3-loop tadpoles with substructure from 12 elliptic curves

David Broadhurst, Open University, UK, 29 May 2023
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The generic **2-loop kite** integral has **5** internal masses. Its completion by a **sixth** propagator gives a **3-loop tadpole** whose substructure involves **12 elliptic curves**. I shall show how to compute all such kites and their tadpoles, with 200 digit precision achieved in seconds, thanks to the procedure of the **arithmetic geometric mean** for complete **elliptic integrals** of the **third** kind. The number theory of 3-loop tadpoles poses challenges for packages such as `HyperInt`.

1. If you know the **discontinuity** σ of f, use a **dispersion relation** to get f.
2. If you know the **derivative** σ', integrate that against a **log**.
3. For the 2-loop **photon** propagator, σ' has logs, so f has **tri-logs**.
4. For the 2-loop **electron** propagator, σ' is **elliptic**, so f is harder to compute.
5. To determine a 3-loop **tadpole**, integrate an **elliptic** σ' against a **dilog**.

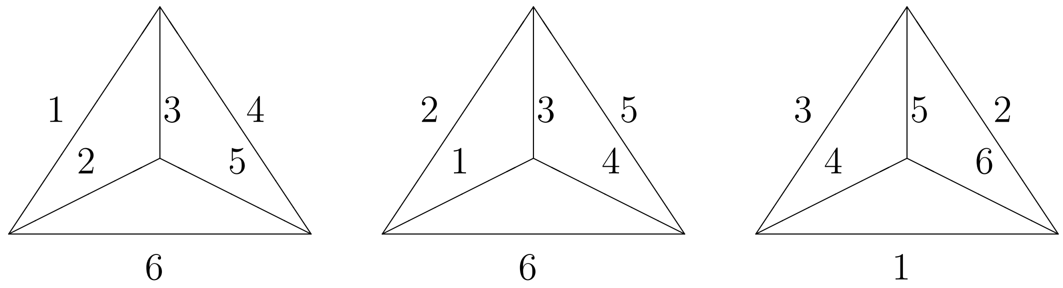
I define the 2-loop **scalar kite** integral in 4-dimensional Minkowski space as

$$I(q^2; m_1^2, m_2^2, m_3^2, m_4^2, m_5^2) = -\frac{q^2}{\pi^4} \int \mathrm{d}^4 l \int \mathrm{d}^4 k \prod_{j=1}^{5} \frac{1}{p_j^2 - m_j^2 - \mathrm{i}\epsilon}, \tag{1}$$

$$(p_1,\, p_2,\, p_3,\, p_4,\, p_5) = (l,\, l-q,\, l-k,\, k,\, k-q). \tag{2}$$

with a **cut** $s \in [s_\mathrm{L}, \infty]$ and a **branch point** s_L that is the lowest of the **thresholds** $\{s_{1,2}, s_{4,5}, s_{2,3,4}, s_{1,3,5}\}$, where $s_{j,k} = (m_j + m_k)^2$ and $s_{i,j,k} = (m_i + m_j + m_k)^2$. On the top lip of the cut, let $\Im I(s + \mathrm{i}\epsilon) = \pi\sigma(s)$. Then we have the **dispersion relation**

$$I(q^2) = -\int_{s_\mathrm{L}}^{\infty} \mathrm{d}s\, \sigma'(s) \log\left(1 - \frac{q^2}{s}\right). \tag{3}$$



Regularization in $4-2\epsilon$ dimensions of tetrahedral **tadpole** formed by joining the external vertices of the kite with a propagator $1/(q^2-m_6^2)$ gives

$$T_{1,2,3}^{5,4,6} = \left(\frac{1}{3\epsilon}+1\right)6\zeta_3 + 3\zeta_4 - F_{1,2,3}^{5,4,6} + O(\epsilon), \tag{4}$$

$$F_{1,2,3}^{5,4,6} = \int_{s_{\mathrm{L}}}^{\infty} \mathrm{d}s\,\sigma'(s;m_1^2,m_2^2,m_3^2,m_4^2,m_5^2)\left(\mathrm{Li}_2\left(1-\frac{m_6^2}{s}\right)+\frac{1}{2}\log^2\left(\frac{\overline{m}^2}{s}\right)\right) \tag{5}$$

where $\overline{m}$ is the scale of **dimensional regularization** and the **dilogarithm** is the analytic continuation of the sum $\mathrm{Li}_2(z) = \sum_{n>0} z^n/n^2$. In the totally massive case we can detect **12 elliptic curves** inherent in the tetrahedron.

In the absence of anomalous thresholds, the **non-elliptic** contribution is

$$\sigma'_{\mathrm{N}}(s) = \Theta(s-s_{1,2})\sigma'_{1,2}(s) + \Theta(s-s_{4,5})\sigma'_{4,5}(s). \tag{6}$$

Denote the **square root** of the symmetric **Källén function** by

$$\Delta(a,b,c) = \sqrt{a^2+b^2+c^2-2(ab+bc+ca)} \tag{7}$$

with convenient **abbreviations** $\Delta_{j,k}(s) = \Delta(s,m_j^2,m_k^2)$ and $\Delta_{i,j,k} = \Delta_{j,k}(m_i^2)$.

$$D_{j,k}(s) = \frac{r}{s-(m_j-m_k)^2}\log\left(\frac{1+r}{1-r}\right), \quad r = \left(\frac{s-(m_j-m_k)^2}{s-(m_j+m_k)^2}\right)^{1/2} \tag{8}$$

provides the **logarithms** in

$$\Delta_{1,2}(s)\sigma'_{1,2}(s) = \Re\left((s+\alpha)D_{4,5}(s) + L_{4,5} + \sum_{i=0,+,-} C_i \frac{D_{4,5}(s)-D_{4,5}(s_i)}{s-s_i}\right) \tag{9}$$

with **constants**

$$C_0 = -(m_1^2-m_2^2)(m_4^2-m_5^2), \quad C_\pm = \alpha s_\pm + \beta, \quad L_{4,5} = \log\left(\frac{m_4 m_5}{m_3^2}\right),$$
$$\alpha = \frac{(m_1^2-m_4^2)(m_2^2-m_5^2)}{m_3^2} - m_3^2, \quad \beta = \frac{(m_1^2m_5^2-m_2^2m_4^2)(m_1^2-m_2^2-m_4^2+m_5^2)}{m_3^2},$$
$$s_0 = 0, \quad s_\pm = \frac{m_1^2+m_2^2-2m_3^2+m_4^2+m_5^2-\alpha}{2} \pm \frac{\Delta_{1,3,4}\Delta_{2,3,5}}{2m_3^2}$$

where $s_\pm$ locate **leading Landau singularities** of triangles that form the kite. To obtain $\sigma'_{4,5}$, exchange (m_1, m_2) with (m_4, m_5).

Elliptic contribution: This comes from **3-particle** intermediate states, giving

$$\sigma'_{\mathrm{E}}(s) = \Theta(s - s_{2,3,4})\sigma'_{2,3,4}(s) + \Theta(s - s_{1,3,5})\sigma'_{1,3,5}(s). \tag{10}$$

It contains **complete** elliptic integrals of the **third kind** of the form

$$P(n,k) = \frac{\Pi(n,k)}{\Pi(0,k)}, \quad \Pi(n,k) = \int_0^{\pi/2} \frac{d\theta}{(1 - n\sin^2\theta)\sqrt{1 - k^2\sin^2\theta}} \tag{11}$$

with $\Pi(0,k) = (\pi/2)/\mathrm{AGM}\,(1, \sqrt{1-k^2})$ determined by the **arithmetic-geometric mean** of Gauss. With $s = w^2$, an integration over the phase space of particles 2, 3 and 4 determines

$$k^2 = 1 - \frac{16 m_2 m_3 m_4 w}{W}, \quad W = (w_+^2 - m_+^2)(w_-^2 - m_-^2) \tag{12}$$

with $w_\pm = w \pm m_2$ and $m_\pm = m_3 \pm m_4$. Then I obtain

$$\sigma'_{2,3,4}(w^2) = \frac{4\pi m_3 m_4}{\mathrm{AGM}\,(\sqrt{16 m_2 m_3 m_4 w}, \sqrt{W})} \Re\left(\sum_{i=+,-} E_i \frac{P(n_i,k) - P(n_1,k)}{t_i - t_1}\right) \tag{13}$$

with coefficients and arguments given, as **compactly** as possible, by

$$
\begin{aligned}
E_\pm &= \frac{m_2^2 - m_3^2 + m_5^2}{2m_5^2} \pm \left(\frac{m_4^2 - m_5^2 - w^2}{2m_5^2}\right) \frac{\Delta_{2,3,5}}{\Delta_{4,5}(w^2)}, \\
t_\pm = \frac{\gamma \pm \Delta_{2,3,5}\Delta_{4,5}(w^2)}{2m_5^2}, &\quad t_1 = m_1^2, \quad n_i = \frac{(w_-^2 - m_+^2)(t_i - m_-^2)}{(w_-^2 - m_-^2)(t_i - m_+^2)}, \\
\gamma = (m_2^2 + m_3^2 + m_4^2 &- m_5^2 + w^2)m_5^2 + (m_2^2 - m_3^2)(m_4^2 - w^2).
\end{aligned}
$$

An **AGM procedure** speedily evaluates $P(n,k) = \Pi(n,k)/\Pi(0,k)$ to high precision:

1. **Initialize** $[a,\, b,\, p,\, q] = [1,\, \sqrt{1-k^2},\, \sqrt{1-n},\, n/(2-2n)]$. Then set $f = 1 + q$.
2. Set $m = ab$ and then $r = p^2 + m$. Compute a vector of **new values** as follows: $[(a+b)/2,\, \sqrt{m},\, r/(2p),\, (r-2m)q/(2r)]$. Then replace $[a,\, b,\, p,\, q]$ by those new values. Then add q to f.
3. If $|q/f|$ is sufficiently **small**, then return $P(n,k) = f$, else go to step 2.

This converges **very** quickly, for $n \notin [1, \infty]$. On the cut with $n \geq 1$, replace n by $n' = k^2/n < 1$, to obtain the **principal value** $\Re P(n,k) = 1 - P(n',k)$.

Criterion for an anomalous contribution: Suppose that $s_{4,5} \geq s_{1,2}$. Then

$$\sigma'(s) = \sigma'_{\mathrm{N}}(s) + \sigma'_{\mathrm{E}}(s) + C_{\mathrm{A}} \frac{\Theta(s - s_{4,5})}{\Delta_{4,5}(s)} \Re\left(\frac{2\pi \mathrm{i} \Delta_{4,5}(s_-)}{s - s_-}\right) \tag{14}$$

with $C_{\mathrm{A}} \neq 0$ **if and only if** $(m_1 + m_2)(m_3^2 + m_1 m_2) < m_1 m_5^2 + m_2 m_4^2$ and at least one of $\Delta_{1,3,4}$ and $\Delta_{2,3,5}$ is imaginary, in which case $C_{\mathrm{A}} = \pm 1$ is the sign of $\Im \Delta_{4,5}(s_-)$.

This value of C_{A} is required by the **elliptic** contribution at high energy. With $L_k = m_k^2 \log(s/m_k^2)$, the large-$s$ behaviour

$$s^2 \sigma'(s) = 2L_3 + \sum_{k=1,2,4,5} (L_k + m_k^2) + O\left(\frac{\log(s)}{s}\right) \tag{15}$$

invariably holds. The elliptic contribution σ'_{E} in (14) is oblivious to the anomalous threshold problem. Its high-energy behaviour determines C_{A}, ensuring (15).

Tadpoles and number theory

The **rescaling** $m_k \to \kappa\, m_k$ gives $F \to F + 12\zeta_3 \log(\kappa)$ for the **finite** part F.
To standardize, I set $\overline{m} = \max(m_k) = 1$.

I define a tetrahedral tadpole to be **perfect** if and only if the Källén function **vanishes** at each of its 4 vertices, thereby avoiding all resolutions of square roots. Promoting the subscripts and superscripts of F to arguments that denote the 6 masses, I define the two-parameter **family of perfect tadpoles**:

$$\widehat{F}(x,y) = F^{(1-y,1-x,|x-y|)}_{(x,y,1)} = \widehat{F}(y,x) = \widehat{F}(1-x,1-y) \tag{16}$$

with symmetries restricting distinct cases to $x \geq y \geq 1-x \geq 0$ and hence $x \in [\frac{1}{2}, 1]$.
In QED, I identified **tetralogarithms** in two perfect **binary** tadpoles, obtaining

$$\widehat{F}(1,0) = F^{(1,1,0)}_{(1,1,0)} = 17\zeta_4 + 16U_{3,1}, \quad \widehat{F}(1,1) = F^{(0,0,0)}_{(1,1,1)} = 12\zeta_4, \tag{17}$$

$$U_{3,1} = \sum_{m>n>0} \frac{(-1)^{m+n}}{m^3 n} = \tfrac{1}{2}\zeta_4 + \tfrac{1}{2}\zeta_2 \log^2(2) - \tfrac{1}{12}\log^4(2) - 2\,\mathrm{Li}_4(\tfrac{1}{2}). \tag{18}$$

Fast elliptic determination of a perfect tadpole

Now consider the elliptic route to evaluating $\widehat{F}(\frac{1}{2},\frac{1}{2})$. With $(m_3,m_6)=(1,0)$ and $m_1=m_2=m_4=m_5=\frac{1}{2}$, I obtained

$$\widehat{F}(\tfrac{1}{2},\tfrac{1}{2}) = \tfrac{1}{2}\int_1^\infty \mathrm{d}s\,(\widehat{\sigma}'_{\mathrm{N}}(s)+\widehat{\sigma}'_{\mathrm{E}}(s))\log^2(s), \tag{19}$$

$$w^2\widehat{\sigma}'_{\mathrm{N}}(w^2) = \Theta(w-1)\left(2\log\left(\frac{r+1}{r-1}\right)-4r\log(2)\right),\ r=\frac{w}{\sqrt{w^2-1}}, \tag{20}$$

$$w^2\widehat{\sigma}'_{\mathrm{E}}(w^2) = \frac{4\pi(1-P(n,k))\Theta(w-2)}{\mathrm{AGM}(2\sqrt{w},(w-1)\sqrt{w^2+2w})},\ n=\frac{w^2-2w}{(w-1)^2},\ \frac{k^2}{n}=\frac{(w+1)^2}{w^2+2w} \tag{21}$$

and readily discovered a **new** reduction of a perfect tadpole to tetralogarithms

$$\widehat{F}(\tfrac{1}{2},\tfrac{1}{2}) = 30\zeta_3\log(2)-16\zeta_4-32U_{3,1}. \tag{22}$$

Relations between tadpoles

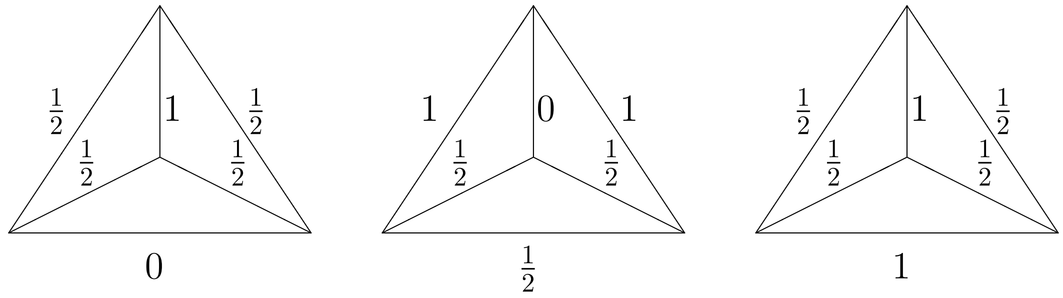


Figure 2: The perfect tadpoles $\widehat{F}(\frac{1}{2},\frac{1}{2})$, $\widehat{F}(1,\frac{1}{2})$ and $\widehat{G}(\frac{1}{2})$ in relation (24)

In addition to the two-parameter family $\widehat{F}(x,y)$, there is a one-parameter family $\widehat{G}(x)=F^{(x,1-x,1)}_{(x,1-x,1)}$ of perfect tadpoles, with $x\in[0,\frac{1}{2}]$ and $\widehat{G}(0)=17\zeta_4+16U_{3,1}$.
I used the efficient AGM of Gauss to obtain **200 digits** of

$$\widehat{G}(\tfrac{1}{2})=F^{(\frac{1}{2},\frac{1}{2},1)}_{(\frac{1}{2},\frac{1}{2},1)}=-\int_1^\infty \mathrm{d}s\,(\widehat{\sigma}'_{\mathrm{N}}(s)+\widehat{\sigma}'_{\mathrm{E}}(s))\,\mathrm{Li}_2(1-s) \tag{23}$$

to which all routes are **elliptic**. This revealed the intriguing **empirical** relation

$$2\widehat{F}(\tfrac{1}{2},\tfrac{1}{2})+2\widehat{F}(1,\tfrac{1}{2})+\widehat{G}(\tfrac{1}{2})=42\zeta_4+24\zeta_3\log(2). \tag{24}$$

A non-elliptic route to $\widehat{F}(1,\frac{1}{2})$ led to multiple polylogarithms in an alphabet of forms, $\mathrm{d}x/(x-a_i)$, with $a_i \in \{0,1,-1,-2\}$. After help with these, from **Steven Charlton**, I then found the integer relation

$$\widehat{G}(\tfrac{1}{2}) = 6\left(2\zeta_4 - 3\mathrm{Li}_4(\tfrac{1}{4})\right) + 8\left(2\zeta_3 - 3\mathrm{Li}_3(\tfrac{1}{4})\right)L - 12\,\mathrm{Li}_2(\tfrac{1}{4})L^2 - 4L^4 \tag{25}$$

with $L = \log(2)$ and **classical** polylogs giving 10000 digits in less than a second.

Binary tadpoles, with $m_k \in \{0,1\}$, evaluate to multiple polylogarithms in an alphabet containing **sixth roots** of unity, with $\lambda = (1+\sqrt{-3})/2$ appearing if three massive edges meet at a vertex, where $\Delta_{i,j,k} = \sqrt{-3}$. For example, with 5 unit edges

$$F^{(1,1,0)}_{(1,1,1)} = \frac{109}{6}\left(\frac{\pi}{3}\right)^4 + 16\Re\left(\frac{\mathrm{Li}_2^2(\lambda)}{6} + \sum_{m>n>0}\frac{\lambda^{3m+2n}}{m^3 n}\right). \tag{26}$$

There are **linear relations** between binary tadpoles, as here:

$$3F^{(1,1,1)}_{(0,0,0)} = F^{(0,0,0)}_{(1,1,1)} + 2F^{(1,0,0)}_{(1,1,0)}\,, \tag{27}$$

$$3F^{(0,0,0)}_{(1,1,0)} = F^{(0,0,0)}_{(1,0,0)} + 2F^{(0,0,0)}_{(1,1,1)}\,, \tag{28}$$

$$F^{(\mathbf{1},\mathbf{1},\mathbf{1})}_{(\mathbf{1},\mathbf{1},\mathbf{1})} + F^{(1,0,0)}_{(1,0,0)} = F^{(1,1,0)}_{(1,1,0)} + F^{(1,1,1)}_{(0,0,0)}\,. \tag{29}$$

Number fields of the alphabets of tadpoles

So far, one might **guess** that a tadpole with rational masses evaluates to multiple tetralogarithms in an alphabet whose number field is no larger than the **compositum** of the **quadratic** number fields associated by Gunnar Källén to the vertices of the tetrahedron, namely the field $Q(\Delta_{1,3,4}, \Delta_{2,3,5}, \Delta_{1,2,6}, \Delta_{4,5,6})$.

Yet that is **not** the case. The **imperfect** binary tadpole $F^{(1,0,0)}_{(1,1,0)}$ involves $\Re\mathrm{Li}^2_2(\lambda)$, but the Källén field is **rational**.

Faced with this rather limited, yet potent, evidence, I arrive at three **suggestions**, each too weak to be dignified as a well-tested conjecture.

1. Every tetrahedral tadpole with rational masses reduces to multiple or single tetralogarithms whose alphabet lies in an algebraic number field.
2. If the tadpole is perfect, the alphabet is rational.
3. If the tadpole is imperfect, the alphabetic field may include the Källén field.

Experimentum crucis: I found an empirical relation between the totallly massive **imperfect** tadpole $F^{(1,1,1)}_{(\frac{1}{2},\frac{1}{2},\frac{1}{2})}$ with Källén field $Q(\sqrt{-3})$ and the perfect tadpole $\widehat{G}(\frac{1}{2})$, already evaluated in terms of classical polylogs:

$$F^{(1,1,1)}_{(\frac{1}{2},\frac{1}{2},\frac{1}{2})} = 3\zeta_3\log(2) - 4U_{3,1} + 10\zeta_4 + 10\mathrm{Cl}_2^2(\pi/3) - \tfrac{1}{2}\widehat{G}(\tfrac{1}{2}) \tag{30}$$

with a Clausen value $\mathrm{Cl}_2(\pi/3) = \Im\mathrm{Li}_2(\lambda)$, from the Källén field. It took less than a minute to validate (30) at 600-digit precision.

Tests and benchmarks for kites and tadpoles

1. Elliptic terms do not depend on the order of phase-space integrations.
2. The derivative of the discontinuity of a kite satisfies the sum rule
$$\int_{s_\mathrm{L}}^{\infty} \mathrm{d}s\,\sigma'(s)\log\left(\frac{s}{s_\mathrm{L}}\right) = 6\zeta_3. \tag{31}$$
3. The high energy behaviour of $s^2\sigma'(s)$ holds irrespective of anomalous thresholds.
4. The same tadpole is obtained by integrating over 6 distinct kites.

These tests were invariably passed, at high precision, in a plethora of cases.

Benchmark 1: A useful benchmark was established by Stefan Bauberger and Manfred Böhm, who gave 6 decimal digits of $B_1 = I(50+\mathrm{i}\epsilon; 1, 2, 3, 4, 5)/50$, with all 4 cuts opened. For B_1, I obtain the value

```
+0.173901219069555460362391997806756419040779085211744093645075
-0.118080028202009293890731446888246675922194086181504660940640*I
```

Benchmark 2: Stephen Martin computed 8 digits of $B_2 = -I(10+\mathrm{i}\epsilon; 1, 3, 5, 2, 4)/10$, in a non-anomalous case with only one open cut. For B_2, I obtain the value

```
+0.7183353533534129653528554796276560425262176802655670356407
+0.390162199972762321424365961074218884677858368327292408622989*I
```

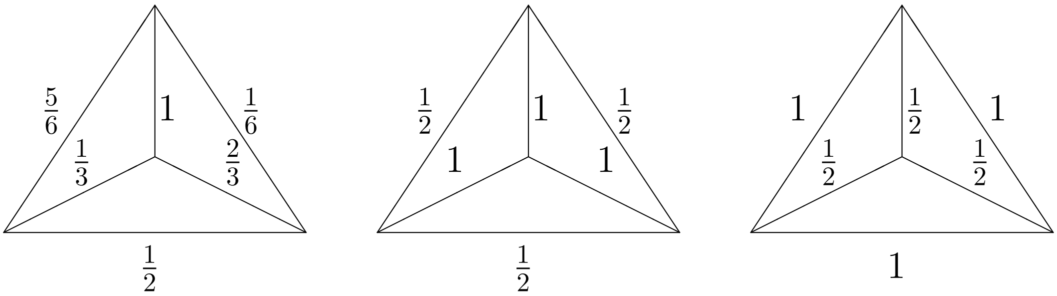


Figure 3: Tadpoles for benchmarks B_3, B_4 and B_5

The benchmarks of Figure 3 are ambitious targets for adept users of `HyperInt`.

Benchmark 3: The first example in Figure 3 is the simplest **perfect** tadpole with 6 **distinct** non-zero rational masses. I suggest that its alphabet may be rational. For its finite part $B_3 = \widehat{F}(\frac{5}{6}, \frac{1}{3})$, I obtain

`13.386145534873902269761545032722855218524865485549746470 8212`

Benchmark 4: The second example is an **imperfect** tadpole The benchmark for $B_4 = F^{(1,1,1)}_{(\frac{1}{2},\frac{1}{2},\frac{1}{2})}$ is

`16.605954281198022808164888007314169734724382432117664354 1089`

Benchmark 5: The third example is **doubly imperfect**. I suggest that its alphabetic field may include $Q(\sqrt{-3}, \sqrt{5})$. The benchmark for $B_5 = F^{(\frac{1}{2},\frac{1}{2},\frac{1}{2})}_{(1,1,1)}$ is

`16.589699907187102254889131728013166971196806164364336112 1466`

Comments and summary

1. Elliptic substructure of 2-loop kites and 3-loop tadpoles is not a problem. The time taken to evaluate a complete elliptic integral, of whatever kind, is commensurate with the time for a logarithm and less than the time for a dilogarithm. Thanks to Gauss, elliptic integrals should be embraced, not feared.

2. Anomalous terms are not problematic. They submit to Gauss, at high energy.

3. The number theory of tadpoles is subtle. They may be polylogarithmic, even in totally massive cases to which every route is elliptic.

4. I have given far-reaching suggestions on the number theory of tadpoles and benchmarks for users of `HyperInt` to investigate those suggestions analytically.

Appendix with zero-mass limits: As $m_3 \to 0$ with $m_1 \neq m_4$ and $m_2 \neq m_5$,

$$\Delta_{1,2}(s)\sigma'_{1,2}(s) \to \Re\left((2s - s_3)D_{4,5}(s) + \widehat{L}_{4,5} + \sum_{i=0,3} C_i \, \frac{D_{4,5}(s) - D_{4,5}(s_i)}{s - s_i}\right),$$

$$s_3 = -\frac{(m_1^2 m_5^2 - m_2^2 m_4^2)(m_1^2 - m_2^2 - m_4^2 + m_5^2)}{M}, \quad M = (m_1^2 - m_4^2)(m_2^2 - m_5^2),$$

$$\widehat{L}_{4,5} = \log\left(\frac{m_4^2 m_5^2}{M}\right), \quad C_3 = -\left(\frac{m_1^2}{u} - m_2^2 u\right)\left(\frac{m_4^2}{u} - m_5^2 u\right), \quad u = \frac{m_1^2 - m_4^2}{m_2^2 - m_5^2}.$$

As $m_3 \to 0$ with $m_1 = m_4$ and $m_2 \neq m_5$

$$\begin{aligned}\Delta_{2,4}(s)\sigma'_{1,2}(s) \to \Re\Bigg((3s - m_2^2 - 2m_4^2 - m_5^2)D_{4,5}(s) + \log\left(\frac{m_4 m_5^3}{(m_2^2 - m_5^2)^2}\right) \\ + (m_2^2 - m_4^2)(m_4^2 - m_5^2)\, \frac{D_{4,5}(s) - D_{4,5}(0)}{s}\Bigg).\end{aligned}$$

The degenerate case with $m_1 = m_4$ and $m_2 = m_5$ will be considered after adding contributions from three-particle cuts.

As $m_3 \to 0$, the three-particle cuts yield logarithms:

$$\sigma'_{2,3,4}(w^2) \to \Re\left(\sum_{i=\pm} E_i \frac{\widehat{P}_{2,4}(t_i,w) - \widehat{P}_{2,4}(m_1^2,w)}{t_i - m_1^2}\right),$$

$$\widehat{P}_{j,k}(t,w) = \frac{(m_k^2 - t)v(t)}{(w-m_j)^2 - t}\log\left(\frac{v(t)+v(m_k^2)}{v(t)-v(m_k^2)}\right), \quad v(t) = \left(\frac{(w-m_j)^2 - t}{(w+m_j)^2 - t}\right)^{1/2}.$$

With $m_1 = m_4$ and $m_2 = m_5$ all four thresholds collide as $m_3 \to 0$, giving

$$\sigma'(s) \to \Theta(s - s_{4,5})\,\frac{2\mu(y_4) + 2\mu(y_5) - 8\mu(y_4 y_5)}{\Delta_{4,5}(s)},$$

$$\mu(y) = \log|1-y| + \frac{y\log|y|}{1-y}, \quad y_k = \frac{-2m_k^2}{s - m_4^2 - m_5^2 + \Delta_{4,5}(s)}.$$

Next, consider cases with $m_3 > 0$ and one of the other masses vanishing. Without loss of generality, take it to be m_4. As $m_4 \to 0$, logarithms from appear in

$$\sigma'_{2,3,4}(w^2) \to \Re\left(\sum_{i=\pm} E_i \frac{\widehat{P}_{2,3}(t_i,w) - \widehat{P}_{2,3}(m_1^2,w)}{t_i - m_1^2}\right).$$

The logarithms for two-particle cuts are modified, as $m_4 \to 0$, to give

$$\Delta_{1,2}(s)\sigma'_{1,2}(s) \to \Re\left((s+\alpha)\widehat{D}_5(s) + \widehat{L}_5 + \sum_{i=0,+,-} C_i\,\frac{\widehat{D}_5(s)-\widehat{D}_5(s_i)}{s-s_i}\right),$$

$$\widehat{D}_5(s) = \frac{1}{s-m_5^2}\log\left(1-\frac{s}{m_5^2}\right), \quad \widehat{L}_5 = \log\left(\frac{m_5^2}{m_3^2}\right).$$

An elliptic contribution persists if two **non-adjacent** edges have vanishing mass. As $m_1 \to 0$ and $m_5 \to 0$,

$$(w^2-m_4^2)\sigma'_{2,3,4}(w^2) \;\to\; -\frac{4\pi m_3 m_4 \Re R(w^2,m_2^2,m_3^2,m_4^2)}{\mathrm{AGM}\left(\sqrt{16m_2m_3m_4w},\sqrt{W}\right)},$$

$$R(s,b,c,d) = P(\widehat{n},k) - \rho P(n_0,k) + (\rho-1)P(n_3,k),$$

$$\widehat{n} = \frac{w_-^2-m_+^2}{w_-^2-m_-^2}, \quad \frac{n_0}{\widehat{n}} = \frac{m_-^2}{m_+^2}, \quad \frac{n_3}{\widehat{n}} = \frac{t_3-m_-^2}{t_3-m_+^2}, \quad t_3 = \frac{(bd-cs)(b-c+d-s)}{(b-c)(d-s)},$$

$$\rho = \left(\frac{d-s}{b-c+d-s}\right)\left(\frac{(b+c)(d-s)+(b-c)(b+d)}{bd-cs}\right),$$

$$R(s,c,c,d) = 2P(\widehat{n},k) - 2P(n_0,k), \quad R(s,d,d,d) = \frac{s-9d}{6d}$$

with a **rational** result for R in the QED case $m_2 = m_3 = m_4$.