Feynman integrals, geometries and differential equations

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Section 1

Introduction

Scattering amplitudes

- We would like to make precise predictions for observables in scattering experiments from (quantum) field theory.
- Any such calculation will involve a scattering amplitude.
- Unfortunately we cannot calculate scattering amplitudes exactly.
- If we have a small parameter like a small coupling, we may use perturbation theory.
- We may organise the perturbative expansion of a scattering amplitude in terms of Feynman diagrams.

Scattering amplitude = sum of all Feynman diagrams

Standard techniques

- Dimensional regularisation ('t Hooft, Veltman '72, Bollini, Giambiagi '72, Ashmore '72):
  $D = 4 - 2\varepsilon$, used to regulate ultraviolet and infrared divergences.

- Integration-by-parts identities (Tkachov '81, Chetyrkin '81):
  leads to master integrals $I = (I_1, I_2, \dots, I_{N_F})$.

- Method of differential equations (Kotikov '90, Remiddi '97, Gehrmann and Remiddi '99):

$$dI = A(x, \varepsilon)\, I$$

- Transformation to ε-factorised form (Henn '13):

$$dI = \varepsilon A(x)\, I$$

Vector bundles

- Fibre spanned by the master integrals $I = (I_1, ..., I_{N_F})$.

  (The master integrals $I_1(x), \dots, I_{N_F}(x)$ can be viewed as local sections, and for each x they define a basis of the vector space in the fibre.)
- Base space with coordinates $x = (x_1, ..., x_{N_B})$ corresponding to kinematic variables.
- Connection defined by the matrix A with differential one-forms $\omega = (\omega_1, ..., \omega_{N_L})$.

We would like to **transform** this vector bundle **to an ε-factorised form** through

- a change of basis in the fibre,
- a coordinate transformation on the base manifold.

Multiple polylogarithms

Consider differential one-forms on $\mathbb{C} \cup \{\infty\}$ (the Riemann sphere) of the form

$$\omega^{\mathrm{mpl}}(z_j) = \frac{d\lambda}{\lambda - z_j}.$$

Definition (Multiple polylogarithms)

$$G(z_1, ..., z_k; \lambda) = \int\limits_0^{\lambda} \frac{d\lambda_1}{\lambda_1 - z_1} \int\limits_0^{\lambda_1} \frac{d\lambda_2}{\lambda_2 - z_2} ... \int\limits_0^{\lambda_{k-1}} \frac{d\lambda_k}{\lambda_k - z_k}, \qquad z_k \neq 0$$

Section 2

Geometry

The base space

Question:

After a suitable coordinate transformation, can we relate the base space to a space known from mathematics?

The base space

- Assume we have $(n-3)$ variables $z_1, \dots, z_{n-3}$ and differential one-forms

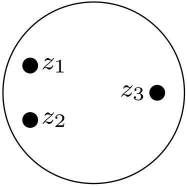
$$\begin{aligned}\omega_k \quad \in \quad & \{ d \ln (z_1), d \ln (z_2), \dots, d \ln (z_{n-3}), \\ & d \ln (z_1 - 1), d \ln (z_2 - 1), \dots, d \ln (z_{n-3} - 1), \\ & d \ln (z_1 - z_2), d \ln (z_1 - z_3), \dots, d \ln (z_{n-2} - z_{n-3}) \}\end{aligned}$$

- The iterated integrals $I_\gamma(\omega_1, \dots, \omega_r; \lambda)$ are multiple polylogarithms.
- We require $z_i \notin \{0, 1, \infty\}$ and $z_i \neq z_j$:
This defines the **moduli space** $\mathcal{M}_{0,n}$: The space of configurations of n points on a Riemann sphere modulo Möbius transformations.
- Usually the z_i are functions of the kinematic variables x and the arguments of the dlog-forms define the Landau singularities.

Multiple polylogarithms again

Take home message:

Feynman integrals, which evaluate to multiple polylogarithms are related to a Riemann sphere (a smooth complex algebraic curve of genus zero).



Section 3

Elliptic curves

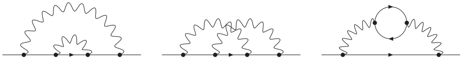
Beyond multiple polylogarithms

- Not every Feynman integral can be expressed in terms of multiple polylogarithms.
- Starting from two-loops, we encounter more complicated functions.
- The next-to-simplest Feynman integrals involve an **elliptic curve**.

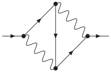
Elliptic curves

We do not have to go very far to encounter elliptic integrals in precision calculations: The simplest example is the two-loop electorn self-energy in QED:

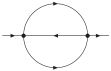
There are three Feynman diagrams contributing to the two-loop electron self-energy in QED with a single fermion:



All master integrals are (sub-) topologies of the **kite graph**:



One sub-topology is the **sunrise graph** with three equal non-zero masses:



(Sabry, '62)

Where is the elliptic curve?

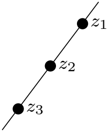
For the sunrise it's very simple: The second graph polynomial defines an elliptic curve in Feynman parameter space:

$$-p^2 a_1 a_2 a_3 + (a_1 + a_2 + a_3)(a_1 a_2 + a_2 a_3 + a_3 a_1) m^2 = 0.$$

Moduli spaces

$\mathcal{M}_{g,n}$: Space of **isomorphism classes of** smooth (complex, algebraic) **curves of genus g with n marked points**.

complex curve



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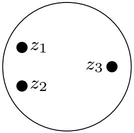
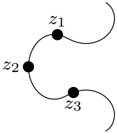
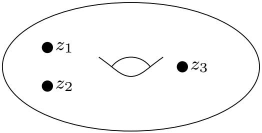


real surface



$\Leftrightarrow$



Coordinates

Genus 0: $\dim \mathcal{M}_{0,n} = n - 3$.
Sphere has a unique shape
Use Möbius transformation to fix $z_{n-2} = 1$, $z_{n-1} = \infty$, $z_n = 0$
Coordinates are $(\mathbf{z_1}, \dots, \mathbf{z_{n-3}})$

Genus 1: $\dim \mathcal{M}_{1,n} = n$.
One coordinate describes the shape of the torus
Use translation to fix $z_n = 0$
Coordinates are $(\tau, \mathbf{z_1}, \dots, \mathbf{z_{n-1}})$

Iterated integrals on $\mathcal{M}_{0,n}$ and $\mathcal{M}_{1,n}$

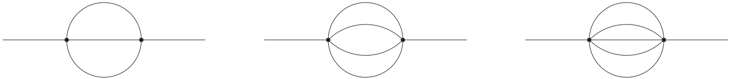
- Iterated integrals on $\mathcal{M}_{0,n}$ with at most simple poles are **multiple polylogarithms**.
  Most of the known Feynman integrals fall into this category.

- Iterated integrals on $\mathcal{M}_{1,n}$ are **iterated integrals of modular forms** and **elliptic multiple polylogarithms** (and mixtures thereof).
  Adams, S.W. '17, Broedel, Duhr, Dulat, Tancredi, '17

  These can be evaluated numerically within `GiNaC` with arbitrary precision.
  Walden, S.W, '20

Generalisations

- We understand by now very well Feynman integrals related to algebraic curves of genus 0 and 1. These correspond to iterated integrals on the moduli spaces $\mathcal{M}_{0,n}$ and $\mathcal{M}_{1,n}$.
- The obvious generalisation is the generalisation to algebraic curves of higher genus g, i.e. iterated integrals on the moduli spaces $\mathcal{M}_{g,n}$.
- However, we also need the generalisation from curves to surfaces and **higher dimensional objects**: The geometry of the banana graphs with equal non-vanishing internal masses



are Calabi-Yau manifolds.

Section 4

Calabi-Yau manifolds

Definition

A Calabi-Yau manifold of complex dimension n is a compact Kähler manifold M with vanishing first Chern class.

Theorem (conjectured by Calabi, proven by Yau)

An equivalent condition is that M has a Kähler metric with vanishing Ricci curvature.

Mirror symmetry

The **mirror map** relates a Calabi-Yau manifold A to another Calabi-Yau manifold B with Hodge numbers $h_B^{p,q} = h_A^{n-p,q}$.

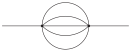
Candelas, De La Ossa, Green, Parkes '91

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & h^{1,1} & & 0 & \\
1 & & h^{2,1} & & h^{2,1} & & 1 \\
 & 0 & & h^{1,1} & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
\qquad\qquad
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & h^{2,1} & & 0 & \\
1 & & h^{1,1} & & h^{1,1} & & 1 \\
 & 0 & & h^{2,1} & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
$$

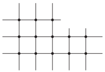
Calabi-Yau manifold A — mirror image B

Fantastic Beasts and Where to Find Them

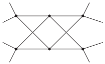
- Bananas



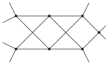
- Fishnets



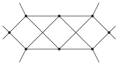
- Amoebas



- Tardigrades



- Paramecia



Aluffi, Marcolli, '09, Bloch, Kerr, Vanhove, '14
Bourjaily, McLeod, von Hippel, Wilhelm, '18
Duhr, Klemm, Loebbert, Nega, Porkert, '22

Bananas

- The l-loop banana integral with (equal) non-zero masses is related to a **Calabi-Yau $(l-1)$-fold**.
- An elliptic curve is a Calabi-Yau 1-fold, this is the geometry at two-loops.
- The system of differential equations for the equal mass l-loop banana integral can be transformed to an **ε-factorised form**.
  - Change of variables from $x = p^2/m^2$ to τ given by mirror map.
  - Transformation constructed from special local normal form of a Calabi-Yau operator.

    M. Bogner '13, D. van Straten '17
- Strong support for the conjecture that a transformation to an ε-factorised differential equation exists for all Feynman integrals.

Section 5

The special local normal form of a Calabi-Yau operator

Special local normal form

Consider a sequence which starts as

$$\begin{array}{rcl} l = 0: & & 1 \\ l = 1: & & \theta \\ l = 2: & & \theta \cdot \theta \\ l = 3: & & \theta \cdot \theta \cdot \theta \end{array}$$

We would like to understand the general term at l loops.

Special local normal form

We first compute the $(l = 4)$-term:

$$\begin{aligned} l &= 0 : & & 1 \\ l &= 1 : & & \theta \\ l &= 2 : & & \theta \cdot \theta \\ l &= 3 : & & \theta \cdot \theta \cdot \theta \\ l &= 4 : & & \theta \cdot \theta \cdot \frac{1}{Y_2} \cdot \theta \cdot \theta \end{aligned}$$

Special local normal form

The general term at l loops is given by

$$\theta \cdot \frac{1}{Y_{l-1}} \cdot \theta \cdot \frac{1}{Y_{l-2}} \cdot \theta \cdot \frac{1}{Y_{l-3}} \cdot \ldots \cdot \frac{1}{Y_3} \cdot \theta \cdot \frac{1}{Y_2} \cdot \theta \cdot \frac{1}{Y_1} \cdot \theta$$

and we have

$$Y_1 = 1$$

and the duality

$$Y_j = Y_{l-j}.$$

Special local normal form

Up to seven loops we therefore have

$$\begin{aligned}
l &= 0: & 1 \\
l &= 1: & \theta \\
l &= 2: & \theta \cdot \theta \\
l &= 3: & \theta \cdot \theta \cdot \theta \\
l &= 4: & \theta \cdot \theta \cdot \frac{1}{Y_2} \cdot \theta \cdot \theta \\
l &= 5: & \theta \cdot \theta \cdot \frac{1}{Y_2} \cdot \theta \cdot \frac{1}{Y_2} \cdot \theta \cdot \theta \\
l &= 6: & \theta \cdot \theta \cdot \frac{1}{Y_2} \cdot \theta \cdot \frac{1}{Y_3} \cdot \theta \cdot \frac{1}{Y_2} \cdot \theta \cdot \theta \\
l &= 7: & \theta \cdot \theta \cdot \frac{1}{Y_2} \cdot \theta \cdot \frac{1}{Y_3} \cdot \theta \cdot \frac{1}{Y_3} \cdot \theta \cdot \frac{1}{Y_2} \cdot \theta \cdot \theta
\end{aligned}$$

Special local normal form

- θ is the **Euler operator** $\theta = q\frac{d}{dq}$ in the variable q, the functions Y_j are called Y-invariants.
- $N = \theta^2 \frac{1}{Y_2} \theta \frac{1}{Y_3} \dots \frac{1}{Y_3} \theta \frac{1}{Y_2} \theta^2$ is the **special local normal form** of a Calabi-Yau operator.
- Operators like N are related to **Picard-Fuchs operators** of Calabi-Yau Feynman integrals.
- From the factorisation of N we may construct the ε-factorised differential equation.

Section 6

The ansatz

The ansatz

- We set $D = 2 - 2\varepsilon$.
- Instead of $x = p^2/m^2$ we work with the variable τ (or q).
- We now **construct master integrals**

$$M = \left(M_0, M_1, \dots, M_l\right)^T,$$

which put the differential equation into an ε-factorised form.
- M_0 is proportional to the l-loop tadpole integral:

$$M_0 = \varepsilon^l I_{1\dots10}.$$

The ansatz

- $I_{1\dots 11}$ has Picard-Fuchs operator $L^{(l)}$, the ε^0-part $L^{(l,0)}$ is of the form

$$L^{(l,0)} = \beta\theta^2 \frac{1}{Y_{l-2}} \theta \frac{1}{Y_{l-3}} \dots \frac{1}{Y_3} \theta \frac{1}{Y_2} \theta^2 \frac{1}{\psi_0}$$

- M_1 should start at order ε^l.
- $L^{(l,0)}$ **annihilates** $I_{1\dots 11}$ modulo ε and modulo tadpoles.
- This suggests

$$M_1 = \frac{\varepsilon^l}{\psi_0} I_{1\dots 11}.$$

The ansatz

- We construct a derivative basis. The **factorisation** of $L^{(l,0)}$ in the variable q suggests for the master integrals $M_2 - M_l$

$$M_j \quad = \quad \frac{\mathbf{1}}{\mathbf{Y_{j-1}}} \left[\frac{1}{2\pi i \varepsilon} \frac{d}{d\tau} M_{j-1} + \text{junk} \right],$$

- **Griffiths transversality**:

$$M_j \quad = \quad \frac{1}{Y_{j-1}} \left[\frac{1}{2\pi i \varepsilon} \frac{d}{d\tau} M_{j-1} - \sum_{k=1}^{j-1} \mathbf{F_{(j-1)k} M_k} \right],$$

with a priori unkown but ε-independent functions $F_{ij}(\tau)$.

Summary of the ansatz

$$
\begin{aligned}
M_0 &= \varepsilon^l I_{1\dots10} \\
M_1 &= \frac{\varepsilon^l}{\psi_0} I_{1\dots11} \\
M_j &= \frac{1}{Y_{j-1}} \left[\frac{1}{2\pi i \varepsilon} \frac{d}{d\tau} M_{j-1} - \sum_{k=1}^{j-1} F_{(j-1)k} M_k \right] \qquad \text{for } j \geq 2
\end{aligned}
$$

The differential equation

The ansatz leads to the differential equation

$$\frac{1}{2\pi i \varepsilon}\frac{d}{d\tau} M \quad = \quad \varepsilon \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & F_{11} & 1 & 0 & 0 & & 0 & 0 \\ 0 & F_{21} & F_{22} & Y_2 & 0 & & 0 & 0 \\ 0 & F_{31} & F_{32} & F_{33} & Y_3 & & 0 & 0 \\ \vdots & & & & & \ddots & & \vdots \\ 0 & F_{(l-2)1} & F_{(l-2)2} & F_{(l-2)3} & F_{(l-2)4} & \dots & Y_{l-2} & 0 \\ 0 & F_{(l-1)1} & F_{(l-1)2} & F_{(l-1)3} & F_{(l-1)4} & \dots & F_{(l-1)(l-1)} & 1 \\ * & * & * & * & * & \dots & * & * \end{pmatrix} M.$$

- The first l rows are in an ε-factorised form.
- Determine the functions F_{ij} such that the $(l+1)$-th row is in ε-factorised form.

The differential equation

The condition that in the $(l+1)$-th row only terms of order ε^1 are present leads to

- differential equations
- **algebraic equations** from self-duality

$$\frac{1}{2\pi i \varepsilon}\frac{d}{d\tau} M = \varepsilon \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & F_{11} & 1 & 0 & 0 & & 0 & 0 \\ 0 & F_{21} & F_{22} & Y_2 & 0 & & 0 & 0 \\ 0 & F_{31} & F_{32} & F_{33} & Y_3 & & 0 & 0 \\ \vdots & & & & & \ddots & & \vdots \\ 0 & F_{(l-2)1} & F_{(l-2)2} & F_{(l-2)3} & F_{(l-2)4} & \dots & Y_{l-2} & 0 \\ 0 & F_{(l-1)1} & F_{(l-1)2} & F_{(l-1)3} & F_{(l-1)4} & \dots & F_{(l-1)(l-1)} & 1 \\ * & * & * & * & * & \dots & * & * \end{pmatrix} M$$

The differential equation

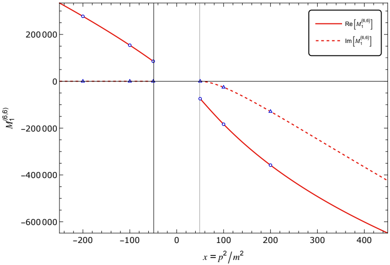
- The equations for F_{ij}'s have a natural triangular structure and can be solved systematically.
- We arrive at the **differential equation in ε-factorised form**:

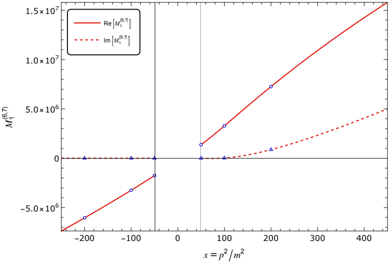
$$dM = \varepsilon A M$$

Section 7

Results and potential applications

Results: Six loops





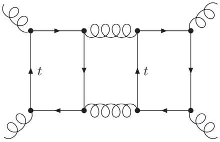
Expansion around $y = 0$ converges at six loops for $|p^2| > 49m^2$.
Agrees with results from `pySecDec`.

The geometry of this Feynman integral is a **Calabi-Yau five-fold**.

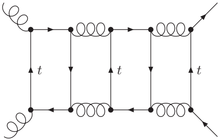
Pögel, Wang, S.W. '22

Examples

- Dijet production at N^{3}LO
  (related to a Calabi-Yau 2-fold).



- Top pair production at N^{4}LO
  (related to a Calabi-Yau 3-fold)



Conclusions

- Feynman integrals are needed for precision calculations in perturbative quantum field theory.
- Method of differential equations is a powerfull tool for computing Feynman integrals.
- It is helpful to relate a Feynman integral to a geometric object (spheres, elliptic curves, Calabi-Yau n-folds, ...).
  Algebraic geometry gives us information on the original Feynman integral.