vol in correlators evaluated in isospin-symm QCD.

+ One does not compute something tiny rather, derivatives wrt α and Δm_{ud}, which may be O(1)
+ Only renormalization in QCD needs to be discussed
= Still a zero-mode prescription for the explicit photon propagator is needed. With some caveats, the approach can be combined with the infinite-volume propagator [X. Feng and L. Jin, Phys.Rev.D 100 (2019) 9, 094509]. Anyhow, much better control as the computation is fixed order in α.
– The expansion produces quark-disconnected diagrams ($\simeq$ those neglected in electroquenched).

Massive QED

$$L_{QED_m} = \frac{1}{4}F_{\mu\nu}^2 + \frac{1}{2}m_\gamma^2 A_\mu^2 + L_f = L_{Proca} + L_f$$

+ is renormalizable by power counting once the Feynman gauge is imposed through the Stückelberg mechanism [see book by Zinn-Justin]
+ it is local, softly breaks gauge symmetry and has a smooth $m_\gamma \to 0$ limit.
+ Clearly the shift-transformation is not a symmetry anymore. The mass term acts as an extra non-derivative gauge-fixing.

= It introduces a new IR scale on top of L. First one should take $L \to \infty$ and then $m_\gamma \to 0$.

+ Finite volume corrections are (exponentially) small, as long as $m_\gamma L \geq 4$ and $m_\gamma \ll m_\pi$.

The mass term introduces a Gaussian damping factor for the zero mode

$$e^{-\frac{1}{2}m_\gamma^2 \tilde{A}_\mu^2(0)} \; e^{-\frac{1}{2}m_\gamma^2 \sum_{p\neq 0} \tilde{A}_\mu^2(p)}$$

in the path integral. The zero-mode vanishes on average and has variance m_γ^{-1} (so in $\lim m_\gamma \to \infty$ one smoothly recovers QED_{TL}).

The fluctuations of the different modes

$$\sigma_{\tilde{A}_\mu(p)} \approx \frac{1}{p^2 + m_\gamma^2} \quad \text{in particular} \quad \sigma_{\tilde{A}_\mu(0)} \approx \frac{1}{m_\gamma^2}$$

allow to distinguish two regimes (smallest non-zero lattice $p = \frac{2\pi}{L}$)

- $m_\gamma << \frac{2\pi}{L}$ i.e. $Lm_\gamma << 2\pi$. The quantum fluctuations are dominated by the zero-mode, which needs to be treated separately (ε_γ regime).
- $m_\gamma >> \frac{2\pi}{L}$ i.e. $Lm_\gamma >> 2\pi$. All modes have similar fluctuations (p_γ regime).

Finite size effects

In the p_γ regime effects are exponential in $m_\gamma L$ [M. Endres et al., Phys.Rev.Lett. 117 (2016)]. Using non-relativistic QED:

$$\delta M^{LO} = 2\pi\alpha Q^2 m_\gamma \mathcal{I}_1(m_\gamma L)$$

in terms of Bessel functions. NLO in the effective theory also available. The computation is very similar to what is done in χPT, e.g. [J. Bijnens et al., JHEP 1401 (2014) 019].

In the ε_γ regime 0-modes contribute and one may expect power-law FSE. However, a conf. with a 0-mode $\tilde{A}_\mu(0) = c_\mu$ has a weight

$$e^{-\frac{1}{2} m_\gamma^2 c_\mu L^3 T}$$

which vanishes if any of the spatial or temporal extents goes to ∞.
$\Rightarrow$ We expect power-like FSE (or any 0-mode effect) to be $\propto (L^3 T)^{-1}$

Tree-level computation in scalar QED in ε_γ regime [J. T. Tsang, A. Shindler et al., LATTICE21, arXiv:2201.03251]

$$A_\mu(x) = q_\mu(x) + B_\mu \, , \quad \text{with} \quad \int d^4 x q_\mu(x) = 0$$

keeping only the non-interacting (with $q_\mu(x)$) part of the Lagrangian

$$\Gamma_2(p) = (p_\mu + eB_\mu)^2 + m^2$$

$$\textit{for} \quad \vec{p} = \vec{0}, \quad \Gamma_2(p) = (p_0 + eB_0)^2 + \omega_B^2 \quad \textit{with} \quad \omega_B^2 = m^2 + e^2|\vec{B}|^2$$

Integrating (non-perturbatively) over B_μ, the 2-pt function reads

$$\langle 0|\Phi_0^*(t)\Phi_0(0)|0\rangle = Z^{-1} \int d^4B \, e^{-\frac{1}{2}m_\gamma^2 B^2 V_4} \int dp_0 \frac{e^{ip_0 t}}{(p_0 + eB_0)^2 + \omega_B^2}$$

$$\propto \int d^3B \; e^{-\frac{1}{2}m_\gamma^2|\vec{B}|^2 V_4} \frac{e^{-\omega_B t}}{2\omega_B} \; \int dB_0 \; e^{-\frac{1}{2}m_\gamma^2 B_0^2 V_4} e^{-ieB_0 t}$$

Fourier transform of a Gaussian is a Gaussian ....

$$\to e^{-\frac{e^2 t^2}{2 m_\gamma^2 V_4}} \int d^3 B \; e^{-\frac{1}{2} m_\gamma^2 |\vec{B}|^2 V_4} \frac{e^{-\omega_B t}}{2\omega_B}$$

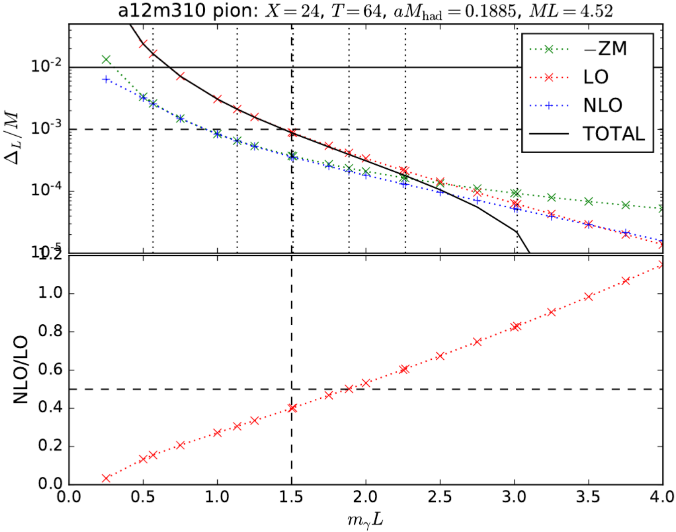
1st non-trivial effect of zero mode: there is a universal term in the correlator falling as e^{-t^2}. The effect is V_4 suppressed (in the effective mass), as expected. [M. Endres et al., Phys.Rev.Lett. 117 (2016) and A. Patella, PoS LATTICE2016 (2017) 020].

The remaining integral by saddle point (exact for $V_4 \to \infty$)

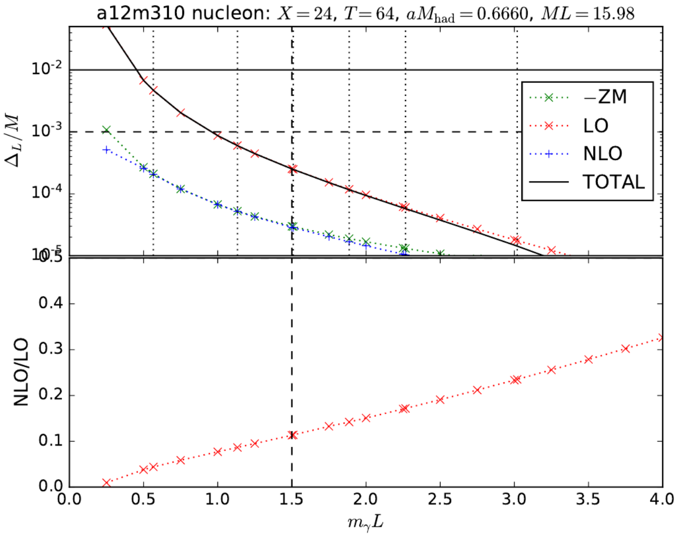
$$\to e^{-\frac{e^2 t^2}{2 m_\gamma^2 V_4}} e^{-m\left(1+\frac{e^2}{2 m^2 m_\gamma^2 V_4}\right) t}$$

2nd non-trivial effect of zero mode: there is a $O(1/V_4)$ FSE correction to the hadron mass.

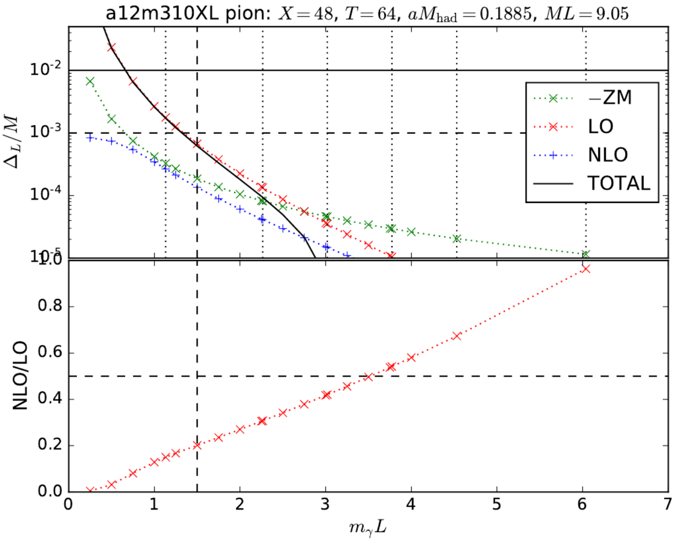
Putting things together



Finite size effects



Finite size effects



We conclude we want

$$m_\gamma L \geq 1.5$$

Suppose at the same time we want

$$m_\gamma \leq \frac{m_\pi}{n}$$

for $m_\gamma \to 0$ extrapolation. All together this means

$$m_\pi L \geq 1.5n$$

with the lower bound $m_\pi L \approx 4$ from QCD FSE.
So we need to understand what n we need to safely extrapolate in m_γ.
From [M. Endres et al., Phys.Rev.Lett. 117 (2016)], the leading effect is linear in m_γ

$$\Delta_\gamma M^{LO} = -\frac{\alpha}{2} Q^2 m_\gamma$$

Results

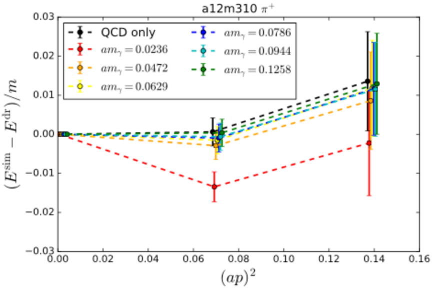
- Mixed action setup [E. Berkovitz et al.,Phys. Rev. D 96, 054513 (2017)]:
  $N_f = 2 + 1 + 1$ HISQ in the sea
  Möbius domain wall in the valence (after gradient flowing confs)
- a12m310 and a12m310XL with $T/a = 64$ and $L/a = 24$ and 48 resp.
- Electroquenched approximation with Feynman gauge and compact formulation
- Preliminary account in [J. T. Tsang, A. Shindler et al., LATTICE21, arXiv:2201.03251]
- Paper(s) in preparation, with measurements collected on a12m130 , a12m220and a09m310 to explore chiral and continuum limit.

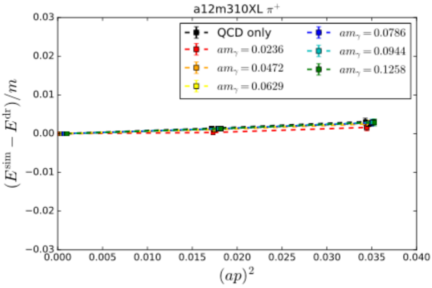
Dispersion relation $E^2 = m^2 + p^2$

As argued in [A. Patella, PoS LATTICE2016 (2017) 020], in the limit $m_\gamma \to 0$ at finite L one gets

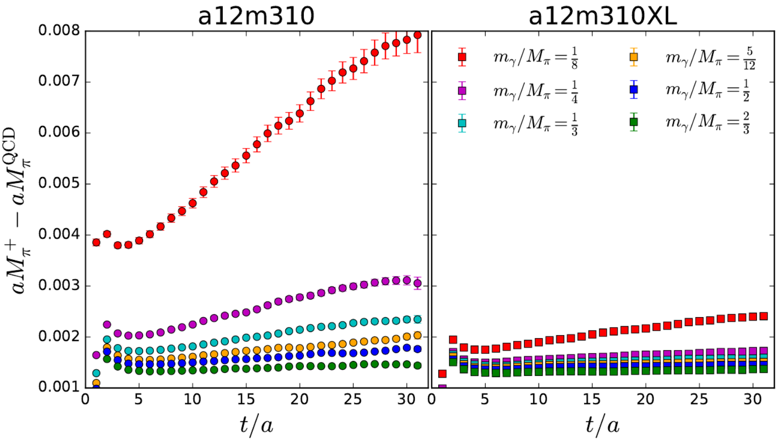
$$\lim_{m_\gamma \to 0} C(t, \vec{p}) \propto e^{-\frac{e^2}{2m_\gamma^2 V_4} t^2} \langle \psi(t, \vec{0}) \bar{\psi}(0) \delta_{Q,0} \rangle_{TL} \left(1 + O(m_\gamma^2)\right)$$

i.e. some *stiffness* to external momenta. We are not in that regime:

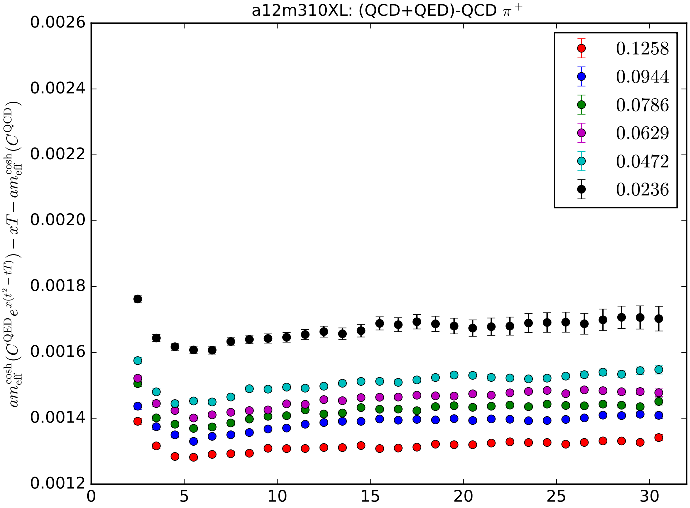




Still, we see zero-mode effects



Those are universal and can be subtracted though.


a12m310XL: (QCD+QED)-QCD π^+
$am_{\rm eff}^{\rm cosh}(C^{\rm QED}e^{x(t^2-tT)}) - xT - am_{\rm eff}^{\rm cosh}(C^{\rm QCD})$
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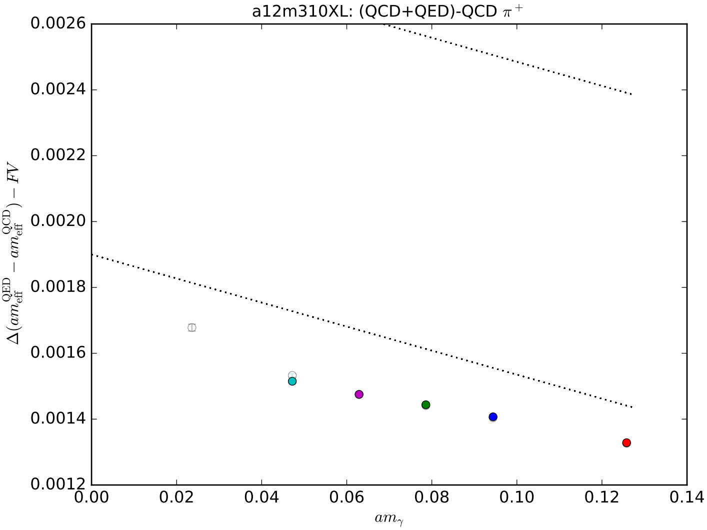
a12m310XL: (QCD+QED)-QCD π^+
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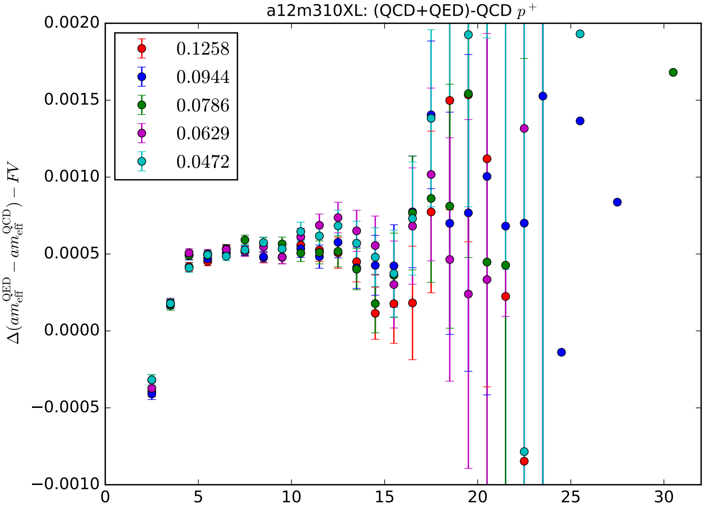
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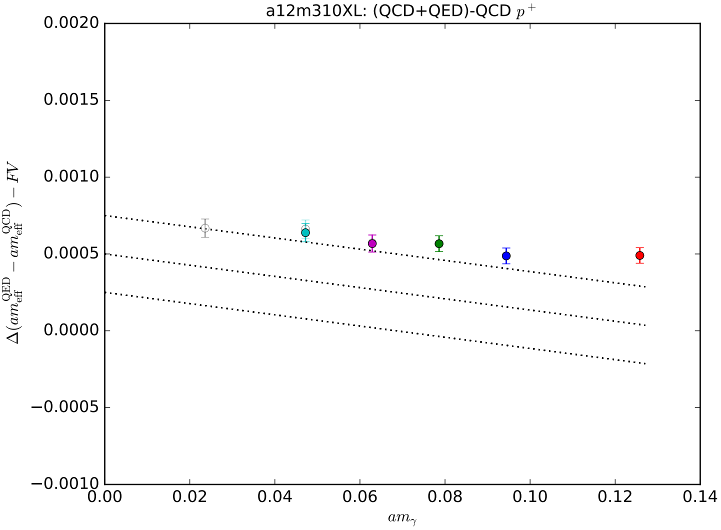
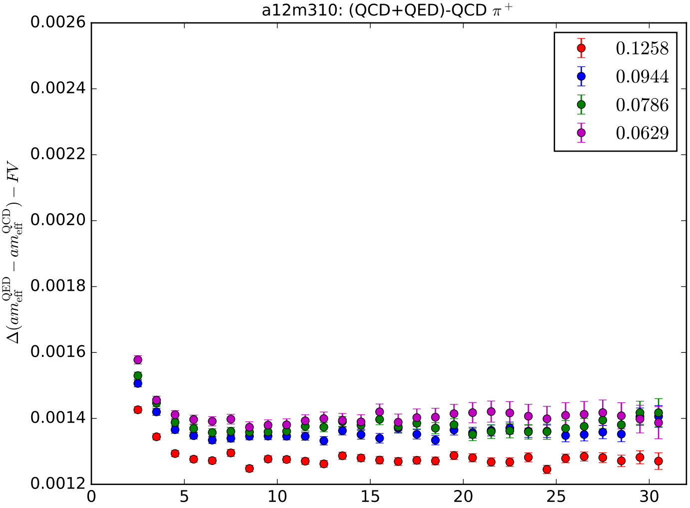


Things look consistent going to $m_\gamma \approx \frac{m_\pi}{4}$. Relaxing to $\frac{m_\pi}{3}$...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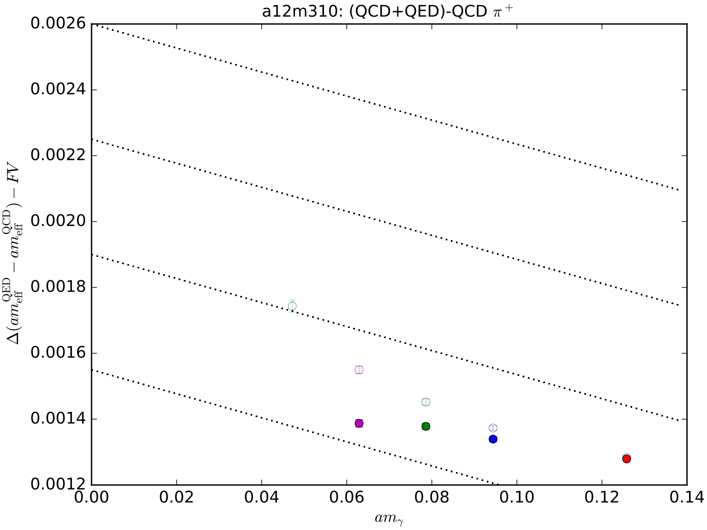
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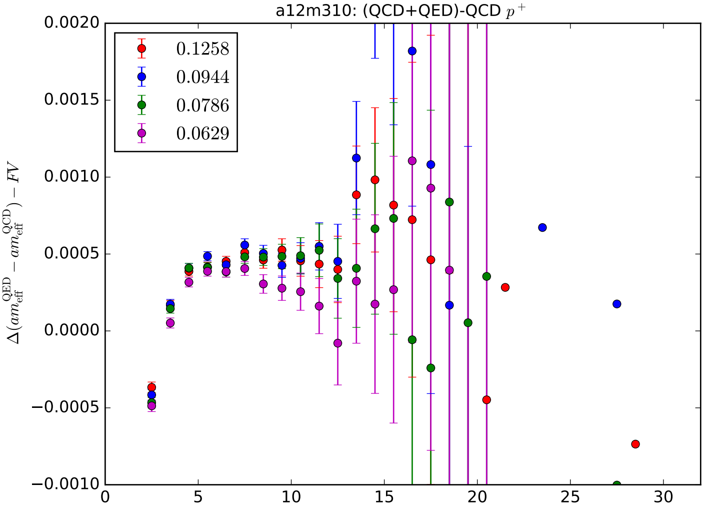
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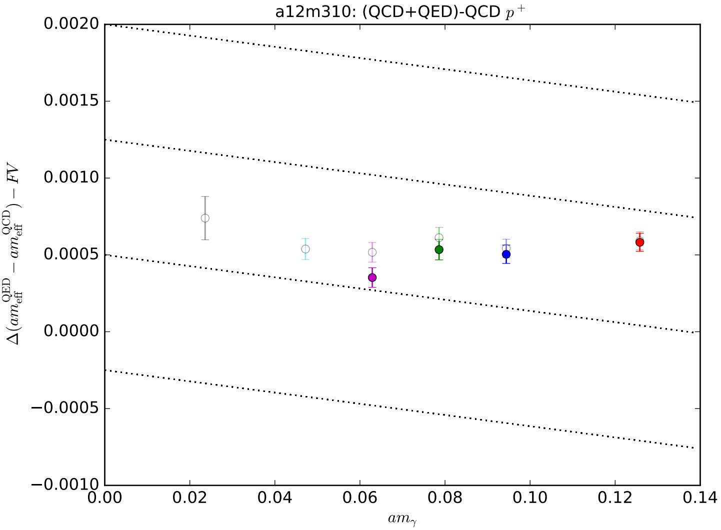
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Conclusions and outlook

- We have shown that QED_M is a viable approach to non-perturbative QED on the lattice with the goal of high precision. We are systematics dominated (error from FSE).
- We completed the *due diligence* by looking at the spectrum.
- We discussed the interplay between m_γ and L and we empirically obtained a rule $m_\gamma L \geq 1.5$ and $m_\pi \geq 4m_\gamma$ for FSE and m_γ-effects to be under control. All together we need $m_\pi L \approx 6$.
- Still, in our simulations we see residual effects of zero-modes, e.g. FSE of $O(1/V_4)$.
- Short run: We plan to include strong-isospin breaking using the 'perturbative' RM123 method.
- Long run: QED corrections to form-factors, starting with g_A.

Short run: Dashen's theorem:

$$(\Delta M_\pi^2)^\gamma = (\Delta M_K^2)^\gamma$$

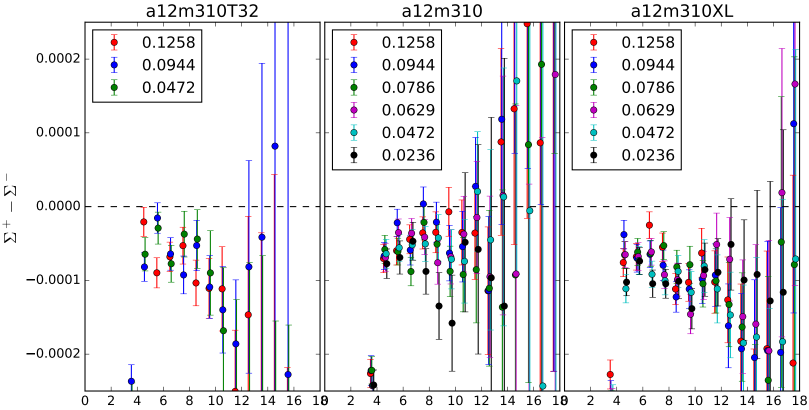
with $\Delta M_X^2 = M_{X^+}^2 - M_{X^0}^2$. Violations are parameterized by

$$\epsilon = \frac{(\Delta M_K^2 - \Delta M_\pi^2)^\gamma}{\Delta M_\pi^2}$$

FLAG 21 gives $\epsilon = 0.79(6)$ for $N_f = 2+1+1$ from 3 computations (RM123, MILC and BMW).

In order to address that we need to define the isospin symmetric point at $\alpha \neq 0$. In [A. Bussone et al., PoS LATTICE2018 (2018) 293] we defined a scheme for that by requiring

$$M_{\Sigma^+} = M_{\Sigma^-}$$



… small valence retuning needed