# Two-loop master integrals and form-factors for pseudo-scalar quarkonia

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based on JHEP09(2022)194 & JHEP02(2023)250
**S. Abreu**, **M. Becchetti**, **C. Duhr**, **M.A. Ozcelik**

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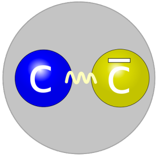
29 May 2023

# Introduction: What is a Quarkonium?

- similar to positronium bound state $\mathrm{e}^+\mathrm{e}^-$ in QED
- bound state of heavy quark and its anti-quark in QCD, e.g. Charmonium (charm quark) and Bottomonium (bottom quark)




[Figure from Wikipedia 'Quarkonium']

- Toponium ($t\bar{t}$) bound state: high mass of top quark $\rightarrow$ decays via weak interaction before formation of bound state
- for light quarks: mixing between (u,d,s) quarks due to low mass difference $\rightarrow$ $\pi$-meson, the $\rho$-meson and the $\eta$-meson

# Motivation: Why study Quarkonia?

- charmonium production allows us to probe QCD at its interplay between the perturbative and non-perturbative regimes

- deeper understanding of confinement (production mechanism)

- access to spin/momentum distribution of gluons in protons
  $\rightarrow$ use quarkonia to constrain the gluon PDFs in
  the proton

- it is interesting to assess the convergence of perturbative expansion in $\alpha_s$ where $\alpha_s(m_c) \sim 0.34$ and $\alpha_s(m_b) \sim 0.22$

# the $\eta_c$ - a good gluon probe

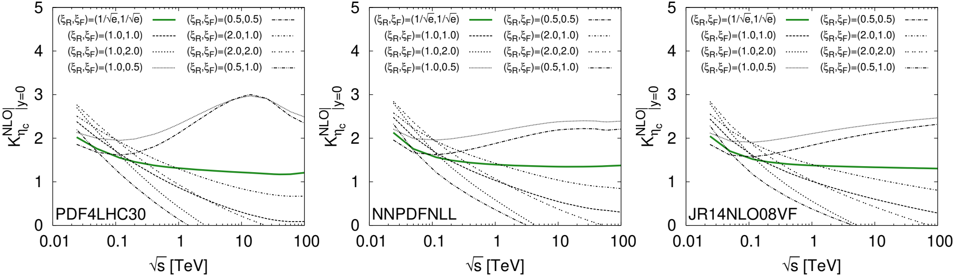
- $\eta_c$ is a gluon probe at low scales at $M_{\eta_c} = 3$ GeV
- is a pseudo-scalar particle and simplest of all quarkonia as far as computation of hadro-production
- $\eta_c$ cross section computation known
  - at NLO since 1992 in collinear factorisation

    **[J. Kühn, E. Mirkes, Phys.Lett. B296 (1992) 425-429]**
  - at LO since 2012 and at NLO since 2013 in TMD factorisation

    **[D. Boer, C. Pisano, Phys.Rev. D86 (2012) 094007]**

    **[J.P. Ma, J.X. Wang, S. Zhao, Phys.Rev. D88 (2013) no.1, 014027]**

# scale variations and negative cross-sections




- large scale uncertainties

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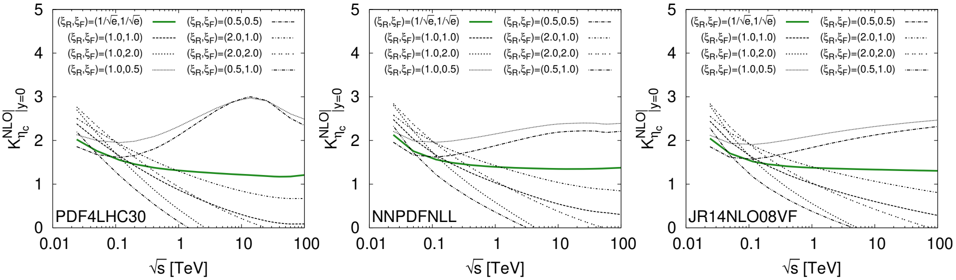


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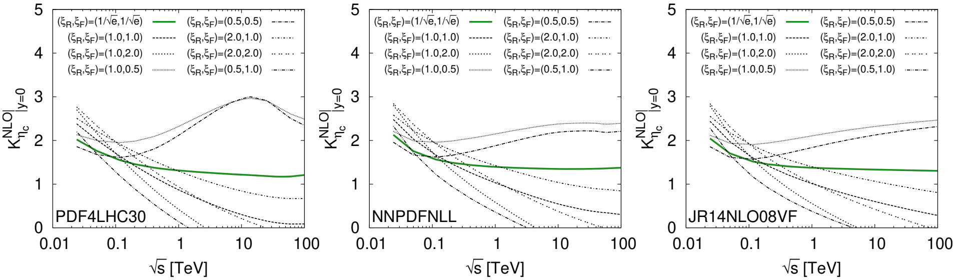


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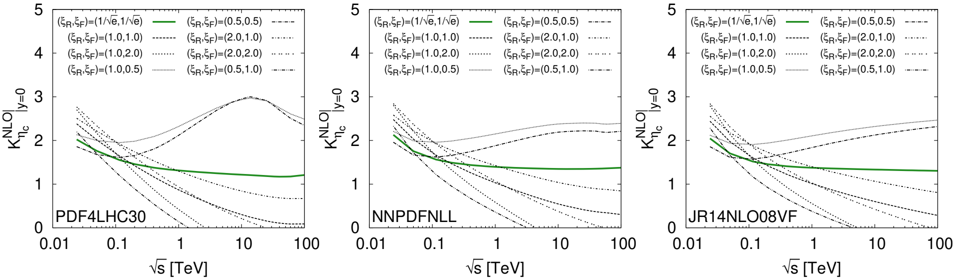


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  - resolved with new scale prescription for $\mu_F$ (green curve)

**[J.-P. Lansberg, Melih A. Ozcelik, Eur.Phys.J.C 81 (2021) 6, 497 (arXiv:2012.00702)]**

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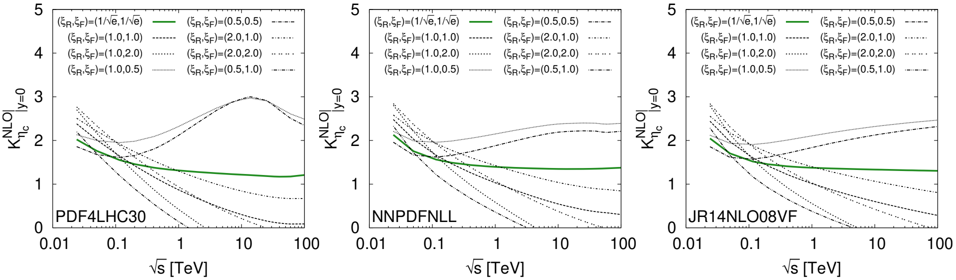


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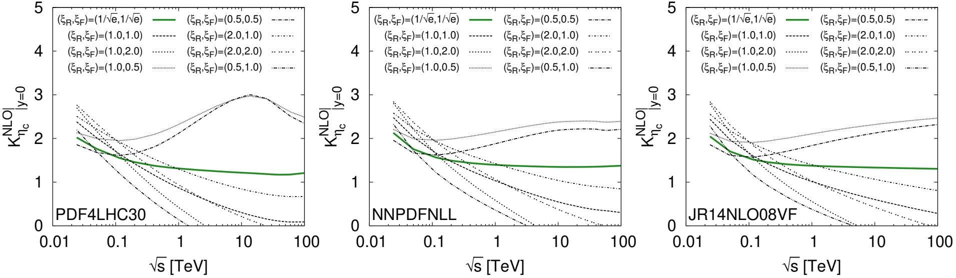


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- for general scale reduction need NNLO calculation $\rightarrow$ need two-loop form-factors

# Form-factors

- compute two-loop form-factors analytically in different channels that contribute at NNLO accuracy
  - $\gamma\gamma \leftrightarrow \eta_Q \left({}^1S_0^{[1]}\right) \rightarrow$ exclusive/inclusive decay
  - $gg \leftrightarrow \eta_Q \left({}^1S_0^{[1]}\right) \rightarrow$ **hadro-production** and hadronic decay width
  - $\gamma g \leftrightarrow {}^1S_0^{[8]} \rightarrow$ colour-octet contribution
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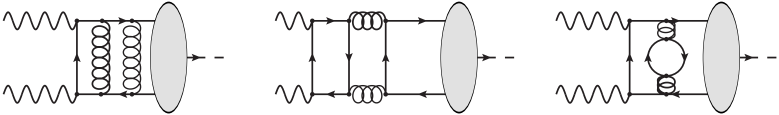
- form-factors applicable to both production and decay
- in the past form-factors have been computed only in numerical form
  - $\eta_Q \rightarrow \gamma\gamma$ **[A. Czarnecki, K. Melnikov, Phys.Lett.B 519 (2001) 212-218] [F. Feng, Y. Jia, W.-L. Sang, Phys.Rev.Lett. 115 (2015) 22, 222001]**
  - para-Positronium $\rightarrow \gamma\gamma$ **[A. Czarnecki, K. Melnikov, A. Yelkhovsky, Phys.Rev.A 61 (2000) 052502]**

# Amplitude generation & partial fraction



$$\gamma(k_1) + \gamma(k_2) \to Q(p_1)\overline{Q}(p_2) \tag{1}$$

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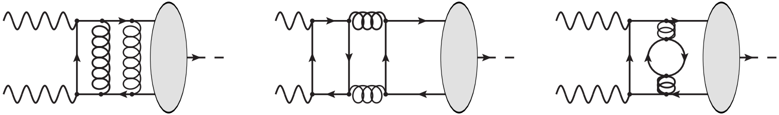

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- $p^2 = m_Q^2$ for final-state heavy quarks with $\boxed{p = p_1 = p_2}$
- $k_1^2 = k_2^2 = 0$ for initial-state photons
- threshold kinematics with $\boxed{\hat{s} = M_{\mathcal{Q}}^2 = 4m_Q^2}$ where $M_{\mathcal{Q}} = 2m_Q$

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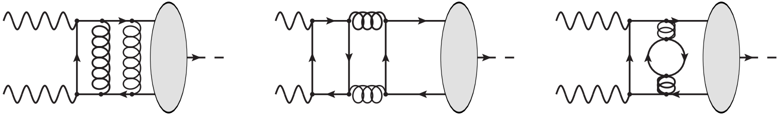

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- generate Feynman diagram with `FeynArts` ($\sim$ 450 diagrams for $gg \leftrightarrow \eta_Q$ case)

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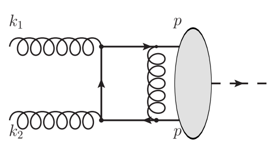
The fact that the two heavy-quark momenta are equal allows us to simplify some integrals beforehand via the procedure of partial fractioning

## Example

Feynman diagram:



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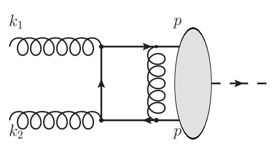
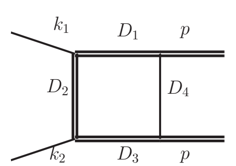



$$I_{\mathrm{Coul.}} = \int d^D q_1 \frac{1}{D_1 D_2 D_3 D_4} = \tag{2}$$



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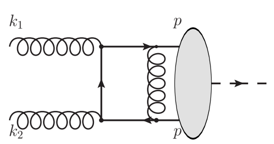


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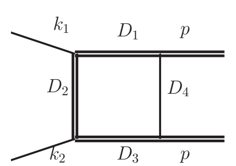


Denominators are linearly dependent: $D_4 = \frac{1}{2}(D_1 + D_3)$

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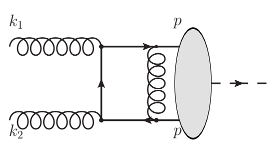
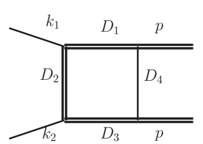
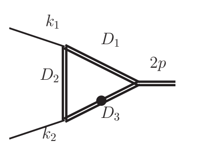



$$I_{\text{Coul.}} = \int d^D q_1 \frac{1}{D_1 D_2 D_3 D_4} = \int d^D q_1 \frac{2}{D_1 D_2 D_3^2} - \int d^D q_1 \frac{1}{D_2 D_3^2 D_4} \tag{3}$$




$= 2$

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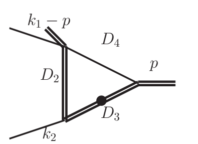


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**[F. Feng, Comput.Phys.Commun. 183 (2012) 2158-2164]**

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  **[F. Feng, Comput.Phys.Commun. 183 (2012) 2158-2164]**

- perform tensor integral decomposition in new basis

- reduce integrals to master integrals via IBP with `FIRE`

  **[A.V. Smirnov, Comput.Phys.Commun. 189 (2015) 182-191]**

# Amplitude

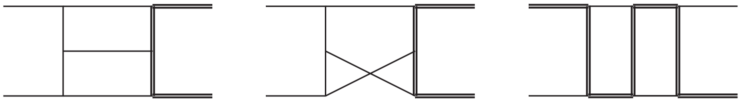
- two-loop Amplitude $\mathcal{A}^{(2)}$:

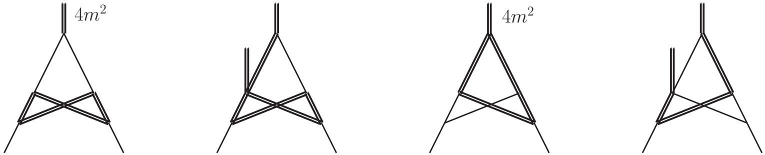
$$\mathcal{A}^{(2)} = \mathcal{A}^{(0)} \sum_{i=1}^{n_{\text{master}}} c_i(\epsilon)\text{MI}[i] \tag{5}$$

  - tree-level Amplitude $\mathcal{A}^{(0)}$
  - coefficient $c_i$ contains information on:
    - rational factor depending on dimensional regulator $\epsilon$
    - colour factor ($C_A$, $C_F$, $T_F$)
    - number of massive ($n_h$) and massless ($n_l$) closed fermion loops (vacuum & light-by-light)
  - need to compute master integrals $\text{MI}[i]$

# Topologies and master integrals

Some examples of topologies:





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  - **Triangle Relations**

# Partial Fraction Identities

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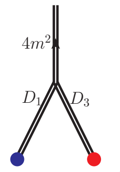
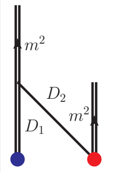



$= \frac{1}{2}$

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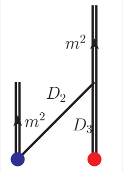


(6)

relation at *integrand* level:

$$\frac{1}{\underbrace{\left[(q+p)^2-m^2\right]}_{D_1}\underbrace{\left[(q-p)^2-m^2\right]}_{D_3}} = \frac{1}{2}\frac{1}{\underbrace{\left[(q+p)^2-m^2\right]}_{D_1}\underbrace{q^2}_{D_2}} + \frac{1}{2}\frac{1}{\underbrace{q^2}_{D_2}\underbrace{\left[(q-p)^2-m^2\right]}_{D_3}}$$

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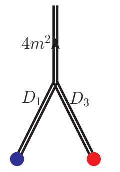


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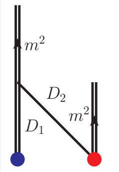


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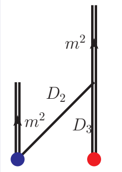


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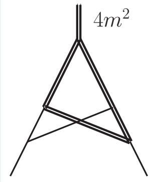


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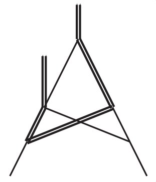

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$$m_{45} = \frac{2\,(3d-11)\,m^2}{(d-3)\,(3d-10)} m_{53} - \frac{8m^4}{(d-3)\,(3d-10)} m_{54} + \frac{(d-2)^2}{4\,(d-3)\,(3d-10)\,m^4} m_{76}$$

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- question for future: can one systematically incorporate partial fraction relations into IBP reduction system (useful for phase-space integrations)?

# Triangle Relations

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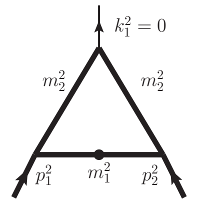


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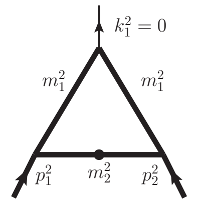


(8)

relation at *integral* level:

$$\int d^d q \frac{1}{\left[q^2 - m_1^2\right]^2 \left[(q+p_1)^2 - m_2^2\right] \left[(q-p_2)^2 - m_2^2\right]} = \int d^d q \,(m_1 \leftrightarrow m_2)$$

no constraint for $p_1$ and $p_2$ (can involve loop momenta), only constraint is that $k_1^2 = 0$

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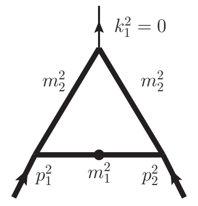


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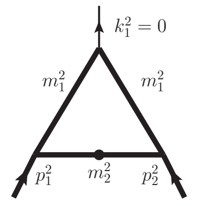


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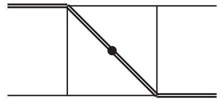

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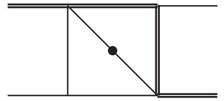

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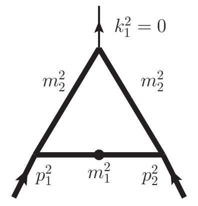


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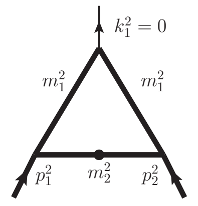
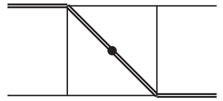



(8)

## Example



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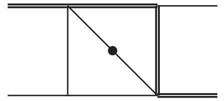

(9)

questions for future: can we systematically incorporate these relations into IBP? And are there more of these relations (box, pentagon integrals)?

# Special functions

- *Multiple Polylogarithms* - points on the *Riemann sphere*
- *elliptic Multiple Polylogarithms* - points on the *torus*
- *iterated integrals of modular forms* - *rational* points on the *torus*

# Multiple Polylogarithms (MPLs)

Multiple Polylogarithms (MPLs) **[Goncharov,Remiddi,Vermaseren]**

$$G(a_1, \dots, a_n; z) = \int_0^z dt \frac{1}{t - a_1} G(a_2, \dots, a_n; t) \tag{10}$$

$$G(0; t) = \log t \tag{11}$$

- weight of function corresponds to number of indices $w = n$
- $m$-loop amplitude usually exhibits functions up to weight of $w = 2m \rightarrow$ will be useful as cross-check of amplitude
- numerical evaluation can be achieved with `GiNaC`-interface

**[Vollinga, Weinzierl]**

# elliptic Multiple Polylogarithms (eMPLs)

elliptic Multiple Polylogarithms (eMPLs) [Brown,Levin;Broedel,Duhr,Dulat,Tancredi;Weinzierl...]

$$E_4\left(\begin{smallmatrix} n_1 \ldots n_m \\ c_1 \ldots c_m \end{smallmatrix}; x,\ \vec{q}\right) = \int_0^x dt\, \psi_{n_1}(c_1, t, \vec{q})\, E_4\left(\begin{smallmatrix} n_2 \ldots n_m \\ c_2 \ldots c_m \end{smallmatrix}; t,\ \vec{q}\right) \tag{12}$$

$$E_4\left(\begin{smallmatrix} \vec{1} \\ \vec{c} \end{smallmatrix}; x,\ \vec{q}\right) = G(\vec{c}; x) \tag{13}$$

- $\vec{q}$ are the roots of the elliptic curve defined by

$$y^2 = (t - q_1)(t - q_2)(t - q_3)(t - q_4) \tag{14}$$

- $\psi_{n_1}(c_1, t, \vec{q})$ are the elliptic kernels
  - e.g. $\psi_0(0, t, \vec{q}) = \frac{c_4}{y}$ where $c_4 = \frac{1}{2}\sqrt{(q_1 - q_3)(q_2 - q_4)}$
  - e.g. $\psi_1(c, t, \vec{q_r}) = \frac{1}{t-c}$
- define weight as $w = \sum_i^m |n_i|$ and length as $l = m$

# elliptic Multiple Polylogarithms (eMPLs)

eMPLs in torus representation **[Brown,Levin;Broedel,Duhr,Dulat,Tancredi;Weinzierl...]**

$$\tilde{\Gamma}\left(\begin{smallmatrix} n_1 \ldots n_m \\ z_1 \ldots z_m \end{smallmatrix}; z, \tau\right) = \int_0^z dz' \, g^{(n_1)}\left(z' - z_1, \tau\right) \tilde{\Gamma}\left(\begin{smallmatrix} n_2 \ldots n_m \\ z_2 \ldots z_m \end{smallmatrix}; z', \tau\right) \tag{15}$$

- a torus is double-periodic and can be defined as a two-dimensional lattice

$$\Lambda_\tau = \mathbb{Z} + \mathbb{Z}\,\tau = \{m + n\,\tau | m, n \in \mathbb{Z}\} \tag{16}$$

- $\tau$ characterises the shape of the torus
- $z$ are the points on the torus within $\Lambda_\tau$

# Iterated integrals of modular forms

if all $z_i$ are rational points on the torus of the form

$$z_i = \frac{r}{N} + \frac{s}{N}\tau \text{ with } 0 \leq r, s < N \text{ and } r, s, N \in \mathbb{N} \tag{17}$$

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$\rightarrow$ can rewrite them in terms of iterated integrals of modular forms

$$I(f_1, \ldots, f_n; \tau) = \int_{i\infty}^{\tau} \frac{d\tau'}{2\pi i} f_1 \, I(f_2, \ldots, f_n; \tau) \tag{18}$$

$$f_i = h_{N,r,s}^{(n)}(\tau) = -\sum_{\substack{(a,b)\in\mathbb{Z}^2 \\ (a,b)\neq(0,0)}} \frac{e^{2\pi i \frac{(bs-ar)}{N}}}{(a\tau + b)^n} \tag{19}$$

# Direct Integration

Feynman integral can be represented via two graph polynomials $\mathcal{U}$ and $\mathcal{F}$ which are the first and second Symanzik polynomial respectively.

$$I = (-1)^a \left(e^{\epsilon\gamma_E}\right)^h \Gamma\left(a - h\frac{D}{2}\right) \int_0^\infty dx_1 \ldots \int_0^\infty dx_m \delta(1-\Delta_H) \times \\ \times \prod_{i=1}^{m} \left(\frac{x_i^{a_i-1}}{\Gamma(a_i)}\right) \frac{\mathcal{U}^{a-(h+1)\frac{D}{2}}}{\mathcal{F}^{a-h\frac{D}{2}}} \tag{20}$$

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- the second Symanzik polynomial $\mathcal{F}$ distinguishes between massive and massless propagators
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  - each **massless** propagator/edge contributes **linearly** to $\mathcal{F}$
  - each **massive** propagator/edge contributes **quadratically** to $\mathcal{F}$
- need to integrate out each single edge $x_i$; one done via Cheng-Wu delta function $\delta(1-\Delta_H)$.

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- **linear reducibility**: an order of integration variables can be found where the integration kernels are all linear

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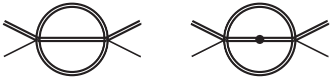
# Master Integrals - Elliptic Curves

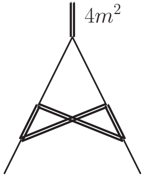
We encounter two different types of elliptic curves,

- one is associated to the elliptic sunrise



$$\vec{q} = \left(\frac{1}{2}\left(1-\sqrt{1+2i}\right), \frac{1}{2}\left(1-\sqrt{1-2i}\right), \frac{1}{2}\left(1+\sqrt{1+2i}\right), \frac{1}{2}\left(1+\sqrt{1-2i}\right)\right) \tag{21}$$

- the other is associated to the master integral




$$\vec{q} = \left(1-\sqrt{5}, 0, 2, 1+\sqrt{5}\right) \tag{22}$$

and appears only in light-by-light scattering contribution

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- PSLQ procedure: find additional relations between elliptic integrals beyond equivalence relations shown earlier

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$\rightarrow$ the one for colour-singlet ($c = 1$) is in agreement with literature, while the colour-octet ($c = 8$) is *new*.

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- QED corrections to para-Positronium result, agreement with existing numerical results in literature **[A. Czarnecki, K. Melnikov, A. Yelkhovsky, Phys.Rev.A 61 (2000) 052502]**

# Summary: Form-factors

- computed all two-loop master integrals **analytically**
- produced high-precision numerics ($>1000$ digits)
- find some interesting equivalence relations
- have complete **analytical** results for form-factors available
- form-factors are finite after UV and IR renormalisation

  $\rightarrow$ ready for phenomenological applications

# Thank you for attention!