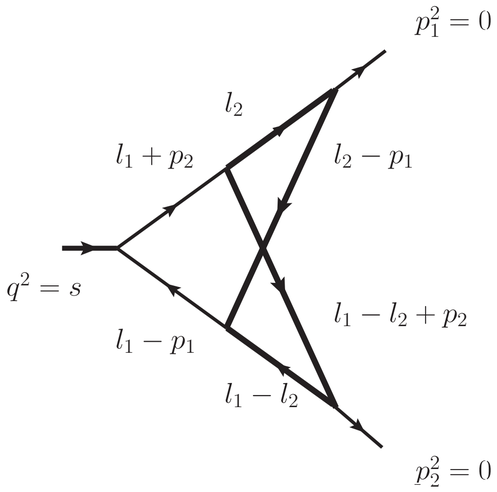
ε-form for non-planar triangles with elliptic curves at two loops

Xing Wang
Technische Universität München

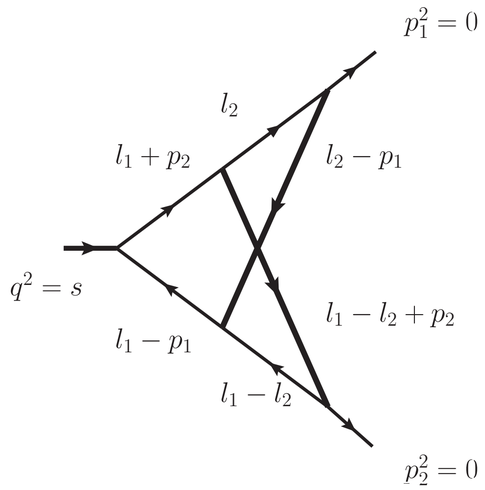
based on 2305.13951 with Xuhang Jiang, Li Lin Yang, and Jingbang Zhao

RADCOR 2023, 28 May 2023 @ Crieff

Setup



$(a) : \kappa = 1$



$(b) : \kappa = 0$

$$I_{\nu_1\nu_2\cdots\nu_7} = e^{2\varepsilon\gamma_E}(m^2)^{\nu-D}\int\frac{d^Dl_1}{i\pi^{D/2}}\int\frac{d^Dl_2}{i\pi^{D/2}}\frac{D_7^{-\nu_7}}{D_1^{\nu_1}D_2^{\nu_2}D_3^{\nu_3}D_4^{\nu_4}D_5^{\nu_5}D_6^{\nu_6}}$$

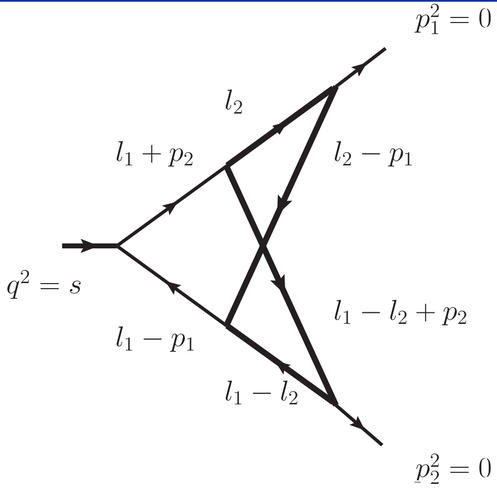
$$D = 4 - 2\varepsilon,$$
$$\nu = \nu_1 + \nu_2 + \cdots + \nu_7,$$
$$y = -\frac{m^2}{s} \in \mathbb{R} + i0,$$
$$x = -\frac{1}{y}.$$

a) Top: **2** MIs: $I_1^{(a)} = I_{1111110}^{(a)}$, $I_2^{(a)} = I_{1111120}^{(a)}$
Sub: 9 MIs;
[IBP via Litered]

b) Top: **3** MIs: $I_1^{(b)} = I_{1111110}^{(b)}$, $I_2^{(b)} = I_{1112110}^{(b)}$, $I_3^{(b)} = I_{1111120}^{(b)}$
Sub: 15 MIs.
[IBP via Litered]

$$D_1 = (l_1 - p_1)^2,\ D_2 = (l_2 - p_1)^2 - m^2,\ D_3 = (l_1 + p_2)^2,$$
$$D_4 = (l_1 - l_2 + p_2)^2 - m^2,\ D_5 = (l_1 - l_2)^2 - \kappa m^2,\ D_6 = l_2^2 - \kappa m^2.$$

Canonical basis for sub-sectors



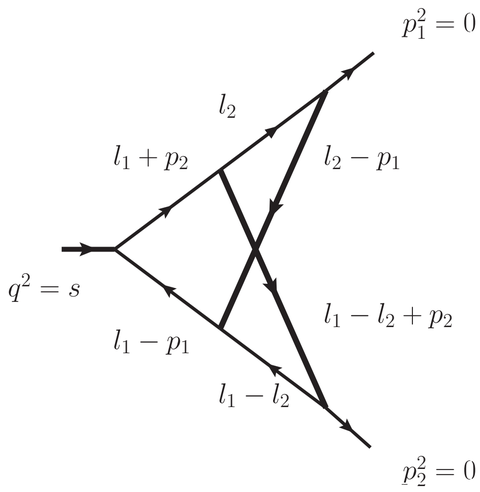
(a)

$$M_3^{(a)} = \varepsilon^4 x\, I^{(a)}_{0111110}, \quad M_4^{(a)} = \varepsilon^4 x\, I^{(a)}_{1011110}, \quad M_5^{(a)} = \varepsilon^2\sqrt{x(4+x)}\left(x\, I^{(a)}_{1122000} - \frac{\varepsilon}{2(1+2\varepsilon)} I^{(a)}_{0202000}\right),$$

$$M_6^{(a)} = \varepsilon^3 x\, I^{(a)}_{1010210}, \quad M_7^{(a)} = \varepsilon^3 x\, I^{(a)}_{0211100}, \quad M_8^{(a)} = \varepsilon^2 x\, I^{(a)}_{0210200},$$

$$M_9^{(a)} = \varepsilon^2\sqrt{x(x-4)}\left(\frac{1}{2} I^{(a)}_{0210200} + I^{(a)}_{0220100}\right), \quad M_{10}^{(a)} = \varepsilon^2 x\, I^{(a)}_{1020020}, \quad M_{11}^{(a)} = \varepsilon^2 I^{(a)}_{0000220}.$$

- Adopt the basis in [von Manteuffel and Tancredi, '17]
- Alphabet: $\left\{y, 1+4y, 1-4y, \sqrt{1-4y}, \sqrt{1+4y}\right\}$



(b)

$$M_4^{(b)} = \varepsilon^4 x\, I^{(b)}_{1111100}, \quad M_5^{(b)} = \varepsilon^3 x^2\, I^{(b)}_{1120110}, \quad M_6^{(b)} = \varepsilon^4 x\, I^{(b)}_{1110110}, \quad M_7^{(b)} = \varepsilon^3 x\, I^{(b)}_{1102110}, \quad M_8^{(b)} = \varepsilon^4 x\, I^{(b)}_{1101110},$$

$$M_9^{(b)} = \varepsilon^2\sqrt{x(4+x)}\left(-x\, I^{(b)}_{1122000} + \frac{\varepsilon}{2(1+2\varepsilon)} I^{(b)}_{0202000}\right), \quad M_{10}^{(b)} = \varepsilon^3 x\, I^{(b)}_{1112000},$$

$$M_{11}^{(b)} = \varepsilon^2\frac{x}{2}\left((1+x) I^{(b)}_{0211200} - (1+x) I^{(b)}_{0121200} - 2 I^{(b)}_{0120200} + \frac{2-x}{x} I^{(b)}_{0202000}\right),$$

$$M_{12}^{(b)} = \varepsilon^2\frac{x}{4}\left((3+x) I^{(b)}_{0121200} + (1-x) I^{(b)}_{0211200} + 2 I^{(b)}_{0220100} + 4 I^{(b)}_{0120200} + I^{(b)}_{0202000}\right), \quad M_{13}^{(b)} = \varepsilon^3 x\, I^{(b)}_{1010120}, \quad M_{14}^{(b)} = \varepsilon^2 x\, I^{(b)}_{1220000},$$

$$M_{15}^{(b)} = \frac{\varepsilon(\varepsilon-1)}{2} I^{(b)}_{1200200}, \quad M_{16}^{(b)} = -\varepsilon^2\frac{x}{2} I^{(b)}_{0220100}, \quad M_{17}^{(b)} = -\varepsilon^2\frac{1-x}{4}\left(I^{(b)}_{0120200} + 2 I^{(b)}_{0220100}\right), \quad M_{18}^{(b)} = \varepsilon^2 I^{(b)}_{0202000},$$

- Via the method in [Chen, Jiang, Xu, Yang, '20]
- Alphabet: $\left\{y, y-1, y+1, 4y+1, \sqrt{1-4y}\right\}$

Canonical basis for sub-sectors

$$\frac{d}{dy}\begin{pmatrix} I_1^{(a)} \\ I_2^{(a)} \end{pmatrix} = A_{\text{top}}^{(a)}(y,\varepsilon)\begin{pmatrix} I_1^{(a)} \\ I_2^{(a)} \end{pmatrix} + \varepsilon^2 A_{\text{mix}}^{(a)}(y)\begin{pmatrix} M_3^{(a)} \\ \vdots \\ M_{11}^{(a)} \end{pmatrix}$$

$$\frac{d}{dy}\begin{pmatrix} M_3^{(a)} \\ \vdots \\ M_{11}^{(a)} \end{pmatrix} = \varepsilon A_{\text{sub}}^{(a)}(y)\begin{pmatrix} M_3^{(a)} \\ \vdots \\ M_{11}^{(a)} \end{pmatrix}$$

$$\frac{d}{dy}\begin{pmatrix} I_1^{(b)} \\ I_2^{(b)} \\ I_3^{(b)} \end{pmatrix} = A_{\text{top}}^{(b)}(y,\varepsilon)\begin{pmatrix} I_1^{(b)} \\ I_2^{(b)} \\ I_3^{(b)} \end{pmatrix} + \varepsilon^2 A_{\text{mix}}^{(b)}(y)\begin{pmatrix} M_3^{(b)} \\ \vdots \\ M_{18}^{(b)} \end{pmatrix}$$

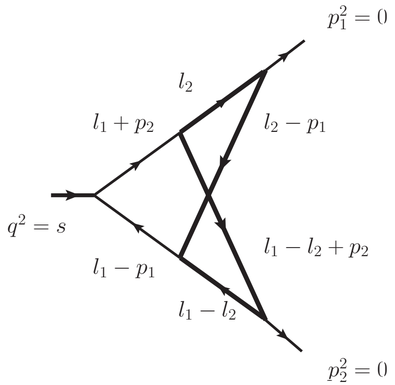
$$\frac{d}{dy}\begin{pmatrix} M_3^{(b)} \\ \vdots \\ M_{18}^{(b)} \end{pmatrix} = \varepsilon A_{\text{sub}}^{(b)}(y)\begin{pmatrix} M_3^{(b)} \\ \vdots \\ M_{18}^{(b)} \end{pmatrix}$$

The next step is to deal with top sectors, which are elliptic.

- **Derivative Ansatz** based on Picard-Fuchs operators, inspired by Sunrise/Banana integrals

  [Weinzierl et al, '15-'17; Pögel, XW, Weinzierl, '22-'23].

Elliptic curves in top sectors



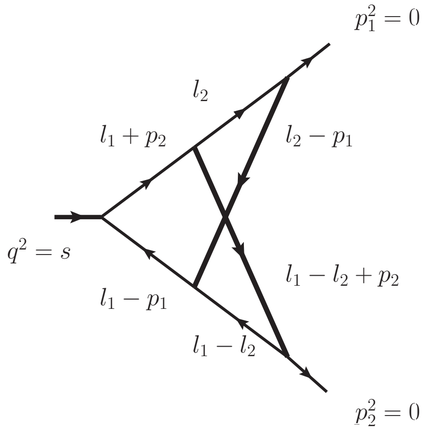
$(a) : \kappa = 1$

$$\mathrm{MaxCut}\left[I^{(a)}_{1111110}\right]$$

$$E: \; v^2 = (u-u_1)\,(u-u_2)\,(u-u_3)\,(u-u_4),$$

$$u_1 = 0\,, \quad u_2 = \frac{1-\sqrt{1-16y}}{2}\,, \quad u_3 = \frac{1+\sqrt{1-16y}}{2}\,, \quad u_4 = 1,$$

$$k^2 = \frac{(u_2-u_1)(u_4-u_3)}{(u_3-u_1)(u_4-u_2)} = \left(\frac{1-\sqrt{1-16y}}{1+\sqrt{1-16y}}\right)^2$$



$(b) : \kappa = 0$

$$\mathrm{MaxCut}\left[I^{(b)}_{1111110}\right]$$

$$E: \; v^2 = (u-u_1)\,(u-u_2)\,(u-u_3)\,(u-u_4),$$

$$u_1 = -1\,, \quad u_2 = -\frac{\left(\sqrt{1-8y}+1\right)^2}{4}\,, \quad u_3 = -\frac{\left(\sqrt{1-8y}-1\right)^2}{4}\,, \quad u_4 = 0,$$

$$k^2 = \frac{(u_2-u_1)(u_4-u_3)}{(u_3-u_1)(u_4-u_2)} = \frac{1-4y-8y^2-\sqrt{1-8y}}{1-4y-8y^2+\sqrt{1-8y}}\,.$$

Picard-Fuchs operators

$$\hat{L}_2^{(a)}(y,\varepsilon)\, I^{(a)}_{1111110} \equiv \sum_{n=0}^{2} r_n^{(a)}(y,\epsilon)\left(\frac{d}{dy}\right)^n I^{(a)}_{1111110} = \varepsilon^2\, \vec{f}^{(a)}_{\text{sub},\,2}(y)\cdot \vec{M}^{(a)}_{\text{sub}}(y),$$

$$\hat{L}_3^{(b)}(y,\varepsilon)\, I^{(b)}_{1111110} \equiv \sum_{n=0}^{3} r_n^{(b)}(y,\epsilon)\left(\frac{d}{dy}\right)^n I^{(b)}_{1111110} = \varepsilon^2\, \vec{f}^{(b)}_{\text{sub},\,2}(y)\cdot \vec{M}^{(b)}_{\text{sub}}(y) + \varepsilon^3\, \vec{f}^{(b)}_{\text{sub},\,3}(y)\cdot \vec{M}^{(b)}_{\text{sub}}(y)\,.$$

Picard-Fuchs operators

$$\hat{L}_2^{(a)}(y,\varepsilon)\, I^{(a)}_{1111110} \equiv \sum_{n=0}^{2} r_n^{(a)}(y,\epsilon)\left(\frac{d}{dy}\right)^n I^{(a)}_{1111110} = \varepsilon^2\, \vec{f}^{(a)}_{\text{sub},\,2}(y)\cdot \vec{M}^{(a)}_{\text{sub}}(y),$$

$$\hat{L}_3^{(b)}(y,\varepsilon)\, I^{(b)}_{1111110} \equiv \sum_{n=0}^{3} r_n^{(b)}(y,\epsilon)\left(\frac{d}{dy}\right)^n I^{(b)}_{1111110} = \varepsilon^2\, \vec{f}^{(b)}_{\text{sub},\,2}(y)\cdot \vec{M}^{(b)}_{\text{sub}}(y) + \varepsilon^3\, \vec{f}^{(b)}_{\text{sub},\,3}(y)\cdot \vec{M}^{(b)}_{\text{sub}}(y)\,.$$

- Maximal cut erases inhomogeneous terms;

Picard-Fuchs operators

$$\hat{L}_2^{(a)}(y,\varepsilon)\, I^{(a)}_{1111110} \equiv \sum_{n=0}^{2} r_n^{(a)}(y,\epsilon)\left(\frac{d}{dy}\right)^n I^{(a)}_{1111110} = \varepsilon^2\, \vec{f}^{(a)}_{\text{sub},\,2}(y)\cdot \vec{M}^{(a)}_{\text{sub}}(y),$$

$$\hat{L}_3^{(b)}(y,\varepsilon)\, I^{(b)}_{1111110} \equiv \sum_{n=0}^{3} r_n^{(b)}(y,\epsilon)\left(\frac{d}{dy}\right)^n I^{(b)}_{1111110} = \varepsilon^2\, \vec{f}^{(b)}_{\text{sub},\,2}(y)\cdot \vec{M}^{(b)}_{\text{sub}}(y) + \varepsilon^3\, \vec{f}^{(b)}_{\text{sub},\,3}(y)\cdot \vec{M}^{(b)}_{\text{sub}}(y)\,.$$

- Maximal cut erases inhomogeneous terms;
- $\hat{L}_2^{(a)}(0,y)$ and $\hat{L}_3^{(b)}(0,y)$ encode geometric information related to the two elliptic curves;

Picard-Fuchs operators

$$\hat{L}_2^{(a)}(y,\varepsilon)\, I^{(a)}_{1111110} \equiv \sum_{n=0}^{2} r_n^{(a)}(y,\epsilon)\left(\frac{d}{dy}\right)^n I^{(a)}_{1111110} = \varepsilon^2\, \vec{f}^{(a)}_{\text{sub},\,2}(y)\cdot \vec{M}^{(a)}_{\text{sub}}(y),$$

$$\hat{L}_3^{(b)}(y,\varepsilon)\, I^{(b)}_{1111110} \equiv \sum_{n=0}^{3} r_n^{(b)}(y,\epsilon)\left(\frac{d}{dy}\right)^n I^{(b)}_{1111110} = \varepsilon^2\, \vec{f}^{(b)}_{\text{sub},\,2}(y)\cdot \vec{M}^{(b)}_{\text{sub}}(y) + \varepsilon^3\, \vec{f}^{(b)}_{\text{sub},\,3}(y)\cdot \vec{M}^{(b)}_{\text{sub}}(y)\,.$$

- Maximal cut erases inhomogeneous terms;
- $\hat{L}_2^{(a)}(0,y)$ and $\hat{L}_3^{(b)}(0,y)$ encode geometric information related to the two elliptic curves;

  ☑ Solutions: periods, which are d.o.f's.

Picard-Fuchs operators

$$\hat{L}_2^{(a)}(y,\varepsilon)\, I^{(a)}_{1111110} \equiv \sum_{n=0}^{2} r_n^{(a)}(y,\epsilon)\left(\frac{d}{dy}\right)^n I^{(a)}_{1111110} = \varepsilon^2\, \vec{f}^{(a)}_{\text{sub},\,2}(y)\cdot \vec{M}^{(a)}_{\text{sub}}(y),$$

$$\hat{L}_3^{(b)}(y,\varepsilon)\, I^{(b)}_{1111110} \equiv \sum_{n=0}^{3} r_n^{(b)}(y,\epsilon)\left(\frac{d}{dy}\right)^n I^{(b)}_{1111110} = \varepsilon^2\, \vec{f}^{(b)}_{\text{sub},\,2}(y)\cdot \vec{M}^{(b)}_{\text{sub}}(y) + \varepsilon^3\, \vec{f}^{(b)}_{\text{sub},\,3}(y)\cdot \vec{M}^{(b)}_{\text{sub}}(y)\,.$$

- Maximal cut erases inhomogeneous terms;
- $\hat{L}_2^{(a)}(0,y)$ and $\hat{L}_3^{(b)}(0,y)$ encode geometric information related to the two elliptic curves;
  - ☑ Solutions: periods, which are d.o.f's.
  - ☑ 2 periods v.s. 3 solutions of $\hat{L}_3^{(b)}(0,y)$? ⇜⇝ no contradiction since $\hat{L}_3^{(b)}(0,y)$ factorises!

Picard-Fuchs operators

$$\hat{L}_2^{(a)}(y,\varepsilon)\, I^{(a)}_{1111110} \equiv \sum_{n=0}^{2} r_n^{(a)}(y,\epsilon)\left(\frac{d}{dy}\right)^n I^{(a)}_{1111110} = \varepsilon^2\, \vec{f}^{(a)}_{\text{sub},\,2}(y)\cdot \vec{M}^{(a)}_{\text{sub}}(y),$$

$$\hat{L}_3^{(b)}(y,\varepsilon)\, I^{(b)}_{1111110} \equiv \sum_{n=0}^{3} r_n^{(b)}(y,\epsilon)\left(\frac{d}{dy}\right)^n I^{(b)}_{1111110} = \varepsilon^2\, \vec{f}^{(b)}_{\text{sub},\,2}(y)\cdot \vec{M}^{(b)}_{\text{sub}}(y) + \varepsilon^3\, \vec{f}^{(b)}_{\text{sub},\,3}(y)\cdot \vec{M}^{(b)}_{\text{sub}}(y)\,.$$

- Maximal cut erases inhomogeneous terms;
- $\hat{L}_2^{(a)}(0,y)$ and $\hat{L}_3^{(b)}(0,y)$ encode geometric information related to the two elliptic curves;
  - ☑ Solutions: periods, which are d.o.f's.
  - ☑ 2 periods v.s. 3 solutions of $\hat{L}_3^{(b)}(0,y)$? ⇜⇝ no contradiction since $\hat{L}_3^{(b)}(0,y)$ factorises!
- Periods (solutions of PF) as d.o.f's to construct the derivative basis in top sectors;

Picard-Fuchs operators

$$\hat{L}_2^{(a)}(y,\varepsilon)\, I^{(a)}_{1111110} \equiv \sum_{n=0}^{2} r_n^{(a)}(y,\epsilon)\left(\frac{d}{dy}\right)^n I^{(a)}_{1111110} = \varepsilon^2\, \vec{f}^{(a)}_{\text{sub},\,2}(y)\cdot \vec{M}^{(a)}_{\text{sub}}(y),$$

$$\hat{L}_3^{(b)}(y,\varepsilon)\, I^{(b)}_{1111110} \equiv \sum_{n=0}^{3} r_n^{(b)}(y,\epsilon)\left(\frac{d}{dy}\right)^n I^{(b)}_{1111110} = \varepsilon^2\, \vec{f}^{(b)}_{\text{sub},\,2}(y)\cdot \vec{M}^{(b)}_{\text{sub}}(y) + \varepsilon^3\, \vec{f}^{(b)}_{\text{sub},\,3}(y)\cdot \vec{M}^{(b)}_{\text{sub}}(y)\,.$$

- Maximal cut erases inhomogeneous terms;
- $\hat{L}_2^{(a)}(0,y)$ and $\hat{L}_3^{(b)}(0,y)$ encode geometric information related to the two elliptic curves;
  - ☑ Solutions: periods, which are d.o.f's.
  - ☑ 2 periods v.s. 3 solutions of $\hat{L}_3^{(b)}(0,y)$? ⇜⇝ no contradiction since $\hat{L}_3^{(b)}(0,y)$ factorises!
- Periods (solutions of PF) as d.o.f's to construct the derivative basis in top sectors;
- Undo maximal cut to include mixing with sub sectors: non-trivial in general!

Frobenius method

$$\hat{L}_3^{(b)}(y,0) = \frac{d^3}{dy^3} + \frac{3(1-2y)}{y(y+1)(8y-1)}\frac{d^2}{dy^2} + \frac{8y^3+30y^2+6y-7}{y^2(y+1)^2(8y-1)}\frac{d}{dy} - \frac{2\left(4y^3+16y^2+2y-4\right)}{y^3(y+1)^2(8y-1)}$$

$$= \underbrace{\left[\frac{d}{dy} + \frac{8}{8y-1}\right]}_{\hat{L}_1^{(b)}(y)} \underbrace{\left[\frac{d^2}{dy^2} + \left(\frac{1}{y+1} + \frac{8}{8y-1} - \frac{3}{y}\right)\frac{d}{dy} + \frac{8y(y+2)-4}{y^2(y+1)(8y-1)}\right]}_{\hat{L}_2^{(b)}(y)}$$

Frobenius method

$$\hat{L}_3^{(b)}(y,0) = \frac{d^3}{dy^3} + \frac{3(1-2y)}{y(y+1)(8y-1)}\frac{d^2}{dy^2} + \frac{8y^3+30y^2+6y-7}{y^2(y+1)^2(8y-1)}\frac{d}{dy} - \frac{2\left(4y^3+16y^2+2y-4\right)}{y^3(y+1)^2(8y-1)}$$

$$= \underbrace{\left[\frac{d}{dy} + \frac{8}{8y-1}\right]}_{\hat{L}_1^{(b)}(y)} \underbrace{\left[\frac{d^2}{dy^2} + \left(\frac{1}{y+1} + \frac{8}{8y-1} - \frac{3}{y}\right)\frac{d}{dy} + \frac{8y(y+2)-4}{y^2(y+1)(8y-1)}\right]}_{\hat{L}_2^{(b)}(y)}$$

- It has four regular singularities: $\{-1, 0, 1/8, \infty\}$, and $y = 0$ is the so-called maximal unipotent monodromy point, where there exists the Frobenius basis with maximal logarithmic hierarchy:

Frobenius method

$$\hat{L}_3^{(b)}(y,0) = \frac{d^3}{dy^3} + \frac{3(1-2y)}{y(y+1)(8y-1)}\frac{d^2}{dy^2} + \frac{8y^3+30y^2+6y-7}{y^2(y+1)^2(8y-1)}\frac{d}{dy} - \frac{2\left(4y^3+16y^2+2y-4\right)}{y^3(y+1)^2(8y-1)}$$

$$= \underbrace{\left[\frac{d}{dy} + \frac{8}{8y-1}\right]}_{\hat{L}_1^{(b)}(y)} \underbrace{\left[\frac{d^2}{dy^2} + \left(\frac{1}{y+1} + \frac{8}{8y-1} - \frac{3}{y}\right)\frac{d}{dy} + \frac{8y(y+2)-4}{y^2(y+1)(8y-1)}\right]}_{\hat{L}_2^{(b)}(y)}$$

- It has four regular singularities: $\{-1, 0, 1/8, \infty\}$, and $y = 0$ is the so-called maximal unipotent monodromy point, where there exists the Frobenius basis with maximal logarithmic hierarchy:

$$\psi_k(y) = \frac{1}{(2\pi i)^k}\sum_{j=0}^{k}\frac{\ln^j y}{j!}\sum_{n=0}^{\infty} a_{k-j,n}\, y^{n+2}, \quad (k = 0,1,2)$$

Frobenius method

$$\hat{L}_3^{(b)}(y,0) = \frac{d^3}{dy^3} + \frac{3(1-2y)}{y(y+1)(8y-1)}\frac{d^2}{dy^2} + \frac{8y^3+30y^2+6y-7}{y^2(y+1)^2(8y-1)}\frac{d}{dy} - \frac{2\left(4y^3+16y^2+2y-4\right)}{y^3(y+1)^2(8y-1)}$$

$$= \underbrace{\left[\frac{d}{dy} + \frac{8}{8y-1}\right]}_{\hat{L}_1^{(b)}(y)} \underbrace{\left[\frac{d^2}{dy^2} + \left(\frac{1}{y+1} + \frac{8}{8y-1} - \frac{3}{y}\right)\frac{d}{dy} + \frac{8y(y+2)-4}{y^2(y+1)(8y-1)}\right]}_{\hat{L}_2^{(b)}(y)}$$

- It has four regular singularities: $\{-1, 0, 1/8, \infty\}$, and $y = 0$ is the so-called maximal unipotent monodromy point, where there exists the Frobenius basis with maximal logarithmic hierarchy:

$$\psi_k(y) = \frac{1}{(2\pi i)^k}\sum_{j=0}^{k}\frac{\ln^j y}{j!}\sum_{n=0}^{\infty} a_{k-j,n}\, y^{n+2}, \quad (k = 0,1,2)$$

- $\hat{L}_2^{(b)}(y)\,\psi_0 = \hat{L}_2^{(b)}(y)\,\psi_1 = 0$ and $\hat{L}_2^{(b)}(y)\psi_2 = \dfrac{1}{8y-1}$.

Modular map

Modular map

- It is natural to trade (ψ_0, ψ_1) to (ψ_0, τ), w/

$$\tau \equiv \frac{\psi_1}{\psi_0}, \quad q \equiv \exp\left[2\pi i \tau\right]$$

Modular map

- It is natural to trade (ψ_0, ψ_1) to (ψ_0, τ), w/

$$\tau \equiv \frac{\psi_1}{\psi_0}, \quad q \equiv \exp\left[2\pi i \tau\right]$$

- y and q are nicely related under this modular map:

$$y(q) = q - 3q^2 + 3q^3 + 5q^4 - 18q^5 + 15q^6 + 24q^7 + \mathcal{O}(q^8)\,,$$

OEIS:A123633

$$\leftarrow y(q) = \frac{\eta(\tau)^3\,\eta(6\tau)^9}{\eta(2\tau)^3\,\eta(3\tau)^9}\,.$$

$$q(y) = y + 3y^2 + 15y^3 + 85y^4 + 522y^5 + 3366y^6 + 22450y^7 + \mathcal{O}(y^8)\,.$$

Modular map

- It is natural to trade (ψ_0, ψ_1) to (ψ_0, τ), w/

$$\tau \equiv \frac{\psi_1}{\psi_0}, \quad q \equiv \exp\left[2\pi i \tau\right]$$

- y and q are nicely related under this modular map:

$$y(q) = q - 3q^2 + 3q^3 + 5q^4 - 18q^5 + 15q^6 + 24q^7 + \mathcal{O}(q^8)\,,$$

OEIS:A123633

$$\leftarrow y(q) = \frac{\eta(\tau)^3\,\eta(6\tau)^9}{\eta(2\tau)^3\,\eta(3\tau)^9}\,.$$

$$q(y) = y + 3y^2 + 15y^3 + 85y^4 + 522y^5 + 3366y^6 + 22450y^7 + \mathcal{O}(y^8)\,.$$

- The Jacobian from y to τ or equivalently q is related to the Wronskian:

$$J(y) \equiv \frac{1}{2\pi i}\frac{dy}{d\tau} = \frac{\psi_0^2(y)}{W(y)}, \text{ w/ } W(y) \equiv 2\pi i\left[\psi_0 \frac{d}{dy}\psi_1 - \psi_1 \frac{d}{dy}\psi_0\right] = \frac{y^3}{(1-8y)(1+y)}.$$

Factorization

Factorization

- The irreducible PF operator factorises furthermore in terms of modular variable:

$$\hat{L}_2^{(b)}(y) = \frac{\psi_0}{J^2}\Theta_q^2\frac{1}{\psi_0}, \quad \text{w/} \quad \Theta_q = q\frac{d}{dq} = \frac{1}{2\pi i}\frac{d}{d\tau}$$

Factorization

- The irreducible PF operator factorises furthermore in terms of modular variable:

$$\hat{L}_2^{(b)}(y) = \frac{\psi_0}{J^2}\Theta_q^2\frac{1}{\psi_0}, \quad \text{w/} \quad \Theta_q = q\frac{d}{dq} = \frac{1}{2\pi i}\frac{d}{d\tau}$$

- $\hat{L}_3^{(b)}(y,0)$ is reducible (factorises in y space), such that ψ_2 can be represented by (ψ_0, τ):

$$\psi_2(\tau) = \psi_0(\tau)\int_{i\infty}^{\tau} d\tau_2 \int_{i\infty}^{\tau_2} d\tau_1 \frac{(1-8y_1(\tau_1))(1+y_1(\tau_1))^2}{y_1(\tau_1)^6}\psi_0(\tau_1)^3$$

Factorization

- The irreducible PF operator factorises furthermore in terms of modular variable:

$$\hat{L}_2^{(b)}(y) = \frac{\psi_0}{J^2}\Theta_q^2\frac{1}{\psi_0}, \quad \text{w/} \quad \Theta_q = q\frac{d}{dq} = \frac{1}{2\pi i}\frac{d}{d\tau}$$

- $\hat{L}_3^{(b)}(y,0)$ is reducible (factorises in y space), such that ψ_2 can be represented by (ψ_0, τ):

$$\psi_2(\tau) = \psi_0(\tau)\int_{i\infty}^{\tau} d\tau_2 \int_{i\infty}^{\tau_2} d\tau_1 \frac{(1-8y_1(\tau_1))(1+y_1(\tau_1))^2}{y_1(\tau_1)^6}\psi_0(\tau_1)^3$$

- Nevertheless, we can use it to construct a Y-invariant (Stefan's talk):

$$Y(\tau) \equiv \frac{\alpha_1}{\alpha_2} = \frac{d^2}{d\tau^2}\frac{\psi_2}{\psi_0} = \frac{\psi_0(y)^3}{(1-8y)W(y)^2}$$

Factorization

- The irreducible PF operator factorises furthermore in terms of modular variable:

$$\hat{L}_2^{(b)}(y) = \frac{\psi_0}{J^2}\Theta_q^2\frac{1}{\psi_0}, \quad \text{w/} \quad \Theta_q = q\frac{d}{dq} = \frac{1}{2\pi i}\frac{d}{d\tau}$$

- $\hat{L}_3^{(b)}(y,0)$ is reducible (factorises in y space), such that ψ_2 can be represented by (ψ_0, τ):

$$\psi_2(\tau) = \psi_0(\tau)\int_{i\infty}^{\tau} d\tau_2 \int_{i\infty}^{\tau_2} d\tau_1 \frac{(1-8y_1(\tau_1))(1+y_1(\tau_1))^2}{y_1(\tau_1)^6}\psi_0(\tau_1)^3$$

- Nevertheless, we can use it to construct a Y-invariant (Stefan's talk):

$$Y(\tau) \equiv \frac{\alpha_1}{\alpha_2} = \frac{d^2}{d\tau^2}\frac{\psi_2}{\psi_0} = \frac{\psi_0(y)^3}{(1-8y)W(y)^2}$$

Then the reducible $\hat{L}_3^{(b)}(y,0)$ has the following pattern:

$$L_3^{(0)}(y) = \frac{\psi_0 Y}{J^3}\Theta_q\frac{1}{Y}\Theta_q^2\frac{1}{\psi_0}$$

Factorization

- The irreducible PF operator factorises furthermore in terms of modular variable:

$$\hat{L}_2^{(b)}(y) = \frac{\psi_0}{J^2}\Theta_q^2\frac{1}{\psi_0}, \quad \text{w/} \quad \Theta_q = q\frac{d}{dq} = \frac{1}{2\pi i}\frac{d}{d\tau}$$

- $\hat{L}_3^{(b)}(y,0)$ is reducible (factorises in y space), such that ψ_2 can be represented by (ψ_0, τ):

$$\psi_2(\tau) = \psi_0(\tau)\int_{i\infty}^{\tau} d\tau_2 \int_{i\infty}^{\tau_2} d\tau_1 \frac{(1-8y_1(\tau_1))(1+y_1(\tau_1))^2}{y_1(\tau_1)^6}\psi_0(\tau_1)^3$$

- Nevertheless, we can use it to construct a Y-invariant (Stefan's talk):

$$Y(\tau) \equiv \frac{\alpha_1}{\alpha_2} = \frac{d^2}{d\tau^2}\frac{\psi_2}{\psi_0} = \frac{\psi_0(y)^3}{(1-8y)W(y)^2}$$

Then the reducible $\hat{L}_3^{(b)}(y,0)$ has the following pattern:

$$\boxed{L_3^{(0)}(y) = \frac{\psi_0 Y}{J^3}\Theta_q\frac{1}{Y}\Theta_q^2\frac{1}{\psi_0}}$$

→ **indicates ansatz in top sector!**

Ansatz

Ansatz

$$L_3^{(b)}(y,0) \propto \Theta_q \frac{1}{Y} \Theta_q^2 \frac{1}{\psi_0} = J \frac{d}{dy} \frac{1}{Y} J \frac{d}{dy} J \frac{d}{dy} \frac{1}{\psi_0}$$

Ansatz

$$L_3^{(b)}(y,0) \propto \Theta_q \frac{1}{Y} \Theta_q^2 \frac{1}{\psi_0} = J \frac{d}{dy} \frac{1}{Y} J \frac{d}{dy} J \frac{d}{dy} \frac{1}{\psi_0}$$

$$M_1 = \varepsilon^4 \frac{I^{(b)}_{1111110}}{\psi_0},$$

$$M_2 = \frac{J(y)}{\varepsilon} \frac{d}{dy} M_1 - F_{11} M_1,$$

$$M_3 = \frac{1}{Y(y)} \left[\frac{J(y)}{\varepsilon} \frac{d}{dy} M_2 - F_{21} M_1 - F_{22} M_2 \right] + \text{sub-sector integrals}.$$

Ansatz

$$L_3^{(b)}(y,0) \propto \Theta_q \frac{1}{Y} \Theta_q^2 \frac{1}{\psi_0} = J \frac{d}{dy} \frac{1}{Y} J \frac{d}{dy} J \frac{d}{dy} \frac{1}{\psi_0}$$

$$M_1 = \varepsilon^4 \frac{I^{(b)}_{1111110}}{\psi_0},$$

$$M_2 = \frac{J(y)}{\varepsilon} \frac{d}{dy} M_1 - F_{11} M_1,$$

$$M_3 = \frac{1}{Y(y)} \left[\frac{J(y)}{\varepsilon} \frac{d}{dy} M_2 - F_{21} M_1 - F_{22} M_2 \right] + \text{sub-sector integrals}.$$

Rotation coeffs F_{ij} are determined by requiring this ansatz is ε-form under maximal cut:

$$F_{22}(y) = F_{11}(y) = \frac{28y^2 + 2y + 1}{3y^4} \psi_0(y)^2,$$

$$F_{21}(y) = \frac{(2y-1)\left(88y^3 + 84y^2 + 66y - 11\right)}{3y^8} \psi_0(y)^4.$$

Ansatz

$$J(y)\frac{d}{dy}\begin{pmatrix} M_1 \\ M_2 \\ M_3 \end{pmatrix} = \varepsilon \begin{pmatrix} \eta_{1,2} & 1 & 0 \\ \eta_4 & \eta_{1,2} & \eta_{1,3} \\ -\frac{64}{27}\eta_{1,3} & 0 & \eta_{2,2} \end{pmatrix} \begin{pmatrix} M_1 \\ M_2 \\ M_3 \end{pmatrix} + \text{subs}$$

$$\eta_{1,2} = \frac{28y^2 + 2y + 1}{3y^4}\psi_0(y)^2,$$

$$\eta_{2,2} = \frac{2(y+1)(8y-1)}{3y^4}\psi_0(y)^2,$$

$$\eta_{1,3} = \frac{(y+1)^2(8y-1)\psi_0(y)^3}{y^6},$$

$$\eta_4 = \frac{(2y-1)\left(88y^3 + 84y^2 + 66y - 11\right)\psi_0(y)^4}{3y^8}.$$

Ansatz

$$J(y)\frac{d}{dy}\begin{pmatrix} M_1 \\ M_2 \\ M_3 \end{pmatrix} = \varepsilon \begin{pmatrix} \eta_{1,2} & 1 & 0 \\ \eta_4 & \eta_{1,2} & \eta_{1,3} \\ -\frac{64}{27}\eta_{1,3} & 0 & \eta_{2,2} \end{pmatrix} \begin{pmatrix} M_1 \\ M_2 \\ M_3 \end{pmatrix} + \text{subs}$$

$$\begin{aligned}
\eta_{1,2} &= 28b_{1,1}^2 + 2b_{1,1}b_{2,1} + b_{2,1}^2\,, \\
\eta_{2,2} &= 16b_{1,1}^2 + 14b_{1,1}b_{2,1} - 2b_{2,1}^2\,, \\
\eta_{1,3} &= -3\sqrt{3}\left[8b_{1,1}^3 + 15b_{1,1}^2b_{2,1} + 6b_{1,1}b_{2,1}^2 - b_{2,1}^3\right], \\
\eta_4 &= 3\left[176b_{1,1}^4 + 80b_{1,1}^3b_{2,1} + 48b_{1,1}^2b_{2,1}^2 - 88b_{1,1}b_{2,1}^3 + 11b_{2,1}^4\right]
\end{aligned}$$

$$b_{1,1} = \frac{1}{\sqrt{3}}\frac{\psi_0}{y}, \quad b_{2,1} = \frac{1}{\sqrt{3}}\frac{\psi_0}{y^2}$$

Analytic continuation of modular map

Analytic continuation of modular map

- Modular map defined via the Frobenius basis only works in the vicinity of $y = 0$. We want to analytically continue it, such that

Analytic continuation of modular map

- Modular map defined via the Frobenius basis only works in the vicinity of $y = 0$. We want to analytically continue it, such that

$$y \in \mathbb{R} + i0 \xrightarrow{y(\tau)} \tau \in \mathbb{H} \cup \{\text{cusps}\} \xrightarrow{\tau(y)} y \in \mathbb{R} + i0.$$

Analytic continuation of modular map

- Modular map defined via the Frobenius basis only works in the vicinity of $y = 0$. We want to analytically continue it, such that

$$y \in \mathbb{R} + i0 \xrightarrow{y(\tau)} \tau \in \mathbb{H} \cup \{\text{cusps}\} \xrightarrow{\tau(y)} y \in \mathbb{R} + i0.$$

- We turn to the elliptic curve, and represent the periods with complete elliptic fns: $\left(K(k^2), iK(1-k^2)\right)$

Analytic continuation of modular map

- Modular map defined via the Frobenius basis only works in the vicinity of $y = 0$. We want to analytically continue it, such that

$$y \in \mathbb{R} + i0 \xrightarrow{y(\tau)} \tau \in \mathbb{H} \cup \{\text{cusps}\} \xrightarrow{\tau(y)} y \in \mathbb{R} + i0.$$

- We turn to the elliptic curve, and represent the periods with complete elliptic fns: $\left(K(k^2), iK(1-k^2)\right)$

- Complete elliptic function $K(k^2)$ has discontinuities when k^2 crossing the branch cut: compensate the discontinuities by rotation:

Analytic continuation of modular map

- Modular map defined via the Frobenius basis only works in the vicinity of $y = 0$. We want to analytically continue it, such that

$$y \in \mathbb{R} + i0 \xrightarrow{y(\tau)} \tau \in \mathbb{H} \cup \{\text{cusps}\} \xrightarrow{\tau(y)} y \in \mathbb{R} + i0.$$

- We turn to the elliptic curve, and represent the periods with complete elliptic fns: $\left(K(k^2), iK(1-k^2)\right)$

- Complete elliptic function $K(k^2)$ has discontinuities when k^2 crossing the branch cut: compensate the discontinuities by rotation:

$$\begin{pmatrix} \bar{\psi}_1(y) \\ \bar{\psi}_0(y) \end{pmatrix} = \frac{2}{\pi} \frac{y^2}{\sqrt{(u_3-u_1)(u_4-u_2)}} \gamma(y) \begin{pmatrix} i\,K(1-k^2) \\ K(k^2) \end{pmatrix}, \quad \gamma(y) = \begin{cases} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, & y < 0 \text{ or } y \geq \dfrac{\sqrt{3}-1}{4}, \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & 0 \leq y < \dfrac{\sqrt{3}-1}{4}. \end{cases}$$

Analytic continuation of modular map

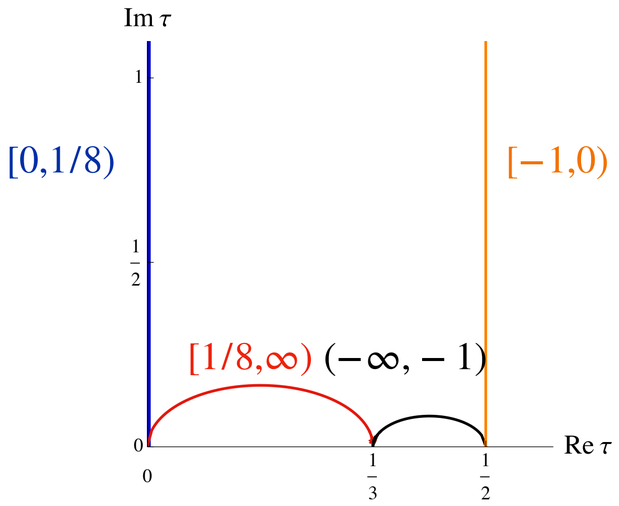
Analytic continuation of modular map

Series expand around $y = 0$, and we can identify: $\psi_0 = \bar{\psi}_0$, $\psi_1 = \bar{\psi}_1/6$;

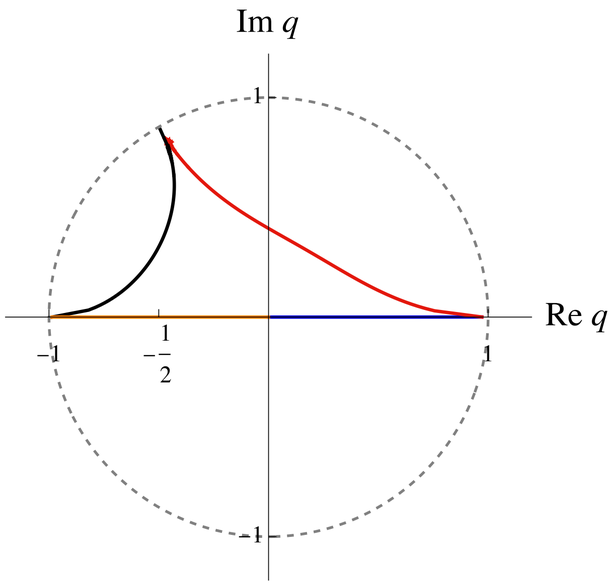
$$\tau(y) = \frac{\bar{\psi}_1(y)}{6\bar{\psi}_0(y)}$$

Analytic continuation of modular map

Series expand around $y = 0$, and we can identify: $\psi_0 = \bar{\psi}_0$, $\psi_1 = \bar{\psi}_1/6$;

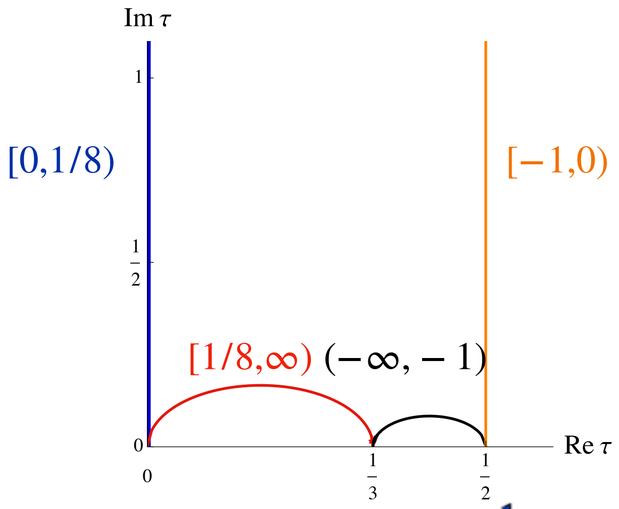


$$\tau(y) = \frac{\bar{\psi}_1(y)}{6\bar{\psi}_0(y)}$$

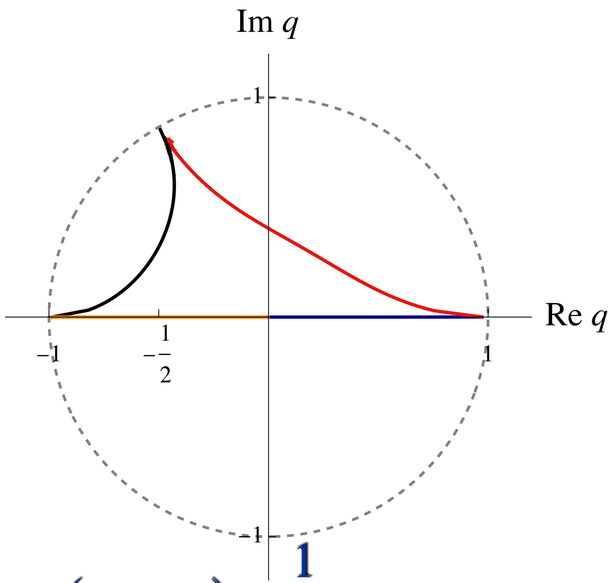


Analytic continuation of modular map

Series expand around $y = 0$, and we can identify: $\psi_0 = \bar{\psi}_0$, $\psi_1 = \bar{\psi}_1/6$;



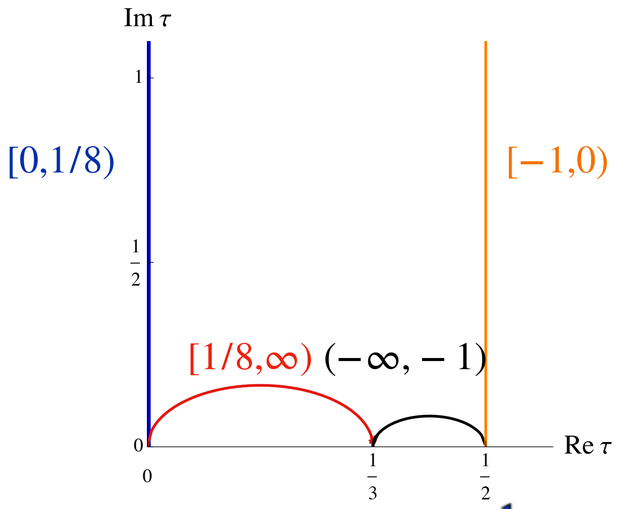
$$\tau(y) = \frac{\bar{\psi}_1(y)}{6\bar{\psi}_0(y)}$$



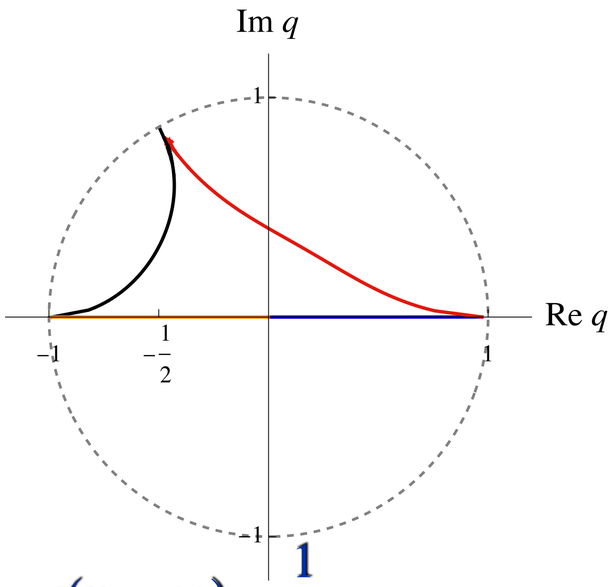
$$\tau(y = -1) = \frac{1}{2}, \quad \tau(y = 0) = i\infty, \quad \tau(y = 1/8) = 0, \quad \tau(y = \infty) = \frac{1}{3}$$

Analytic continuation of modular map

Series expand around $y = 0$, and we can identify: $\psi_0 = \bar{\psi}_0$, $\psi_1 = \bar{\psi}_1/6$;



$$\tau(y) = \frac{\bar{\psi}_1(y)}{6\bar{\psi}_0(y)}$$

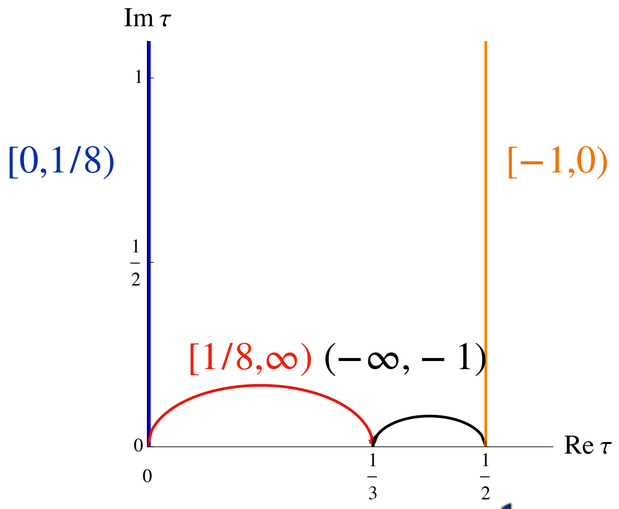


$$\tau(y = -1) = \frac{1}{2}, \quad \tau(y = 0) = i\infty, \quad \tau(y = 1/8) = 0, \quad \tau(y = \infty) = \frac{1}{3}$$

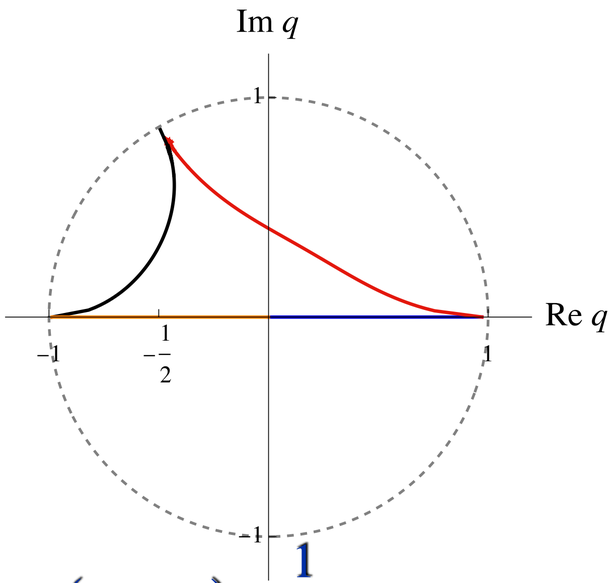
↪ cusps of $\Gamma_1(6)$

Analytic continuation of modular map

Series expand around $y = 0$, and we can identify: $\psi_0 = \bar{\psi}_0$, $\psi_1 = \bar{\psi}_1/6$;



$$\tau(y) = \frac{\bar{\psi}_1(y)}{6\bar{\psi}_0(y)}$$



$$\tau(y = -1) = \frac{1}{2}, \quad \tau(y = 0) = i\infty, \quad \tau(y = 1/8) = 0, \quad \tau(y = \infty) = \frac{1}{3}$$

↪ cusps of $\Gamma_1(6)$

$$y(\tau) = \frac{\eta(\tau)^3\,\eta(6\tau)^9}{\eta(2\tau)^3\,\eta(3\tau)^9}\,.$$

Include sub-sectors

Include sub-sectors

$$\hat{L}_2^{(a)}(y,\varepsilon)\, I^{(a)}_{1111110} \equiv \sum_{n=0}^{2} r_n^{(a)}(y,\epsilon)\left(\frac{d}{dy}\right)^n I^{(a)}_{1111110} = \varepsilon^2\, \vec{f}^{(a)}_{\text{sub},\,2}(y)\cdot \vec{M}^{(a)}_{\text{sub}}(y),$$

$$\hat{L}_3^{(b)}(y,\varepsilon)\, I^{(b)}_{1111110} \equiv \sum_{n=0}^{3} r_n^{(b)}(y,\epsilon)\left(\frac{d}{dy}\right)^n I^{(b)}_{1111110} = \varepsilon^2\, \vec{f}^{(b)}_{\text{sub},\,2}(y)\cdot \vec{M}^{(b)}_{\text{sub}}(y) + \varepsilon^3\, \vec{f}^{(b)}_{\text{sub},\,3}(y)\cdot \vec{M}^{(b)}_{\text{sub}}(y)\,.$$

Include sub-sectors

$$\hat{L}_2^{(a)}(y,\varepsilon)\, I^{(a)}_{1111110} \equiv \sum_{n=0}^{2} r_n^{(a)}(y,\epsilon)\left(\frac{d}{dy}\right)^n I^{(a)}_{1111110} = \varepsilon^2\, \vec{f}^{(a)}_{\text{sub},\,2}(y)\cdot \vec{M}^{(a)}_{\text{sub}}(y),$$

$$\hat{L}_3^{(b)}(y,\varepsilon)\, I^{(b)}_{1111110} \equiv \sum_{n=0}^{3} r_n^{(b)}(y,\epsilon)\left(\frac{d}{dy}\right)^n I^{(b)}_{1111110} = \varepsilon^2\, \vec{f}^{(b)}_{\text{sub},\,2}(y)\cdot \vec{M}^{(b)}_{\text{sub}}(y) + \varepsilon^3\, \vec{f}^{(b)}_{\text{sub},\,3}(y)\cdot \vec{M}^{(b)}_{\text{sub}}(y)\,.$$

- When using (M_1, M_2, M_3) defined previously, $\vec{f}^{(b)}_{\text{sub},2}$ makes its dependence on sub-sector MIs not ε -factorized.

Include sub-sectors

$$\hat{L}_2^{(a)}(y,\varepsilon)\, I^{(a)}_{1111110} \equiv \sum_{n=0}^{2} r_n^{(a)}(y,\epsilon)\left(\frac{d}{dy}\right)^n I^{(a)}_{1111110} = \varepsilon^2\, \vec{f}^{(a)}_{\text{sub},\,2}(y)\cdot \vec{M}^{(a)}_{\text{sub}}(y),$$

$$\hat{L}_3^{(b)}(y,\varepsilon)\, I^{(b)}_{1111110} \equiv \sum_{n=0}^{3} r_n^{(b)}(y,\epsilon)\left(\frac{d}{dy}\right)^n I^{(b)}_{1111110} = \varepsilon^2\, \vec{f}^{(b)}_{\text{sub},\,2}(y)\cdot \vec{M}^{(b)}_{\text{sub}}(y) + \varepsilon^3\, \vec{f}^{(b)}_{\text{sub},\,3}(y)\cdot \vec{M}^{(b)}_{\text{sub}}(y)\,.$$

- When using (M_1, M_2, M_3) defined previously, $\vec{f}^{(b)}_{\text{sub},2}$ makes its dependence on sub-sector MIs not ε -factorized.

- ☑ This can be cured by a "minimal" subtraction for the last MI in top sector:

$$M_3 = \frac{1}{Y(y)}\left[\frac{J(y)}{\varepsilon}\frac{d}{dy}M_2 - F_{21}(y)M_1 - F_{22}(y)M_2\right] - \vec{g}_{\text{sub},2}(y)\cdot \vec{M}_{\text{sub}}.$$

Include sub-sectors

$$\hat{L}_2^{(a)}(y,\varepsilon)\, I^{(a)}_{1111110} \equiv \sum_{n=0}^{2} r_n^{(a)}(y,\epsilon)\left(\frac{d}{dy}\right)^n I^{(a)}_{1111110} = \varepsilon^2\, \vec{f}^{(a)}_{\text{sub},\,2}(y)\cdot \vec{M}^{(a)}_{\text{sub}}(y),$$

$$\hat{L}_3^{(b)}(y,\varepsilon)\, I^{(b)}_{1111110} \equiv \sum_{n=0}^{3} r_n^{(b)}(y,\epsilon)\left(\frac{d}{dy}\right)^n I^{(b)}_{1111110} = \varepsilon^2\, \vec{f}^{(b)}_{\text{sub},\,2}(y)\cdot \vec{M}^{(b)}_{\text{sub}}(y) + \varepsilon^3\, \vec{f}^{(b)}_{\text{sub},\,3}(y)\cdot \vec{M}^{(b)}_{\text{sub}}(y)\,.$$

- When using (M_1, M_2, M_3) defined previously, $\vec{f}^{(b)}_{\text{sub},2}$ makes its dependence on sub-sector MIs not ε -factorized.

- ☑ This can be cured by a "minimal" subtraction for the last MI in top sector:

$$M_3 = \frac{1}{Y(y)}\left[\frac{J(y)}{\varepsilon}\frac{d}{dy}M_2 - F_{21}(y)M_1 - F_{22}(y)M_2\right] - \vec{g}_{\text{sub},2}(y)\cdot \vec{M}_{\text{sub}}.$$

- ☑ And it is sufficient that

$$\frac{d}{dy}\vec{g}_{\text{sub},2}(y) = (1-8y)\vec{f}_{\text{sub},2}(y).$$

ε-form

ε-form

$$\frac{1}{2\pi i}\frac{d}{d\tau}\begin{pmatrix} M_1 \\ M_2 \\ M_3 \end{pmatrix} = \varepsilon \begin{pmatrix} \eta_{1,2} & 1 & 0 \\ \eta_4 & \eta_{1,2} & \eta_{1,3} \\ -\frac{64}{27}\eta_{1,3} & 0 & \eta_{2,2} \end{pmatrix} \begin{pmatrix} M_1 \\ M_2 \\ M_3 \end{pmatrix} + \varepsilon \begin{pmatrix} 0 \\ g_{\text{sub},2} \\ g_{\text{sub},3} \end{pmatrix}$$

$$\begin{aligned} g_{\text{sub},2} = {} & 10\eta_{2,3}M_5 - 10\eta_{2,3}M_6 - 8\eta_{2,3}M_7 - 8\eta_{2,3}M_8 + \varrho M_9 - 9\eta_{2,3}M_{10} + 10\eta_{2,3}M_{11} \\ & + 12\eta_{2,3}M_{12} + 8\eta_{2,3}M_{13} - 4\eta_{2,3}M_{14} \end{aligned}$$

w/

$$\begin{aligned} g_{\text{sub},3} = {} & -4\eta_{3,2}M_5 + 4\eta_{3,2}M_6 + 2\eta_{3,2}M_7 + 2\eta_{3,2}M_8 - 4\vartheta M_9 - (4\eta_{3,2} + 6\varpi)M_{11} + 4\eta_{3,2}M_{13} \\ & -2\eta_{3,2}M_{14} + \eta_{3,2}M_{15} - (8\eta_{3,2} + 24\varpi)M_{16} - (6\eta_{3,2} + 12\varpi)M_{17} + (5\eta_{3,2} + 9\varpi)M_{18} \end{aligned}$$

letters

letters

$$\varrho = \frac{7-8y}{\sqrt{1-4y}}\eta_{2,3}\,, \quad \vartheta = \frac{1+y}{\sqrt{1-4y}}\eta_{3,2}\,, \quad \varpi = \frac{\eta_{3,2}}{y-1}$$

Letters with modular weight 2 are particular $\left(y = \frac{t}{(1+t)^2}\,, \quad t = \frac{1-\sqrt{1-4y}}{1+\sqrt{1-4y}} \right)$:

$$\eta_{1,2} \cdot 2\pi i d\tau = \frac{1}{3} d\log y - d\log(y+1) - \frac{1}{2} d\log(1-8y)\,,$$

$$\eta_{2,2} \cdot 2\pi i d\tau = -\frac{2}{3} d\log y, \quad \eta_{2,2} \cdot 2\pi i d\tau = \frac{16}{3}\big(d\log y - d\log(1+y)\big)\,,$$

$$\vartheta \cdot 2\pi i d\tau = \frac{16}{3} d\log\frac{1-\sqrt{1-4y}}{1+\sqrt{1-4y}} = \frac{16}{3} d\log t\,,$$

$$\varpi \cdot 2\pi i d\tau = \frac{8}{3}\big[d\log(1-y) + d\log(1+y) - 2\,d\log y\big]\,.$$

Boundary

Boundary

Mellin-Barnes technique:

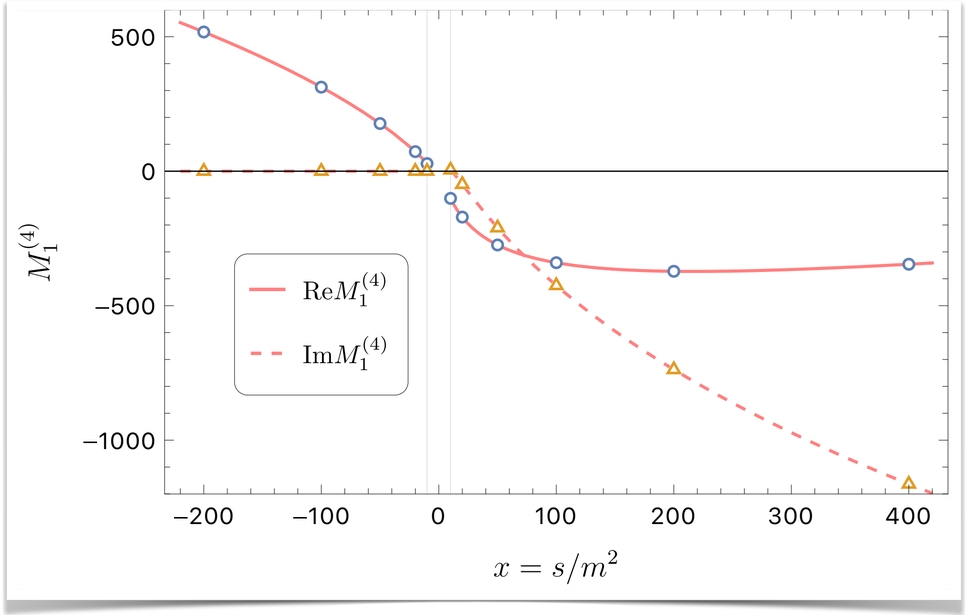
$$M_1\,|_{y\to 0}^{\text{boundary}} = \varepsilon^4\left[\frac{7}{12}L_q^4 - \zeta_2 L_q^2 - 20\zeta_3 L_q - \frac{31\zeta_4}{2}\right] + \varepsilon^5\left[\frac{3}{20}L_q^5 - 9\zeta_2 L_q^3 - 34\zeta_3 L_q^2 - \frac{125\zeta_4}{2}L_q + 40\zeta_2\zeta_3 + 8\zeta_5\right] + \mathcal{O}(\varepsilon^6)\,,$$

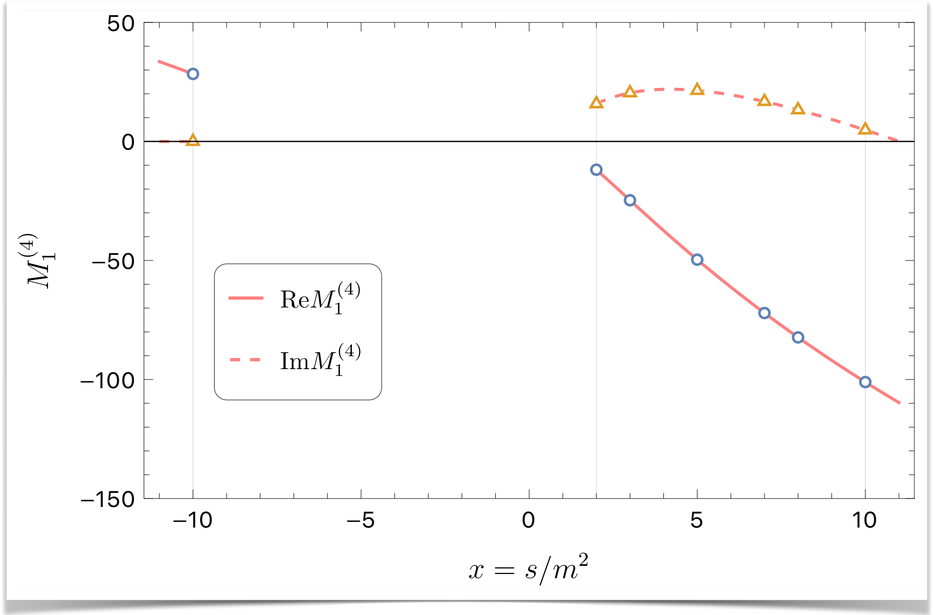
$$M_2\,|_{y\to 0}^{\text{boundary}} = \varepsilon^3\left[\frac{7L_q^3}{3} - 2\zeta_2 L_q - 20\zeta_3\right] + \varepsilon^4\left[\frac{5}{9}L_q^4 - \frac{80}{3}\zeta_2 L_q^2 - \frac{184\zeta_3 L_q}{3} - \frac{172\zeta_4}{3}\right] + \mathcal{O}(\varepsilon^5)\,,$$

$$M_3\,|_{y\to 0}^{\text{boundary}} = \varepsilon^2\left[0\right] + \varepsilon^3\left[\frac{112}{3}\zeta_3 + \frac{16}{3}\zeta_2 L_q - \frac{8}{9}L_q^3\right] + \mathcal{O}(\varepsilon^4)\,,$$

Numerics

$$M_1 = \varepsilon^4 \Bigg[-\frac{31\zeta_4}{2} - 20\zeta_3 \ln q + \zeta_2 I(1,6\eta_{2,3} - \rho;\ i\infty, \tau) + \frac{9}{32} I\left(1,\rho,\vartheta,\eta_{2,2};\ i\infty,\tau\right)$$
$$-9I\left(1,\eta_{2,3},\eta_{2,2},\eta_{2,2} + \frac{9}{16}\eta_{3,2};\ i\infty,\tau\right)\Bigg] + \mathcal{O}(\varepsilon^5)\,.$$





- Curves: evaluate via q-expansion, convergent very fast (seconds); points via AMFLOW [Liu, Ma, Wang, '20]
- The convergence of q-expansion is determined by nearest singularity in sub-sectors ($y = 1$)!

Conclusion and outlook

Conclusion and outlook

- We demonstrate that the method develops for Bananas also applies to more general FIs;

Conclusion and outlook

- We demonstrate that the method develops for Bananas also applies to more general FIs;
- We put more evidence that FIs have ε-forms;

Conclusion and outlook

- We demonstrate that the method develops for Bananas also applies to more general FIs;
- We put more evidence that FIs have ε-forms;
- The letters in the top sectors are modular forms of $\Gamma_1(4)$ and $\Gamma_1(6)$ respectively;

Conclusion and outlook

- We demonstrate that the method develops for Bananas also applies to more general FIs;
- We put more evidence that FIs have ε-forms;
- The letters in the top sectors are modular forms of $\Gamma_1(4)$ and $\Gamma_1(6)$ respectively;
- Poles from sub-sectors set the convergent radius of q expansion;

Conclusion and outlook

- We demonstrate that the method develops for Bananas also applies to more general FIs;
- We put more evidence that FIs have ε-forms;
- The letters in the top sectors are modular forms of $\Gamma_1(4)$ and $\Gamma_1(6)$ respectively;
- Poles from sub-sectors set the convergent radius of q expansion;
- Extend to FIs with more scales?

Conclusion and outlook

- We demonstrate that the method develops for Bananas also applies to more general FIs;
- We put more evidence that FIs have ε-forms;
- The letters in the top sectors are modular forms of $\Gamma_1(4)$ and $\Gamma_1(6)$ respectively;
- Poles from sub-sectors set the convergent radius of q expansion;
- Extend to FIs with more scales?
- Analytically continue the mixing part to cover the whole regime, expand around other points?

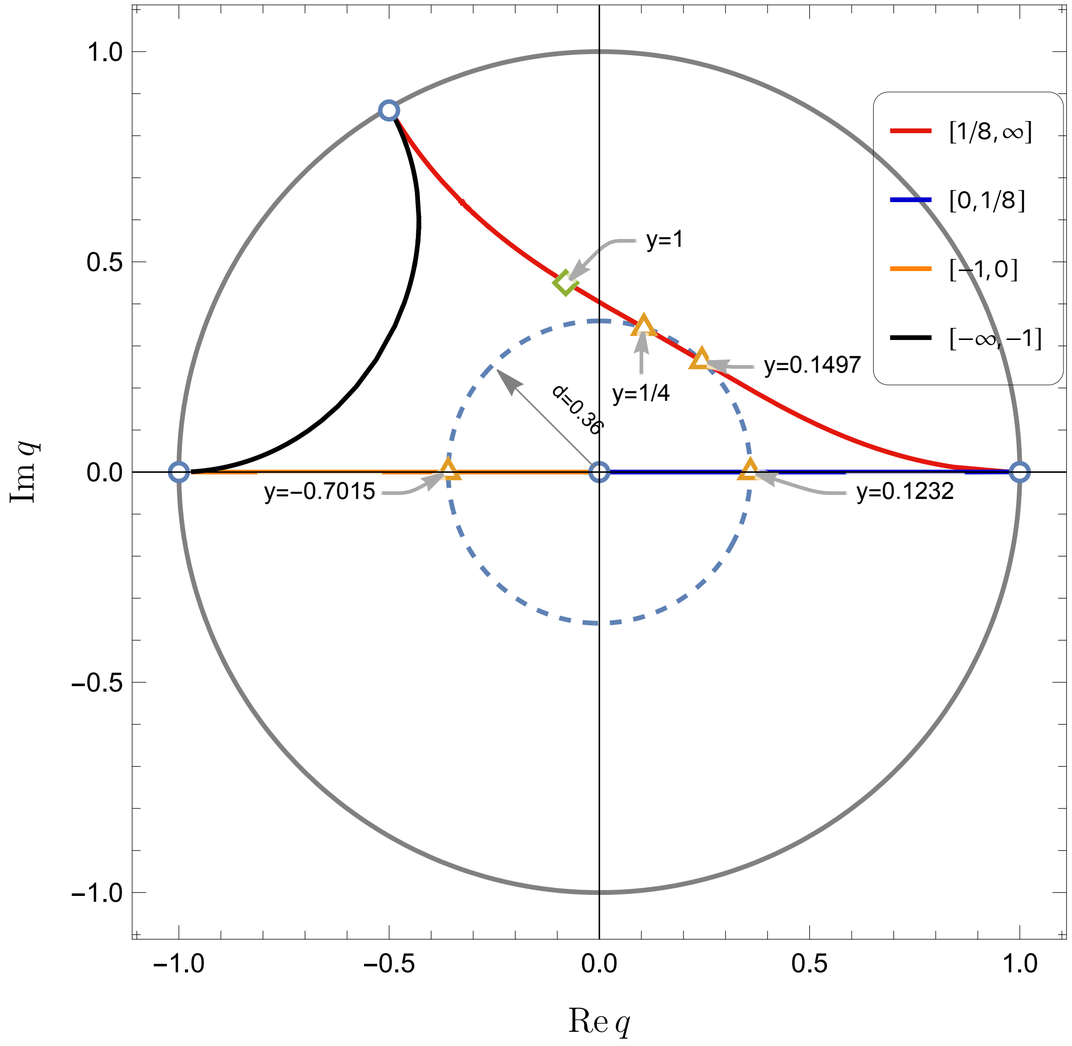
Conclusion and outlook

- We demonstrate that the method develops for Bananas also applies to more general FIs;
- We put more evidence that FIs have ε-forms;
- The letters in the top sectors are modular forms of $\Gamma_1(4)$ and $\Gamma_1(6)$ respectively;
- Poles from sub-sectors set the convergent radius of q expansion;
- Extend to FIs with more scales?
- Analytically continue the mixing part to cover the whole regime, expand around other points?

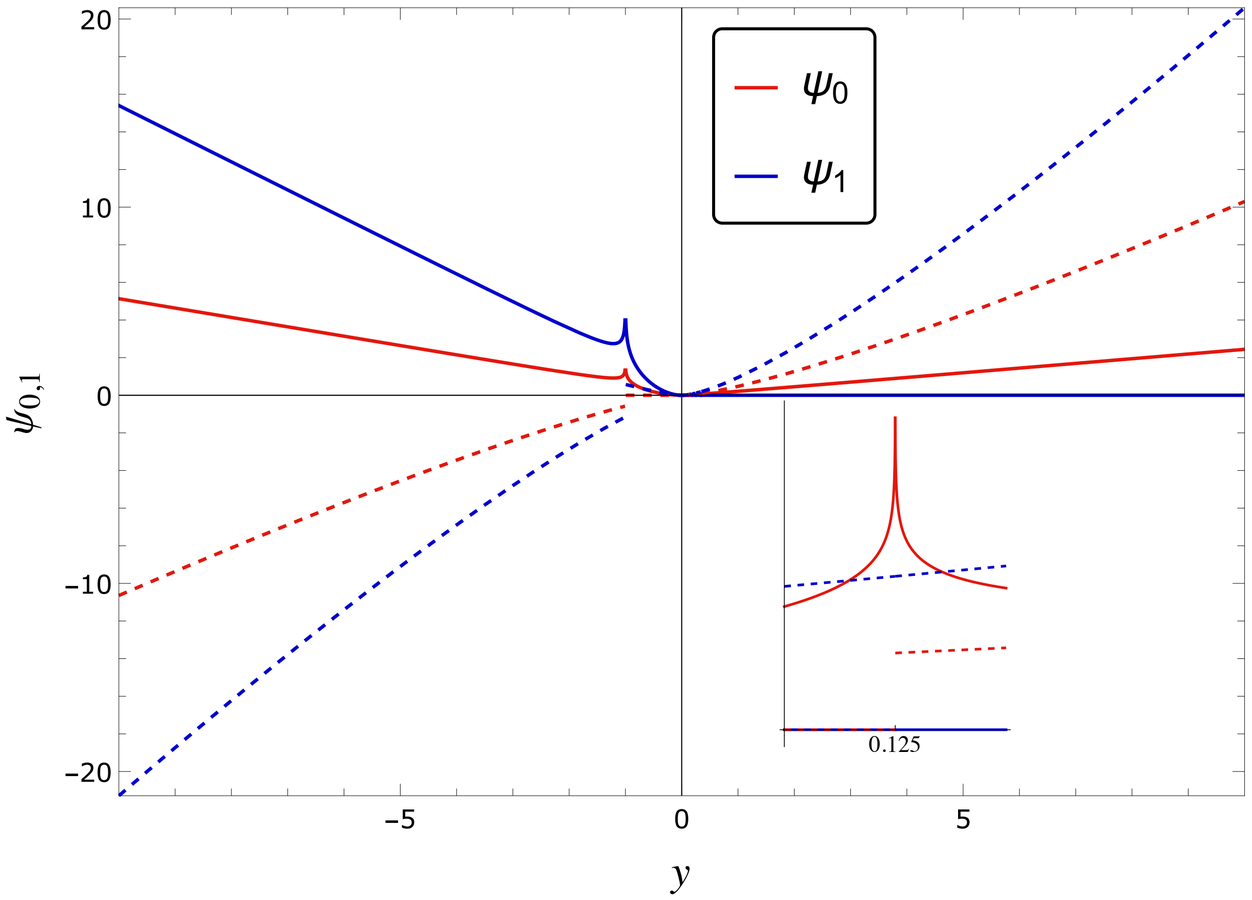
Thank you for your attention!

Backup

Convergent radius



Periods of family (a)



τ and q-path of family (a)