Mixed QCD-EW two-loop amplitudes for CC DY

Narayan Rana
NISER Bhubaneswar

in collaboration with T. Armadillo, R. Bonciani, S. Devoto, A. Vicini

RADCOR 2023

29th May 2023

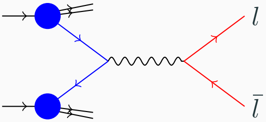
The Standard Model has been extremely successful
in describing the properties and interactions of the known elementary particles.

But, as the queries like origin of EWSB, neutrino mass, existence of dark matter etc. remain unanswered in the domain of the SM, it empowers the search for BSM physics.

The signature of BSM scenarios still remains a secret! The probability of striking and macroscopic new physics signatures with a moderate increase in energy appears low. We will probably have to disentangle small distortions from large SM backgrounds.

A huge amount of data will be accumulated in the HL-LHC. It is clear that an alternative path to uncover possible new physics is the search for small deviations from the predictions of the SM, and that **precision is the key**.

Drell-Yan



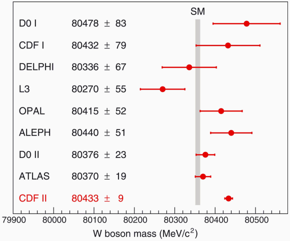
Standard model precision studies

Precise measurement of m_W ($<$ 10 MeV)!

Determination of $\sin^2\theta_W$ is becoming competitive with LEP result.

BSM studies

Precise determination of the SM background is crucial for BSM studies!
Requires control of the SM prediction at the $\mathcal{O}(0.5\%)$ level in the TeV region.



Perturbative expansion

Parton model

$$\sigma_{tot}(z) = \sum_{i,j\in q,\bar{q},g,\gamma} \int \mathrm{d}x_1 \mathrm{d}x_2\; f_i(x_1,\mu_F) f_j(x_2,\mu_F) \sigma_{ij}(z,\varepsilon,\mu_F)$$

In the full QCD-EW SM, we have a double series expansion of the partonic cross sections in the electromagnetic and strong coupling constants, α and α_s, respectively:

$$\begin{aligned}
\sigma_{ij}(z) &= \sigma_{ij}^{(0)} \sum_{m,n=0}^{\infty} \alpha_s^m \alpha^n \; \sigma_{ij}^{(m,n)}(z) \\
&= \sigma_{ij}^{(0)} \Big[\sigma_{ij}^{(0,0)}(z) \\
&\quad + \alpha_s \sigma_{ij}^{(1,0)}(z) + \alpha \sigma_{ij}^{(0,1)}(z) \\
&\quad + \alpha_s^2 \sigma_{ij}^{(2,0)}(z) + \alpha\alpha_s \sigma_{ij}^{(1,1)}(z) + \alpha^2 \sigma_{ij}^{(0,2)}(z) \\
&\quad + \alpha_s^3 \sigma_{ij}^{(3,0)}(z) + \alpha\alpha_s^2 \sigma_{ij}^{(2,1)}(z) + \alpha^2\alpha_s \sigma_{ij}^{(1,2)}(z) + \alpha^3 \sigma_{ij}^{(0,3)}(z) + \cdots \Big]
\end{aligned}$$

Perturbative expansion : QCD corrections

$$\sigma_{ij}(z) = \sigma_{ij}^{(0)} \Big[\sigma_{ij}^{(0,0)}(z) + \alpha_s \sigma_{ij}^{(1,0)}(z) + \alpha \sigma_{ij}^{(0,1)}(z) + \alpha_s^2 \sigma_{ij}^{(2,0)}(z) + \alpha \alpha_s \sigma_{ij}^{(1,1)}(z) + \alpha^2 \sigma_{ij}^{(0,2)}(z) + \alpha_s^3 \sigma_{ij}^{(3,0)}(z) + \alpha \alpha_s^2 \sigma_{ij}^{(2,1)}(z) + \alpha^2 \alpha_s \sigma_{ij}^{(1,2)}(z) + \alpha^3 \sigma_{ij}^{(0,3)}(z) + \cdots \Big]$$

NLO

Altarelli, Ellis, Martinelli (1979);

NNLO

Hamberg, Matsuura, van Neerven (1991);
Anastasiou, Dixon, Melnikov, Petriello (2003);
Catani, Cieri, Ferrera, de Florian, Grazzini (2009);

N^{3}LO

Ahmed, Mahakhud, NR, Ravindran (2014); Duhr, Dulat, Mistlberger (2020); Chen, Gehrmann, Glover, Huss, Yang, Zhu (2021); Camarda, Cieri, Ferrera (2021); Chen, Gehrmann, Glover, Huss, Monni, Re, Rottoli, Torrielli (2022)

Perturbative expansion : EW corrections

$$
\begin{aligned}
\sigma_{ij}(z) = \sigma_{ij}^{(0)} \Big[& \sigma_{ij}^{(0,0)}(z) \\
& + \alpha_s \sigma_{ij}^{(1,0)}(z) + \alpha \sigma_{ij}^{(0,1)}(z) \\
& + \alpha_s^2 \sigma_{ij}^{(2,0)}(z) + \alpha\alpha_s \sigma_{ij}^{(1,1)}(z) + \alpha^2 \sigma_{ij}^{(0,2)}(z) \\
& + \alpha_s^3 \sigma_{ij}^{(3,0)}(z) + \alpha\alpha_s^2 \sigma_{ij}^{(2,1)}(z) + \alpha^2\alpha_s \sigma_{ij}^{(1,2)}(z) + \alpha^3 \sigma_{ij}^{(0,3)}(z) + \cdots \Big]
\end{aligned}
$$

NLO
Baur, Brein, Hollik, Schappacher, Wackeroth (2002);
Carloni Calame, Montagna, Nicrosini, Vicini (2007);
Dittmaier, Huber (2010);

NNLO (approximated)
Jantzen, Kühn, Penin, Smirnov (2005);

Perturbative expansion : mixed corrections

$$\begin{aligned}\sigma_{ij}(z) = \sigma_{ij}^{(0)} \Big[& \sigma_{ij}^{(0,0)}(z) \\ & + \alpha_s \sigma_{ij}^{(1,0)}(z) + \alpha \sigma_{ij}^{(0,1)}(z) \\ & + \alpha_s^2 \sigma_{ij}^{(2,0)}(z) + \alpha\alpha_s \sigma_{ij}^{(1,1)}(z) + \alpha^2 \sigma_{ij}^{(0,2)}(z) \\ & + \alpha_s^3 \sigma_{ij}^{(3,0)}(z) + \alpha\alpha_s^2 \sigma_{ij}^{(2,1)}(z) + \alpha^2\alpha_s \sigma_{ij}^{(1,2)}(z) + \alpha^3 \sigma_{ij}^{(0,3)}(z) + \cdots \Big]\end{aligned}$$

NLO QCD and NLO EW corrections are separately large. What about the mixed corrections, particularly $\sigma_{ij}^{(1,1)}(z)$?

Recent progress in the NNLO mixed QCDxEW corrections

On-shell Z/W production

- Pole approximation : Dittmaier, Huss, Schwinn;
- Analytic QCDxQED corrections : de Florian, Der, Fabre;
- p_T^Z distribution in QCDxQED including p_T resummation : Cieri, Ferrera, Sborlini;
- Differential on-shell Z production including QCDxQED : Delto, Jaquier, Melnikov, Roentsch;
- Total QCDxEW corrections to Z production (fully analytic):
Bonciani, Buccioni, **NR**, Triscari, Vicini; Bonciani, Buccioni, NR, Vicini;
- Differential on-shell Z/W production including QCDxEW :
Behring, Buccioni, Caola, Delto, Jaquier, Melnikov, Roentsch;

Technical developments

- Master integrals : Aglietti, Bonciani; Bonciani, Di Vita, Mastrolia, Schubert; Heller, von Manteuffel, Schabinger; Long, Zhang, Ma, Jiang, Han, Li, Wang; Liu, Ma;
- Mixed QCD-QED splitting functions : de Florian, Sborlini, Rodrigo;
- Renormalisation : Degrassi, Vicini; Dittmaier, Schmidt, Schwarz; Dittmaier;

Complete Drell-Yan

- neutrino pair production in QCDxQED : Cieri, de Florian, Der, Mazzitelli;
- $pp \to l\nu_l + X$ in QCDxEW : Buonocore, Grazzini, Kallweit, Savoini, Tramontano;
- two-loop amplitudes: Heller, von Manteuffel, Schabinger; Armadillo, Bonciani, Devoto, **NR**, Vicini;
- Complete NNLO QCDxEW corrections to neutral current Drell-Yan:
Bonciani, Buonocore, Grazzini, Kallweit, **NR**, Tramontano, Vicini;
Buccioni, Caola, Chawdhry, Devoto, Heller, von Manteuffel, Melnikov, Röntsch, Signorile-Signorile;

Why $\sigma_{ij}^{(1,1)}(z)$ is important?

$\alpha_s(m_Z) \simeq 0.118$	$\alpha(m_Z) \simeq 0.0078$	$\frac{\alpha_s(m_Z)}{\alpha(m_Z)} \simeq 15.1$	$\frac{\alpha_s^3(m_Z)}{\alpha_s(m_Z)\alpha(m_Z)} \simeq 1.8$

1. From naive argument of coupling strength, N^{3}LO QCD $\sim$ mixed NNLO QCD$\otimes$EW.
2. However, in specific phase-space points, fixed order EW corrections can become very large because of logarithmic (weak and QED Sudakov type) enhancement. These effects are large for W mass measurements. On the other hand, these corrections suffer from large uncertainties coming from unphysical scales.
3. N^{3}LO QCD corrections control the uncertainties arising from the unphysical scales, but they lack the large EW effects.
4. The EW corrections reduce the input scheme dependence (from 3.53% to 0.23%).

The NNLO mixed QCD-EW corrections

- have similar magnitude as N^{3}LO QCD,
- contain the large EW effects,
- reduce the theoretical uncertainties.
- reduce the input scheme dependence.

NNLO QCD$\otimes$EW corrections extremely important for high ($\mathcal{O}(10^{-4})$) precision pheno.

NNLO contributions to NC/CC Drell-Yan

Pure Virtual

Real-Virtual

Double Real

Each individual contribution is divergent : $\frac{1}{\epsilon}$ in dimensional regularization

NNLO contributions to NC/CC Drell-Yan

Pure Virtual

$- \; S^{(1,1)}$

Real-Virtual

Double Real

$\Big\} + d\sigma_{CT}^{(1,1)}$

Subtraction : $\quad S^{(1,1)} \sim \int d\sigma_{CT}^{(1,1)} \quad \Rightarrow \quad$ The two sets are separately finite!

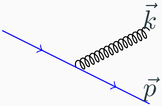
NNLO contributions to NC/CC Drell-Yan

Pure Virtual

The two-loop virtual amplitudes contain divergences of two types

(a) **Ultraviolet divergences** : UV renormalization of fields and couplings
(b) **Infrared divergences** : Soft (soft gluons & photons) & collinear (collinear partons)

$$\frac{1}{(k+p)^2} = \frac{1}{2k.p} = \frac{1}{2k^0 p^0 (1-\cos\theta)}$$



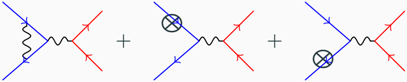
$k^0 \to 0$ Soft divergence
$\theta \to 0$ Collinear divergence

The infrared structure of scattering amplitudes is universal!

Ultraviolet renormalization

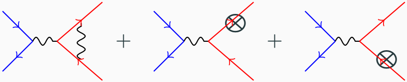
⊛ The Born contribution is zeroth order in α_s, hence no α_s renormalization is needed.

⊛ Renormalization of quark wave function receives one-loop EW and two-loop mixed QCD⊗EW contributions in the on-shell scheme.



$\Rightarrow$ UV finite

⊛ Renormalization of lepton wave function receives one-loop EW contributions.



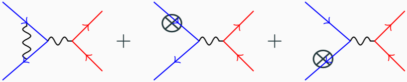
$\Rightarrow$ UV finite

We consider massive leptons, but small mass limit. In that case, the QED part of the renormalization constant is with massive lepton. On the other hand, the weak part can be computed using massless lepton.

Ultraviolet renormalization

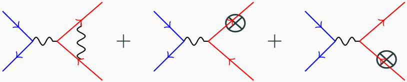
⊛ The Born contribution is zeroth order in α_s, hence no α_s renormalization is needed.

⊛ Renormalization of quark wave function receives one-loop EW and two-loop mixed QCD⊗EW contributions in the on-shell scheme.



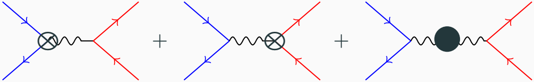
$\Rightarrow$ UV finite

⊛ Renormalization of lepton wave function receives one-loop EW contributions.



$\Rightarrow$ UV finite

⊛ The computation is performed in background field gauge, with the advantage that the vertex and propagator contributions are separately UV finite.



$\Rightarrow$ UV finite

The infrared divergences and lepton mass : NC DY

The infrared structure of scattering amplitudes is universal!

$$\mathcal{M}_{\rm fin}^{(1,1)} = \mathcal{M}^{(1,1)} - \mathcal{I}^{(1,1)}\mathcal{M}^{(0)} - \mathcal{I}^{(0,1)}\mathcal{M}_{\rm fin}^{(1,0)} - \mathcal{I}^{(1,0)}\mathcal{M}_{\rm fin}^{(0,1)}$$

The q_T subtraction requires the final state emitters (leptons) to be massive!

The full computation with lepton mass is extremely difficult!

Divergence regulator	massless lepton : $\frac{1}{\epsilon}$	massive lepton : $\log m_l$

The infrared divergences and lepton mass : NC DY

The infrared structure of scattering amplitudes is universal!

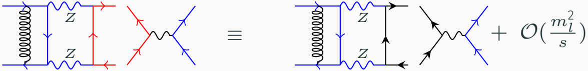
$$\mathcal{M}_{\text{fin}}^{(1,1)} = \mathcal{M}^{(1,1)} - \mathcal{I}^{(1,1)}\mathcal{M}^{(0)} - \mathcal{I}^{(0,1)}\mathcal{M}_{\text{fin}}^{(1,0)} - \mathcal{I}^{(1,0)}\mathcal{M}_{\text{fin}}^{(0,1)}$$

The q_T subtraction requires the final state emitters (leptons) to be massive!

The full computation with lepton mass is extremely difficult!

Divergence regulator	massless lepton : $\frac{1}{\epsilon}$	massive lepton : $\log m_l$

(a) When the lepton is attached to a massive boson, it does not generate any collinear divergence. Hence, in all such cases, we can safely assume a massless lepton.



The infrared divergences and lepton mass : NC DY

The infrared structure of scattering amplitudes is universal!

$$\mathcal{M}^{(1,1)}_{\text{fin}} = \mathcal{M}^{(1,1)} - \mathcal{I}^{(1,1)}\mathcal{M}^{(0)} - \mathcal{I}^{(0,1)}\mathcal{M}^{(1,0)}_{\text{fin}} - \mathcal{I}^{(1,0)}\mathcal{M}^{(0,1)}_{\text{fin}}$$

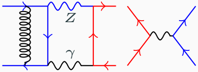
The q_T subtraction requires the final state emitters (leptons) to be massive!

The full computation with lepton mass is extremely difficult!

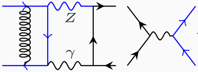
Divergence regulator massless lepton : $\frac{1}{\epsilon}$ massive lepton : $\log m_l$

(a) When the lepton is attached to a massive boson, it does not generate any collinear divergence. Hence, in all such cases, we can safely assume a massless lepton.

(b) In a single box diagram, where lepton is attached to one photon and one Z boson, it generates a collinear singularity. However, thanks to [Frenkel, Taylor], once all diagrams are summed up, the collinear divergences cancel.



$\equiv$



$$- \mathcal{I}(\epsilon) + \mathcal{C}_l \log\left(\frac{m_l^2}{s}\right) + \mathcal{O}\left(\frac{m_l^2}{s}\right)$$

It is also reflected in the subtraction formula e.g. for the QED box part

$$\left[H(-1, y_l) - H(-1, z_l)\right]|_{m_l \to 0} \equiv \log(t/u)$$

The infrared divergences and lepton mass : NC DY

The infrared structure of scattering amplitudes is universal!

$$\mathcal{M}^{(1,1)}_{\text{fin}} = \mathcal{M}^{(1,1)} - \mathcal{I}^{(1,1)}\mathcal{M}^{(0)} - \mathcal{I}^{(0,1)}\mathcal{M}^{(1,0)}_{\text{fin}} - \mathcal{I}^{(1,0)}\mathcal{M}^{(0,1)}_{\text{fin}}$$

The q_T subtraction requires the final state emitters (leptons) to be massive!

The full computation with lepton mass is extremely difficult!

Divergence regulator	massless lepton : $\frac{1}{\epsilon}$	massive lepton : $\log m_l$

(a) When the lepton is attached to a massive boson, it does not generate any collinear divergence. Hence, in all such cases, we can safely assume a massless lepton.

(b) In a single box diagram, where lepton is attached to one photon and one Z boson, it generates a collinear singularity. However, thanks to [Frenkel, Taylor], once all diagrams are summed up, the collinear divergences cancel.

(c) Hence, the collinear singularities from leptons ($\log m_l$) come from only the QED-type corrections to the lepton vertex, which we compute with full lepton mass dependence.

The infrared divergences and lepton mass : CC DY

The infrared structure of scattering amplitudes is universal!

$$\mathcal{M}_{\rm fin}^{(1,1)} = \mathcal{M}^{(1,1)} - \mathcal{I}^{(1,1)}\mathcal{M}^{(0)} - \mathcal{I}^{(0,1)}\mathcal{M}_{\rm fin}^{(1,0)} - \mathcal{I}^{(1,0)}\mathcal{M}_{\rm fin}^{(0,1)}$$

The q_T subtraction requires the final state emitters (leptons) to be massive!

The full computation with lepton mass is extremely difficult!

Divergence regulator	massless lepton : $\frac{1}{\epsilon}$	massive lepton : $\log m_l$

The infrared divergences and lepton mass : CC DY

The infrared structure of scattering amplitudes is universal!

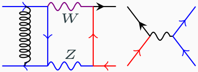
$$\mathcal{M}^{(1,1)}_{\mathrm{fin}} = \mathcal{M}^{(1,1)} - \mathcal{I}^{(1,1)}\mathcal{M}^{(0)} - \mathcal{I}^{(0,1)}\mathcal{M}^{(1,0)}_{\mathrm{fin}} - \mathcal{I}^{(1,0)}\mathcal{M}^{(0,1)}_{\mathrm{fin}}$$

The q_T subtraction requires the final state emitters (leptons) to be massive!

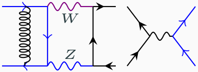
The full computation with lepton mass is extremely difficult!

Divergence regulator massless lepton : $\frac{1}{\epsilon}$ massive lepton : $\log m_l$

(a) When the lepton is attached to a massive boson, it does not generate any collinear divergence. Hence, in all such cases, we can safely assume a massless lepton.



$\equiv$



$+ \ \mathcal{O}(\frac{m_l^2}{s})$

The infrared divergences and lepton mass : CC DY

The infrared structure of scattering amplitudes is universal!

$$\mathcal{M}_{\rm fin}^{(1,1)} = \mathcal{M}^{(1,1)} - \mathcal{I}^{(1,1)}\mathcal{M}^{(0)} - \mathcal{I}^{(0,1)}\mathcal{M}_{\rm fin}^{(1,0)} - \mathcal{I}^{(1,0)}\mathcal{M}_{\rm fin}^{(0,1)}$$

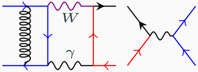
The q_T subtraction requires the final state emitters (leptons) to be massive!

The full computation with lepton mass is extremely difficult!

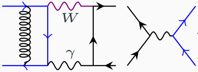
Divergence regulator massless lepton : $\frac{1}{\epsilon}$ massive lepton : $\log m_l$

(a) When the lepton is attached to a massive boson, it does not generate any collinear divergence. Hence, in all such cases, we can safely assume a massless lepton.

(b) In a single box diagram, where lepton is attached to one photon and one W boson, it generates a collinear singularity. However, there is either "t" or "u" channel diagrams. Hence, unlike the NC DY, the collinear divergences do not cancel once all diagrams are summed up.



$\equiv$



$- \mathcal{I}(\epsilon) \; + \; \mathcal{C}_l \log\left(\frac{m_l^2}{s}\right) + \mathcal{O}\left(\frac{m_l^2}{s}\right)$

The infrared divergences and lepton mass : CC DY

The infrared structure of scattering amplitudes is universal!

$$\mathcal{M}^{(1,1)}_{\text{fin}} = \mathcal{M}^{(1,1)} - \mathcal{I}^{(1,1)}\mathcal{M}^{(0)} - \mathcal{I}^{(0,1)}\mathcal{M}^{(1,0)}_{\text{fin}} - \mathcal{I}^{(1,0)}\mathcal{M}^{(0,1)}_{\text{fin}}$$

The q_T subtraction requires the final state emitters (leptons) to be massive!

The full computation with lepton mass is extremely difficult!

Divergence regulator	massless lepton : $\frac{1}{\epsilon}$	massive lepton : $\log m_l$

(a) When the lepton is attached to a massive boson, it does not generate any collinear divergence. Hence, in all such cases, we can safely assume a massless lepton.

(b) In a single box diagram, where lepton is attached to one photon and one W boson, it generates a collinear singularity. However, there is either "t" or "u" channel diagrams. Hence, unlike the NC DY, the collinear divergences do not cancel once all diagrams are summed up.

(c) The corrections to the lepton vertex also contains these collinear singularities.

We compute everything considering massless leptons and then do **massification**:

$$|\mathcal{M}_m\rangle = \mathcal{J}_m \mathcal{J}_0^{-1} |\mathcal{M}_0\rangle$$

Computational procedure

$\boxed{d = 4 - 2\epsilon}$

- Diagrammatic approach -> `QGRAF`/`FeynArts` to generate Feynman diagrams
- In-house `FORM`/`Mathematica` routines for algebraic simplification :
  Lorentz, Dirac and Color algebra
- Decomposition of the dot products to obtain scalar integrals
- Identity relations among scalar integrals : IBPs, LIs & SRs
- Algebraic linear system of equations relating the integrals

$\Downarrow$

Master integrals (MIs)

- Computation of MIs : Method of differential equation & SeaSyde
- Ultraviolet renormalization
- Subtraction of the universal infrared poles ($S^{(1,1)}$).
- Numerical evaluation of the hard function to prepare the grid.

Computational procedure : γ_5

γ_5 is inherently a four-dimensional object.
How can we use it in dimensional regularization?

	Anti-commutation $\{\gamma_\mu, \gamma_5\} = 0$	Cyclicity of the trace
't Hooft and Veltmann	X	✓
Kreimer et al.	✓	X

For the mixed QCD-EW corrections to the NCDY, the two prescriptions yield

- Different one- and two-loop scattering amplitudes
- Same finite remainder after subtraction

[Heller, von Manteuffel, Schabinger, Spiesberger]

Computational procedure : γ_5

γ_5 is inherently a four-dimensional object.
How can we use it in dimensional regularization?

	Anti-commutation $\{\gamma_\mu, \gamma_5\} = 0$	Cyclicity of the trace
't Hooft and Veltmann	X	✓
Kreimer et al.	✓	X

Our approach :

- Consider a fixed point to start the Dirac trace.
- Use anti-commutation relation, bring all γ_5 at the end and use $\gamma_5^2 = 1$.
- Use $\gamma_5 = \frac{i}{24!}\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma$ for the single leftover γ_5.

The method of differential equations

A Feynman integral is a function of spacetime dimension d and kinematic invariant x, y.

$$J_i \sim \int \frac{d^d l_1}{(2\pi)^d} \frac{d^d l_2}{(2\pi)^d} \frac{1}{l_1^2 l_2^2((l_1-l_2)^2-m^2)(l_1-p_1-p_2)^2(l_2-p_3)^2} \equiv f(d,\, x,\, y)$$

The idea is to obtain differential eqns. for the integral $w.r.t.\ x,\ y$ and solve it.

$$d_x \begin{pmatrix} J_1 \\ J_2 \\ J_3 \\ J_4 \\ \vdots \\ J_n \end{pmatrix} = \begin{bmatrix} \bullet & \bullet & \bullet & \bullet & \cdots & \bullet \\ 0 & \bullet & \bullet & \bullet & \cdots & \bullet \\ 0 & \bullet & \bullet & \bullet & \cdots & \bullet \\ 0 & 0 & 0 & \bullet & \cdots & \bullet \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \bullet \end{bmatrix} \begin{pmatrix} J_1 \\ J_2 \\ J_3 \\ J_4 \\ \vdots \\ J_n \end{pmatrix}$$

To solve such a system, we need to perform series expansion in ϵ and to organize the matrix in each order of ϵ in such a way that it diagonalizes, or at least it takes a block-triangular form. Now, it can be solved using bottom-up approach.

The homogeneous solutions are in general $\log$ or Li_2. Because of the ϵ expansion, the non-homogeneous solutions are recursive integral over the homogeneous solutions.

The results are obtained in terms of iterated integrals (GPLs).

Iterated integrals

From Feynman integrals to iterated integrals : What do we gain?

Direct numerical integration of Feynman integrals is tedious, unstable and challenging to obtain precise results.

Iterated integrals

From Feynman integrals to iterated integrals : What do we gain?

Iterated integrals are one-dimensional. They can be computed with great precision in a short amount of time. Besides, they have the following properties:

(a) **Shuffle algebra** : Allows to obtain a basis for a set of iterated integrals. Reduction to such a basis is extremely effective to reduce the computation time by few times.

(b) **Scaling invariance** : Allows to convert the limit of these integrals from kinematical variables (z) to constants (1). This makes the integration really precise.

Iterated integrals

From Feynman integrals to iterated integrals : What do we gain?

Iterated integrals are one-dimensional. They can be computed with great precision in a short amount of time. Besides, they have the following properties:

(a) **Shuffle algebra** : Allows to obtain a basis for a set of iterated integrals. Reduction to such a basis is extremely effective to reduce the computation time by few times.

(b) **Scaling invariance** : Allows to convert the limit of these integrals from kinematical variables (z) to constants (1). This makes the integration really precise.

MIs available for NC DY

- Form factor type MIs : Aglietti, Bonciani; Bonciani, Buccioni, NR, Vicini;
- Box type ($\gamma\gamma$ with massive lepton) : Bonciani, Ferroglia, Gehrmann, Maitre, Studerus;
- Box type (γZ & ZZ with massless lepton) :

Bonciani, Di Vita, Mastrolia, Schubert; Heller, von Manteuffel, Schabinger

5 among the 36 two-same-mass MIs of Bonciani *et al.* contain Chen iterated integrals!

The 36 two-same-mass master integrals for NC DY

Fully analytic

- Most MIs are solved in GPLs.
- Five MIs are solved in terms of Chen's iterated integrals! Numerical evaluation possible only in the non-physical region.

Fully numerical

- Evaluation of the MIs in physical region is demanding! (using Fiesta/pySecDec)
- Specially for those five MIs, achieving a single digit precision in the physical region is extremely challenging!

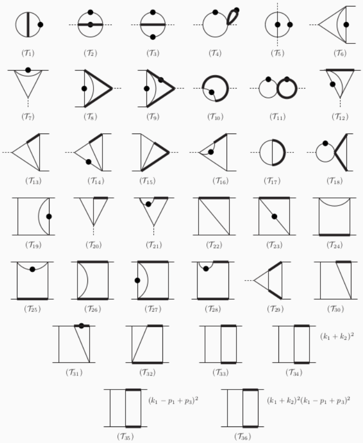


Fig from Roberto et al.

Can we find a mixed approach?

Our semi-analytic approach for NC DY

What do we need for the two-loop virtual amplitudes?

Our semi-analytic approach for NC DY

What do we need for the two-loop virtual amplitudes?

(a) An analytic formula for the singular part, to perform the infrared subtraction.

(b) A formula for the finite part which should be numerically stable and precise.

(i) The universal subtraction operator indicates that the singular part of the amplitude contains only simple GPLs.

(ii) The individual contribution from the five MIs to the single pole of the matrix element contains the Chen iterated integrals, which cancel after summing them.

(iii) Certain internal combinations of the MIs (at the lowest order in ϵ) can be found which can be solved in terms of simple GPLs.

So, only simple GPLs in the singular part! SOLVED!

Our semi-analytic approach for NC DY

What do we need for the two-loop virtual amplitudes?

(a) An analytic formula for the singular part, to perform the infrared subtraction.

(b) **A formula for the finite part which should be numerically stable & precise.**

Most of the MIs are known in terms of GPLs. Few MIs (32-36), which contain Chen iterated integrals, we solve them using series expansion through `SeaSyde`.

Implemented also in the Mathematica package `DiffExp`.

[F. Moriello (2019), M. Hidding (2020)]

Our semi-analytic approach for NC DY

What do we need for the two-loop virtual amplitudes?

(a) An analytic formula for the singular part, to perform the infrared subtraction.

(b) **A formula for the finite part which should be numerically stable & precise.**

Most of the MIs are known in terms of GPLs. Few MIs (32-36), which contain Chen iterated integrals, we solve them using series expansion through SeaSyde.

(i) We consider the system of differential equations for all the 36 MIs. Given a boundary point, the system can be solved using series expansion for a nearby point.
(ii) The solution in this new point can now be considered as boundary and thus we can go forward along a path to obtain solution in any phase space point.

The difference between NC & CC DY

Single mass scale (m_Z or m_W)

- 36 MIs (variations with mass & kinematics).
- Most MIs are solved in GPLs.
- Five MIs are solved in terms of Chen's iterated integrals!

31 MIs : GPLs
5 MIs : SeaSyde

- Full analytic expressions for poles.
- Semi-analytic expressions for finite part.

Two mass scales (both m_Z & m_W)

- 56 MIs (variations with mass & kinematics).
- Most MIs are not known analytically.
- Some MIs (γW boxes and sub-topologies of ZW boxes) are with single mass scale and are known in terms of GPLs.

Our semi-analytic approach for CC DY

What do we need for the two-loop virtual amplitudes?

Our semi-analytic approach for CC DY

What do we need for the two-loop virtual amplitudes?

(a) An analytic formula for the singular part, to perform the infrared subtraction.

(b) A formula for the finite part which should be numerically stable and precise.

(i) The two-different-mass MIs are generally less divergent (maximum $\frac{1}{\epsilon^2}$ pole)!
Also, the $\frac{1}{\epsilon^2}$ pole of these MIs have been computed.

$$\Downarrow$$

The expressions for all except the single pole are analytic!

(ii) We compute the MIs expanding m_W around m_Z in terms of δ_m and check the single pole analytically up to $\mathcal{O}(\delta_m^2)$.

(iii) Finally, we compute all the MIs using SeaSyde and check the single pole numerically at several phase-space points.

Our semi-analytic approach for CC DY

What do we need for the two-loop virtual amplitudes?

(a) An analytic formula for the singular part, to perform the infrared subtraction.

(b) **A formula for the finite part which should be numerically stable & precise.**

We solve all the MIs using SeaSyde. We use NC DY grids as our initial conditions, and solve the differential equations with respect to the mass.

We start from both the limits (m_Z, m_Z) and (m_W, m_W) and arrive at the same results (m_W, m_Z) by using corresponding sets of differential equations.

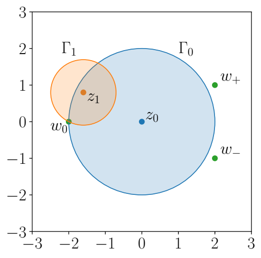
All the known MIs are used to cross-check the SeaSyde result.

SEASYDE

Series Expansion Approach for SYstems of Differential Equations

We have implemented the series expansion method generalizing it with complex variables $\Rightarrow$ complex plane!

- The radius of convergence of the series is limited by the presence of poles.
- Transport from one point to another needs to consider branch-cuts.

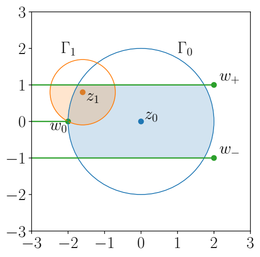


SEASYDE

Series Expansion Approach for SYstems of Differential Equations

We have implemented the series expansion method generalizing it with complex variables $\Rightarrow$ complex plane!

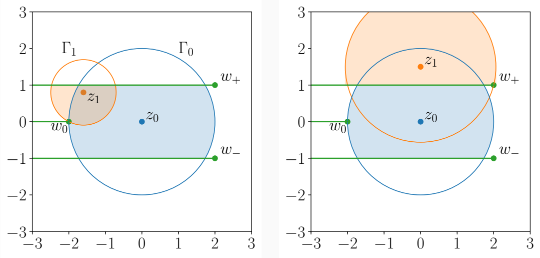
- The radius of convergence of the series is limited by the presence of poles.
- Transport from one point to another needs to consider branch-cuts.



Series Expansion Approach for SYstems of Differential Equations

We have implemented the series expansion method generalizing it with complex variables $\Rightarrow$ complex plane!

- The radius of convergence of the series is limited by the presence of poles.
- Transport from one point to another needs to consider branch-cuts.

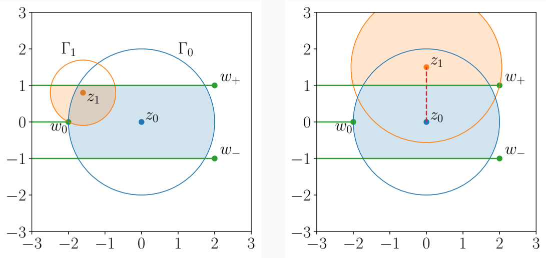


SEASYDE

Series Expansion Approach for SYstems of Differential Equations

We have implemented the series expansion method generalizing it with complex variables $\Rightarrow$ complex plane!

- The radius of convergence of the series is limited by the presence of poles.
- Transport from one point to another needs to consider branch-cuts.

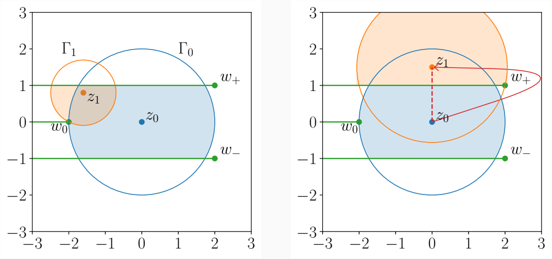


SEASYDE

Series Expansion Approach for SYstems of Differential Equations

We have implemented the series expansion method generalizing it with complex variables $\Rightarrow$ complex plane!

- The radius of convergence of the series is limited by the presence of poles.
- Transport from one point to another needs to consider branch-cuts.



* We transport solution along the red solid line (The corresponding circles are not drawn.)

SEASYDE

Series Expansion Approach for SYstems of Differential Equations

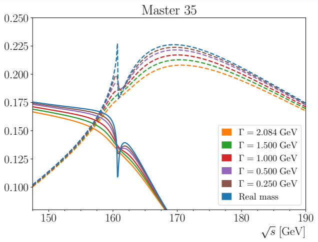
- We consider the full system of differential equations and solve it with SeaSyde.
- The solution can be obtained with **arbitrary number of significant digits**.
- The latest release of SeaSyde can also **solve coupled differential equations**.
- Numerical evaluations of analytically known MIs using `GiNaC` provide crucial checks. Also several other checks with `Fiesta`, `pySecDec`, `DiffExp`.
- Because of using complex variable in `SeaSyde`, it is possible to use complex mass scheme which smoothens the behaviour at threshold.

SEASYDE

Series Expansion Approach for SYstems of Differential Equations

- We consider the full system of differential equations and solve it with SeaSyde.
- The solution can be obtained with arbitrary number of significant digits.
- The latest release of SeaSyde can also solve coupled differential equations.
- Numerical evaluations of analytically known MIs using `GiNaC` provide crucial checks. Also several other checks with `Fiesta`, `pySecDec`, `DiffExp`.
- Because of using complex variable in `SeaSyde`, it is possible to use complex mass scheme which smoothens the behaviour at threshold.

The small width limit reproduces the result in real mass with Feynman prescription



Finally

We obtain the two-loop virtual amplitude:
(a) The singular part is (mostly) analytic and contains GPLs. This allows us to successfully check with the universal infrared behaviour of the scattering amplitudes.

(b) The finite part after performing the infrared subtraction contains:

in case of NC DY: GPLs and a few MIs which have been computed using SeaSyde.

in case of CC DY: γW diagrams in GPLs, ZW diagrams contain MIs computed using SeaSyde.

Finally

We obtain the two-loop virtual amplitude:
(a) The singular part is (mostly) analytic and contains GPLs. This allows us to successfully check with the universal infrared behaviour of the scattering amplitudes.

(b) The finite part after performing the infrared subtraction contains:

in case of NC DY: GPLs and a few MIs which have been computed using SeaSyde.

in case of CC DY: γW diagrams in GPLs, ZW diagrams contain MIs computed using SeaSyde.

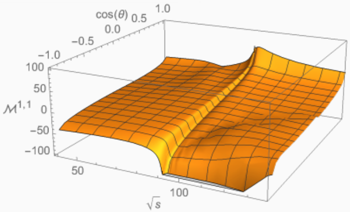
Next? Numerical evaluation of the subtracted finite part

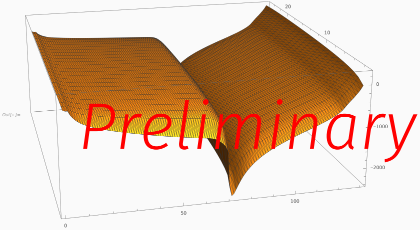
NC DY: there are $\sim$ 11000 GPLs in the full expression. Production of the grid (3250 points) for the MIs required $\mathcal{O}$(12h) on a 32-cores machine. Evaluation of the GPLs on a single phase-space point, for 40 digits precision, ranges from few minutes to $\sim$ 20 minutes, depending on the phase-space point. Evaluation time substantially goes down for smaller number of digits.

CC DY: Production of the grid (3250 points) for the MIs required 2-3 days on a 26-cores machine. Once we obtain the MIs, it is a matter of minutes.

Numerical grids

To obtain a fast compilation and successful numerical evaluation, we divide the contributions from various Feynman diagrams in a gauge invariant way by the presence of different EW vector bosons (γ, Z, W), and further by different topologies. These subdivisions allow us to parallelize the computation.





Plots for some partial contributions in NC and CC DY.

Remarks!

- Precision physics is at the current frontier of particle physics research.
- Precise experimental measurements with precise theoretical predictions, can provide full understanding of the SM and shed light on BSM physics.

————————————————

- The precision measurement of the EW parameters like $m_W, \sin\theta_W$ etc. are sensitive to beyond the SM physics.
- The mixed QCD$\otimes$EW corrections to Drell-Yan production, is going to be a milestone in these precision measurements. It will be an important ingredient for the future Monte-Carlo event generators.

————————————————

- Our semi-analytic approach has opened the possibilities to compute more difficult but important processes;

Thank you for your attention!