Loop calculations with graphical functions

Oliver Schnetz

Department Mathematik
Cauerstraße 11
91058 Erlangen

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Setup

The graphical functions method works for

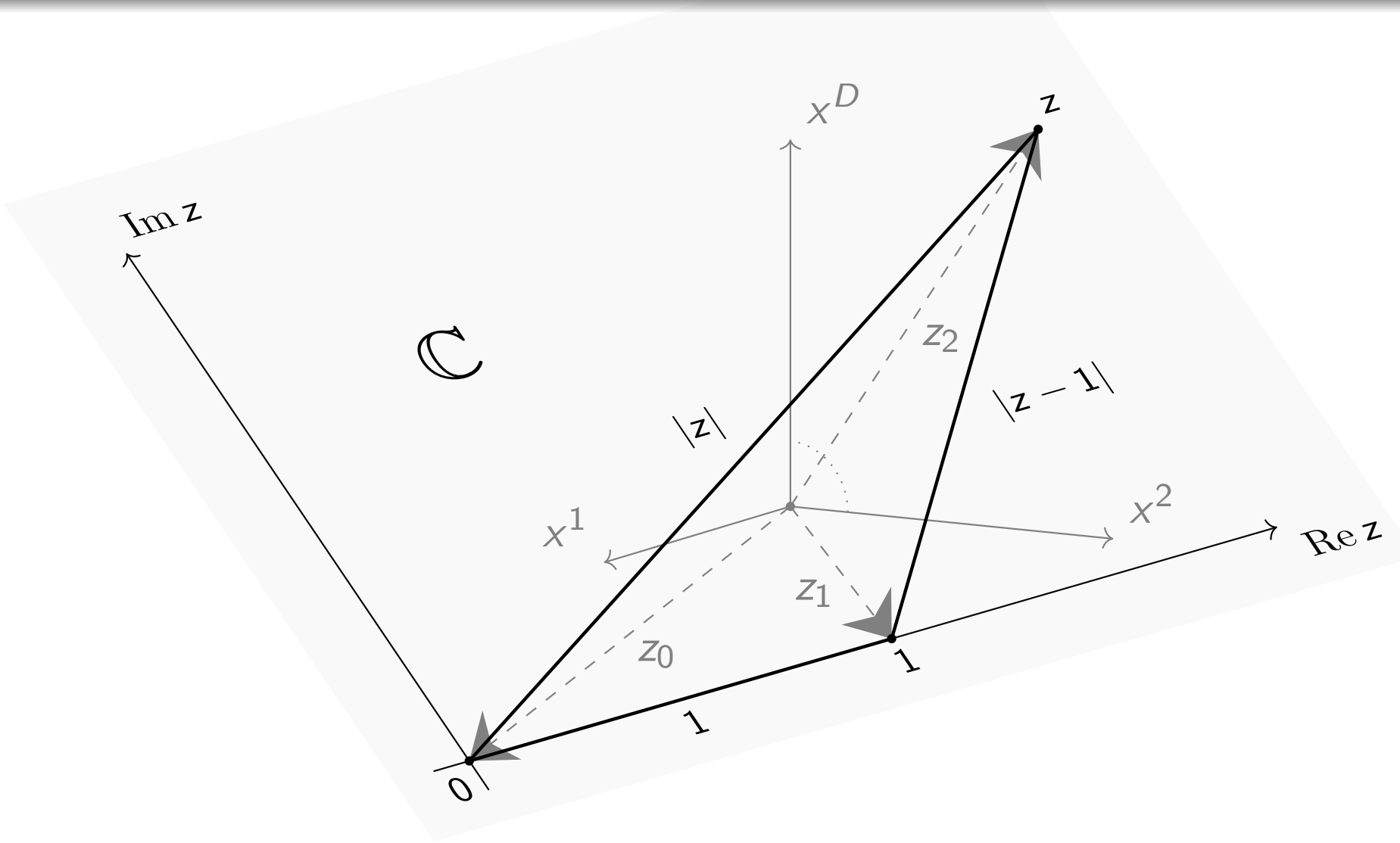
- massless,
- 2pt, 3pt, or convergent (conformal) 4pt amplitudes
- in even dimensions ≥ 4.

Ideal playground: renormalization functions $\beta(g)$, $\gamma(g)$, $\gamma_m(g)$.

Idea

- Massless 2pt amplitudes are scalars (periods).
- More structure in functions.
- Massless 3pt amplitudes (or 4pt conformal) are the simplest functions in QFT (two-scale).
- Generalize 3pt amplitudes by allowing the Feynman graph to have any number of edges at the three external vertices (points).
- Use position space. Three points span a plane in $\mathbb{R}^d$. Consider this plane as $\mathbb{C}$.
- Study the amplitude as a function on $\mathbb{C}$ using the theory of complex functions.

Picture (by M. Borinsky)



Definition

$$A_G(z_0, z_1, z_2) = ||z_1 - z_0||^{-2\lambda N_G} f_G(z),$$

with $\lambda = D/2 - 1$, invariants

$$\frac{\|z_2 - z_0\|^2}{\|z_1 - z_0\|^2} = z\overline{z}, \quad \frac{\|z_2 - z_1\|^2}{\|z_1 - z_0\|^2} = (z-1)(\overline{z}-1),$$

and the scaling weight (superficial degree of divergence)

$$N_G = \Big(\sum_e \nu_e\Big) - \frac{\lambda+1}{\lambda} V_G^{\mathrm{int}}.$$

General properties of graphical functions

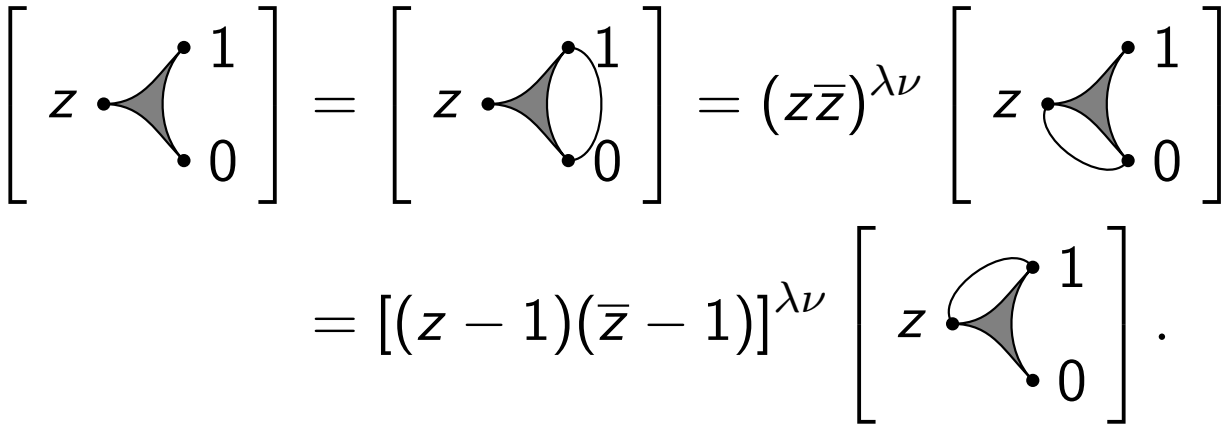
- Reflection symmetry $f_G(z) = f_G(\overline{z})$.
- f_G is a real-analytic single-valued function on $\mathbb{C}\backslash\{0,1\}$ (with M. Golz, E. Panzer).
- There exist single-valued log-Laurent expansions for the ϵ^k coefficients of $f_G(z)$ at the singular points 0, 1, ∞.

$$\sum_{\ell\geq 0}\sum_{m,n=M_s}^{\infty} c^{s,k}_{\ell,m,n}[\log(z-s)(\overline{z}-s)]^{\ell}(z-s)^m(\overline{z}-s)^n \quad \text{if } |z-s|<1,$$

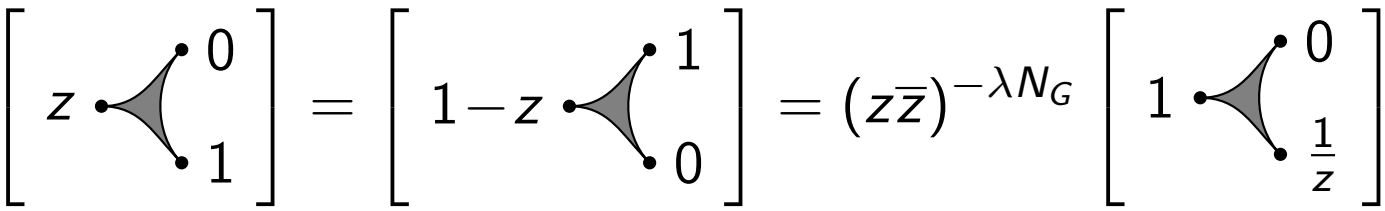
$$\sum_{\ell\geq 0}\sum_{m,n=-\infty}^{M_\infty} c^{\infty,k}_{\ell,m,n}(\log z\overline{z})^{\ell} z^m \overline{z}^n \quad \text{if } |z|>1.$$

Construction of graphical functions

- Add edges between external vertices



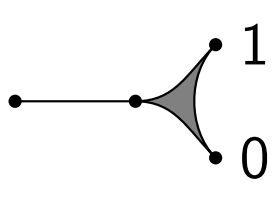
- Permute external vertices



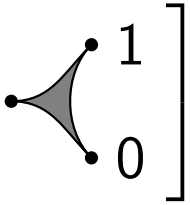
Construction of graphical functions

- Invert the effective Laplace equation for an isolated edge of weight 1 at vertex z,

$$\left(\Delta_n + \frac{\varepsilon/2}{z-\overline{z}}(\partial_z - \partial_{\overline{z}})\right)$$



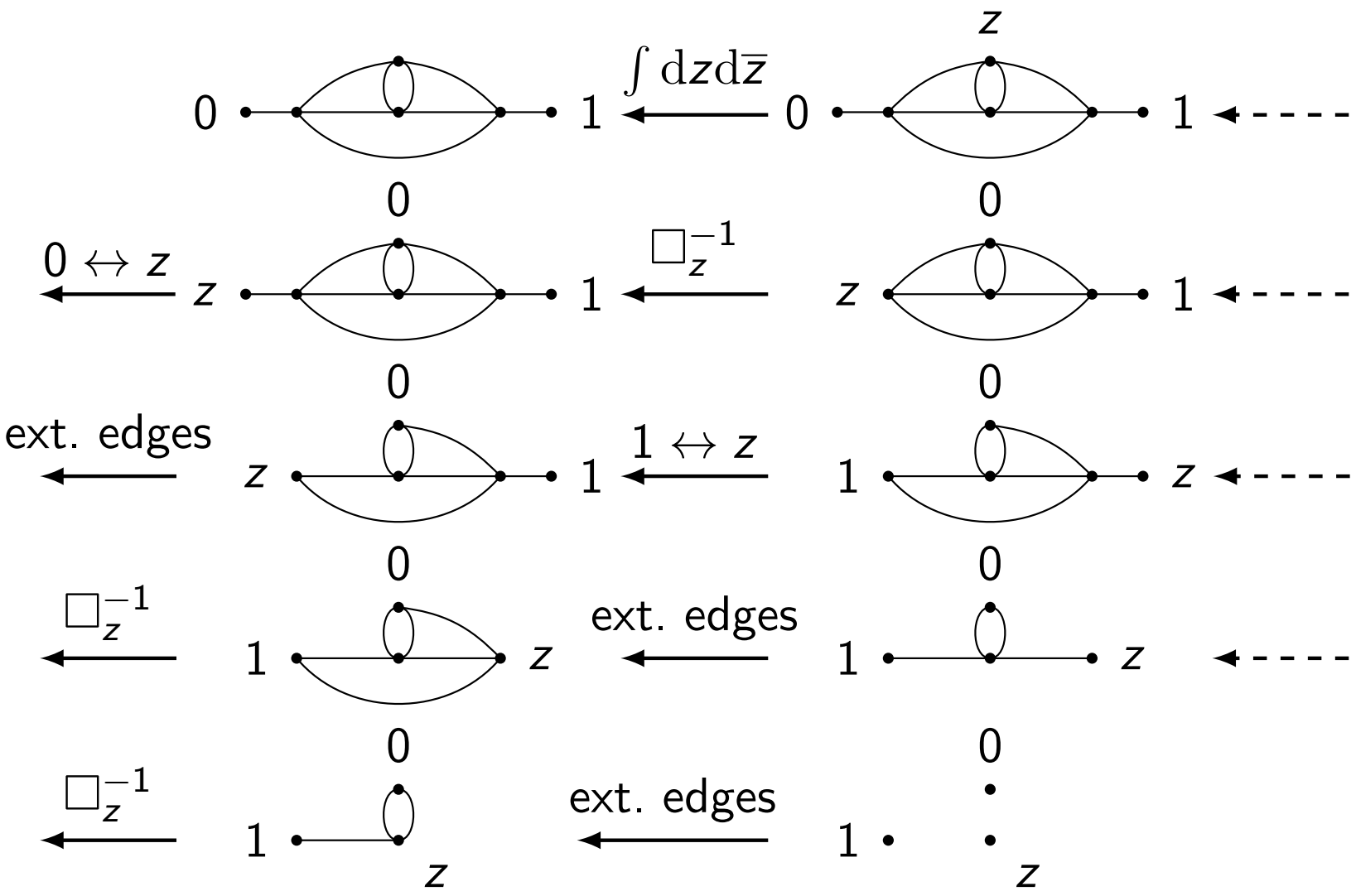
$$= -\frac{1}{\Gamma(\lambda)}$$



$$\text{with} \quad \Delta_n = \frac{1}{(z-\overline{z})^{n+1}}\partial_z\partial_{\overline{z}}(z-\overline{z})^{n+1} + \frac{n(n+1)}{(z-\overline{z})^2},$$

$$\text{where} \quad D = 2n + 4 - \epsilon. \tag{1}$$

Picture (by M. Borinsky)



The 4 miracles of graphical functions

Proved with M. Borinsky:

- For $\epsilon = 0$ there exists a closed solution of the effective Laplace equation by taking single-valued primitives. (This is trivial in $D = 4$ dimensions.)
- For $\epsilon = 0$ the solution is unique in the space of graphical functions.
- The inversion of the effective Laplace equation fully generalizes to non-integer dimensions $\epsilon \neq 0$.
- There exists a function space which is closed under inverting the effective Laplace equation. The inversion is efficient in this function space.

Definition

Generalized single-valued hyperlogarithms (GSVHs) are iterated primitives of differential forms

$$\frac{\mathrm{d}z}{az\overline{z} + bz + c\overline{z} + d}, \qquad a, b, c, d \in \mathbb{C},$$

on the punctured (!) Riemann sphere $\mathbb{C}\backslash\{0, s_1, \ldots, s_n\}$.
Example (C. Duhr et al.).

$$\int_{\mathrm{sv}} \frac{D(z)\mathrm{d}z}{z - \overline{z}},$$

where $D(z)$ is the Bloch-Wigner dilogarithm,

$$D(z) = \mathrm{Im}\,(\mathrm{Li}_2(z) + \log(1 - z)\log|z|).$$

The commutative hexagon

GSVHs can be constructed with a commutative hexagon:

$$
\begin{tikzcd}
 & \mathcal{G} & \\
\partial_z \mathcal{G} \arrow[ur, "\int_{\mathrm{sv}} \mathrm{d}z"] & & \partial_{\overline{z}} \mathcal{G} \arrow[ul, "\int_{\mathrm{sv}} \mathrm{d}\overline{z}"'] \\
\partial_z \mathcal{G} \arrow[u, "\pi_{\partial_z}"] & & \partial_{\overline{z}} \mathcal{G} \arrow[u, "\pi_{\partial_{\overline{z}}}"'] \\
 & \partial_{\overline{z}} \partial_z \mathcal{G} \arrow[ul, "\partial_{\overline{z}}"] \arrow[ur, "\int_{\mathrm{sv}} \mathrm{d}z"'] &
\end{tikzcd}
$$

$2n + 4 - \epsilon$ dimensions

- Taylor coefficients of convergent graphical functions in non-integer dimensions are obtained by a straight forward expansion method.
- For sinular graphical functions a sophisticated subtraction method is necessary to obtain the Laurent coefficients. (Problem: uniqueness of the inversion of the effective Laplace equation.)
- There exists a large toolbox for calculating low order Laurent coefficients of (singular) graphical functions.

Comparism with classical techniques

- Momentum space techniques are more general (masses, Npt functions).
- Momentum space techniques can also be applied to graphical functions (master integrals).
- The theory of graphical functions performs the integrations.
- The large set of constructible graphs is always computable with graphical functions (to sensible orders in ϵ).
- One always obtains a reduction of complexitiy by integrating out a subset of the edges of the Feynman graph.

Finished

- ϕ^4 theory (4 dimensions): 8 loops field anomalous dimension γ.
  7 loops β function, mass anomalous dimension γ_m, self-energy Σ.
- ϕ^3 theory (6 dimensions): 6 loops field anomalous dimension γ, β function, mass anomalous dimension γ_m.
  5 loops self-energy Σ.

Preliminary

$$
\begin{aligned}
\beta_6^{\phi^3} &= \frac{245045}{144}\zeta(9) + 37\zeta(3)^3 + \frac{3357}{40}\zeta(5,3) - \frac{11}{3}\zeta(5)\zeta(3) \\
&- \frac{81733}{2016000}\pi^8 - \frac{456443}{1152}\zeta(7) + \frac{99}{800}\pi^4\zeta(3) - \frac{2425}{384}\zeta(3)^2 \\
&+ \frac{176425}{2612736}\pi^6 - \frac{24878747}{34560}\zeta(5) + \frac{42654751}{74649600}\pi^4 \\
&- \frac{85523425}{186624}\zeta(3) - \frac{173655397121}{3224862720} \\
&= -241.455497609497\ldots
\end{aligned}
$$

($\zeta(5,3) = \sum_{k_1>k_2\geq 1} \frac{1}{k_1^5 k_2^3}$, May 19, 2023, to be checked)

Future

- ϕ^4 theory: 8 loops β function (93% complete), mass anomalous dimension γ_m.
- Gauge theories: 6 loops QED, QCD, ...
- Yukawa theory, ...
- Extensions: odd dimensions, masses, ...