Isospin-breaking corrections to $\Gamma(K_{\mu 2})/\Gamma(\pi_{\mu 2})$ with close-to-physical chiral fermions

Matteo Di Carlo

Workshop "QED in weak decays"
24 June 2022

Higgs Centre for Theoretical Physics
THE UNIVERSITY of EDINBURGH

RBC/UKQCD Collaboration

in particular

Peter Boyle (BNL & Edinburgh)
Matteo Di Carlo (Edinburgh)
Felix Erben (Edinburgh)
Vera Gülpers (Edinburgh)
Maxwell T. Hansen (Edinburgh)
Tim Harris (Edinburgh)
Nils Hermansson-Truedsson (Lund)
Raoul Hodgson (Edinburgh)
Andreas Jüttner (CERN & Southampton)
Antonin Portelli (Edinburgh)
James Richings (Edinburgh)
Andrew Yong (Edinburgh)

Outline of the talk

- QED corrections to leptonic decays on the lattice   see Chris Sachrajda's talk
- The RBC/UKQCD way to $\delta R_{K\pi}$
- First (preliminary) results with chiral fermions close to the physical point
- Comparison with RM123+Soton lattice calculation
- Final remarks on $|V_{us}/V_{ud}|$

The goal: testing the Standard Model

Indirect searches of new physics using CKM matrix unitarity constraints

$$V_{\mathrm{CKM}} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}$$

in the Standard Model:

$$|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1$$

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Matrix elements can be extracted e.g. from **leptonic** and **semileptonic** decays of mesons

$$\underbrace{\frac{\Gamma[K \to \ell\nu_\ell(\gamma)]}{\Gamma[\pi \to \ell\nu_\ell(\gamma)]}}_{\text{experiments}} \propto \left|\frac{V_{us}}{V_{ud}}\right|^2 \underbrace{\left(\frac{f_K}{f_\pi}\right)^2}_{\text{QCD}}$$

$$\underbrace{\Gamma[K \to \pi\ell\nu_\ell(\gamma)]}_{\text{experiments}} \propto |V_{us}|^2 \underbrace{|f_+^{K\pi}(0)|^2}_{\text{QCD}}$$

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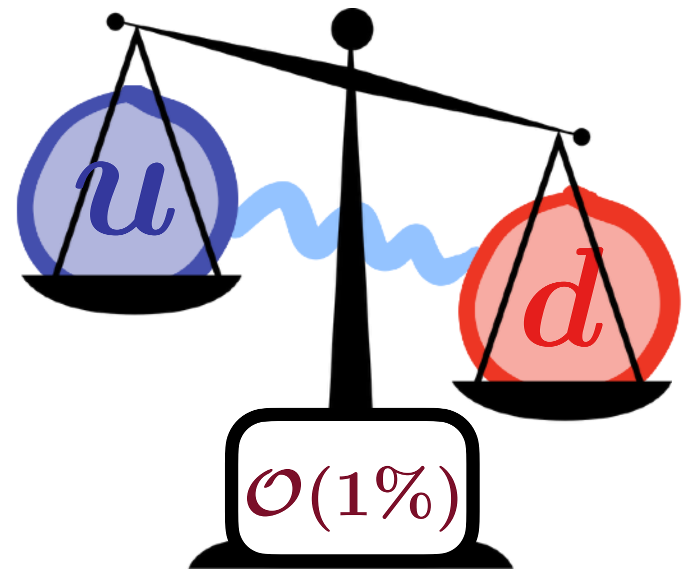
FLAG 2021 Flavour Lattice Averaging Group

f_K/f_π and $f_+^{K\pi}(0)$ determined from lattice QCD with **sub percent precision**!

Isospin-breaking effects & the lattice

Current level of precision requires the inclusion of isospin breaking (IB) corrections:

- ○ strong effects $[m_u - m_d]_{\text{QCD}} \neq 0$
- ○ electromagnetic effects $\alpha \neq 0$



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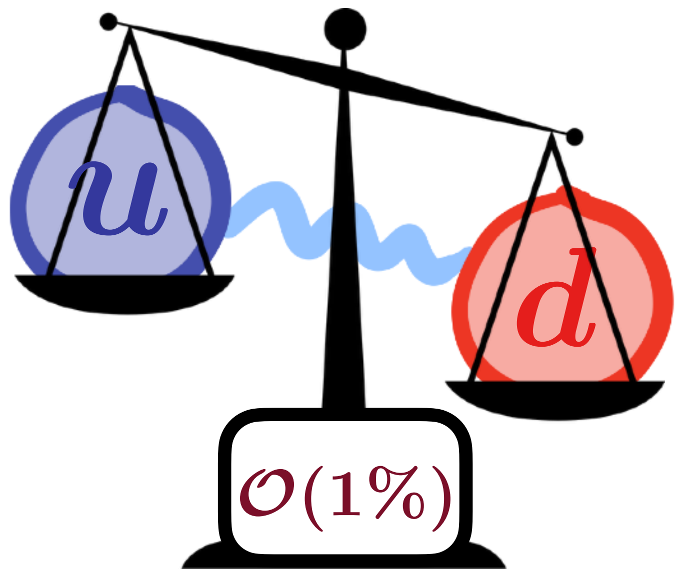
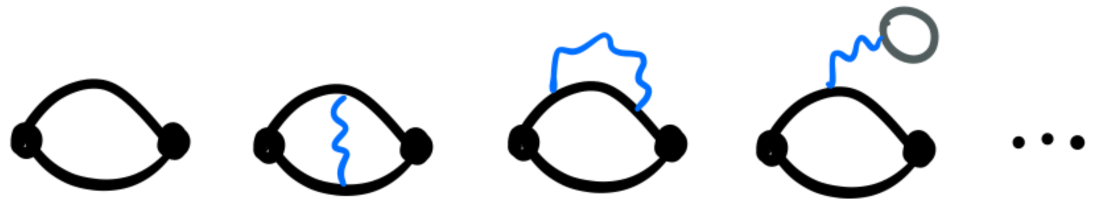
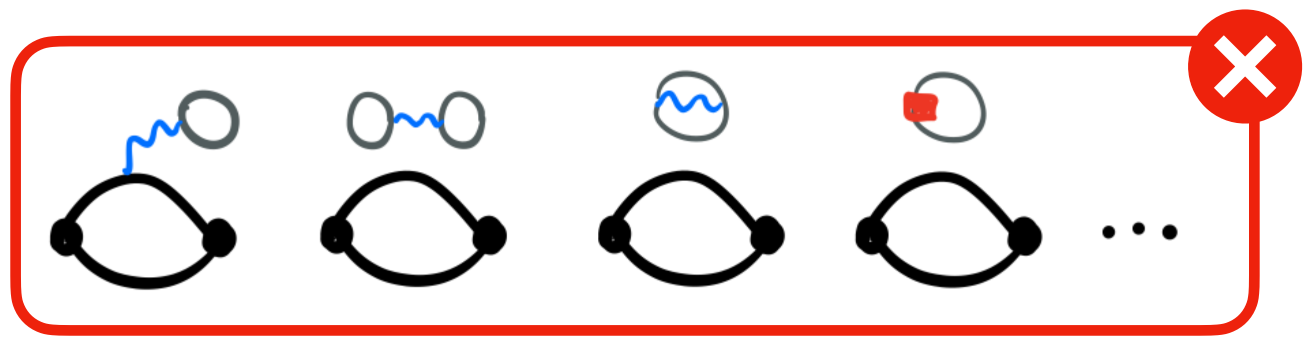


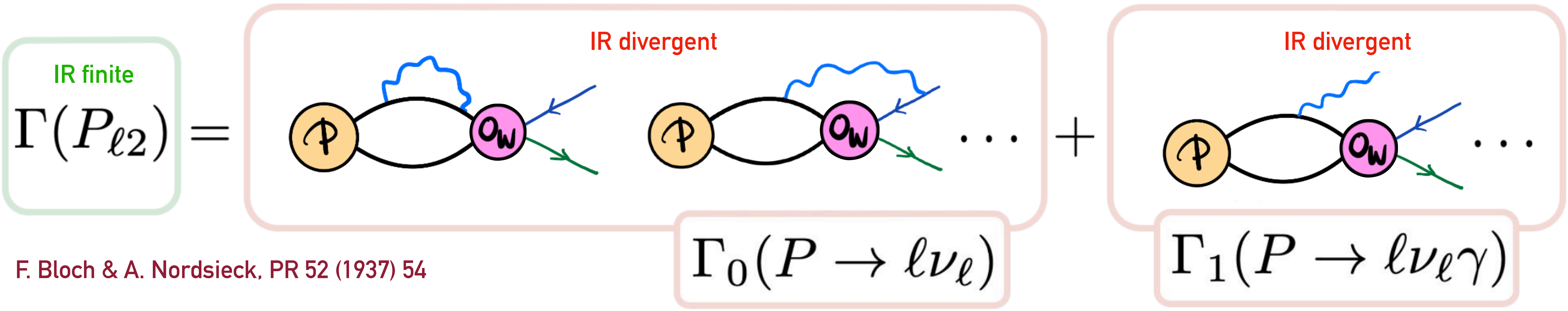
Different ways to include them on the lattice, **in this calculation:**

- **RM123 approach** perturbative expansion of the path integral: 
- Photon regularised using **QED$_L$** prescription: $\sum_{\mathbf{k}} \rightarrow \sum_{\mathbf{k}\neq 0}$ & power-like FVE's
- **Electro-quenched** approximation (neutral sea quarks): 

Studying the decay rate at $\mathcal{O}(\alpha)$

Many subtleties arise, for example

- **IR divergences** in intermediate steps



- new **UV divergences**: add QED in renormalisation of operators

- the decay constant f_P becomes an unphysical quantity:
  need to introduce a **scheme** to give sense to "QCD"

The RM123+Soton recipe

N. Carrasco et al., PRD 91 (2015)
V. Lubicz et al., PRD 95 (2017)
D. Giusti et al., PRL 120 (2018)
MDC et al., PRD 100 (2019)

- $\Gamma_1(P \to \ell\nu_\ell\gamma) \sim \Gamma_1^{\mathrm{pt}}(\Delta E_\gamma)$ for sufficiently soft photons (small ΔE_γ)
- $\Gamma(P_{\ell 2}) = \underbrace{\lim_{L\to\infty}\left(\Gamma_0(L) - \Gamma_0^{\mathrm{pt}}(L)\right)}_{\text{IR finite}} + \underbrace{\lim_{m_\gamma\to 0}\left(\Gamma_0^{\mathrm{pt}}(m_\gamma) + \Gamma_1^{\mathrm{pt}}(\Delta E_\gamma, m_\gamma)\right)}_{\text{IR finite}}$

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... and further improvements

G.M. de Divitiis et al., [1908.10160]
A. Desiderio et al., PRD 102 (2021)
R. Frezzotti et al., PRD 103 (2021)

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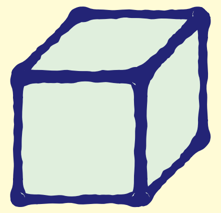
see C. Sachrajda's talk

MDC et al., PRD 105 (2022)

- $\Gamma(P_{\ell 2}) = \underbrace{\lim_{L\to\infty}\left(\Gamma_0(L) - \Gamma_0^{(n)}(L)\right)}_{\text{IR finite}} + \underbrace{\lim_{m_\gamma\to 0}\left(\Gamma_0^{\rm pt}(m_\gamma) + \Gamma_1^{\rm pt}(\Delta E_\gamma, m_\gamma)\right)}_{\text{IR finite}}$ $\quad \Gamma_0^{(n)}(L) = \Gamma_0^{\rm pt}(L) + \Delta\Gamma_0^{(n)}(L)$

see N. Hermansson-Truedsson's talk

Virtual decay rate at $\mathcal{O}(\alpha)$

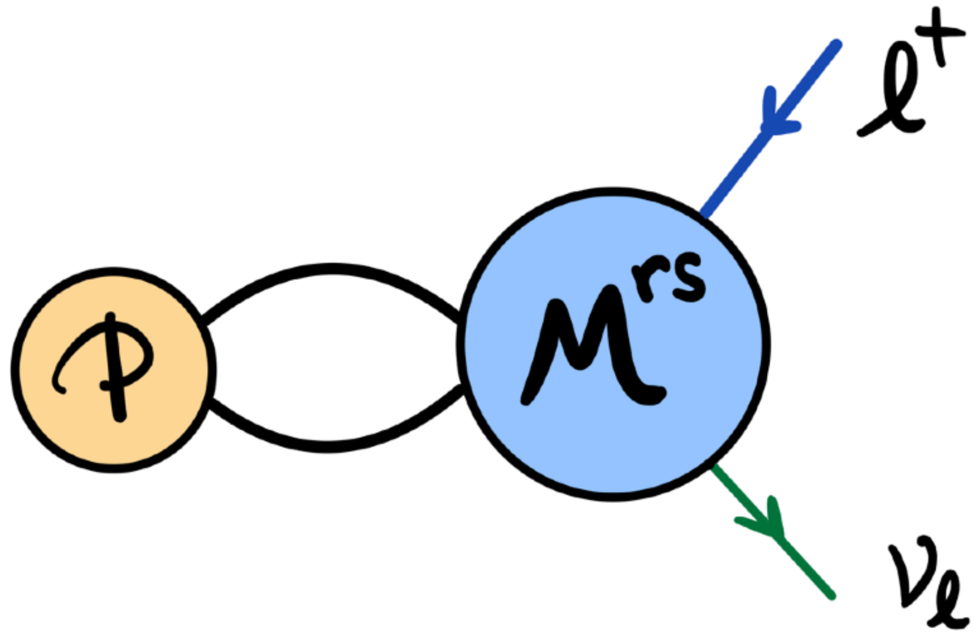


an IR regulator
is in place here

$$\Gamma(P_{\ell 2}) = \mathcal{K}|\mathcal{M}|^2$$

$$\mathcal{K} = \frac{G_F^2}{16\pi}|V_{q_1 q_2}|^2 \frac{1}{2m_P}\left(1 - \frac{m_\ell^2}{m_P^2}\right)$$

from phase space integral



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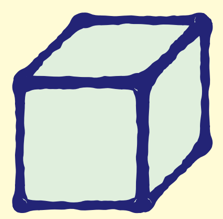

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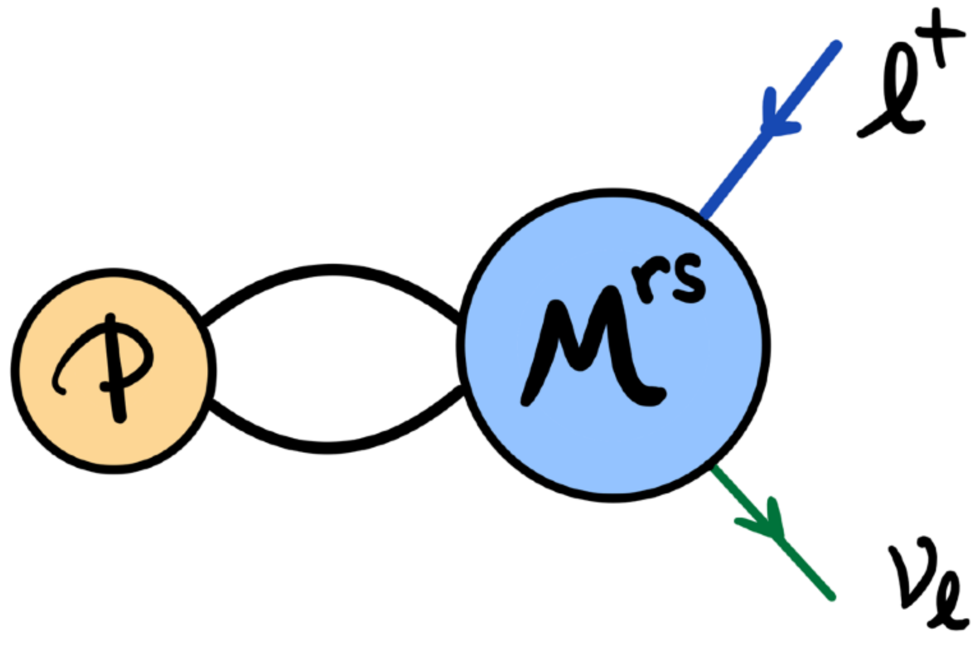


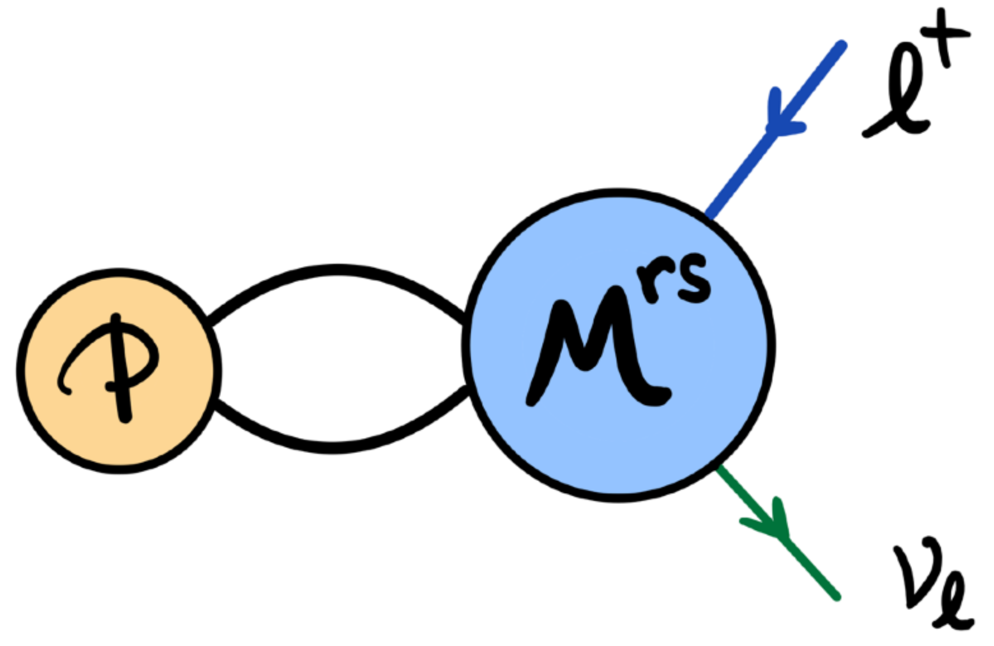
$$\Gamma(P_{\ell 2}) = \Gamma_P^{\text{tree}}\left(1 + \delta R_P\right)$$

PDG convention:

$$\Gamma_P^{\text{tree}} = \frac{G_F^2}{8\pi}|V_{q_1 q_2}|^2 m_\ell^2 \left(1 - \frac{m_\ell^2}{m_P^2}\right)^2 m_P \left[f_{P,0}\right]^2$$

Virtual decay rate at $\mathcal{O}(\alpha)$

an IR regulator is in place here



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$$\delta R_P = \frac{\Gamma(P_{\ell 2}) - \Gamma_P^{\text{tree}}}{\Gamma_P^{\text{tree}}} = 2\left(\frac{\delta \mathcal{A}_P}{\mathcal{A}_{P,0}} - \frac{\delta m_P}{m_{P,0}} + \frac{\delta \mathcal{Z}}{\mathcal{Z}_0}\right)$$

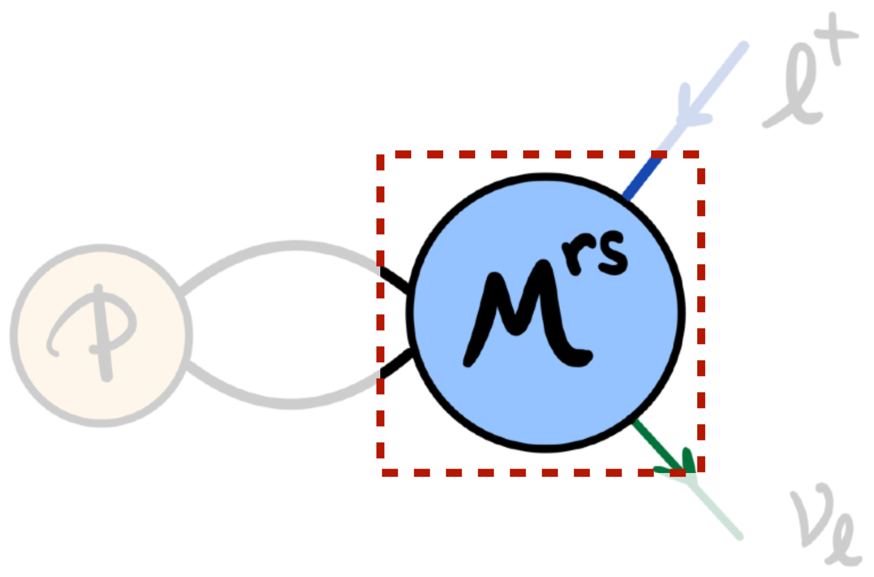
Our target: $$\delta R_{K\pi} = 2\left(\frac{\delta \mathcal{A}_K}{\mathcal{A}_{K,0}} - \frac{\delta m_K}{m_{K,0}}\right) - 2\left(\frac{\delta \mathcal{A}_\pi}{\mathcal{A}_{\pi,0}} - \frac{\delta m_\pi}{m_{\pi,0}}\right)$$

(assuming a mass independent scheme is used to renormalised the operators)

Reduction formula

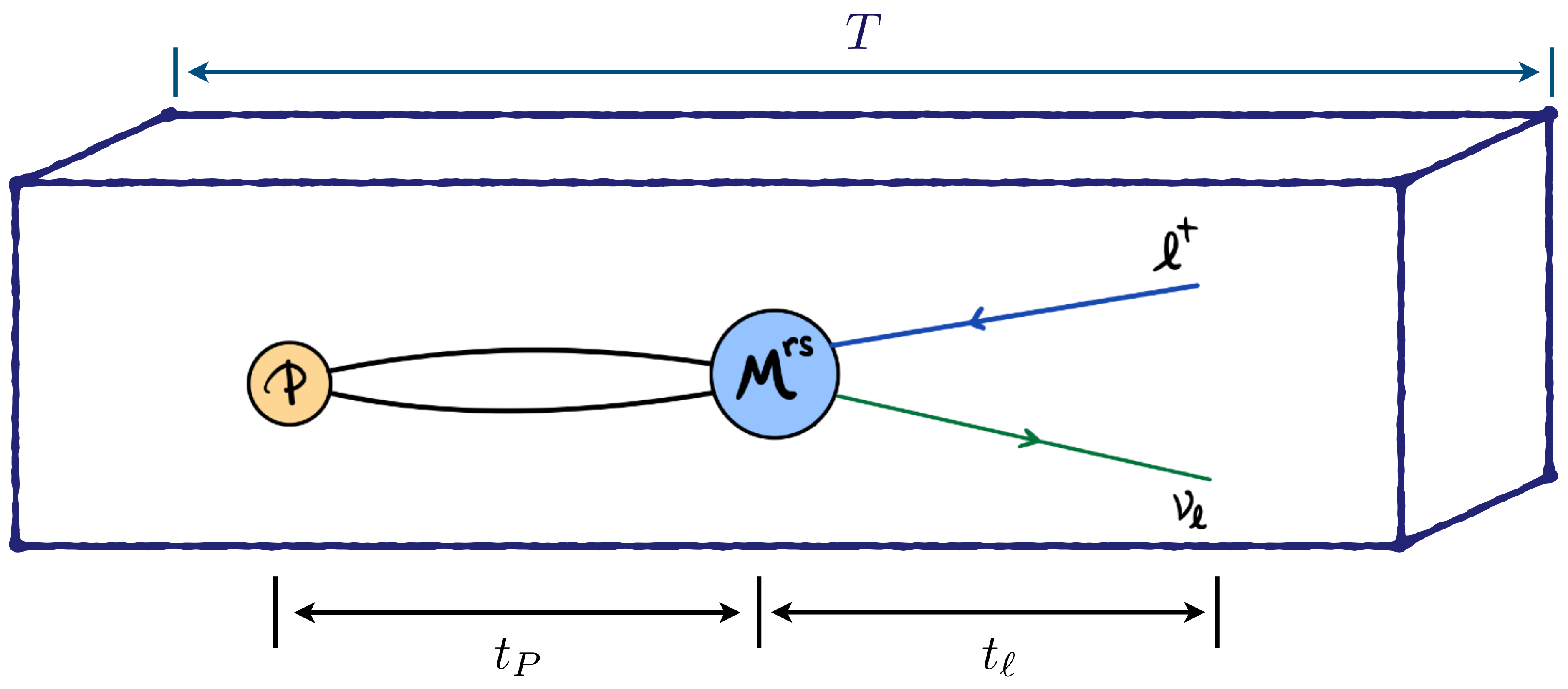
$$\mathcal{M}^{rs} = \lim_{\text{on-shell}} \left\{ Z_P^{-1} S_P(p)^{-1}\ \bar{u}_\nu^r(p_\nu) S_\nu(p_\nu)^{-1}\ \boldsymbol{C_W(p, p_\ell)}\ S_\ell(p_\ell)^{-1} v_\ell^s(p_\ell) \right\}$$

Our goal:



- amputate external states
- take on-shell limit

How we realise it:



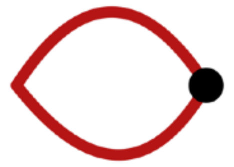
- evaluate correlator in a box in time-momentum representation
- pull interpolating operators far away from weak operator

(finite *T* effects should be treated carefully)

From correlators to matrix elements

Warm up: the tree-level amplitude

$$|\mathcal{M}_0(\mathbf{p}_\ell)|^2 = |\mathcal{A}_{P,0}|^2 |\mathcal{L}_0(\mathbf{p}_\ell)|^2 =$$ 

 $= \mathcal{A}_{P,0} = \langle 0|A^0|P, \mathbf{p} = \mathbf{0}\rangle_0 = i m_{P,0}\,[f_{P,0}]$

 $= |\mathcal{L}_0|^2 = \sum_{r,s} |\bar{u}^r_\nu(-\mathbf{p}_\ell)\,\gamma^0(1-\gamma_5)\,v^s_\ell(\mathbf{p}_\ell)|^2$

 $= \sum_s v^s_\ell(\mathbf{p}_\ell)\bar{v}^s_\ell(\mathbf{p}_\ell)$  $= \sum_r u^r_\nu(-\mathbf{p}_\ell)\bar{u}^r_\nu(-\mathbf{p}_\ell)$

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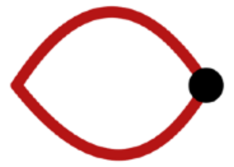 $= \mathcal{A}_{P,0} = \langle 0|A^0|P,\mathbf{p}=\mathbf{0}\rangle_0 = i m_{P,0}\,[f_{P,0}]$

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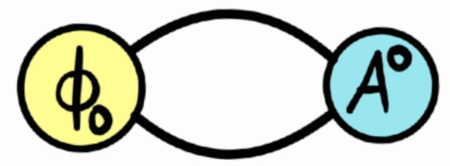
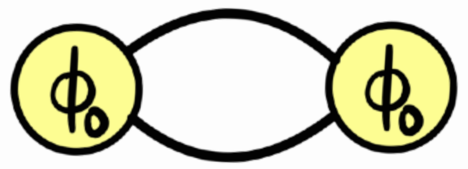
Define Euclidean **lattice correlators** and extract $\mathcal{A}_{P,0}$ by combining them:



$= \langle 0|A^0(0)\phi^\dagger(-t)|0\rangle \quad \to \frac{Z_{P,0}\mathcal{A}_{P,0}}{2m_{P,0}} \quad \mathrm{e}^{-m_{P,0}t}$

$Z_{P,0} = \langle P,\mathbf{p}=\mathbf{0}|\phi^\dagger|0\rangle_0$



$= \langle 0|\phi(0)\phi^\dagger(-t)|0\rangle \quad \to \frac{Z^2_{P,0}}{2m_{P,0}} \quad \mathrm{e}^{-m_{P,0}t}$

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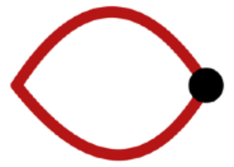 $= \mathcal{A}_{P,0} = \langle 0|A^0|P, \mathbf{p}=\mathbf{0}\rangle_0 = i m_{P,0}\,[f_{P,0}]$

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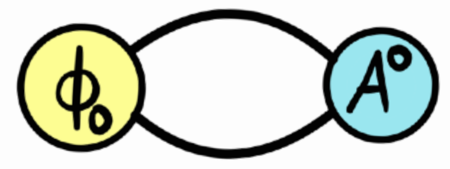
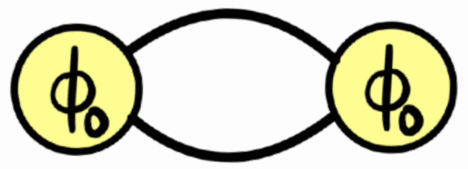


$$= \langle 0|A^0(0)\phi^\dagger(-t)|0\rangle_T \rightarrow \frac{Z_{P,0}\mathcal{A}_{P,0}}{2m_{P,0}}\left\{e^{-m_{P,0}t} - e^{-m_{P,0}(T-t)}\right\}$$

$$Z_{P,0} = \langle P, \mathbf{p}=\mathbf{0}|\phi^\dagger|0\rangle_0$$



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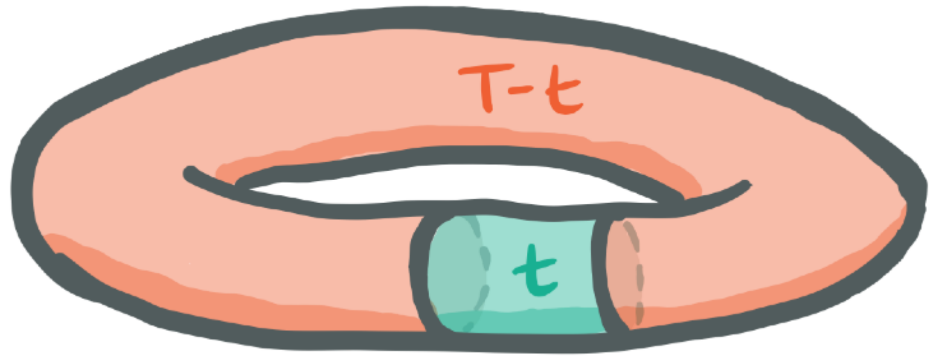


From correlators to matrix elements

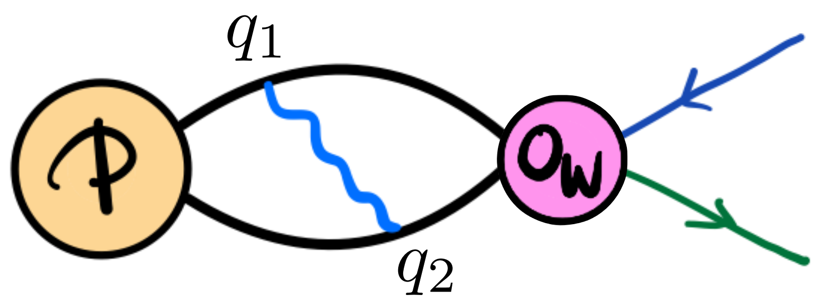
IB corrections

We can distinguish two kinds of corrections:

$$|\mathcal{M}(\mathbf{p}_\ell)|^2 = |\mathcal{M}_0(\mathbf{p}_\ell)|^2 + \delta_{\text{fact}}|\mathcal{M}(\mathbf{p}_\ell)|^2 + \delta_{\text{non-fact}}|\mathcal{M}(\mathbf{p}_\ell)|^2$$

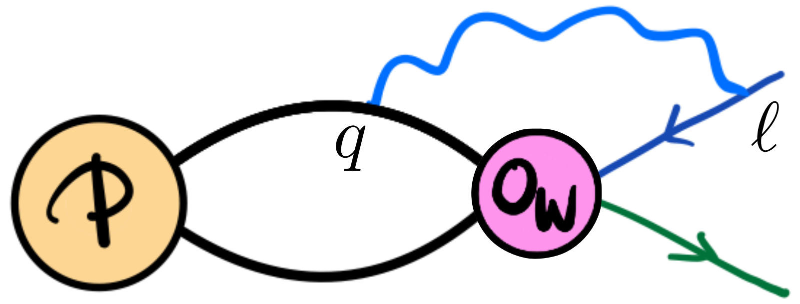
- **Factorisable**

$$\delta_{\text{fact}}|\mathcal{M}(\mathbf{p}_\ell)|^2 = 2\sum_{q_1,q_2} \text{Re}\left\{\delta_{q_1 q_2}\mathcal{M}(\mathbf{p}_\ell)\ [\mathcal{M}_0(\mathbf{p}_\ell)]^\dagger\right\}$$

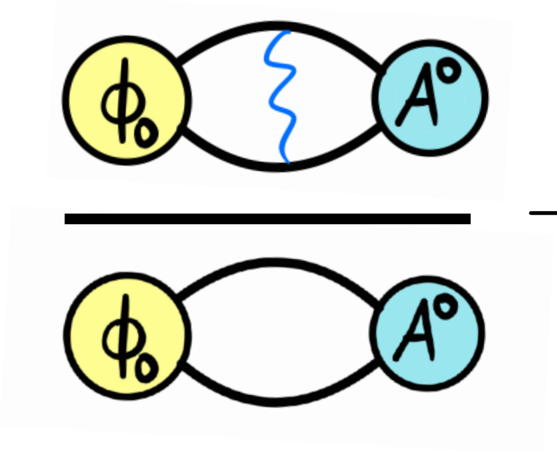


- **Non-factorisable**

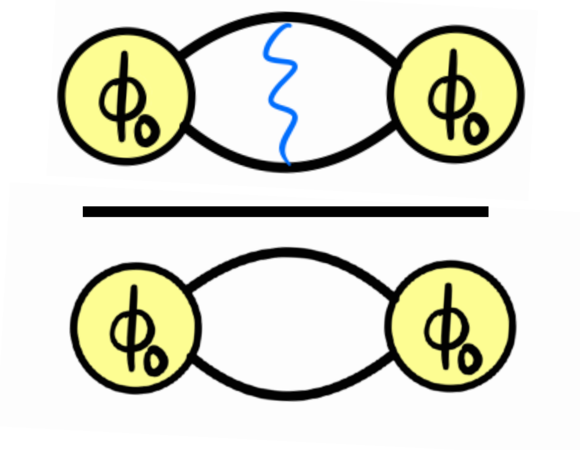
$$\delta_{\text{non-fact}}|\mathcal{M}(\mathbf{p}_\ell)|^2 = 2\sum_{q} \text{Re}\left\{\delta_{q\ell}\mathcal{M}(\mathbf{p}_\ell)\ [\mathcal{M}_0(\mathbf{p}_\ell)]^\dagger\right\}$$



Factorisable QED corrections



$$\rightarrow \frac{\delta_{\text{fact}}\mathcal{A}_P}{\mathcal{A}_{P,0}} + \frac{\delta Z_P}{Z_{P,0}} - \frac{\delta m_P}{m_{P,0}} f_{\text{PA}}(t,T)$$



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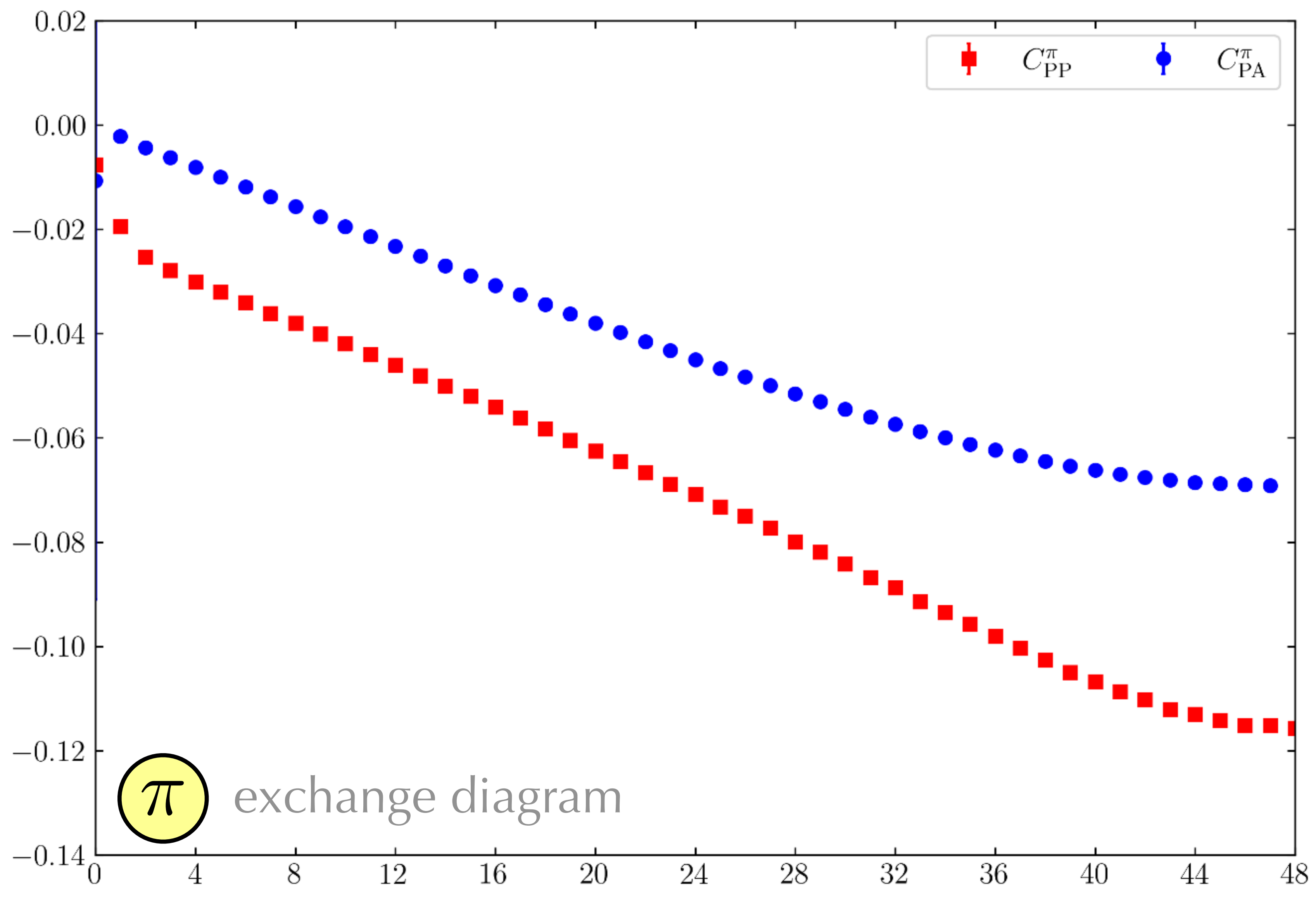
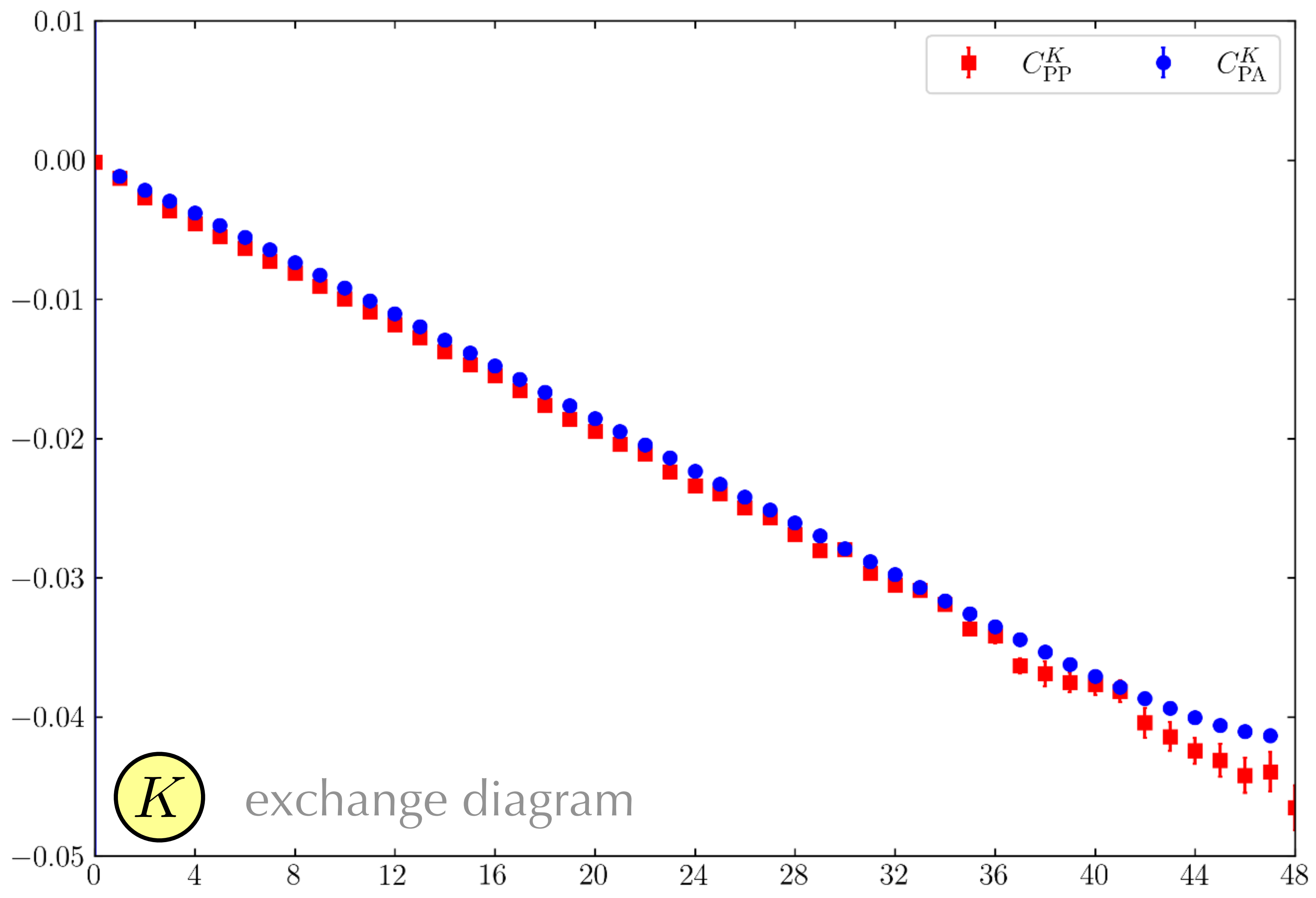


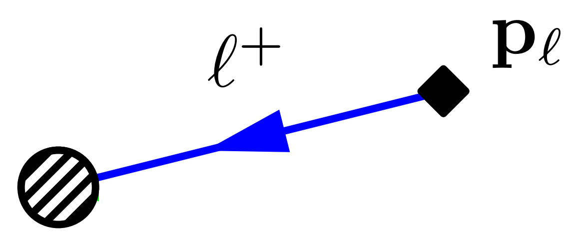


$$f_{\text{PA}}(t,T) = 1 + m_{P,0}\left\{\frac{T}{2} - \left(t - \frac{T}{2}\right)\coth\left[m_{P,0}\left(t - \frac{T}{2}\right)\right]\right\} \approx 1 + m_{P,0}t$$

$$f_{\text{PP}}(t,T) = 1 + m_{P,0}\left\{\frac{T}{2} - \left(t - \frac{T}{2}\right)\tanh\left[m_{P,0}\left(t - \frac{T}{2}\right)\right]\right\} \approx 1 + m_{P,0}t$$

Non-factorisable QED corrections

The lepton in a finite volume



$$= \quad S(0|t,\mathbf{p}_\ell) = \sum_r \left\{ -\mathrm{e}^{-tE_\ell} \frac{v_r(\mathbf{p}_\ell)\bar{v}_r(\mathbf{p}_\ell)}{2\Omega_\ell} \right\}$$

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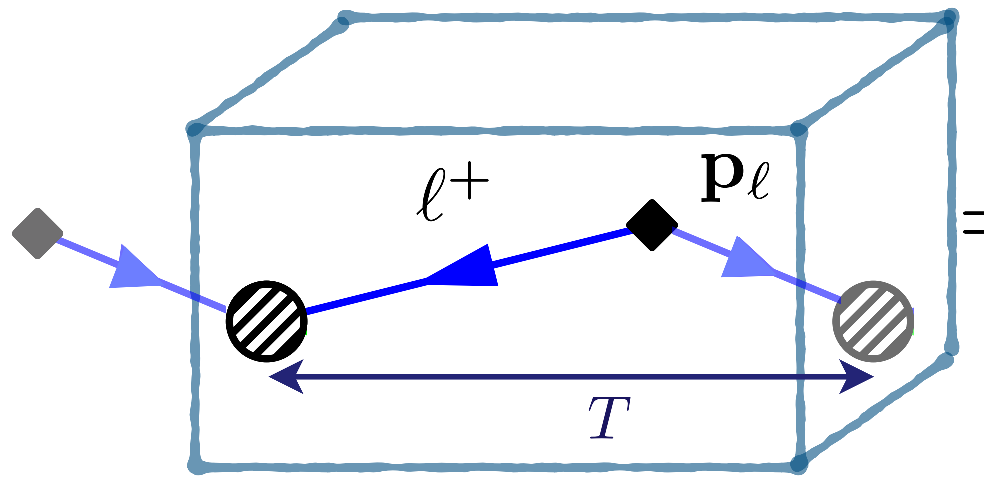

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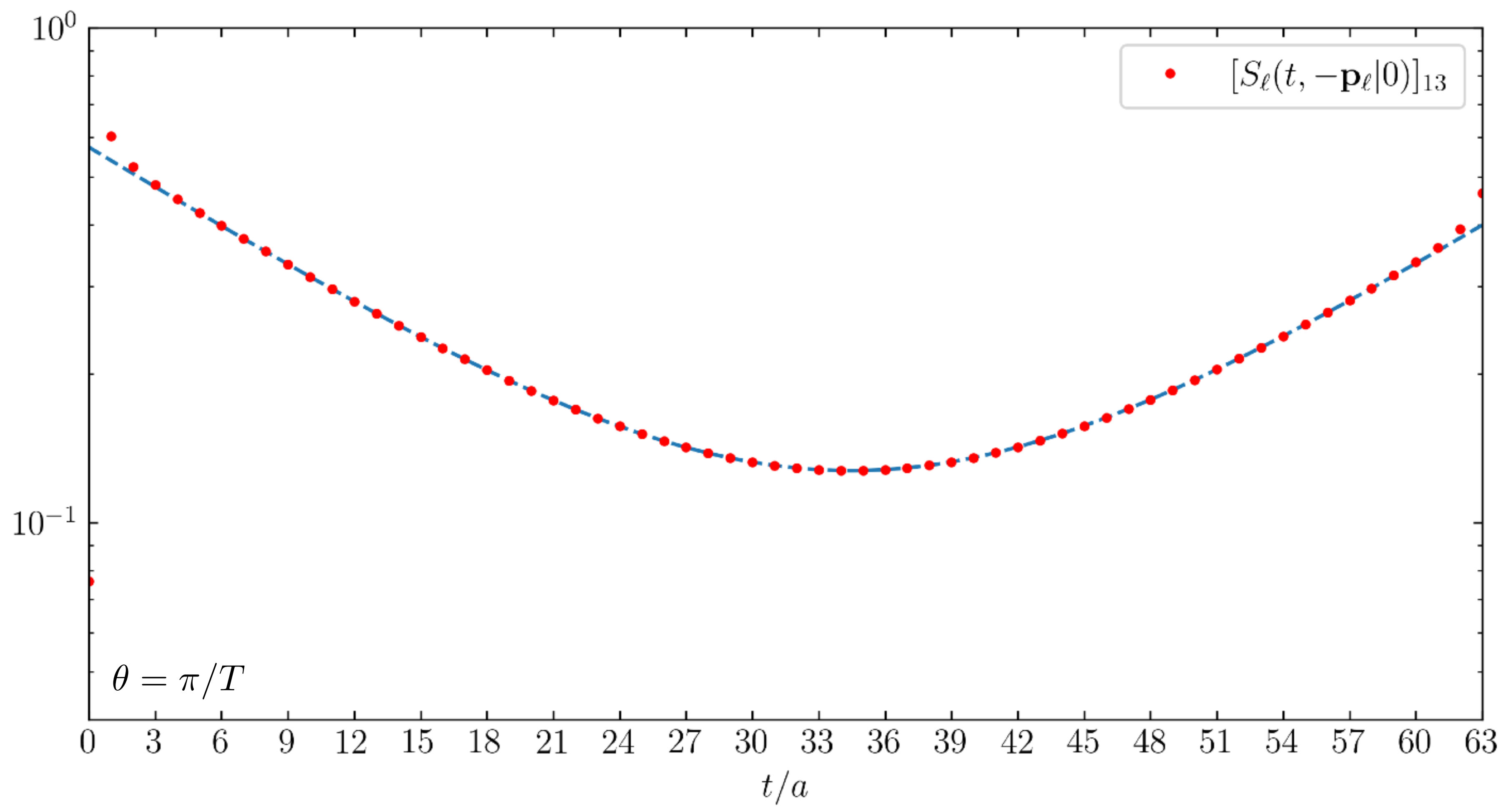

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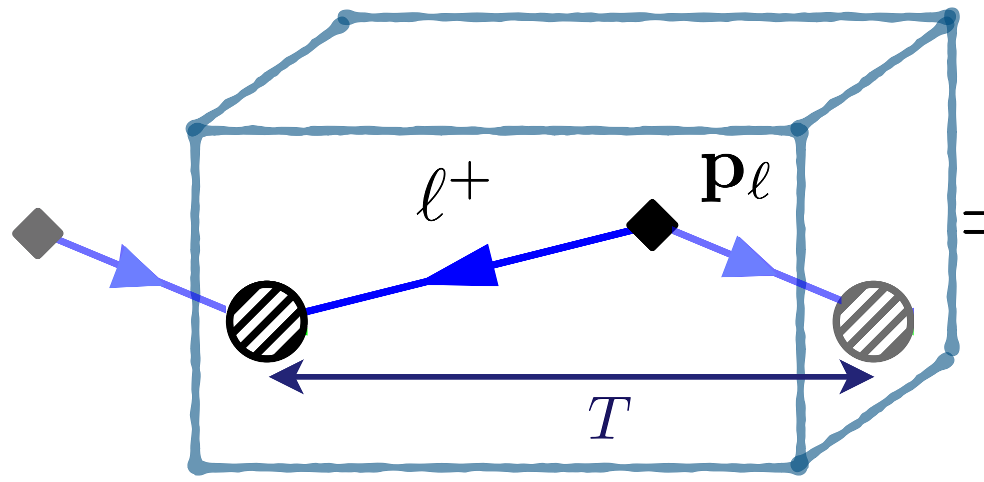
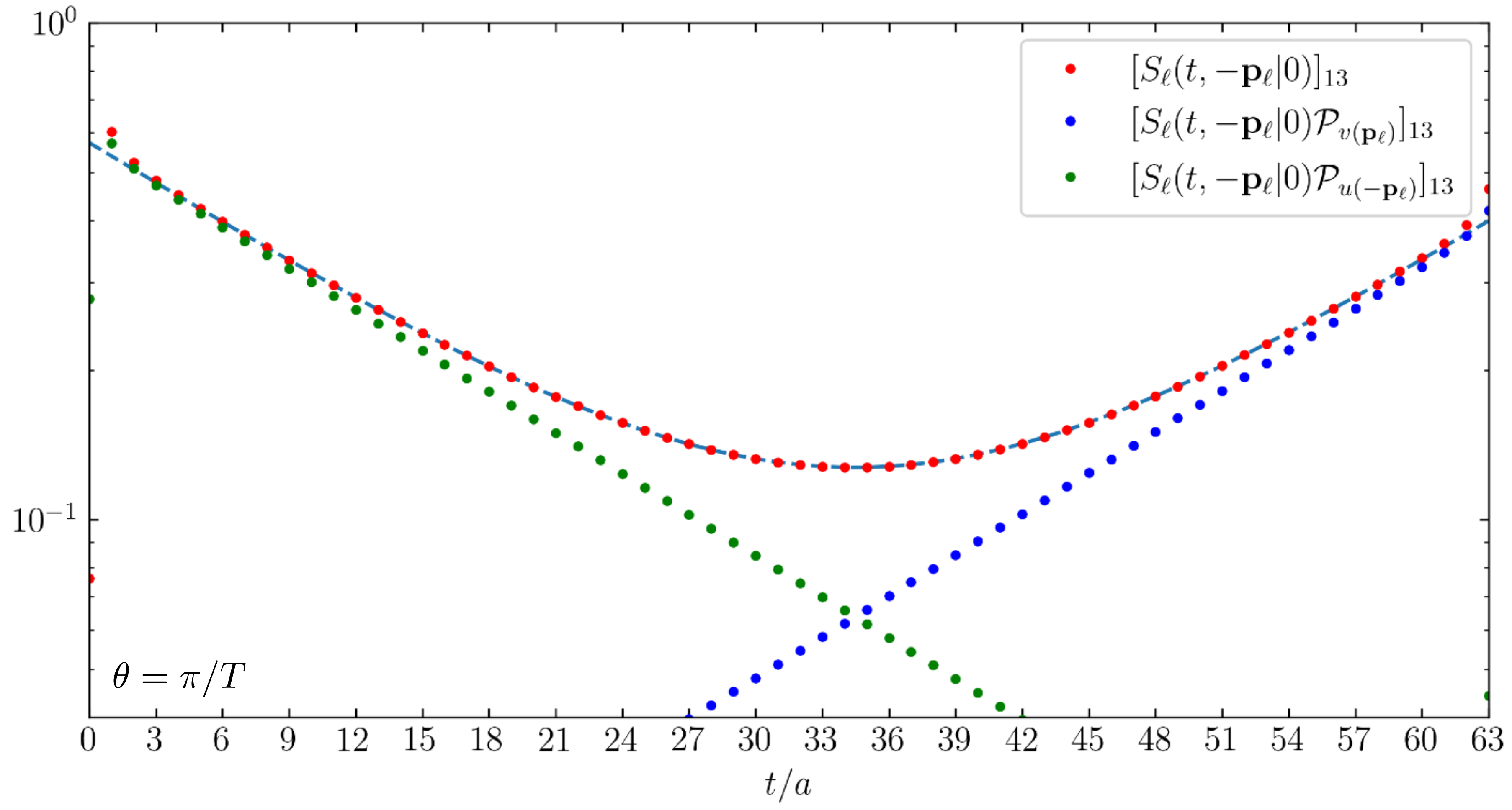
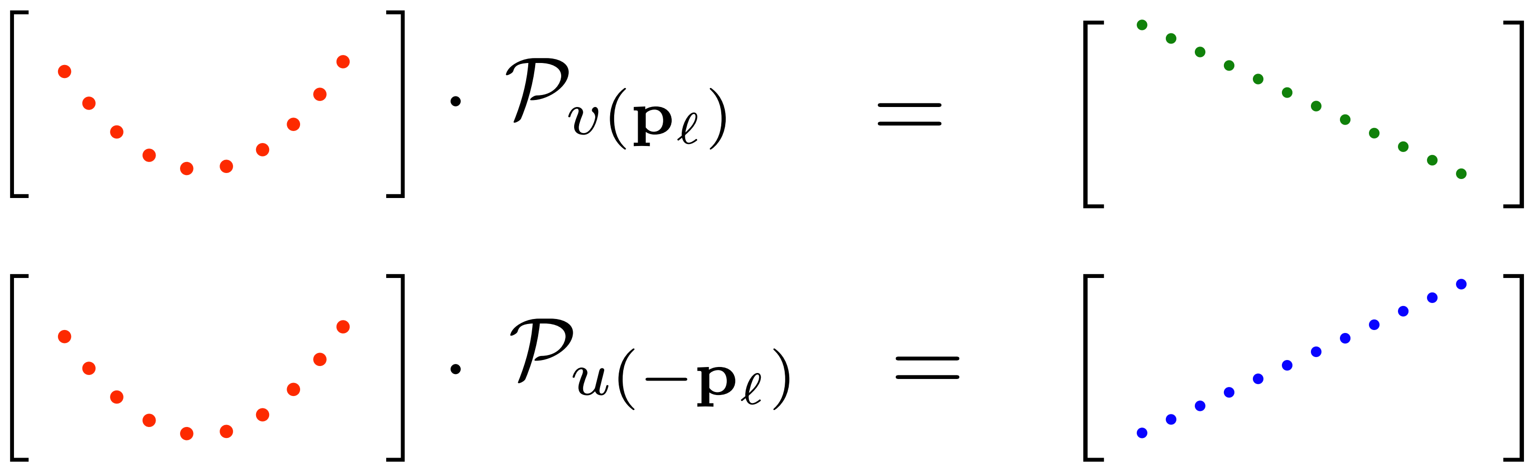
The lepton in a finite volume



$$= S(0|t,\mathbf{p}_\ell) = \sum_r \left\{ -\mathrm{e}^{-tE_\ell} \frac{v_r(\mathbf{p}_\ell)\bar{v}_r(\mathbf{p}_\ell)}{2\Omega_\ell} + \mathrm{e}^{\mathrm{i}\theta T}\mathrm{e}^{-(T-t)E_\ell} \frac{u_r(-\mathbf{p}_\ell)\bar{u}_r(-\mathbf{p}_\ell)}{2\Omega_\ell} \right\} \times \frac{1}{1-\mathrm{e}^{-TE_\ell}\mathrm{e}^{\mathrm{i}\theta T}}$$



We can select specific components using projectors:

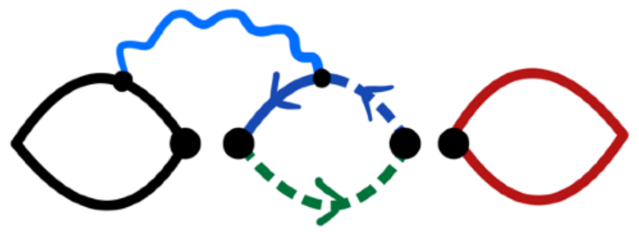


$$\mathcal{P}_{v(\mathbf{p}_\ell)} = \{u_t(-\mathbf{p}_\ell)\bar{u}_t(-\mathbf{p}_\ell) + v_s(\mathbf{p}_\ell)\bar{v}_s(\mathbf{p}_\ell)\}^{-1}\,[v_r(\mathbf{p}_\ell)\bar{v}_r(\mathbf{p}_\ell)]$$

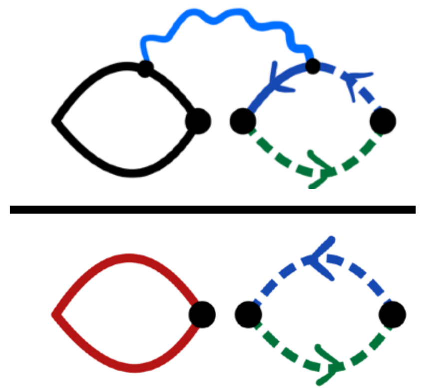
$$\mathcal{P}_{u(-\mathbf{p}_\ell)} = \{u_t(-\mathbf{p}_\ell)\bar{u}_t(-\mathbf{p}_\ell) + v_s(\mathbf{p}_\ell)\bar{v}_s(\mathbf{p}_\ell)\}^{-1}\,[u_r(-\mathbf{p}_\ell)\bar{u}_r(-\mathbf{p}_\ell)]$$

Non-factorisable QED corrections

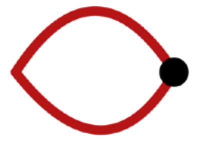
$$\delta_{\text{non-fact}}|\mathcal{M}(\mathbf{p}_\ell)|^2 = 2\sum_q \mathrm{Re}\left\{\delta_{q\ell}\mathcal{M}^{rs}(\mathbf{p}_\ell)\;[\mathcal{M}_0^{rs}(\mathbf{p}_\ell)]^\dagger\right\}$$

$$= 2\sum_q \mathrm{Re}\Big\{$$  $$\Big\}$$

$\longrightarrow$

$$\frac{\delta_{\text{non-fact}}\mathcal{A}_P}{\mathcal{A}_{P,0}} \equiv \sum_q \mathrm{Re}\Big\{$$  $$\Big\}$$

$|\mathcal{M}_0(\mathbf{p}_\ell)|^2 =$ 

 $= \mathcal{A}_{P,0}$

 $= \sum_s v_\ell^s(\mathbf{p}_\ell)\bar{v}_\ell^s(\mathbf{p}_\ell)$

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Non-factorisable QED corrections

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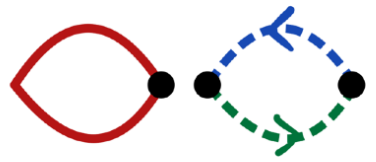

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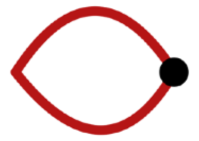 $= \mathcal{A}_{P,0}$

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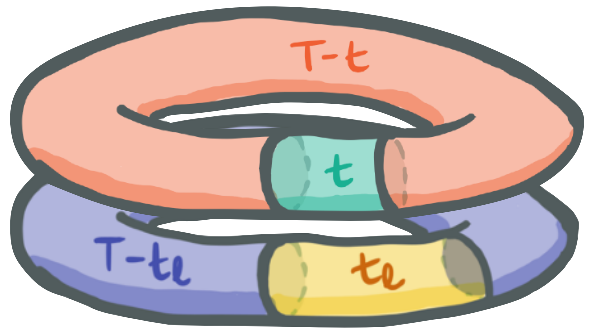

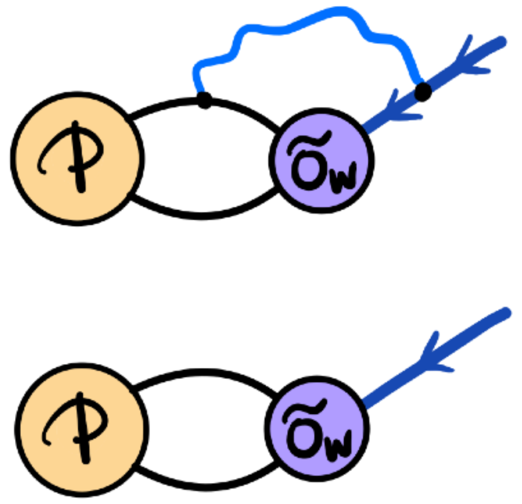

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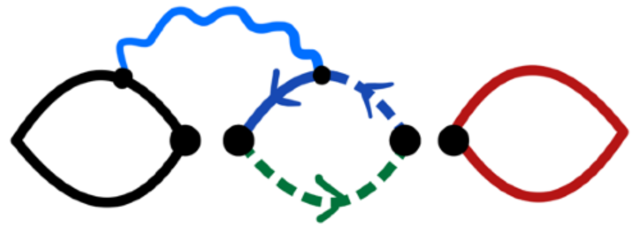

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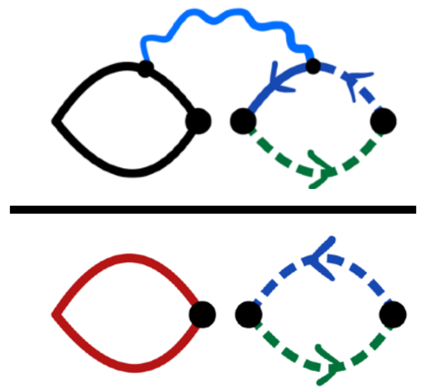

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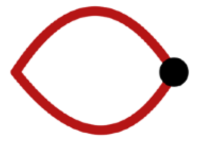 $= \mathcal{A}_{P,0}$

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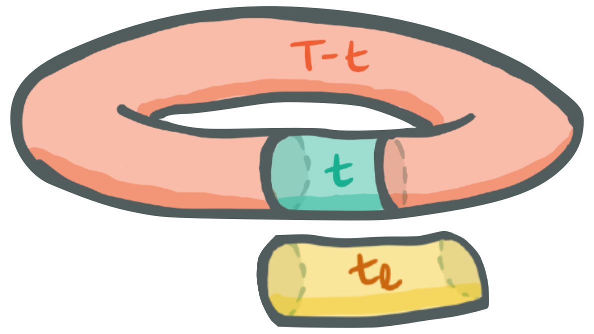

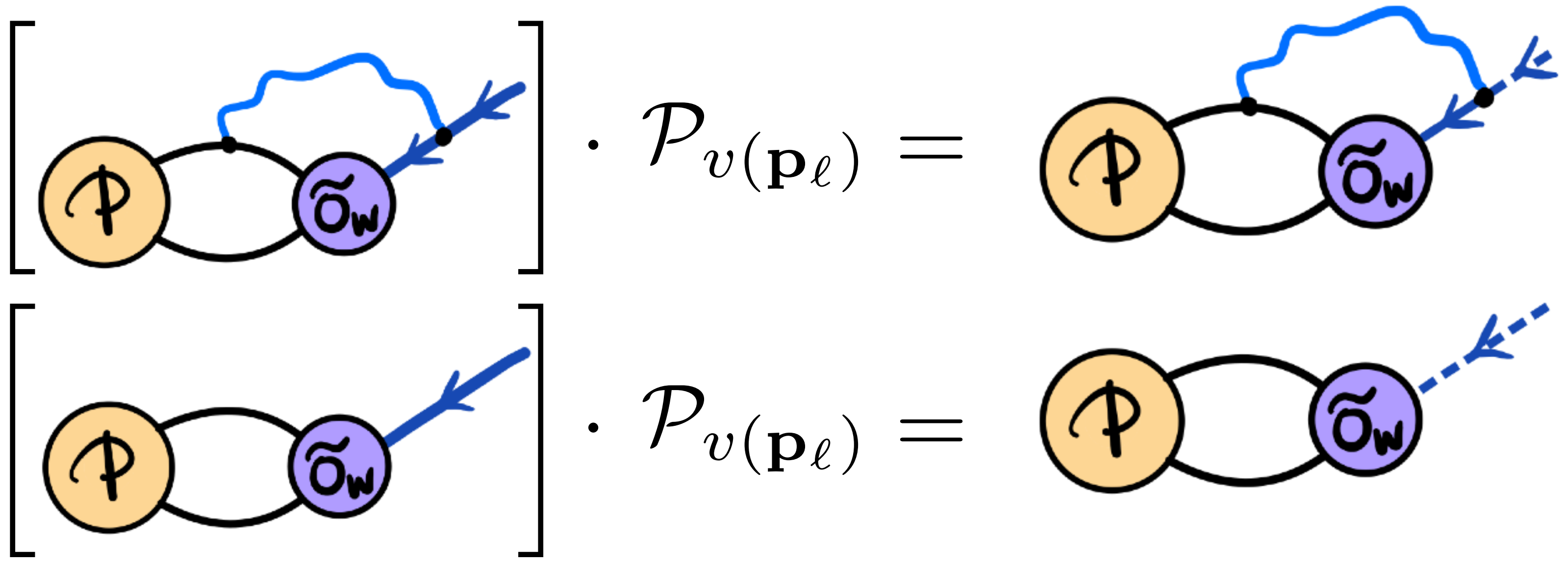

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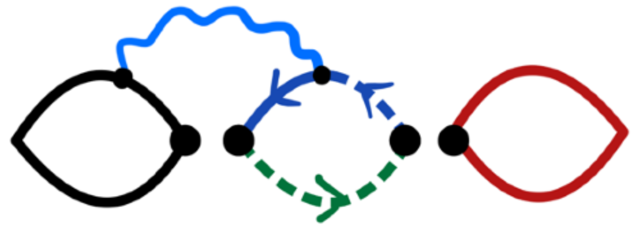
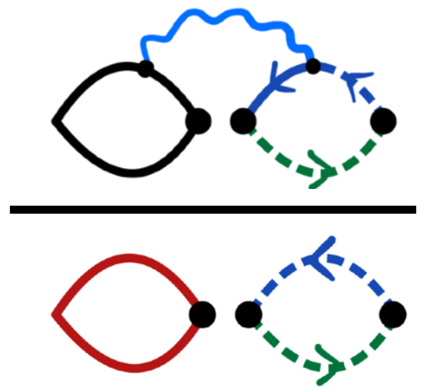
$$= 2\sum_q \mathrm{Re}\Big\{$$  $$\Big\}$$

→ $$\frac{\delta_{\text{non-fact}}\mathcal{A}_P}{\mathcal{A}_{P,0}} \equiv \sum_q \mathrm{Re}\Big\{$$  $$\Big\}$$

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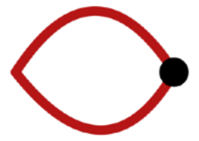 $= \mathcal{A}_{P,0}$

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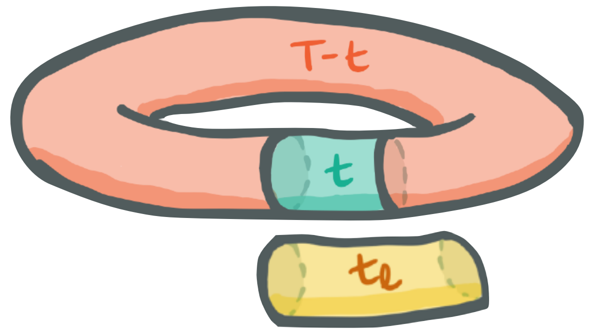
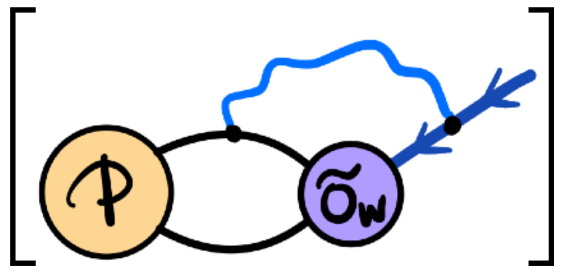
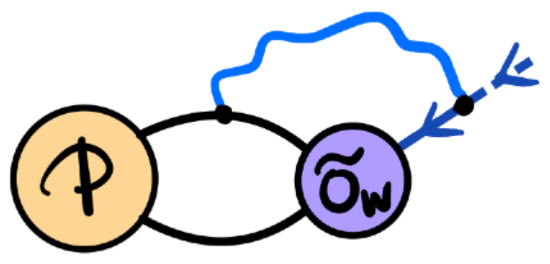
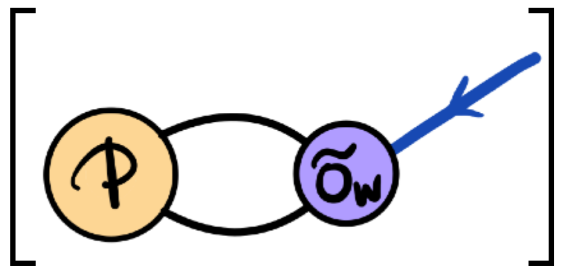
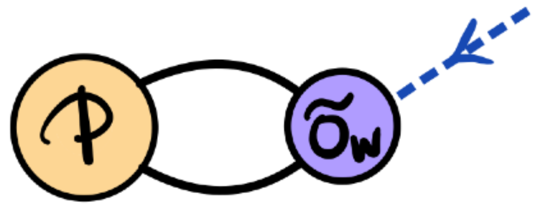
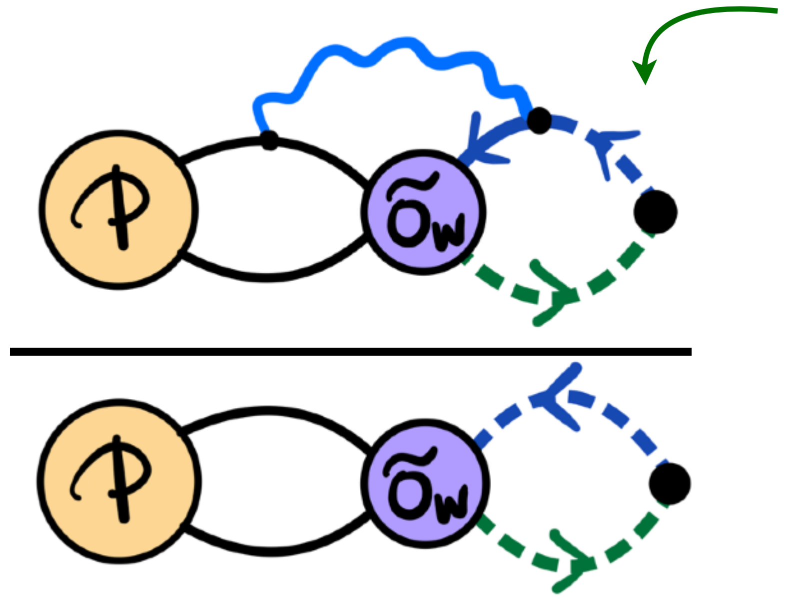


 $\cdot\, \mathcal{P}_{v(\mathbf{p}_\ell)} =$ 

 $\cdot\, \mathcal{P}_{v(\mathbf{p}_\ell)} =$ 

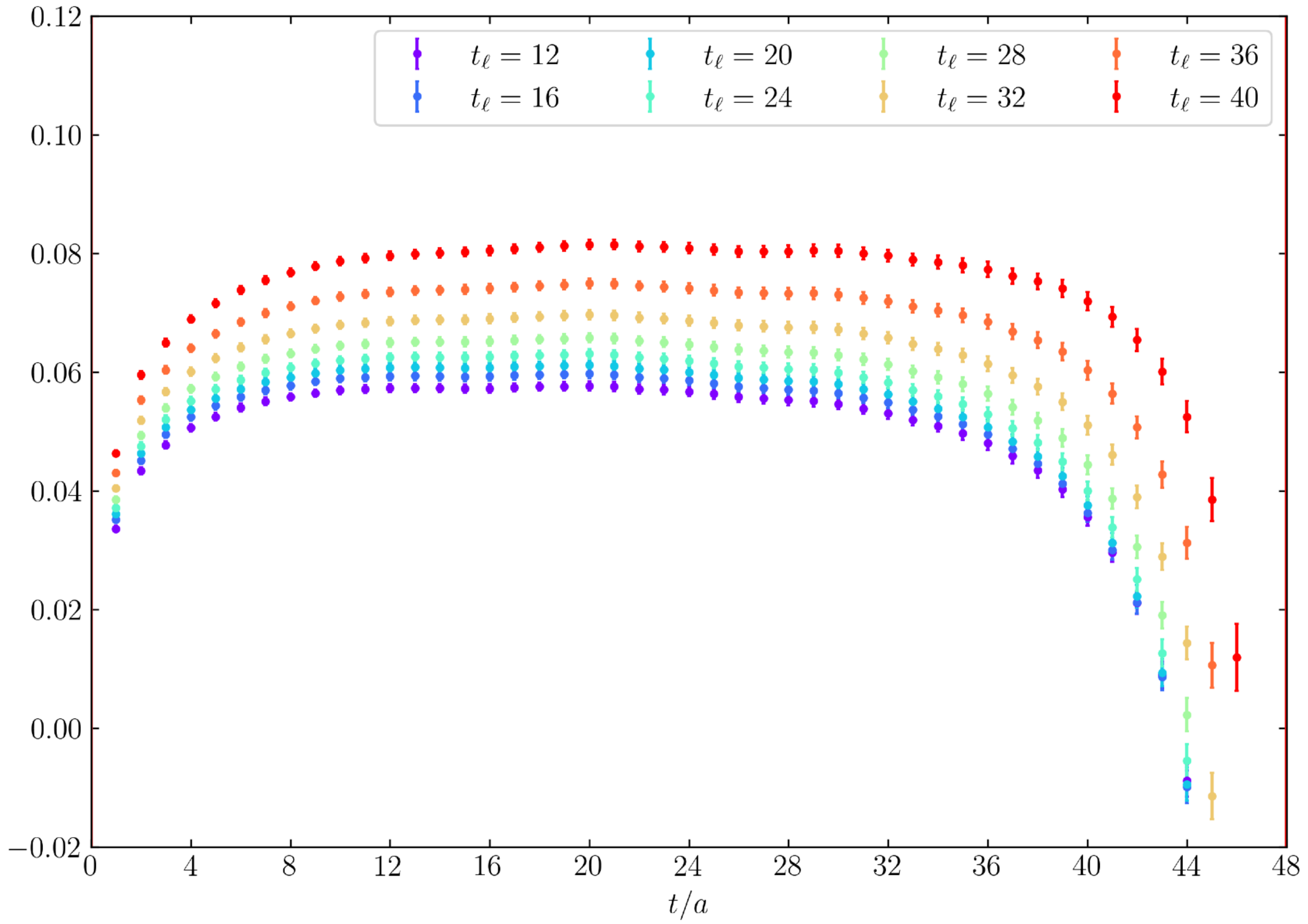
→ 

$$= \frac{\delta_{\text{non-fact}}\mathcal{A}_P}{\mathcal{A}_{P,0}}\, f_{\mathrm{P}\ell}(t,T)$$

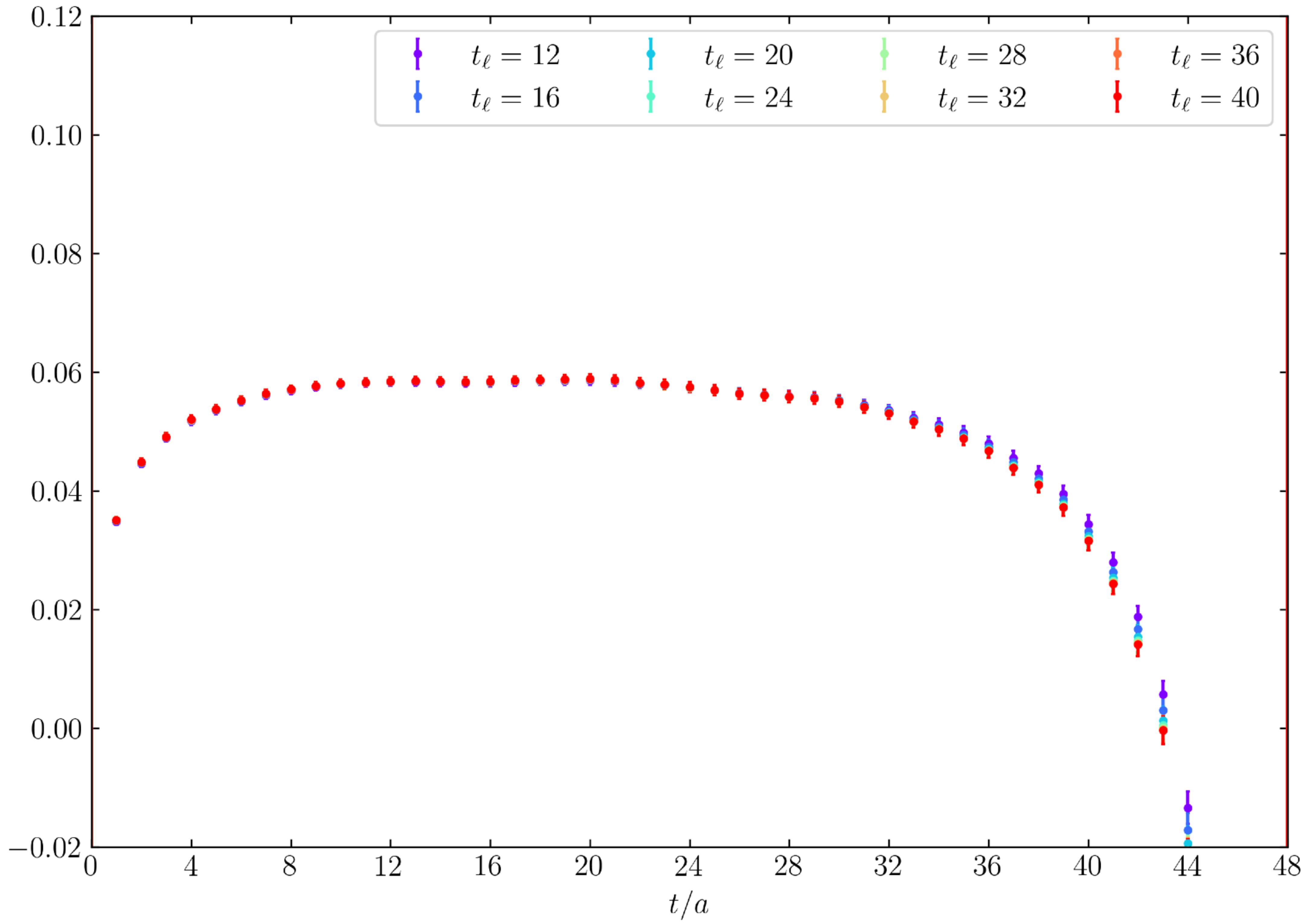
$$f_{\mathrm{P}\ell}(t,T) = \frac{1}{2}\left\{(1+\kappa_\ell) - (1-\kappa_\ell)\coth\left[m_{P,0}\left(t-\frac{T}{2}\right)\right]\right\} \approx 1$$

Non-factorisable QED corrections

Pion



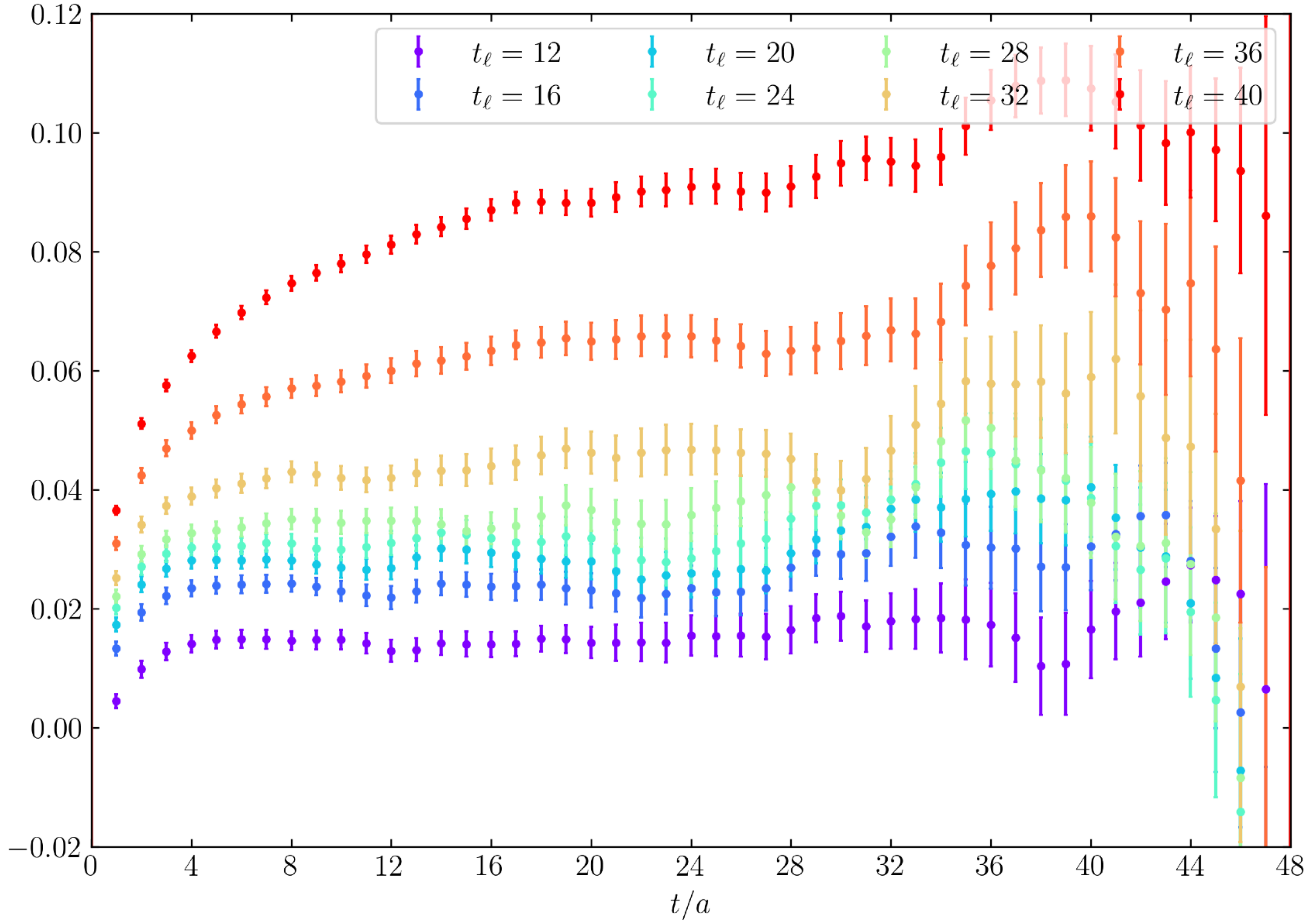
without projection



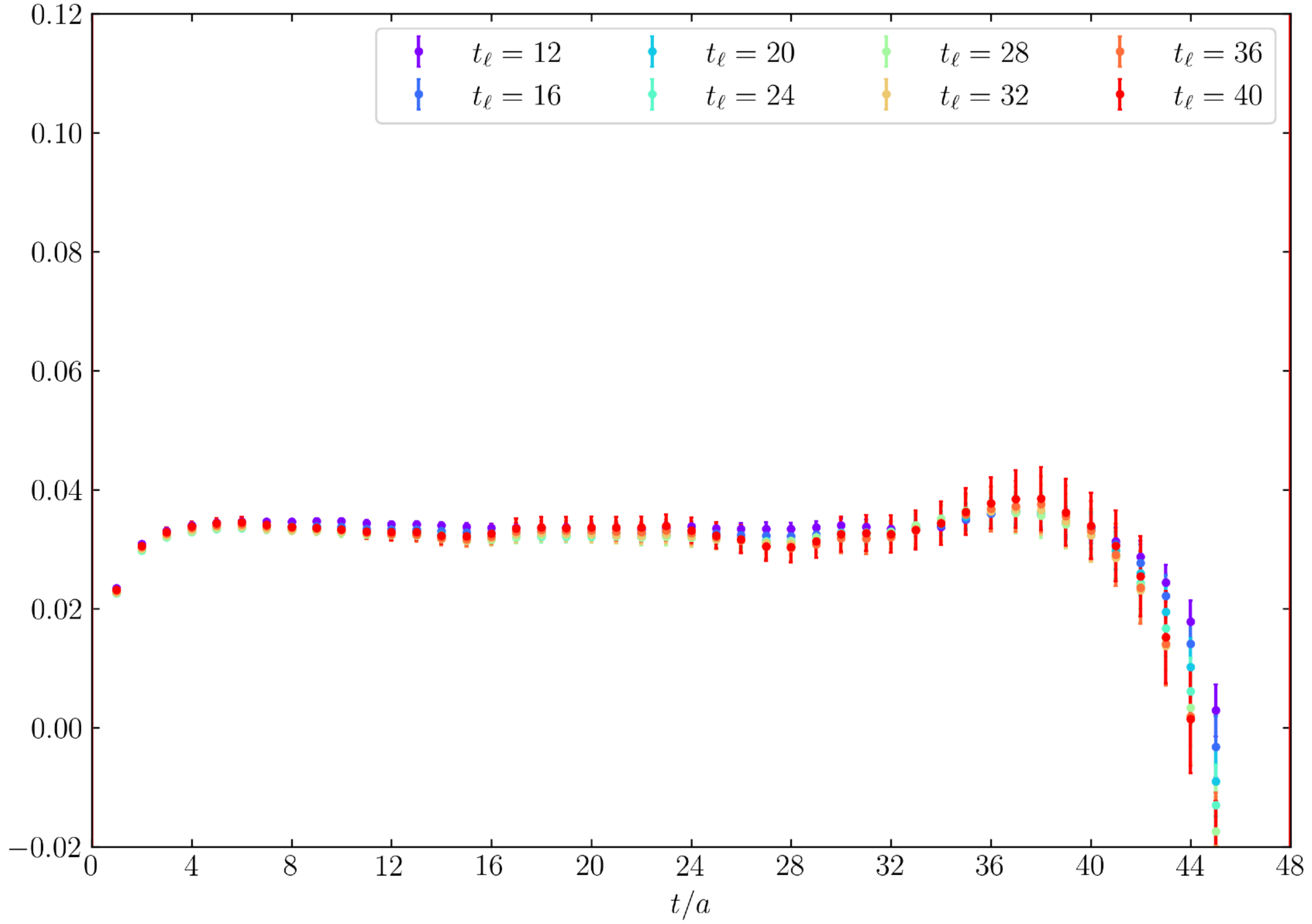
with projection

Non-factorisable QED corrections

Kaon



without projection



with projection

A general comparison of the calculations

	RBC/UKQCD	RM123+Soton
physical masses	✓ physical point simulation	extrapolation needed
chiral symmetry	✓ at finite lattice spacing	recovered in the continuum
fermionic action	Domain Wall	Twisted Mass
continuum limit	single lattice spacing	✓ continuum limit (3)
infinite volume limit	single volume	✓ multiple volumes
IB scheme	BMW [a]	GRS [b]
QED prescription	QED_L	QED_L
sea effects	electro-quenching	electro-quenching

[a] BMW, PRL 111 (2013); BMW, PRL 117 (2016)

[b] Gasser, Rusetsky & Scimemi, EPJC 32 (2003); RM123, PRD 87 (2013)

Defining the iso-symmetric theory

Full QCD+QED theory: fully specified in the continuum by $N_f + 1$ inputs, as well as α^{phys}

At finite lattice spacing:
$(am_u, am_d, am_s, \dots | g, \alpha^{\text{phys}})$

$$\frac{aM_i}{a\Lambda} = \left(\frac{M_i}{\Lambda}\right)^{\text{phys}} \quad i = 1, \dots, N_f \longrightarrow a\boldsymbol{m}^{\text{phys}}(g) \quad \& \quad a(g) = \frac{a\Lambda(a\boldsymbol{m}^{\text{phys}}, g, \alpha^{\text{phys}})}{\Lambda^{\text{phys}}}$$

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Isosymmetric QCD theory: as above but constrain $\alpha = 0$ and $m_u = m_d$

At finite lattice spacing:
$(a_0 m_{ud},\, a_0 m_{ud},\, a_0 m_s,\, \ldots | g_0, 0)$

$$\frac{a_0\mathrm{M}_j}{a_0\Lambda} = \left(\frac{\mathrm{M}_j}{\Lambda}\right)^{\text{iso}} \quad j = 1, \ldots, N_f - 1 \quad \longrightarrow \quad a_0\boldsymbol{m}^{\text{iso}}(g_0) \quad \& \quad a_0(g_0) = \frac{a_0\Lambda(a_0\boldsymbol{m}^{\text{iso}}, g_0, 0)}{\Lambda^{\text{iso}}}$$

\+ specify a *bridge* between the two theories, e.g. $g_0 = \mathcal{G}(g)$

Defining the iso-symmetric theory

Full QCD+QED theory: fully specified in the continuum by $N_f + 1$ inputs, as well as α^{phys}

$$\mathbf{M} = \{M_{\pi^+}, M_{K^+}, M_{K^0}\} \qquad \longleftrightarrow \qquad \mathbf{M}^{\mathsf{phys}} = \{M_{\pi^+}^{\mathsf{PDG}}, M_{K^+}^{\mathsf{PDG}}, M_{K^0}^{\mathsf{PDG}}\}$$

$$\Lambda = M_{\Omega^-} \qquad \qquad \Lambda^{\mathsf{phys}} = M_{\Omega^-}^{\mathsf{PDG}}$$

Isosymmetric QCD theory: as above but constrain $\alpha = 0$ and $m_u = m_d$

At finite lattice spacing:
$(a_0 m_{ud},\, a_0 m_{ud},\, a_0 m_s,\, \ldots | g_0, 0)$

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Isosymmetric QCD theory: as above but constrain $\alpha = 0$ and $m_u = m_d$

$$a_0(g_0) = a(g)$$

$$+ \quad g_0 = g$$

$$\mathbf{M} = \{M_{\text{ud}}^2, \Delta M_{\text{ud}}^2, M_{K\chi}^2\} \quad \longleftrightarrow \quad \mathbf{M}^{\text{iso}} = \left\{(M_{\text{ud}}^2)^{\text{phys}}, 0, (M_{K\chi}^2)^{\text{phys}}\right\}$$

$$M_{\text{ud}}^2 = \frac{1}{2}\left(M_{\bar{\text{u}}\text{u}}^2 + M_{\bar{\text{d}}\text{d}}^2\right) \quad M_{K\chi}^2 = \frac{1}{2}\left(M_{K^+}^2 + M_{K^0}^2 - M_{\pi^+}^2\right) \quad \Delta M_{\text{ud}}^2 = M_{\bar{\text{u}}\text{u}}^2 - M_{\bar{\text{d}}\text{d}}^2$$

BMW, PRL 111 (2013)
BMW, PRL 117 (2016)

Extracting results from data

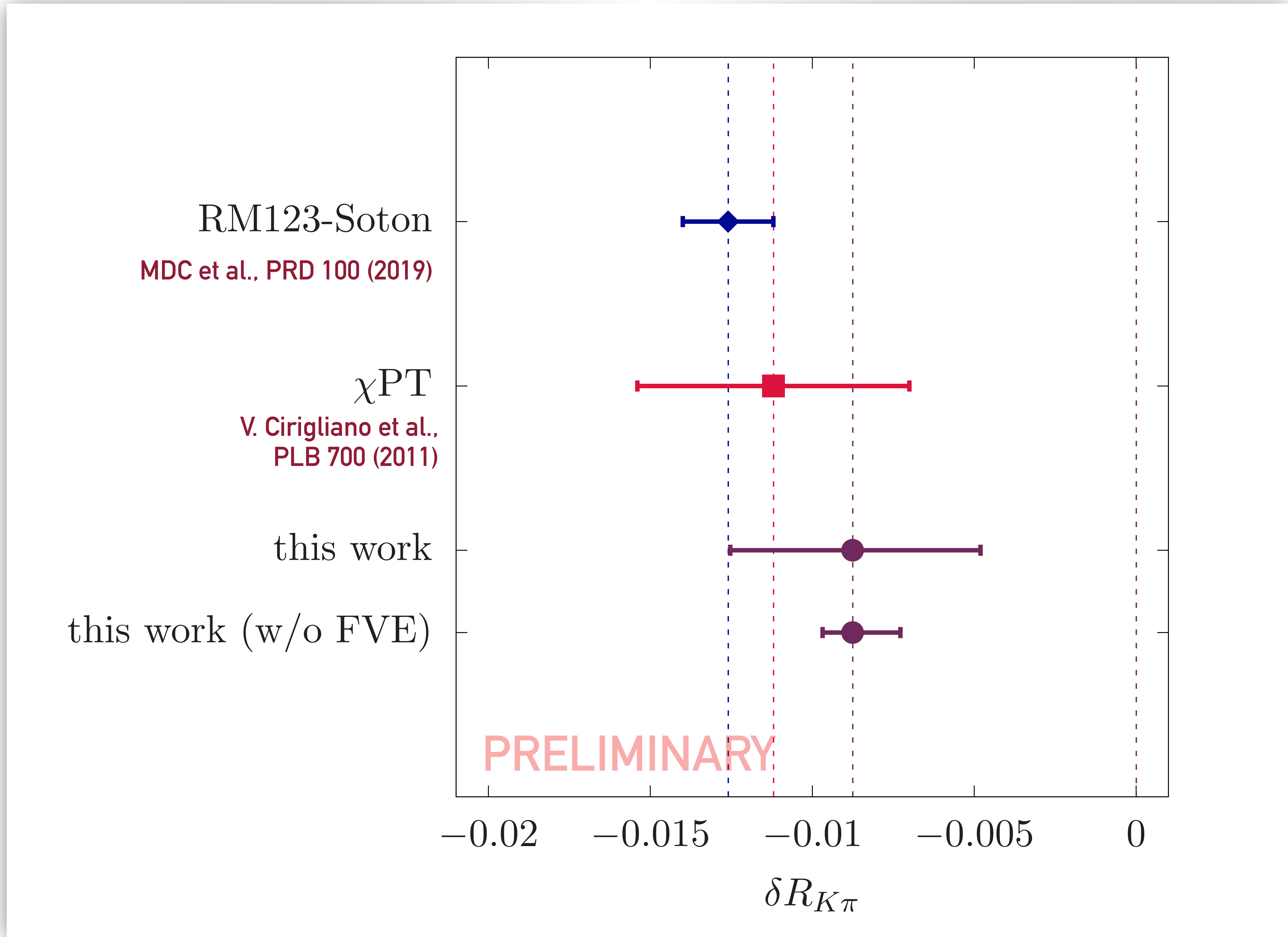
- Simultaneous correlated fit of factorisable and non factorisable correlators
- Strategy for fit scan and selection of good fit range candidates

- Take results from all these fits & assemble the ingredients to get $\delta R_{K\pi}$
- Study distribution of $\delta R_{K\pi}$ results

- Determine $\delta R_{K\pi}$ as median of the distribution
- Estimate associated statistical and systematic uncertainty

Results for $\delta R_{K\pi}$

Results for $\delta R_{K\pi}$



$$\delta R_{K\pi} = -0.0088\,\binom{+15}{-9}$$

total	$\binom{+40}{-38}$
total (w/o FVE)	$\binom{+15}{-9}$

FVE	(± 37)
statistical	$\binom{+6}{-5}$
QED quenching	(± 5)
fit	$\binom{+11}{-4}$
discretisation	(± 5)

PRELIMINARY

RM123-Soton: $\delta R_{K\pi} = -0.0126\,(14)$

χPT: $\delta R_{K\pi} = -0.0112\,(21)$

Our main systematic uncertainty

Finite volume effects

$$\Gamma_0(L) = \Gamma_0^{\text{tree}} \left\{ 1 + 2\, \frac{\alpha}{4\pi}\, Y(L) \right\}$$

$$Y(L) = Y_{\log}(L) + Y_0 + \frac{1}{m_P L}\, Y_1 + \frac{1}{(m_P L)^2}\, Y_2 + \frac{1}{(m_P L)^3}\, Y_3$$

see N. Hermansson-Truedsson's talk

Our main systematic uncertainty

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$$Y^{(1)}_{K\pi}(L/a = 48) \approx -3.96$$

see N. Hermansson-Truedsson's talk

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$$Y_{K\pi}^{(1)}(L/a = 48) \approx -3.96$$

57%

$$Y_{K\pi}^{(2)}(L/a = 48) \approx -6.20$$

see N. Hermansson-Truedsson's talk

Our main systematic uncertainty

Finite volume effects

$$\Gamma_0(L) = \Gamma_0^{\text{tree}} \left\{ 1 + 2\,\frac{\alpha}{4\pi}\, Y(L) \right\}$$

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$Y^{(1)}_{K\pi}(L/a = 48) \approx -3.96$

57%

$Y^{(2)}_{K\pi}(L/a = 48) \approx -6.20$

−54%

$Y^{(3),\text{pt}}_{K\pi}(L/a = 48) \approx -2.83$

point-like approximation

Significant correction from pointlike $1/L^3$!

Central value: $1/L^2$ subtracted result

Systematic error: conservative ~50% error

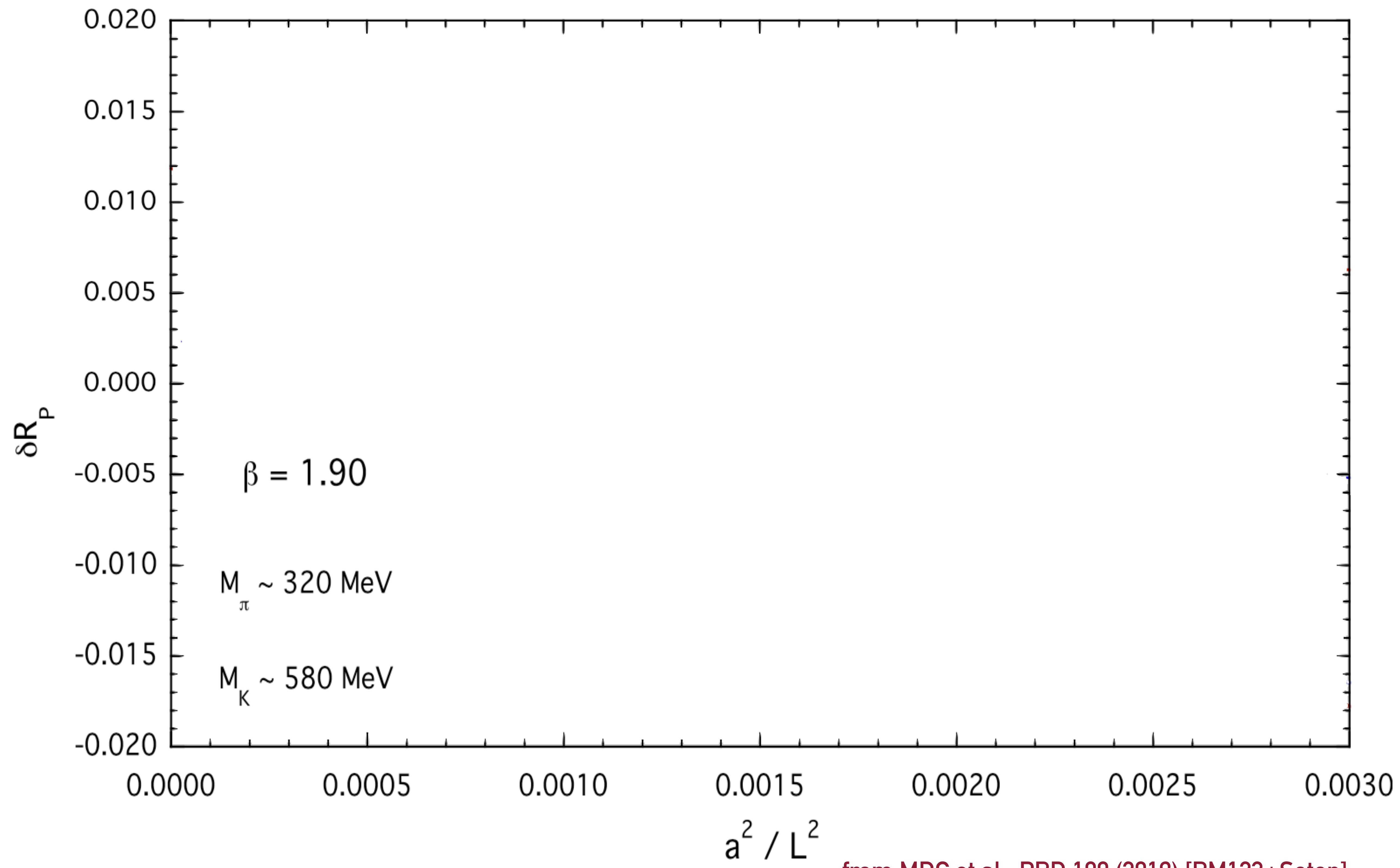
↓

FV scaling should be carefully studied!

see N. Hermansson-Truedsson's talk

Comparing with RM123+Soton result

crucial role of finite volume effects?



from MDC et al., PRD 100 (2019) [RM123+Soton]

Comparing with RM123+Soton result

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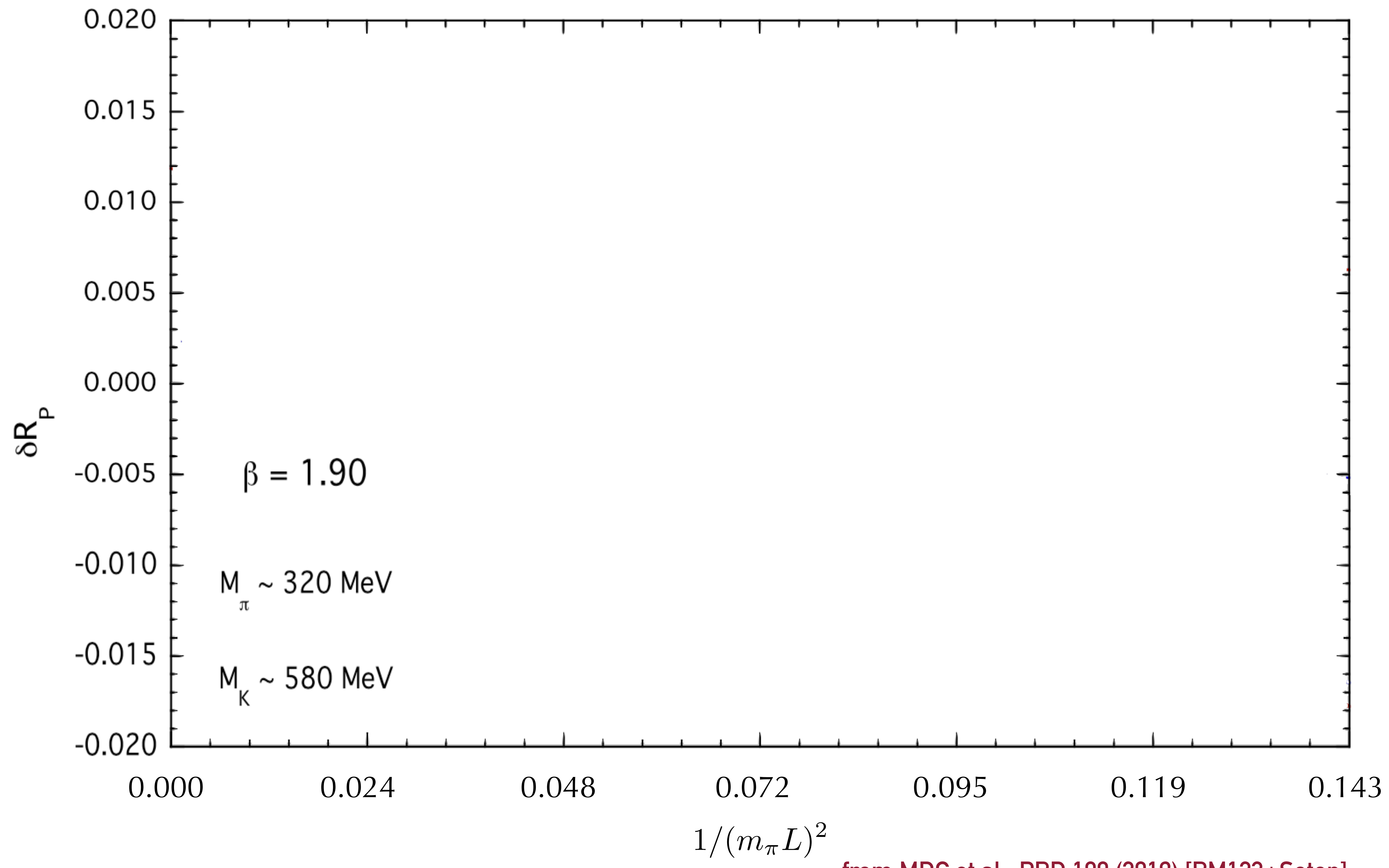

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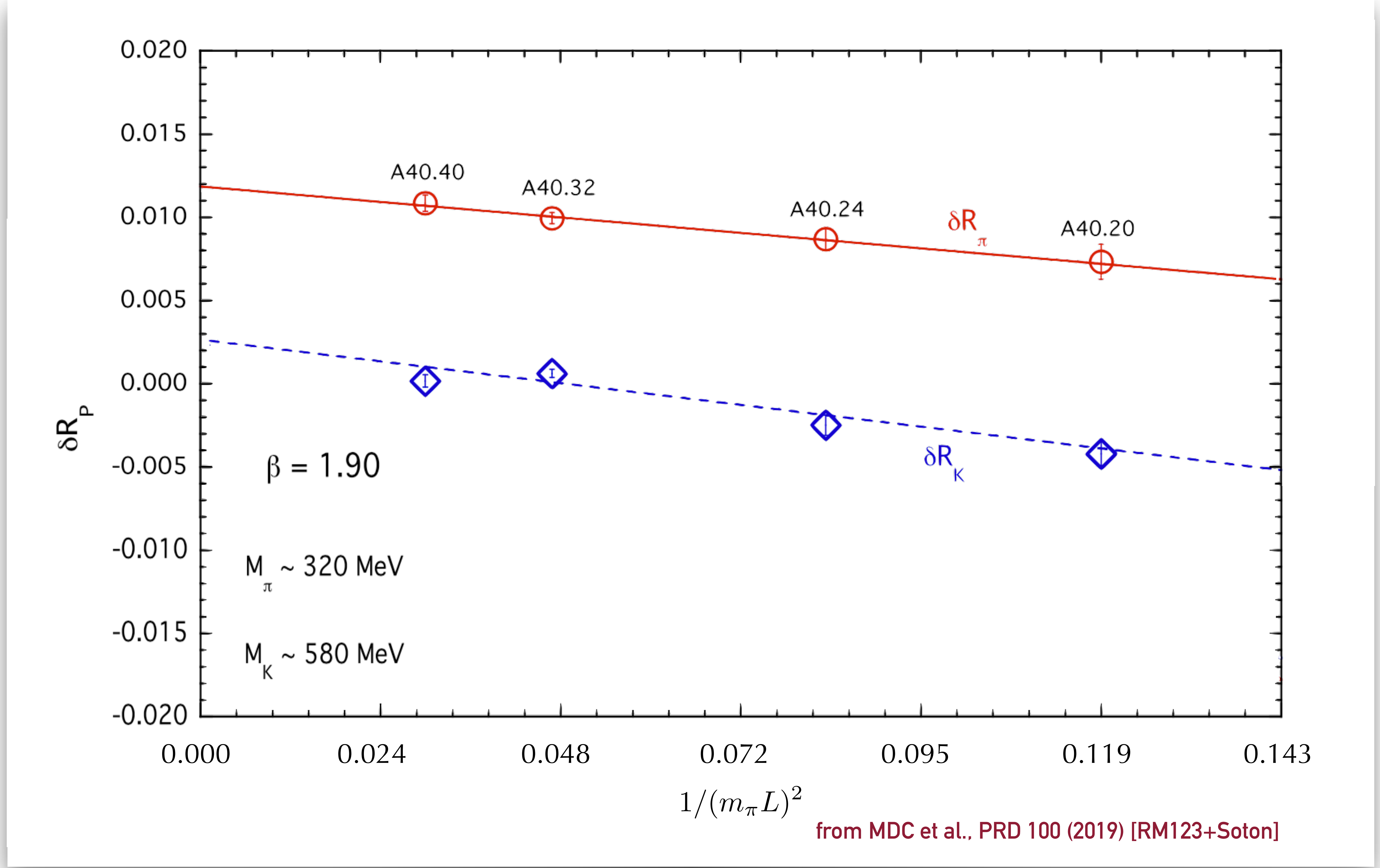
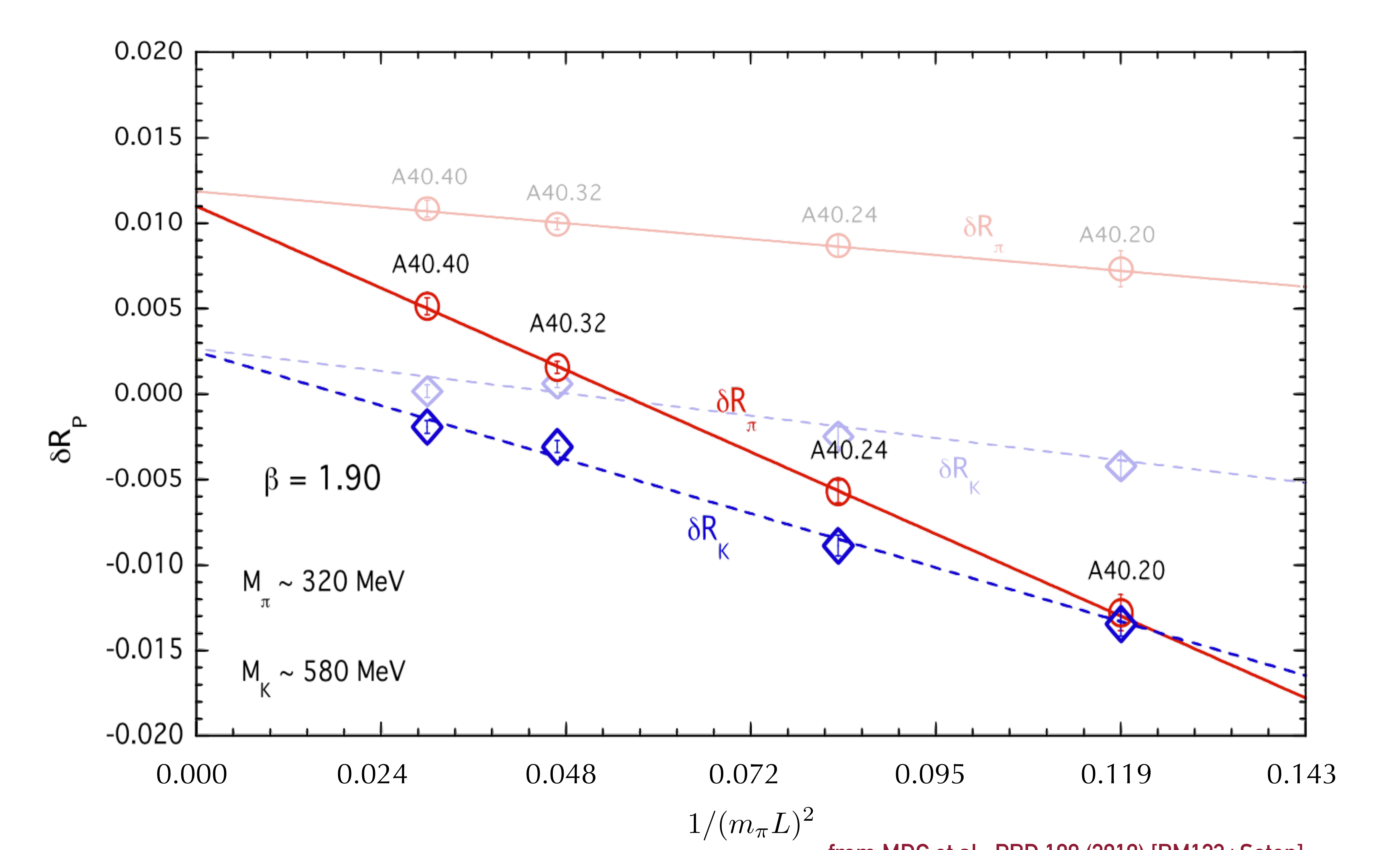


Subtracting:

(a) universal FVEs

Comparing with RM123+Soton result

crucial role of finite volume effects?



from MDC et al., PRD 100 (2019) [RM123+Soton]

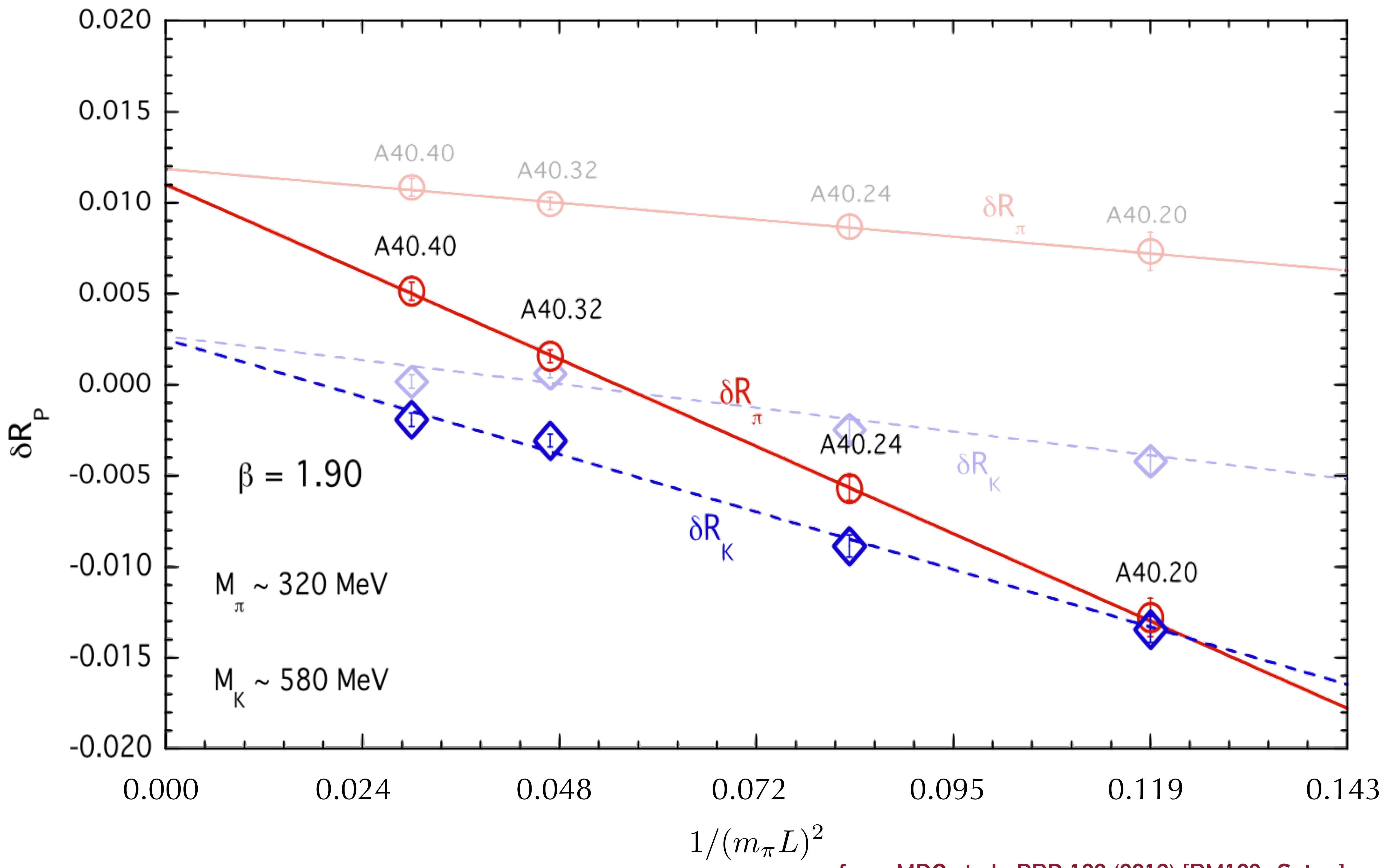
Subtracting:

(a) universal FVEs

(b) + point-like $1/L^2$

Comparing with RM123+Soton result

crucial role of finite volume effects?



from MDC et al., PRD 100 (2019) [RM123+Soton]

Subtracting:

(a) universal FVEs

(b) + point-like $1/L^2$

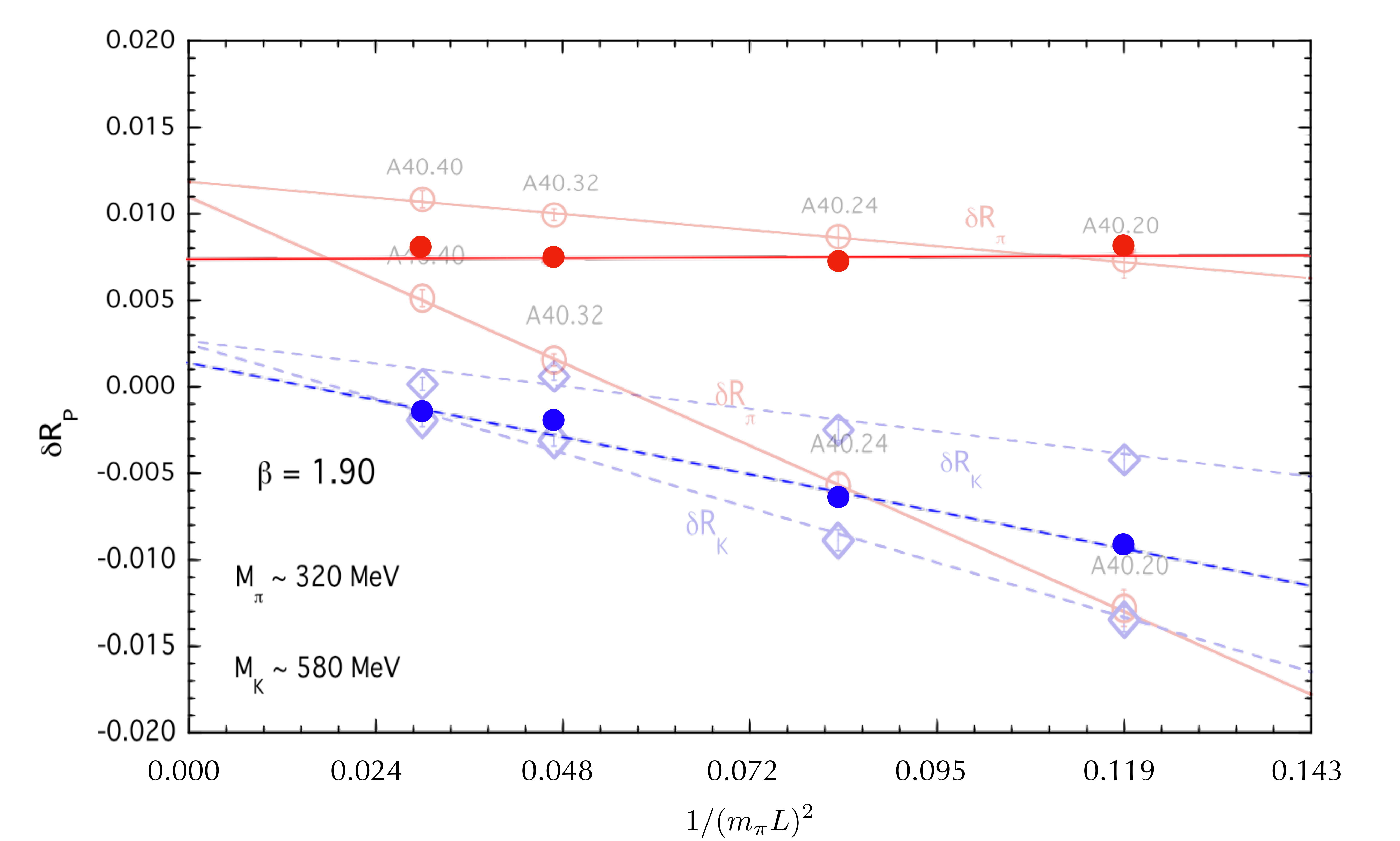
(c) + structure-dependent $1/L^2$

include the pointlike limit $Y^{(2)}_{\text{pt}}(L)$ setting $F^\pi_A = 0$, and notice that the structure-dependent contribution at $\mathcal{O}(1/L^2)$ is negligible with respect to the pointlike one. In total, there

MDC et al., PRD 105 (2022)

Comparing with RM123+Soton result

crucial role of finite volume effects?



from MDC et al., PRD 100 (2019) [RM123+Soton]

Subtracting:

(a) universal FVEs

(b) + point-like $1/L^2$

(c) + structure-dependent $1/L^2$

include the pointlike limit $Y^{(2)}_{\rm pt}(L)$ setting $F_A^\pi = 0$, and notice that the structure-dependent contribution at $\mathcal{O}(1/L^2)$ is negligible with respect to the pointlike one. In total, there

MDC et al., PRD 105 (2022)

(d) + point-like $1/L^3$

$$\Delta^{(3,\rm pt)}(\delta R_P) = \left(\frac{2\alpha}{4\pi}\right) \frac{32\pi^2 c_0 (2 + r_\ell^2)}{(m_P L)^3 (1 + r_\ell^2)^3}$$

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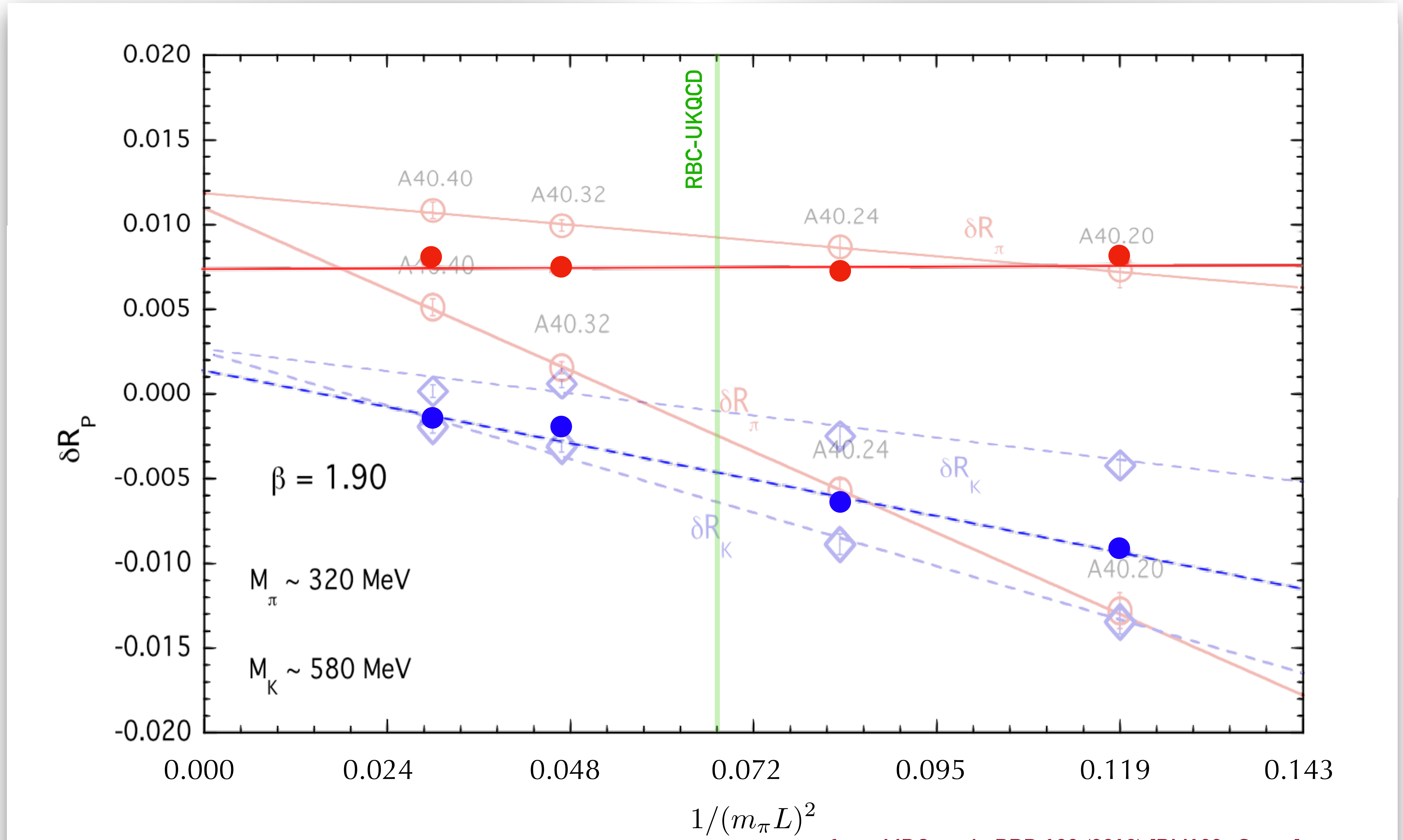

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Results for $|V_{us}/V_{ud}|$

A speculative exercise on the error budget

$$\left|\frac{V_{us}}{V_{ud}}\right|^2 = \left[\frac{\Gamma(K_{\ell 2})}{\Gamma(\pi_{\ell 2})}\frac{M_{K^+}^3}{M_{\pi^+}^3}\frac{(M_{K^+}^2 - M_{\mu^+}^2)^2}{(M_{\pi^+}^2 - M_{\mu^+}^2)^2}\right]_{\mathrm{exp}} \cdot \left[\frac{f_{K,0}}{f_{\pi,0}}\right]^2 (1 + \delta R_{K\pi})$$

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- Let us use $\delta R_{K\pi} = -0.00875\,(^{+147}_{-93})$ (assume FV issue solved)

$[f_{K,0}/f_{\pi,0}]$		$\lvert V_{us}/V_{ud}\rvert$
this work	1.1940 (11)*	$0.23137\ (28)_{\rm exp}\ (14)_{\delta R}\ (22)^*_{f_P}$
FLAG 2+1 best [a]	1.1945 (45)	$0.23127\ (28)_{\rm exp}\ (14)_{\delta R}\ (87)_{f_P}$
FLAG 2+1 average	1.1930 (33)	$0.23155\ (28)_{\rm exp}\ (14)_{\delta R}\ (65)_{f_P}$
FLAG 2+1+1 best [b]	1.1980 (15)	$0.23060\ (28)_{\rm exp}\ (14)_{\delta R}\ (28)_{f_P}$
FLAG 2+1+1 average	1.1966 (18)	$0.23086\ (28)_{\rm exp}\ (14)_{\delta R}\ (35)_{f_P}$

[a] RBC/UKQCD14 48I [b] FNAL/MILC 17 * error due to discretisation effects not included

- From RM123+Soton calculation $\delta R_{K\pi} = -0.0126\,(14)$

$[f_{K,0}/f_{\pi,0}]$		$\lvert V_{us}/V_{ud}\rvert$
FLAG 2+1+1 average	1.1966 (18)	$0.23131\ (28)_{\rm exp}\ (17)_{\delta R}\ (35)_{f_P}$

- the uncertainty on $[f_{K,0}/f_{\pi,0}]$ dominates in the error budget
- if improved, precision from **lattice** starts being competitive with the **experimental** one

Conclusions

- We presented new results for the radiative correction $\delta R_{K\pi}$ from calculation with Domain Wall fermions at the physical point

- Finite volume effects have to be carefully studied, including order $1/L^3$ (looking forward to seeing results with different prescriptions: $\mathrm{QED_C}$, $\mathrm{QED_m}$, $\mathrm{QED_\infty}$)

- *@lattice community:* including radiative corrections is necessary to claim precision, but let's not forget about the iso-QCD part. Might be main source of uncertainty on $|V_{us}/V_{ud}|$

- With small further improvement in lattice calculations, we're very close to be competitive with the experimental precision on $|V_{us}/V_{ud}|$

Thank you