

# Lattice Correlation Functions

from

## Differential Equations

Federico Gasparotto



Based on joined work w/

A. Rapakoulias & S. Weinzierl  
S. Weinzierl & X. Xu

arXiv: [2210.16052](https://arxiv.org/abs/2210.16052)

arXiv: [2305.05447](https://arxiv.org/abs/2305.05447)

Correlation functions are key objects in QFT

$$G_n(x_1, \dots, x_n) = \frac{\int \mathcal{D}\phi \phi(x_1) \dots \phi(x_n) \exp(iS)}{\int \mathcal{D}\phi \exp(iS)}$$

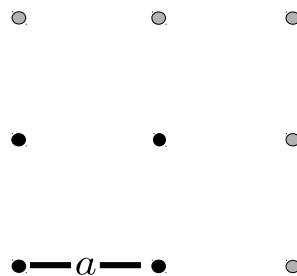
We focus on scalar action– $\lambda\phi^4$  model

$$S = \int d^Dx \frac{1}{2} \partial^\mu \phi(x) \partial_\mu \phi(x) - \frac{m^2}{2} \phi^2(x) - \lambda \phi^4(x)$$

Problem tough  $\rightsquigarrow$  Lattice regularization

lattice  $\Lambda$ , spacing  $a$ ,  $L_\mu$  pts  $\mu$ -direction + periodicity

$$D=2 \quad L_0=L_1=2 \quad N=4$$



$$N = \prod_{\mu=0}^{D-1} L_\mu = \# \text{ lattice pts}$$

$$\begin{aligned} \int d^Dx &\rightsquigarrow a^D \sum_{x \in \Lambda} \\ \partial_\mu \phi(x) &\rightsquigarrow \frac{\phi(x + a\hat{e}_\mu) - \phi(x)}{a} \end{aligned}$$

Minkowskian signature

$$S_M = i \sum_{x \in \Lambda} \left[ \phi(x) \phi(x+a\hat{e}_0) - \sum_{\mu=1}^{D-1} \phi(x) \phi(x+a\hat{e}_\mu) + \left( D + \frac{m^2}{2} - 2 \right) \phi^2(x) + \lambda \phi^4(x) \right]$$

Euclidean signature

$$S_E = \sum_{x \in \Lambda} \left[ - \sum_{\mu=0}^{D-1} \phi(x) \phi(x+a\hat{e}_\mu) + \left( D + \frac{m^2}{2} \right) \phi^2(x) + \lambda \phi^4(x) \right]$$

Schematically

$S_\bullet = \text{“polynomial in fields } \phi(x_i) \text{”}$

$$= S_\bullet^{\text{next neigh.}} + S_\bullet^{(2)} + \lambda S_\bullet^{(4)}$$

$\bullet = M, E$

Everything boils down to computation of finite dimensional integrals

$$I_{\nu_1 \dots \nu_N} = \int_{\mathbb{R}^N} \exp(-S_\bullet) \phi^{\nu_1}(x_1) \dots \phi^{\nu_N}(x_N) d^N \phi$$

$\nu_i \in \mathbb{N}$   
 $\bullet = M, E$   
 $\lambda \text{ not small}$

Correlation functions recovered as

$$G_{\nu_1 \dots \nu_N} = \frac{I_{\nu_1 \dots \nu_N}}{I_{0 \dots 0}}$$

$\bullet = E$  traditionally via Monte Carlo,  $\bullet = M$  harder

In this talk, apply methods from perturbation theory

integral reduction  $\oplus$  differential eqs

in order to describe non-perturbative physics

Integral family

$$I_{\nu_1 \dots \nu_N} = \int_{\mathbb{R}^N} \exp(-S_\bullet) \phi^{\nu_1}(x_1) \dots \phi^{\nu_N}(x_N) d^N \phi$$

$\nu_i \in \mathbb{N}$   
 $\bullet = M, E$   
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Framework of Twisted Co-Homology

[Aomoto]

$$\begin{aligned} & \int_{\mathcal{C}} u \Phi \quad \xrightarrow{\hspace{10em}} \quad u = \exp(-S_\bullet) \\ & \xleftarrow{\hspace{10em}} \quad \Phi = \phi^{\nu_1}(x_1) \dots \phi^{\nu_N}(x_N) d^N \phi \\ & \mathcal{C} = \mathbb{R}^N \text{ } \mathcal{C} \text{ st } u|_{\partial \mathcal{C}} = 0 \end{aligned}$$

Introduce “auxiliary flow”  $t$

(eventually  $t = 1$ )

$$S_\bullet \rightsquigarrow S_\bullet(t) = t S_\bullet^{\text{next neigh.}} + S_\bullet^{(2)} + S_\bullet^{(4)}$$

$\bullet = M, E$

# Integration by parts (twisted Co-Homology POV)

[Mastrolia & Mizera]

$$\begin{aligned} 0 &= \int_{\mathcal{C}} d(u\xi) = \int_{\mathcal{C}} u (d + d \log u \wedge) \xi \\ &= \int_{\mathcal{C}} u \nabla_{\omega} \xi \end{aligned}$$

$$\begin{aligned} \nabla_{\omega}(\bullet) &= d(\bullet) + \omega \wedge \bullet \\ \omega &= d \log u = -dS_{\bullet} \end{aligned}$$

- $\Phi$  &  $\Phi + \nabla_{\omega} \xi$  integrate same result  $\int u \bullet$        $\langle \Phi | : \Phi \sim \Phi + \nabla_{\omega} \xi$
- $\nabla_{\omega} \circ \nabla_{\omega} = 0$

$$\langle \Phi | \in H^N = \frac{\text{Ker } \nabla_{\omega}}{\text{Im } \nabla_{\omega}} \rightsquigarrow \text{twisted Co-Homology group}$$

$$\text{Study } \int_{\mathcal{C}} u \Phi \rightsquigarrow \text{Study } H^N$$

- Dimension Co-Homology group

[Weinzierl]

$$\dim H^N = 3^N \quad N = \# \text{ lattice pts}$$

- Basis

$$\phi^{\nu_1}(x_1) \dots \phi^{\nu_N}(x_N) d^N \phi, \quad \nu_1, \dots, \nu_N < 3$$

- “Proof”

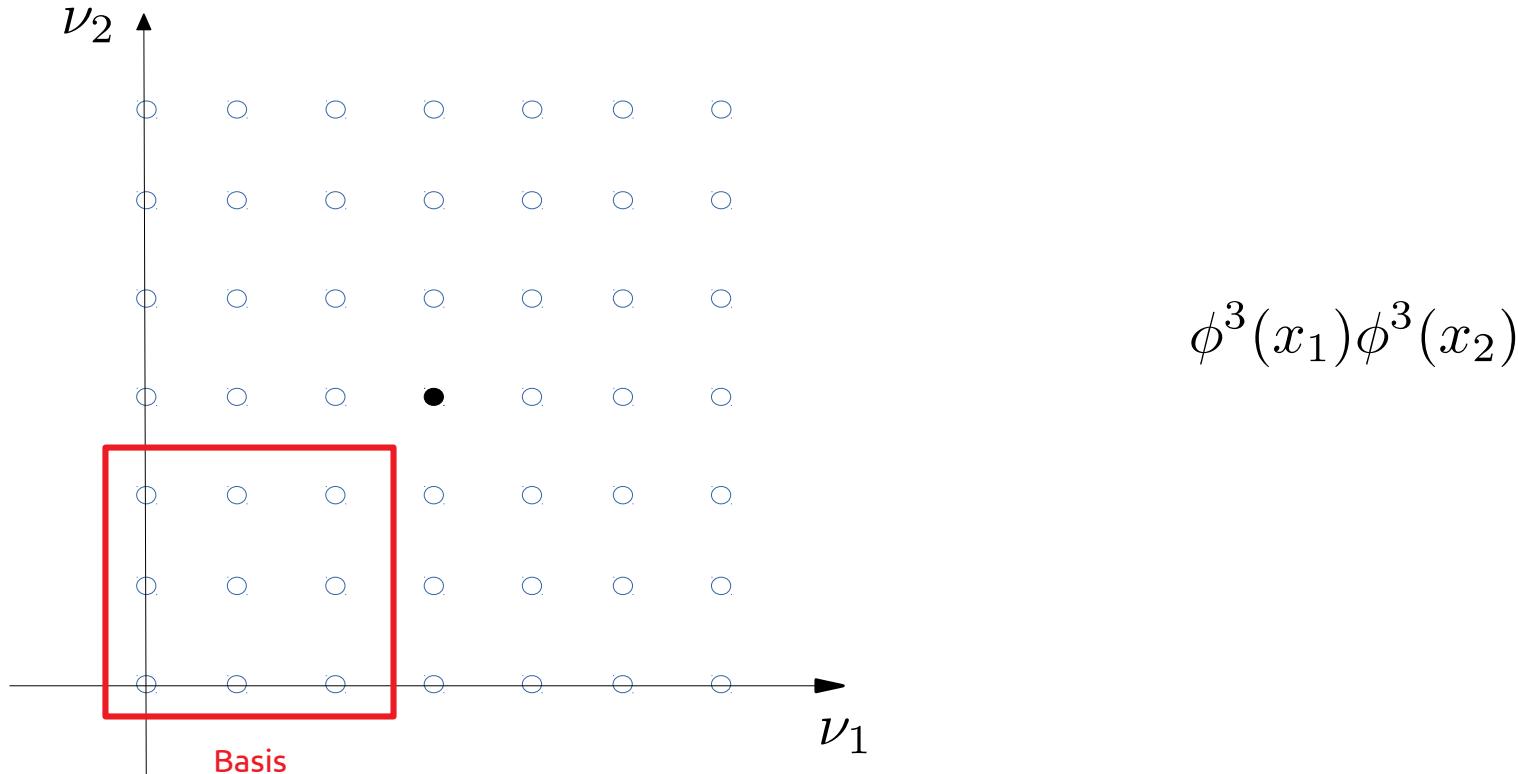
Given:  $\Phi = \phi^{\nu_1}(x_1) \dots \phi^{\nu_k}(x_k) \dots \phi^{\nu_N}(x_N) d^N \phi, \quad \nu_k \geq 3$

Choose:  $\xi \propto \phi^{\nu_1}(x_1) \dots \phi^{\nu_k-3}(x_k) \dots \phi^{\nu_N}(x_N) d^{N-1} \phi$

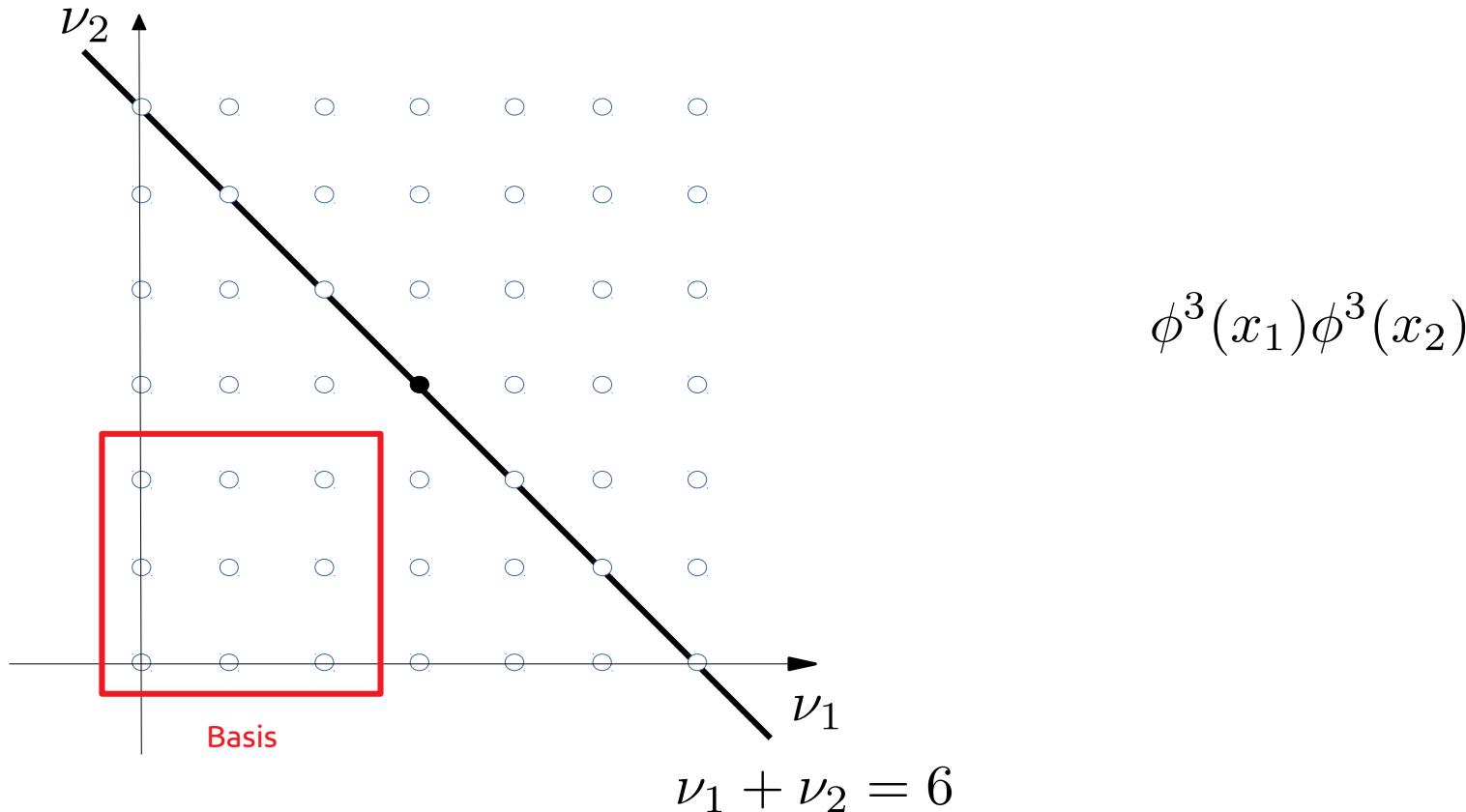
$$S_\bullet \propto \phi^4 + \text{“lower degree”} \rightsquigarrow \omega = -dS_\bullet \propto -\phi^3 + \text{“lower degree”}$$

$$\Phi \sim \Phi + \nabla_\omega \xi = \Phi - \Phi + \sum \text{“lower degree”}$$

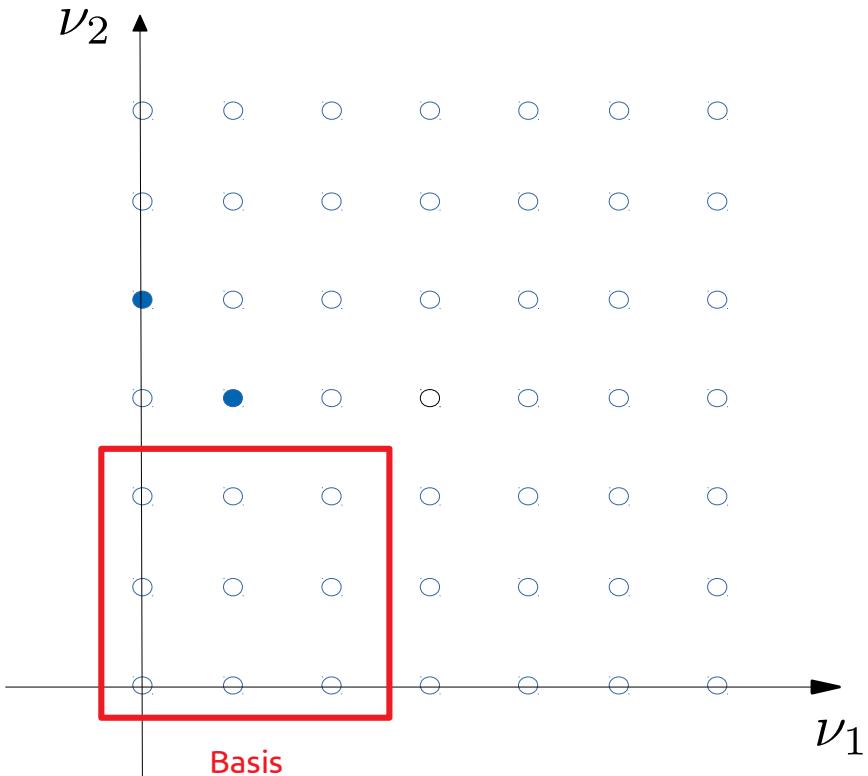
Example  $D = 1$  Euclidean with  $L = N = 2$



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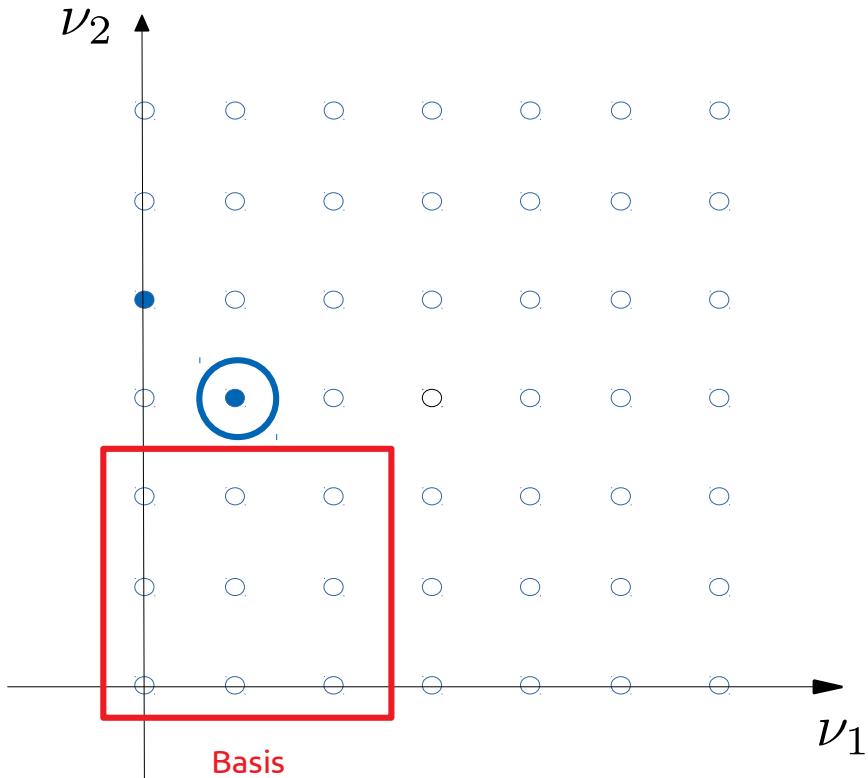


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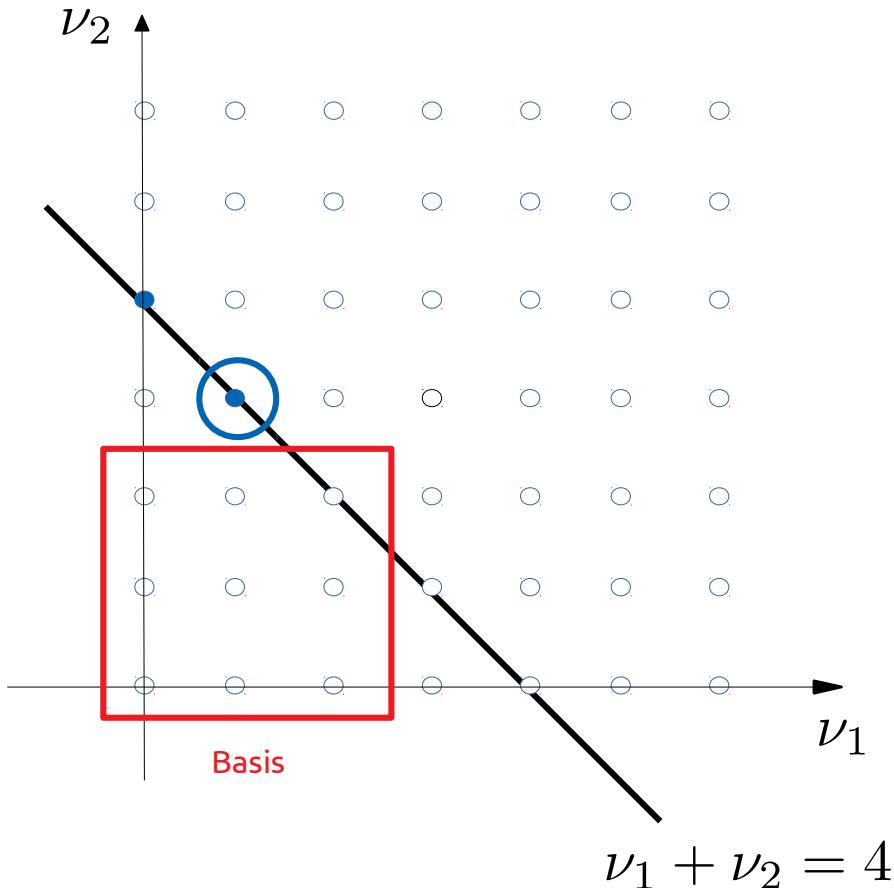
$$-\frac{(2+m^2)}{4\lambda} \phi(x_1)\phi^3(x_2) + \frac{t}{2\lambda} \phi^4(x_2)$$

Example  $D = 1$  Euclidean with  $L = N = 2$



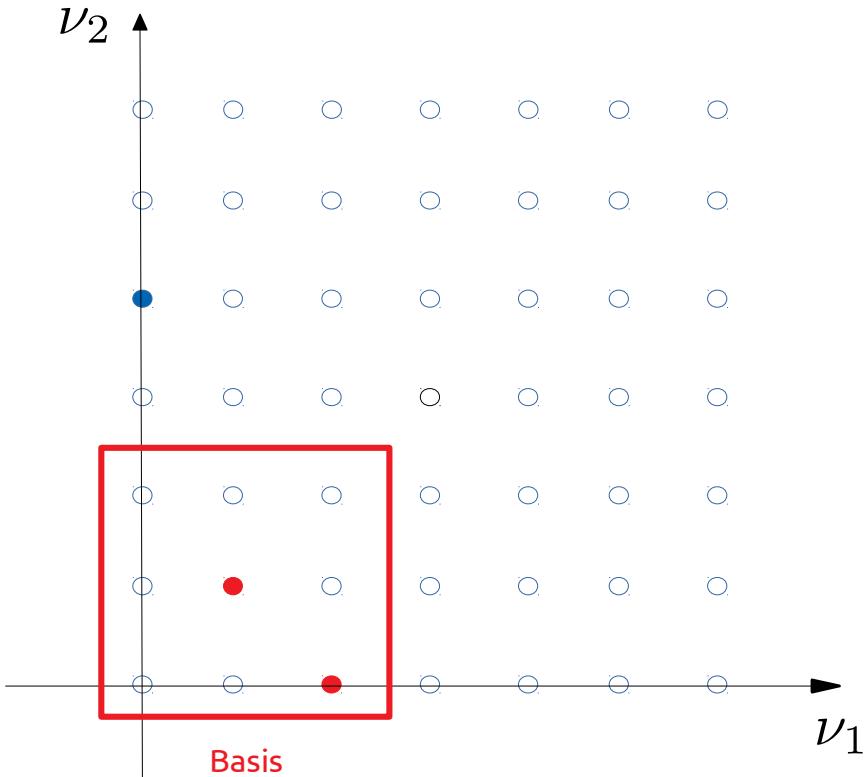
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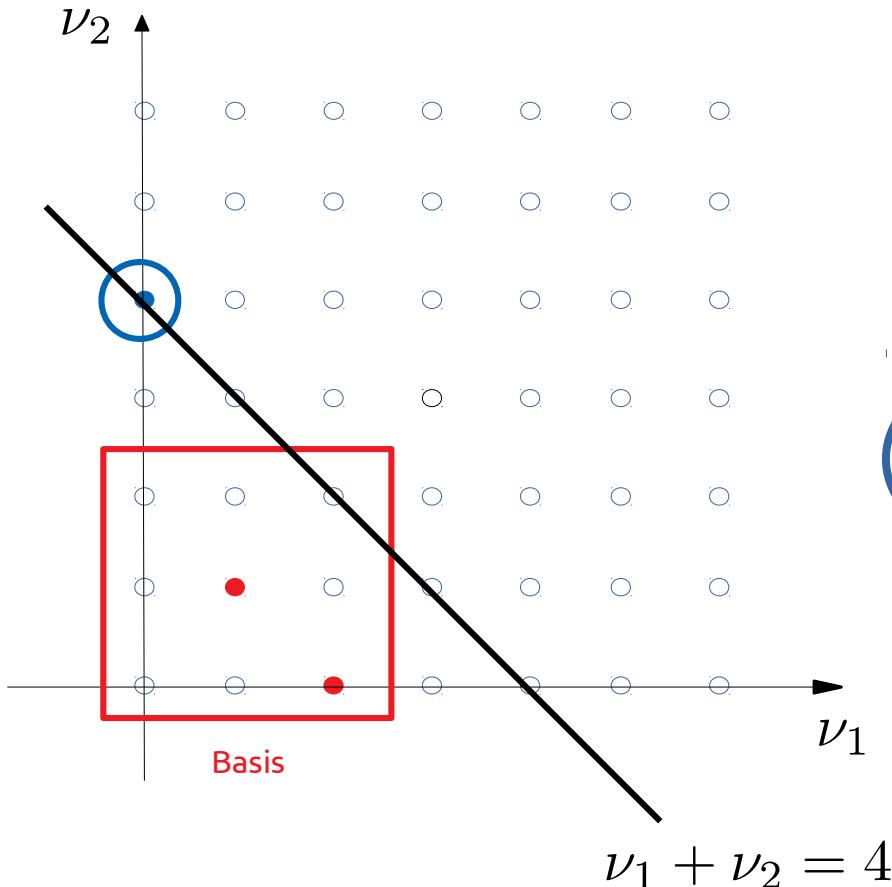
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Example  $D = 1$  Euclidean with  $L = N = 2$



$$+\frac{(2+m^2)}{16\lambda^2} \left( -2t\varphi^2(x_1) + (2+m^2)\varphi(x_1)\varphi(x_2) \right)$$
$$+\frac{t}{2\lambda}\varphi^4(x_2)$$

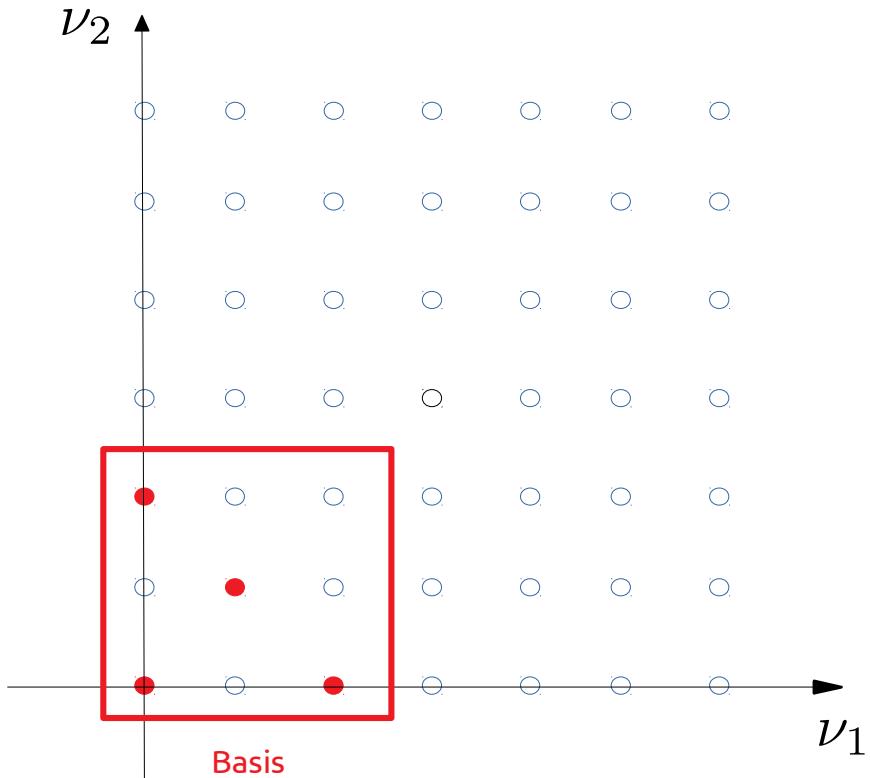
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$$+ \frac{t}{2\lambda}\varphi^4(x_2)$$

Example  $D = 1$  Euclidean with  $L = N = 2$



$$\begin{aligned}
 & + \frac{(2 + m^2)}{8\lambda^2} (-t\varphi^2(x_1) - t\varphi^2(x_2)) \\
 & + \frac{1}{16\lambda^2} (2t \mathbf{1} + ((2 + m^2)^2 + 4t^2)\varphi(x_1)\varphi(x_2))
 \end{aligned}$$

Comment: never produce powers of  $t$  in the denominator

( $L = 2$  each direction)

$D$  vs dim H

D	1	2	3	4
# lattice pts	2	4	8	16
# indep. forms	9	81	6561	43 046 721

( $L = 2$  each direction)

### $D$ vs dim $H$

D	1	2	3	4
# lattice pts	2	4	8	16
# indep. forms	9	81	6561	43046721

Another source of redundancy: symmetry relations!

(not seen by diff. forms, seen by ints.)

Example  $D = 1$  Euclidean with  $L = N = 2$

$$S_E = -2t\phi(x_1)\phi(x_2) + \left(1 + \frac{m^2}{2}\right)(\phi^2(x_1) + \phi^2(x_2)) + \lambda(\phi^4(x_1) + \phi^4(x_2))$$

$\mathbb{Z}_2$

$$\phi(x_i) \rightarrow -\phi(x_i)$$

Permutation

$$\phi(x_1) \leftrightharpoons \phi(x_2)$$

( $L = 2$  each direction)

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$\mathbb{Z}_2$

Permutation

$$\phi(x_i) \rightarrow -\phi(x_i)$$

$$\phi(x_1) \leftrightharpoons \phi(x_2)$$

$$I_{\nu_1 \nu_2} = -I_{\nu_1 \nu_2} = 0$$

$\nu_1 + \nu_2 \quad \text{odd}$

$$I_{\nu_1 \nu_2} = I_{\nu_2 \nu_1}$$

( $L = 2$  each direction)

$D$  vs dim H

D	1	2	3	4
# lattice pts	2	4	8	16
# indep. forms	9	81	6561	43046721

Another source of redundancy: symmetry relations!

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$D$  vs indep. ints

D	1	2	3	4
# lattice pts	2	4	8	16
# indep. ints	4	13	147	66524

Embed integrals into a vector

$$\mathbf{I} = (I_1, \dots, I_{\#\text{indep. ints}})^\top$$

Use reductions & symmetries to obtain system 1<sup>st</sup> order DEQ

$$\frac{d \mathbf{I}}{d(\bullet)} = A_\bullet(m^2, \lambda, t) \mathbf{I} \quad \bullet = m^2, \lambda, t$$

Face (at least) 2 problems:

- i) Singularities along integration path
- ii) Boundary constants:  $\mathbf{I}_0$

Auxiliary parameter  $t$  solves both problems!  $\bullet = t$

By construction:  $A_t$  polynomial in  $t$  (# finite =  $2^D$  for  $L = 2$  each direction)

$$A_t(m^2, \lambda, t) = \sum_{j=0}^{\# \text{ finite}} \mathcal{A}_j(m^2, \lambda) t^j$$

Singularities only at  $t = \infty$ !

Integrate  $t \in [0, 1]$  (no singularities)



Boundary constants  $\mathbf{I}_0$  @  $t = 0$ : product 1-fold ints!  $(\exp(\sum) = \prod \exp)$

$$S_\bullet(t = 0) = 0 \cdot S_\bullet^{\text{next neigh.}} + S_\bullet^{(2)} + \lambda S_\bullet^{(4)}$$

~~$S_\bullet^{\text{next neigh.}}$~~

The strategy (inspired by “auxiliary mass flow”) [Liu & Ma & Wang]

- Fix values for  $(m^2, \lambda) \rightsquigarrow$  compute  $\mathbf{I}_0$
- $A_t(t)$  holomorphic on  $\mathbb{C} \Rightarrow \mathbf{I}(t)$  holomorphic on  $\mathbb{C}$  [Wasow]

$$\mathbf{I}(t) = \sum_{k=0}^{\infty} \mathbf{I}_k t^k$$

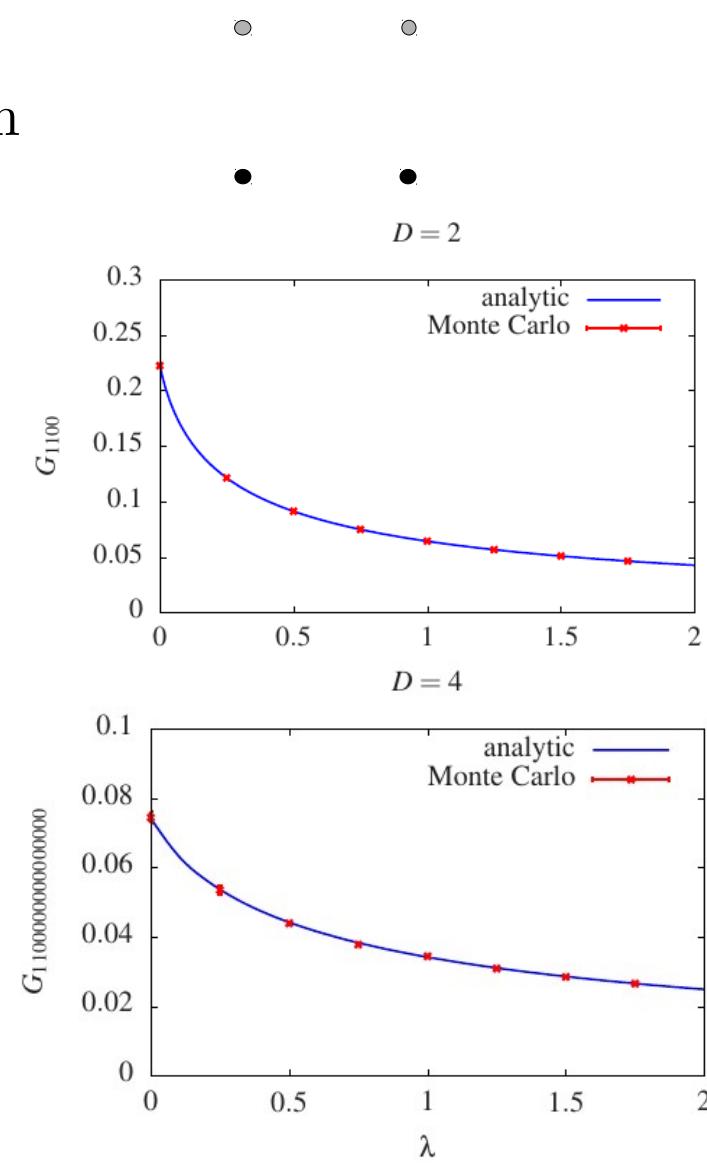
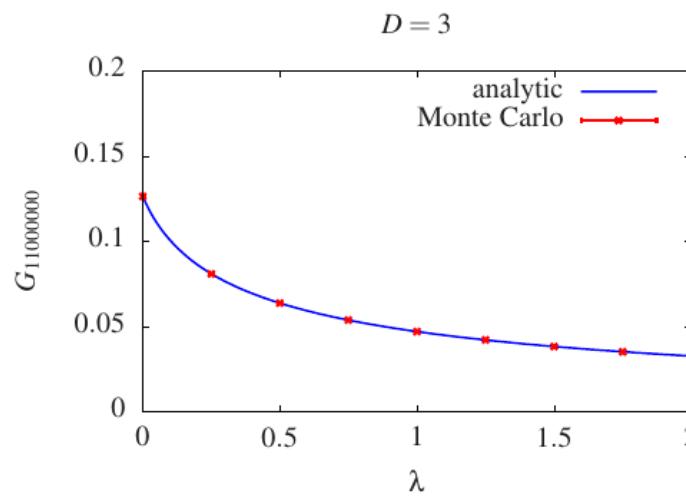
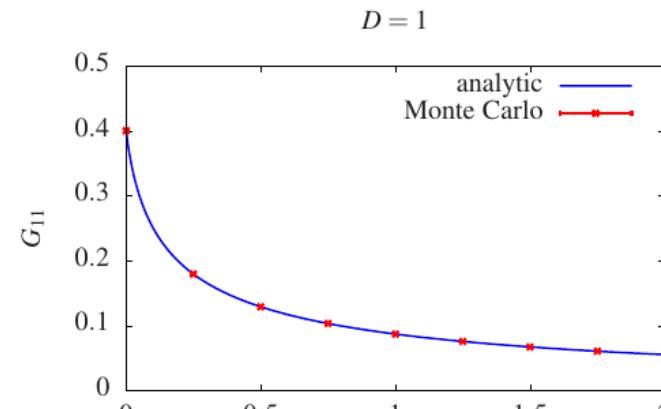
- System 1<sup>st</sup> order DEQ gives recursion

$$\mathbf{I}_k = \frac{1}{k} \sum_{j=0}^{\# \text{ finite}} \mathcal{A}_j \mathbf{I}_{k-j-1} \quad \text{with } A_t(t) = \sum_{j=0}^{\# \text{ finite}} \mathcal{A}_j t^j$$

- Stop recursion at desired  $k$ , evaluate  $\mathbf{I}(t = 1)$
- Change values for  $(m^2, \lambda)$  and iterate

$(L = 2 \text{ each direction}, m^2 = 1)$

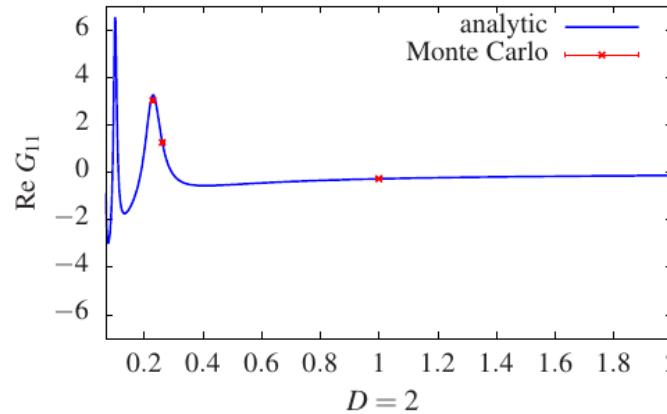
$$G_{110\ldots 0} = \frac{I_{110\ldots 0}}{I_{000\ldots 0}} \quad \text{Euclidean}$$



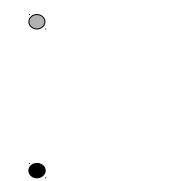
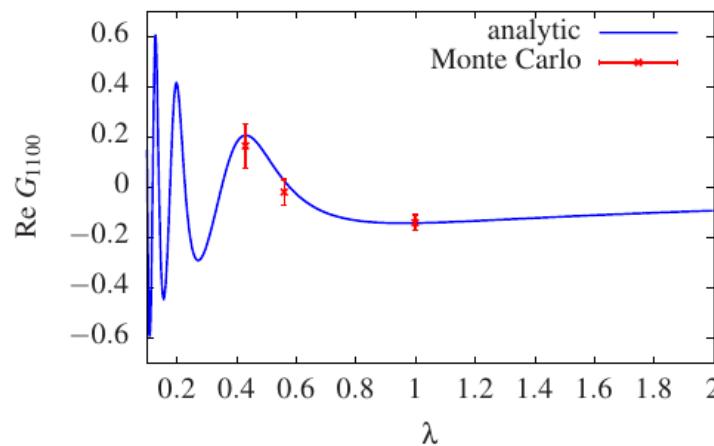
$(L = 2 \text{ each direction}, m^2 = 1)$

$$G_{110\ldots0} = \frac{I_{110\ldots0}}{I_{000\ldots0}} \quad \text{Minkowskian}$$

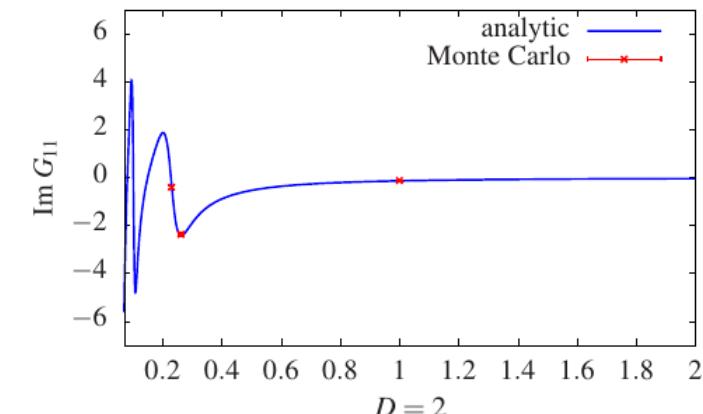
$D = 1$



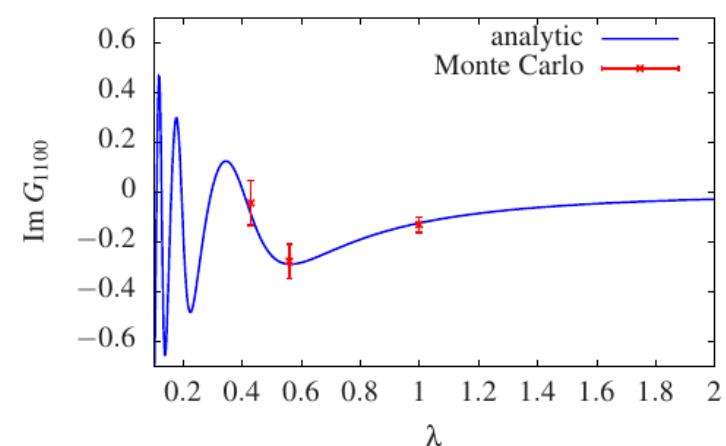
$D = 2$



$D = 1$



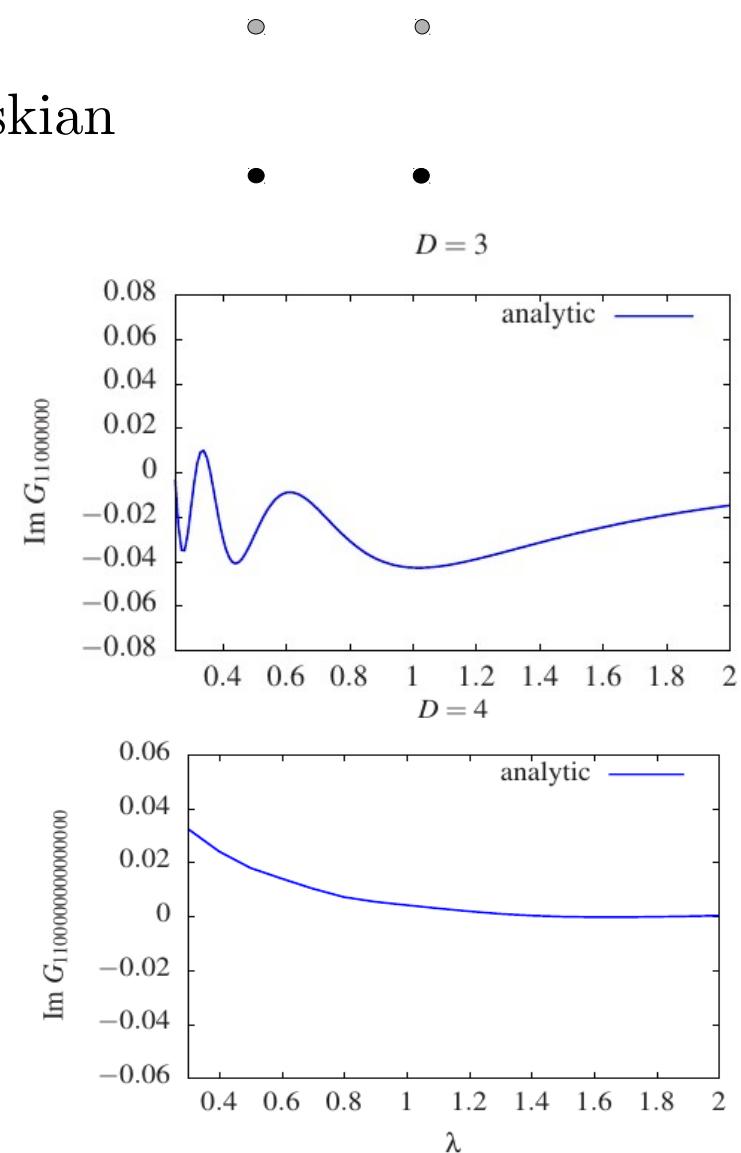
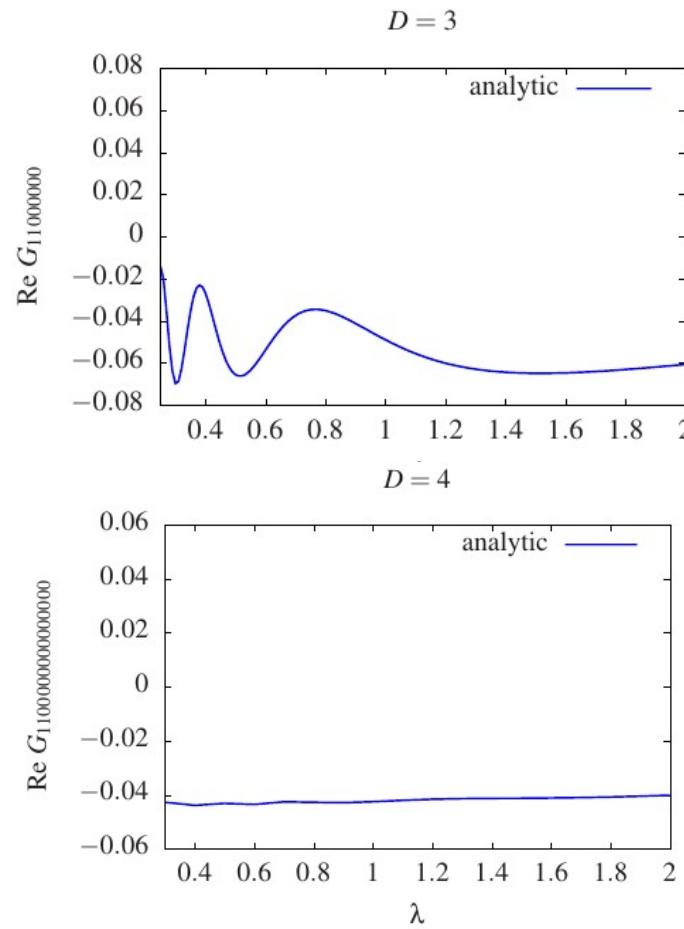
$D = 2$



$(L = 2 \text{ each direction}, m^2 = 1)$

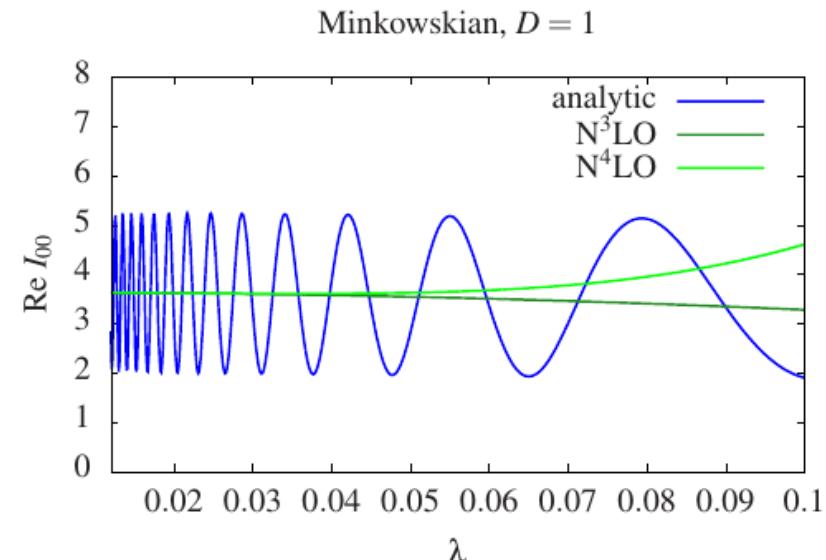
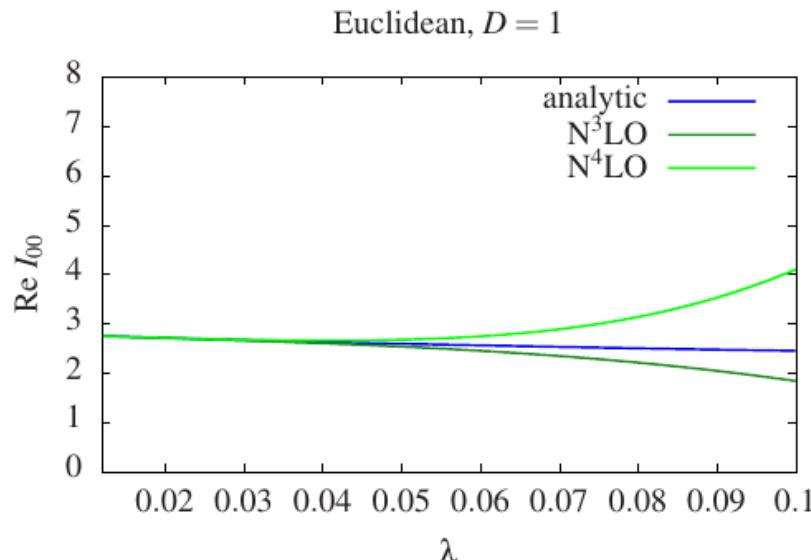
$$G_{110\ldots 0} = \frac{I_{110\ldots 0}}{I_{000\ldots 0}}$$

Minkowskian



$(L = 2 \text{ each direction}, m^2 = 1)$

## Small Coupling $\rightsquigarrow$ comparison with perturbation theory



Qualitative behaviour

$$A + Be^{-\frac{c}{\lambda}} \quad (c > 0)$$

$$A + Be^{i\frac{c}{\lambda}} \quad (c > 0)$$

$(A \text{ from perturbation theory})$

## Conclusions

- Lattice correlation functions studied within twisted Co-Homology
- Methods from perturbation theory transferred to non-perturbative physics
  - reduction to integral basis  $\oplus$  system 1<sup>st</sup> order DEQ
    - “auxiliary flow”  $t$
- Applicable to both Euclidean *and* Minkowskian signature

## Future directions

- Organize calculation more efficiently for bigger lattices
- Apply to Yang-Mills theory