3-loop tadpoles with substructure from 12 elliptic curves David Broadhurst, Open University, UK, 29 May 2023 at RadCor23 in Crieff, Scotland

The generic **2-loop kite** integral has **5** internal masses. Its completion by a **sixth** propagator gives a **3-loop tadpole** whose substructure involves **12 elliptic curves**. I shall show how to compute all such kites and their tadpoles, with 200 digit precision achieved in seconds, thanks to the procedure of the **arithmetic geometric mean** for complete **elliptic integrals** of the **third** kind. The number theory of 3-loop tadpoles poses challenges for packages such as HyperInt.

- 1. If you know the **discontinuity** σ of f, use a **dispersion relation** to get f.
- 2. If you know the **derivative** σ' , integrate that against a log.
- 3. For the 2-loop **photon** propagator, σ' has logs, so f has **tri-logs**.
- 4. For the 2-loop electron propagator, σ' is elliptic, so f is harder to compute.
- 5. To determine a 3-loop tadpole, integrate an elliptic σ' against a dilog.

I define the 2-loop scalar kite integral in 4-dimensional Minkowski space as

$$I(q^2; m_1^2, m_2^2, m_3^2, m_4^2, m_5^2) = -\frac{q^2}{\pi^4} \int d^4l \int d^4k \prod_{j=1}^5 \frac{1}{p_j^2 - m_j^2 - i\epsilon},$$
 (1)

$$(p_1, p_2, p_3, p_4, p_5) = (l, l-q, l-k, k, k-q).$$
 (2)

with a cut $s \in [s_L, \infty]$ and a branch point s_L that is the lowest of the thresholds $\{s_{1,2}, s_{4,5}, s_{2,3,4}, s_{1,3,5}\}$, where $s_{j,k} = (m_j + m_k)^2$ and $s_{i,j,k} = (m_i + m_j + m_k)^2$. On the top lip of the cut, let $\Im I(s + i\epsilon) = \pi \sigma(s)$. Then we have the **dispersion relation**

$$I(q^2) = -\int_{s_{\rm L}}^{\infty} \mathrm{d}s \,\sigma'(s) \log\left(1 - \frac{q^2}{s}\right). \tag{3}$$



Regularization in $4 - 2\epsilon$ dimensions of tetrahedral **tadpole** formed by joining the external vertices of the kite with a propagator $1/(q^2 - m_6^2)$ gives

$$T_{1,2,3}^{5,4,6} = \left(\frac{1}{3\epsilon} + 1\right) 6\zeta_3 + 3\zeta_4 - F_{1,2,3}^{5,4,6} + O(\epsilon), \tag{4}$$

$$F_{1,2,3}^{5,4,6} = \int_{s_{\rm L}}^{\infty} \mathrm{d}s \, \sigma'(s; m_1^2, m_2^2, m_3^2, m_4^2, m_5^2) \left(\mathrm{Li}_2 \left(1 - \frac{m_6^2}{s} \right) + \frac{1}{2} \log^2 \left(\frac{\overline{m}^2}{s} \right) \right) \tag{5}$$

where \overline{m} is the scale of **dimensional regularization** and the **dilogarithm** is the analytic continuation of the sum $\text{Li}_2(z) = \sum_{n>0} z^n/n^2$. In the totally massive case we can detect **12 elliptic curves** inherent in the tetrahedron.

In the absence of anomalous thresholds, the **non-elliptic** contribution is

$$\sigma'_{\rm N}(s) = \Theta(s - s_{1,2})\sigma'_{1,2}(s) + \Theta(s - s_{4,5})\sigma'_{4,5}(s).$$
(6)

Denote the square root of the symmetric Källén function by

$$\Delta(a, b, c) = \sqrt{a^2 + b^2 + c^2 - 2(ab + bc + ca)}$$
(7)

with convenient **abbreviations** $\Delta_{j,k}(s) = \Delta(s, m_j^2, m_k^2)$ and $\Delta_{i,j,k} = \Delta_{j,k}(m_i^2)$.

$$D_{j,k}(s) = \frac{r}{s - (m_j - m_k)^2} \log\left(\frac{1+r}{1-r}\right), \quad r = \left(\frac{s - (m_j - m_k)^2}{s - (m_j + m_k)^2}\right)^{1/2}$$
(8)

provides the **logarithms** in

$$\Delta_{1,2}(s)\sigma_{1,2}'(s) = \Re\left((s+\alpha)D_{4,5}(s) + L_{4,5} + \sum_{i=0,+,-}C_i\frac{D_{4,5}(s) - D_{4,5}(s_i)}{s-s_i}\right)$$
(9)

with constants

$$C_{0} = -(m_{1}^{2} - m_{2}^{2})(m_{4}^{2} - m_{5}^{2}), \quad C_{\pm} = \alpha s_{\pm} + \beta, \quad L_{4,5} = \log\left(\frac{m_{4}m_{5}}{m_{3}^{2}}\right),$$

$$\alpha = \frac{(m_{1}^{2} - m_{4}^{2})(m_{2}^{2} - m_{5}^{2})}{m_{3}^{2}} - m_{3}^{2}, \quad \beta = \frac{(m_{1}^{2}m_{5}^{2} - m_{2}^{2}m_{4}^{2})(m_{1}^{2} - m_{2}^{2} - m_{4}^{2} + m_{5}^{2})}{m_{3}^{2}},$$

$$s_{0} = 0, \quad s_{\pm} = \frac{m_{1}^{2} + m_{2}^{2} - 2m_{3}^{2} + m_{4}^{2} + m_{5}^{2} - \alpha}{2} \pm \frac{\Delta_{1,3,4}\Delta_{2,3,5}}{2m_{3}^{2}}$$

where s_{\pm} locate **leading Landau singularities** of triangles that form the kite. To obtain $\sigma'_{4,5}$, exchange (m_1, m_2) with (m_4, m_5) .

Elliptic contribution: This comes from 3-particle intermediate states, giving

$$\sigma'_{\rm E}(s) = \Theta(s - s_{2,3,4})\sigma'_{2,3,4}(s) + \Theta(s - s_{1,3,5})\sigma'_{1,3,5}(s).$$
(10)

It contains **complete** elliptic integrals of the **third kind** of the form

$$P(n,k) = \frac{\Pi(n,k)}{\Pi(0,k)}, \quad \Pi(n,k) = \int_0^{\pi/2} \frac{d\theta}{(1-n\sin^2\theta)\sqrt{1-k^2\sin^2\theta}}$$
(11)

with $\Pi(0,k) = (\pi/2)/\text{AGM}(1,\sqrt{1-k^2})$ determined by the **arithmetic-geometric mean** of Gauss. With $s = w^2$, an integration over the phase space of particles 2, 3 and 4 determines

$$k^{2} = 1 - \frac{16m_{2}m_{3}m_{4}w}{W}, \quad W = (w_{+}^{2} - m_{+}^{2})(w_{-}^{2} - m_{-}^{2})$$
 (12)

with $w_{\pm} = w \pm m_2$ and $m_{\pm} = m_3 \pm m_4$. Then I obtain

$$\sigma_{2,3,4}'(w^2) = \frac{4\pi m_3 m_4}{\text{AGM}\left(\sqrt{16m_2 m_3 m_4 w}, \sqrt{W}\right)} \Re\left(\sum_{i=+,-} E_i \frac{P(n_i, k) - P(n_1, k)}{t_i - t_1}\right)$$
(13)

with coefficients and arguments given, as **compactly** as possible, by

$$E_{\pm} = \frac{m_2^2 - m_3^2 + m_5^2}{2m_5^2} \pm \left(\frac{m_4^2 - m_5^2 - w^2}{2m_5^2}\right) \frac{\Delta_{2,3,5}}{\Delta_{4,5}(w^2)},$$

$$t_{\pm} = \frac{\gamma \pm \Delta_{2,3,5} \Delta_{4,5}(w^2)}{2m_5^2}, \quad t_1 = m_1^2, \quad n_i = \frac{(w_-^2 - m_+^2)(t_i - m_-^2)}{(w_-^2 - m_-^2)(t_i - m_+^2)},$$

$$\gamma = (m_2^2 + m_3^2 + m_4^2 - m_5^2 + w^2)m_5^2 + (m_2^2 - m_3^2)(m_4^2 - w^2).$$

An **AGM procedure** speedily evaluates $P(n, k) = \Pi(n, k) / \Pi(0, k)$ to high precision:

- 1. Initialize $[a, b, p, q] = [1, \sqrt{1-k^2}, \sqrt{1-n}, n/(2-2n)]$. Then set f = 1+q.
- 2. Set m = ab and then $r = p^2 + m$. Compute a vector of **new values** as follows: $[(a+b)/2, \sqrt{m}, r/(2p), (r-2m)q/(2r)]$. Then replace [a, b, p, q] by those new values. Then add q to f.
- 3. If |q/f| is sufficiently small, then return P(n,k) = f, else go to step 2.

This converges **very** quickly, for $n \notin [1, \infty]$. On the cut with $n \ge 1$, replace n by $n' = k^2/n < 1$, to obtain the **principal value** $\Re P(n, k) = 1 - P(n', k)$.

Criterion for an anomalous contribution: Suppose that $s_{4,5} \ge s_{1,2}$. Then

$$\sigma'(s) = \sigma'_{\rm N}(s) + \sigma'_{\rm E}(s) + C_{\rm A} \frac{\Theta(s - s_{4,5})}{\Delta_{4,5}(s)} \,\Re\left(\frac{2\pi i \Delta_{4,5}(s_{-})}{s - s_{-}}\right) \tag{14}$$

with $C_A \neq 0$ if and only if $(m_1 + m_2)(m_3^2 + m_1m_2) < m_1m_5^2 + m_2m_4^2$ and at least one of $\Delta_{1,3,4}$ and $\Delta_{2,3,5}$ is imaginary, in which case $C_A = \pm 1$ is the sign of $\Im \Delta_{4,5}(s_-)$. This value of C_A is required by the **elliptic** contribution at high energy. With $L_k = m_k^2 \log(s/m_k^2)$, the large-s behaviour

$$s^{2}\sigma'(s) = 2L_{3} + \sum_{k=1,2,4,5} (L_{k} + m_{k}^{2}) + O\left(\frac{\log(s)}{s}\right)$$
(15)

invariably holds. The elliptic contribution $\sigma'_{\rm E}$ in (14) is oblivious to the anomalous threshold problem. Its high-energy behaviour determines $C_{\rm A}$, ensuring (15).

Tadpoles and number theory

The rescaling $m_k \to \kappa m_k$ gives $F \to F + 12\zeta_3 \log(\kappa)$ for the finite part F. To standardize, I set $\overline{m} = \max(m_k) = 1$.

I define a tetrahedral tadpole to be **perfect** if and only if the Källén function **vanishes** at each of its 4 vertices, thereby avoiding all resolutions of square roots. Promoting the subscripts and superscripts of F to arguments that denote the 6 masses, I define the two-parameter **family of perfect tadpoles**:

$$\widehat{F}(x,y) = F_{(x,y,1)}^{(1-y,1-x,|x-y|)} = \widehat{F}(y,x) = \widehat{F}(1-x,1-y)$$
(16)

with symmetries restricting distinct cases to $x \ge y \ge 1 - x \ge 0$ and hence $x \in [\frac{1}{2}, 1]$. In QED, I identified **tetralogarithms** in two perfect **binary** tadpoles, obtaining

$$\widehat{F}(1,0) = F_{(1,1,0)}^{(1,1,0)} = 17\zeta_4 + 16U_{3,1}, \quad \widehat{F}(1,1) = F_{(1,1,1)}^{(0,0,0)} = 12\zeta_4, \tag{17}$$

$$U_{3,1} = \sum_{m>n>0} \frac{(-1)^{m+n}}{m^3 n} = \frac{1}{2}\zeta_4 + \frac{1}{2}\zeta_2 \log^2(2) - \frac{1}{12}\log^4(2) - 2\operatorname{Li}_4(\frac{1}{2}).$$
(18)

Fast elliptic determination of a perfect tadpole

Now consider the elliptic route to evaluating $\widehat{F}(\frac{1}{2},\frac{1}{2})$. With $(m_3,m_6) = (1,0)$ and $m_1 = m_2 = m_4 = m_5 = \frac{1}{2}$, I obtained

$$\widehat{F}(\frac{1}{2},\frac{1}{2}) = \frac{1}{2} \int_{1}^{\infty} \mathrm{d}s \left(\widehat{\sigma}'_{\mathrm{N}}(s) + \widehat{\sigma}'_{\mathrm{E}}(s)\right) \log^{2}(s), \tag{19}$$

$$w^{2}\widehat{\sigma}_{N}'(w^{2}) = \Theta(w-1)\left(2\log\left(\frac{r+1}{r-1}\right) - 4r\log(2)\right), \ r = \frac{w}{\sqrt{w^{2}-1}},$$
(20)

$$w^{2}\widehat{\sigma}_{\rm E}'(w^{2}) = \frac{4\pi(1 - P(n,k))\Theta(w-2)}{\mathrm{AGM}(2\sqrt{w}, (w-1)\sqrt{w^{2} + 2w})}, \ n = \frac{w^{2} - 2w}{(w-1)^{2}}, \ \frac{k^{2}}{n} = \frac{(w+1)^{2}}{w^{2} + 2w}$$
(21)

and readily discovered a \mathbf{new} reduction of a perfect tadpole to tetralogarithms

$$\widehat{F}(\frac{1}{2}, \frac{1}{2}) = 30\zeta_3 \log(2) - 16\zeta_4 - 32U_{3,1}.$$
(22)

Relations between tadpoles



Figure 2: The perfect tadpoles $\widehat{F}(\frac{1}{2},\frac{1}{2})$, $\widehat{F}(1,\frac{1}{2})$ and $\widehat{G}(\frac{1}{2})$ in relation (24)

In addition to the two-parameter family $\widehat{F}(x, y)$, there is a one-parameter family $\widehat{G}(x) = F_{(x,1-x,1)}^{(x,1-x,1)}$ of perfect tadpoles, with $x \in [0, \frac{1}{2}]$ and $\widehat{G}(0) = 17\zeta_4 + 16U_{3,1}$. I used the efficient AGM of Gauss to obtain **200 digits** of

$$\widehat{G}(\frac{1}{2}) = F_{(\frac{1}{2},\frac{1}{2},1)}^{(\frac{1}{2},\frac{1}{2},1)} = -\int_{1}^{\infty} \mathrm{d}s \left(\widehat{\sigma}_{\mathrm{N}}'(s) + \widehat{\sigma}_{\mathrm{E}}'(s)\right) \mathrm{Li}_{2}(1-s)$$
(23)

to which all routes are elliptic. This revealed the intriguing empirical relation

$$2\widehat{F}(\frac{1}{2},\frac{1}{2}) + 2\widehat{F}(1,\frac{1}{2}) + \widehat{G}(\frac{1}{2}) = 42\zeta_4 + 24\zeta_3\log(2).$$
(24)

A non-elliptic route to $\widehat{F}(1,\frac{1}{2})$ led to multiple polylogarithms in an alphabet of forms, $dx/(x-a_i)$, with $a_i \in \{0, 1, -1, -2\}$. After help with these, from **Steven Charlton**, I then found the integer relation

$$\widehat{G}(\frac{1}{2}) = 6\left(2\zeta_4 - 3\operatorname{Li}_4(\frac{1}{4})\right) + 8\left(2\zeta_3 - 3\operatorname{Li}_3(\frac{1}{4})\right)L - 12\operatorname{Li}_2(\frac{1}{4})L^2 - 4L^4$$
(25)

with $L = \log(2)$ and **classical** polylogs giving 10000 digits in less than a second.

Binary tadpoles, with $m_k \in \{0, 1\}$, evaluate to multiple polylogarithms in an alphabet containing sixth roots of unity, with $\lambda = (1 + \sqrt{-3})/2$ appearing if three massive edges meet at a vertex, where $\Delta_{i,j,k} = \sqrt{-3}$. For example, with 5 unit edges

$$F_{(1,1,1)}^{(1,1,0)} = \frac{109}{6} \left(\frac{\pi}{3}\right)^4 + 16\Re\left(\frac{\text{Li}_2^2(\lambda)}{6} + \sum_{m>n>0} \frac{\lambda^{3m+2n}}{m^3n}\right).$$
 (26)

There are **linear relations** between binary tadpoles, as here:

$$3F_{(0,0,0)}^{(1,1,1)} = F_{(1,1,1)}^{(0,0,0)} + 2F_{(1,1,0)}^{(1,0,0)}, \qquad (27)$$

$$3F_{(1,1,0)}^{(0,0,0)} = F_{(1,0,0)}^{(0,0,0)} + 2F_{(1,1,1)}^{(0,0,0)}, \qquad (28)$$

$$F_{(\mathbf{1},\mathbf{1},\mathbf{1})}^{(\mathbf{1},\mathbf{1},\mathbf{1})} + F_{(1,0,0)}^{(1,0,0)} = F_{(1,1,0)}^{(1,1,0)} + F_{(0,0,0)}^{(1,1,1)} .$$
(29)

Number fields of the alphabets of tadpoles

So far, one might **guess** that a tadpole with rational masses evaluates to multiple tetralogarithms in an alphabet whose number field is no larger than the **compositum** of the **quadratic** number fields associated by Gunnar Källén to the vertices of the tetrahedron, namely the field $Q(\Delta_{1,3,4}, \Delta_{2,3,5}, \Delta_{1,2,6}, \Delta_{4,5,6})$.

Yet that is **not** the case. The **imperfect** binary tadpole $F_{(1,1,0)}^{(1,0,0)}$ involves $\Re \text{Li}_2^2(\lambda)$, but the Källén field is **rational**.

Faced with this rather limited, yet potent, evidence, I arrive at three **suggestions**, each too weak to be dignified as a well-tested conjecture.

- 1. Every tetrahedral tadpole with rational masses reduces to multiple or single tetralogarithms whose alphabet lies in an algebraic number field.
- 2. If the tadpole is perfect, the alphabet is rational.
- 3. If the tadpole is imperfect, the alphabetic field may include the Källén field.

Experimentum crucis: I found an empirical relation between the totally massive **imperfect** tadpole $F_{(\frac{1}{2},\frac{1}{2},\frac{1}{2})}^{(1,1,1)}$ with Källén field $Q(\sqrt{-3})$ and the perfect tadpole $\widehat{G}(\frac{1}{2})$, already evaluated in terms of classical polylogs:

$$F_{(\frac{1}{2},\frac{1}{2},\frac{1}{2})}^{(1,1,1)} = 3\zeta_3 \log(2) - 4U_{3,1} + 10\zeta_4 + 10\text{Cl}_2^2(\pi/3) - \frac{1}{2}\widehat{G}(\frac{1}{2})$$
(30)

with a Clausen value $Cl_2(\pi/3) = \Im Li_2(\lambda)$, from the Källén field. It took less than a minute to validate (30) at 600-digit precision.

Tests and benchmarks for kites and tadpoles

- 1. Elliptic terms do not depend on the order of phase-space integrations.
- 2. The derivative of the discontinuity of a kite satisfies the sum rule

$$\int_{s_{\rm L}}^{\infty} \mathrm{d}s \, \sigma'(s) \log\left(\frac{s}{s_{\rm L}}\right) = 6\zeta_3. \tag{31}$$

- 3. The high energy behaviour of $s^2 \sigma'(s)$ holds irrespective of anomalous thresholds.
- 4. The same tadpole is obtained by integrating over 6 distinct kites.

These tests were invariably passed, at high precision, in a plethora of cases.

Benchmark 1: A useful benchmark was established by Stefan Bauberger and Manfred Böhm, who gave 6 decimal digits of $B_1 = I(50 + i\epsilon; 1, 2, 3, 4, 5)/50$, with all 4 cuts opened. For B_1 , I obtain the value

+0.173901219069555460362391997806756419040779085211744093645075

-0.118080028202009293890731446888246675922194086181504660940640* I

Benchmark 2: Stephen Martin computed 8 digits of $B_2 = -I(10+i\epsilon; 1, 3, 5, 2, 4)/10$, in a non-anomalous case with only one open cut. For B_2 , I obtain the value +0.718335353535353534129653528554796276560425262176802655670356407 +0.390162199972762321424365961074218884677858368327292408622989*I



Figure 3: Tadpoles for benchmarks B_3 , B_4 and B_5

The benchmarks of Figure 3 are ambitious targets for adept users of HyperInt.

Benchmark 3: The first example in Figure 3 is the simplest **perfect** tadpole with 6 **distinct** non-zero rational masses. I suggest that its alphabet may be rational. For its finite part $B_3 = \hat{F}(\frac{5}{6}, \frac{1}{3})$, I obtain

13.3861455348739022697615450327228552185248654855497464708212

Benchmark 4: The second example is an **imperfect** tadpole The benchmark for $B_4 = F_{(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})}^{(1,1,1)}$ is

 $16.6059\bar{5}4\bar{2}811980228081648880073141697347243824321176643541089$

Benchmark 5: The third example is **doubly imperfect**. I suggest that its alphabetic field may include $Q(\sqrt{-3}, \sqrt{5})$. The benchmark for $B_5 = F_{(1,1,1)}^{(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})}$ is 16.5896999071871022548891317280131669711968061643643361121466

Comments and summary

- 1. Elliptic substructure of 2-loop kites and 3-loop tadpoles is not a problem. The time taken to evaluate a complete elliptic integral, of whatever kind, is commensurate with the time for a logarithm and less than the time for a dilogarithm. Thanks to Gauss, elliptic integrals should be embraced, not feared.
- 2. Anomalous terms are not problematic. They submit to Gauss, at high energy.
- 3. The number theory of tadpoles is subtle. They may be polylogarithmic, even in totally massive cases to which every route is elliptic.
- 4. I have given far-reaching suggestions on the number theory of tadpoles and benchmarks for users of HyperInt to investigate those suggestions analytically.

Appendix with zero-mass limits: As $m_3 \to 0$ with $m_1 \neq m_4$ and $m_2 \neq m_5$,

$$\Delta_{1,2}(s)\sigma_{1,2}'(s) \to \Re\left((2s-s_3)D_{4,5}(s) + \hat{L}_{4,5} + \sum_{i=0,3} C_i \frac{D_{4,5}(s) - D_{4,5}(s_i)}{s-s_i}\right),$$

$$s_3 = -\frac{(m_1^2m_5^2 - m_2^2m_4^2)(m_1^2 - m_2^2 - m_4^2 + m_5^2)}{M}, \quad M = (m_1^2 - m_4^2)(m_2^2 - m_5^2),$$

$$\hat{L}_{4,5} = \log\left(\frac{m_4^2m_5^2}{M}\right), \quad C_3 = -\left(\frac{m_1^2}{u} - m_2^2u\right)\left(\frac{m_4^2}{u} - m_5^2u\right), \quad u = \frac{m_1^2 - m_4^2}{m_2^2 - m_5^2}.$$

As $m_3 \to 0$ with $m_1 = m_4$ and $m_2 \neq m_5$

$$\begin{split} \Delta_{2,4}(s)\sigma_{1,2}'(s) &\to \Re\left((3s - m_2^2 - 2m_4^2 - m_5^2)D_{4,5}(s) + \log\left(\frac{m_4m_5^3}{(m_2^2 - m_5^2)^2}\right) \\ &+ (m_2^2 - m_4^2)(m_4^2 - m_5^2)\frac{D_{4,5}(s) - D_{4,5}(0)}{s}\right). \end{split}$$

The degenerate case with $m_1 = m_4$ and $m_2 = m_5$ will be considered after adding contributions from three-particle cuts.

As $m_3 \rightarrow 0$, the three-particle cuts yield logarithms:

$$\sigma_{2,3,4}'(w^2) \to \Re\left(\sum_{i=\pm} E_i \frac{\widehat{P}_{2,4}(t_i, w) - \widehat{P}_{2,4}(m_1^2, w)}{t_i - m_1^2}\right),$$
$$\widehat{P}_{j,k}(t, w) = \frac{(m_k^2 - t)v(t)}{(w - m_j)^2 - t} \log\left(\frac{v(t) + v(m_k^2)}{v(t) - v(m_k^2)}\right), \ v(t) = \left(\frac{(w - m_j)^2 - t}{(w + m_j)^2 - t}\right)^{1/2}.$$

With $m_1 = m_4$ and $m_2 = m_5$ all four thresholds collide as $m_3 \to 0$, giving

$$\sigma'(s) \to \Theta(s - s_{4,5}) \frac{2\mu(y_4) + 2\mu(y_5) - 8\mu(y_4y_5)}{\Delta_{4,5}(s)},$$
$$\mu(y) = \log|1 - y| + \frac{y\log|y|}{1 - y}, \quad y_k = \frac{-2m_k^2}{s - m_4^2 - m_5^2 + \Delta_{4,5}(s)}$$

Next, consider cases with $m_3 > 0$ and one of the other masses vanishing. Without loss of generality, take it to be m_4 . As $m_4 \rightarrow 0$, logarithms from appear in

$$\sigma'_{2,3,4}(w^2) \to \Re\left(\sum_{i=\pm} E_i \frac{\widehat{P}_{2,3}(t_i, w) - \widehat{P}_{2,3}(m_1^2, w)}{t_i - m_1^2}\right).$$

The logarithms for two-particle cuts are modified, as $m_4 \rightarrow 0$, to give

$$\Delta_{1,2}(s)\sigma_{1,2}'(s) \to \Re\left((s+\alpha)\widehat{D}_5(s) + \widehat{L}_5 + \sum_{i=0,+,-} C_i \frac{\widehat{D}_5(s) - \widehat{D}_5(s_i)}{s-s_i}\right),$$
$$\widehat{D}_5(s) = \frac{1}{s-m_5^2}\log\left(1-\frac{s}{m_5^2}\right), \quad \widehat{L}_5 = \log\left(\frac{m_5^2}{m_3^2}\right).$$

An elliptic contribution persists if two **non-adjacent** edges have vanishing mass. As $m_1 \rightarrow 0$ and $m_5 \rightarrow 0$,

$$\begin{split} (w^2 - m_4^2)\sigma'_{2,3,4}(w^2) &\to -\frac{4\pi m_3 m_4 \Re R(w^2, m_2^2, m_3^2, m_4^2)}{\operatorname{AGM}\left(\sqrt{16m_2 m_3 m_4 w}, \sqrt{W}\right)},\\ R(s, b, c, d) &= P(\widehat{n}, k) - \rho P(n_0, k) + (\rho - 1)P(n_3, k),\\ \widehat{n} &= \frac{w_-^2 - m_+^2}{w_-^2 - m_-^2}, \quad \frac{n_0}{\widehat{n}} = \frac{m_-^2}{m_+^2}, \quad \frac{n_3}{\widehat{n}} = \frac{t_3 - m_-^2}{t_3 - m_+^2}, \quad t_3 = \frac{(bd - cs)(b - c + d - s)}{(b - c)(d - s)},\\ \rho &= \left(\frac{d - s}{b - c + d - s}\right) \left(\frac{(b + c)(d - s) + (b - c)(b + d)}{bd - cs}\right),\\ R(s, c, c, d) &= 2P(\widehat{n}, k) - 2P(n_0, k), \quad R(s, d, d, d) = \frac{s - 9d}{6d} \end{split}$$

with a **rational** result for R in the QED case $m_2 = m_3 = m_4$.