## 3-loop tadpoles with substructure from 12 elliptic curves

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The generic 2-loop kite integral has 5 internal masses. Its completion by a sixth propagator gives a 3-loop tadpole whose substructure involves 12 elliptic curves. I shall show how to compute all such kites and their tadpoles, with 200 digit precision achieved in seconds, thanks to the procedure of the arithmetic geometric mean for complete elliptic integrals of the third kind. The number theory of 3-loop tadpoles poses challenges for packages such as Hyper Int.

1. If you know the discontinuity $\sigma$ of $f$, use a dispersion relation to get $f$.
2. If you know the derivative $\sigma^{\prime}$, integrate that against a log.
3. For the 2-loop photon propagator, $\sigma^{\prime}$ has logs, so $f$ has tri-logs.
4. For the 2-loop electron propagator, $\sigma^{\prime}$ is elliptic, so $f$ is harder to compute.
5. To determine a 3-loop tadpole, integrate an elliptic $\sigma^{\prime}$ against a dilog.

I define the 2-loop scalar kite integral in 4-dimensional Minkowski space as

$$
\begin{align*}
I\left(q^{2} ; m_{1}^{2}, m_{2}^{2}, m_{3}^{2}, m_{4}^{2}, m_{5}^{2}\right) & =-\frac{q^{2}}{\pi^{4}} \int \mathrm{~d}^{4} l \int \mathrm{~d}^{4} k \prod_{j=1}^{5} \frac{1}{p_{j}^{2}-m_{j}^{2}-\mathrm{i} \epsilon}  \tag{1}\\
\left(p_{1}, p_{2}, p_{3}, p_{4}, p_{5}\right) & =(l, l-q, l-k, k, k-q) \tag{2}
\end{align*}
$$

with a cut $s \in\left[s_{\mathrm{L}}, \infty\right]$ and a branch point $s_{\mathrm{L}}$ that is the lowest of the thresholds $\left\{s_{1,2}, s_{4,5}, s_{2,3,4}, s_{1,3,5}\right\}$, where $s_{j, k}=\left(m_{j}+m_{k}\right)^{2}$ and $s_{i, j, k}=\left(m_{i}+m_{j}+m_{k}\right)^{2}$. On the top lip of the cut, let $\Im I(s+\mathrm{i} \epsilon)=\pi \sigma(s)$. Then we have the dispersion relation

$$
\begin{equation*}
I\left(q^{2}\right)=-\int_{s_{\mathrm{L}}}^{\infty} \mathrm{d} s \sigma^{\prime}(s) \log \left(1-\frac{q^{2}}{s}\right) \tag{3}
\end{equation*}
$$



Regularization in $4-2 \epsilon$ dimensions of tetrahedral tadpole formed by joining the external vertices of the kite with a propagator $1 /\left(q^{2}-m_{6}^{2}\right)$ gives

$$
\begin{gather*}
T_{1,2,3}^{5,4,6}=\left(\frac{1}{3 \epsilon}+1\right) 6 \zeta_{3}+3 \zeta_{4}-F_{1,2,3}^{5,4,6}+O(\epsilon)  \tag{4}\\
F_{1,2,3}^{5,4,6}=\int_{s_{\mathrm{L}}}^{\infty} \mathrm{d} s \sigma^{\prime}\left(s ; m_{1}^{2}, m_{2}^{2}, m_{3}^{2}, m_{4}^{2}, m_{5}^{2}\right)\left(\operatorname{Li}_{2}\left(1-\frac{m_{6}^{2}}{s}\right)+\frac{1}{2} \log ^{2}\left(\frac{\bar{m}^{2}}{s}\right)\right) \tag{5}
\end{gather*}
$$

where $\bar{m}$ is the scale of dimensional regularization and the dilogarithm is the analytic continuation of the sum $\operatorname{Li}_{2}(z)=\sum_{n>0} z^{n} / n^{2}$. In the totally massive case we can detect 12 elliptic curves inherent in the tetrahedron.

In the absence of anomalous thresholds, the non-elliptic contribution is

$$
\begin{equation*}
\sigma_{\mathrm{N}}^{\prime}(s)=\Theta\left(s-s_{1,2}\right) \sigma_{1,2}^{\prime}(s)+\Theta\left(s-s_{4,5}\right) \sigma_{4,5}^{\prime}(s) \tag{6}
\end{equation*}
$$

Denote the square root of the symmetric Källén function by

$$
\begin{equation*}
\Delta(a, b, c)=\sqrt{a^{2}+b^{2}+c^{2}-2(a b+b c+c a)} \tag{7}
\end{equation*}
$$

with convenient abbreviations $\Delta_{j, k}(s)=\Delta\left(s, m_{j}^{2}, m_{k}^{2}\right)$ and $\Delta_{i, j, k}=\Delta_{j, k}\left(m_{i}^{2}\right)$.

$$
\begin{equation*}
D_{j, k}(s)=\frac{r}{s-\left(m_{j}-m_{k}\right)^{2}} \log \left(\frac{1+r}{1-r}\right), \quad r=\left(\frac{s-\left(m_{j}-m_{k}\right)^{2}}{s-\left(m_{j}+m_{k}\right)^{2}}\right)^{1 / 2} \tag{8}
\end{equation*}
$$

provides the logarithms in

$$
\begin{equation*}
\Delta_{1,2}(s) \sigma_{1,2}^{\prime}(s)=\Re\left((s+\alpha) D_{4,5}(s)+L_{4,5}+\sum_{i=0,+,-} C_{i} \frac{D_{4,5}(s)-D_{4,5}\left(s_{i}\right)}{s-s_{i}}\right) \tag{9}
\end{equation*}
$$

with constants

$$
\begin{gathered}
C_{0}=-\left(m_{1}^{2}-m_{2}^{2}\right)\left(m_{4}^{2}-m_{5}^{2}\right), \quad C_{ \pm}=\alpha s_{ \pm}+\beta, \quad L_{4,5}=\log \left(\frac{m_{4} m_{5}}{m_{3}^{2}}\right), \\
\alpha=\frac{\left(m_{1}^{2}-m_{4}^{2}\right)\left(m_{2}^{2}-m_{5}^{2}\right)}{m_{3}^{2}}-m_{3}^{2}, \quad \beta=\frac{\left(m_{1}^{2} m_{5}^{2}-m_{2}^{2} m_{4}^{2}\right)\left(m_{1}^{2}-m_{2}^{2}-m_{4}^{2}+m_{5}^{2}\right)}{m_{3}^{2}}, \\
s_{0}=0, \quad s_{ \pm}=\frac{m_{1}^{2}+m_{2}^{2}-2 m_{3}^{2}+m_{4}^{2}+m_{5}^{2}-\alpha}{2} \pm \frac{\Delta_{1,3,4} \Delta_{2,3,5}}{2 m_{3}^{2}}
\end{gathered}
$$

where $s_{ \pm}$locate leading Landau singularities of triangles that form the kite. To obtain $\sigma_{4,5}^{\prime}$, exchange $\left(m_{1}, m_{2}\right)$ with $\left(m_{4}, m_{5}\right)$.

Elliptic contribution: This comes from 3-particle intermediate states, giving

$$
\begin{equation*}
\sigma_{\mathrm{E}}^{\prime}(s)=\Theta\left(s-s_{2,3,4}\right) \sigma_{2,3,4}^{\prime}(s)+\Theta\left(s-s_{1,3,5}\right) \sigma_{1,3,5}^{\prime}(s) \tag{10}
\end{equation*}
$$

It contains complete elliptic integrals of the third kind of the form

$$
\begin{equation*}
P(n, k)=\frac{\Pi(n, k)}{\Pi(0, k)}, \quad \Pi(n, k)=\int_{0}^{\pi / 2} \frac{d \theta}{\left(1-n \sin ^{2} \theta\right) \sqrt{1-k^{2} \sin ^{2} \theta}} \tag{11}
\end{equation*}
$$

with $\Pi(0, k)=(\pi / 2) / \operatorname{AGM}\left(1, \sqrt{1-k^{2}}\right)$ determined by the arithmetic-geometric mean of Gauss. With $s=w^{2}$, an integration over the phase space of particles 2, 3 and 4 determines

$$
\begin{equation*}
k^{2}=1-\frac{16 m_{2} m_{3} m_{4} w}{W}, \quad W=\left(w_{+}^{2}-m_{+}^{2}\right)\left(w_{-}^{2}-m_{-}^{2}\right) \tag{12}
\end{equation*}
$$

with $w_{ \pm}=w \pm m_{2}$ and $m_{ \pm}=m_{3} \pm m_{4}$. Then I obtain

$$
\begin{equation*}
\sigma_{2,3,4}^{\prime}\left(w^{2}\right)=\frac{4 \pi m_{3} m_{4}}{\operatorname{AGM}\left(\sqrt{16 m_{2} m_{3} m_{4} w}, \sqrt{W}\right)} \Re\left(\sum_{i=+,-} E_{i} \frac{P\left(n_{i}, k\right)-P\left(n_{1}, k\right)}{t_{i}-t_{1}}\right) \tag{13}
\end{equation*}
$$

with coefficients and arguments given, as compactly as possible, by

$$
\begin{gathered}
E_{ \pm}=\frac{m_{2}^{2}-m_{3}^{2}+m_{5}^{2}}{2 m_{5}^{2}} \pm\left(\frac{m_{4}^{2}-m_{5}^{2}-w^{2}}{2 m_{5}^{2}}\right) \frac{\Delta_{2,3,5}}{\Delta_{4,5}\left(w^{2}\right)} \\
t_{ \pm}=\frac{\gamma \pm \Delta_{2,3,5} \Delta_{4,5}\left(w^{2}\right)}{2 m_{5}^{2}}, \quad t_{1}=m_{1}^{2}, \quad n_{i}=\frac{\left(w_{-}^{2}-m_{+}^{2}\right)\left(t_{i}-m_{-}^{2}\right)}{\left(w_{-}^{2}-m_{-}^{2}\right)\left(t_{i}-m_{+}^{2}\right)}, \\
\gamma=\left(m_{2}^{2}+m_{3}^{2}+m_{4}^{2}-m_{5}^{2}+w^{2}\right) m_{5}^{2}+\left(m_{2}^{2}-m_{3}^{2}\right)\left(m_{4}^{2}-w^{2}\right)
\end{gathered}
$$

An AGM procedure speedily evaluates $P(n, k)=\Pi(n, k) / \Pi(0, k)$ to high precision:

1. Initialize $[a, b, p, q]=\left[1, \sqrt{1-k^{2}}, \sqrt{1-n}, n /(2-2 n)\right]$. Then set $f=1+q$.
2. Set $m=a b$ and then $r=p^{2}+m$. Compute a vector of new values as follows: $[(a+b) / 2, \sqrt{m}, r /(2 p),(r-2 m) q /(2 r)]$. Then replace $[a, b, p, q]$ by those new values. Then add $q$ to $f$.
3. If $|q / f|$ is sufficiently small, then return $P(n, k)=f$, else go to step 2 .

This converges very quickly, for $n \notin[1, \infty]$. On the cut with $n \geq 1$, replace $n$ by $n^{\prime}=k^{2} / n<1$, to obtain the principal value $\Re P(n, k)=1-P\left(n^{\prime}, k\right)$.

Criterion for an anomalous contribution: Suppose that $s_{4,5} \geq s_{1,2}$. Then

$$
\begin{equation*}
\sigma^{\prime}(s)=\sigma_{\mathrm{N}}^{\prime}(s)+\sigma_{\mathrm{E}}^{\prime}(s)+C_{\mathrm{A}} \frac{\Theta\left(s-s_{4,5}\right)}{\Delta_{4,5}(s)} \Re\left(\frac{2 \pi \mathrm{i} \Delta_{4,5}\left(s_{-}\right)}{s-s_{-}}\right) \tag{14}
\end{equation*}
$$

with $C_{\mathrm{A}} \neq 0$ if and only if $\left(m_{1}+m_{2}\right)\left(m_{3}^{2}+m_{1} m_{2}\right)<m_{1} m_{5}^{2}+m_{2} m_{4}^{2}$ and at least one of $\Delta_{1,3,4}$ and $\Delta_{2,3,5}$ is imaginary, in which case $C_{\mathrm{A}}= \pm 1$ is the sign of $\Im \Delta_{4,5}\left(s_{-}\right)$. This value of $C_{\mathrm{A}}$ is required by the elliptic contribution at high energy. With $L_{k}=m_{k}^{2} \log \left(s / m_{k}^{2}\right)$, the large- $s$ behaviour

$$
\begin{equation*}
s^{2} \sigma^{\prime}(s)=2 L_{3}+\sum_{k=1,2,4,5}\left(L_{k}+m_{k}^{2}\right)+O\left(\frac{\log (s)}{s}\right) \tag{15}
\end{equation*}
$$

invariably holds. The elliptic contribution $\sigma_{\mathrm{E}}^{\prime}$ in (14) is oblivious to the anomalous threshold problem. Its high-energy behaviour determines $C_{\mathrm{A}}$, ensuring (15).

## Tadpoles and number theory

The rescaling $m_{k} \rightarrow \kappa m_{k}$ gives $F \rightarrow F+12 \zeta_{3} \log (\kappa)$ for the finite part $F$. To standardize, I set $\bar{m}=\max \left(m_{k}\right)=1$.
I define a tetrahedral tadpole to be perfect if and only if the Källén function vanishes at each of its 4 vertices, thereby avoiding all resolutions of square roots. Promoting the subscripts and superscripts of $F$ to arguments that denote the 6 masses, I define the two-parameter family of perfect tadpoles:

$$
\begin{equation*}
\widehat{F}(x, y)=F_{(x, y, 1)}^{(1-y, 1-x,|x-y|)}=\widehat{F}(y, x)=\widehat{F}(1-x, 1-y) \tag{16}
\end{equation*}
$$

with symmetries restricting distinct cases to $x \geq y \geq 1-x \geq 0$ and hence $x \in\left[\frac{1}{2}, 1\right]$. In QED, I identified tetralogarithms in two perfect binary tadpoles, obtaining

$$
\begin{gather*}
\widehat{F}(1,0)=F_{(1,1,0)}^{(1,1,0)}=17 \zeta_{4}+16 U_{3,1}, \quad \widehat{F}(1,1)=F_{(1,1,1)}^{(0,0,0)}=12 \zeta_{4},  \tag{17}\\
U_{3,1}=\sum_{m>n>0} \frac{(-1)^{m+n}}{m^{3} n}=\frac{1}{2} \zeta_{4}+\frac{1}{2} \zeta_{2} \log ^{2}(2)-\frac{1}{12} \log ^{4}(2)-2 \operatorname{Li}_{4}\left(\frac{1}{2}\right) . \tag{18}
\end{gather*}
$$

## Fast elliptic determination of a perfect tadpole

Now consider the elliptic route to evaluating $\widehat{F}\left(\frac{1}{2}, \frac{1}{2}\right)$. With $\left(m_{3}, m_{6}\right)=(1,0)$ and $m_{1}=m_{2}=m_{4}=m_{5}=\frac{1}{2}$, I obtained

$$
\begin{gather*}
\widehat{F}\left(\frac{1}{2}, \frac{1}{2}\right)=\frac{1}{2} \int_{1}^{\infty} \mathrm{d} s\left(\widehat{\sigma}_{\mathrm{N}}^{\prime}(s)+\widehat{\sigma}_{\mathrm{E}}^{\prime}(s)\right) \log ^{2}(s)  \tag{19}\\
w^{2} \widehat{\sigma}_{\mathrm{N}}^{\prime}\left(w^{2}\right)=\Theta(w-1)\left(2 \log \left(\frac{r+1}{r-1}\right)-4 r \log (2)\right), r=\frac{w}{\sqrt{w^{2}-1}}  \tag{20}\\
w^{2} \widehat{\sigma}_{\mathrm{E}}^{\prime}\left(w^{2}\right)=\frac{4 \pi(1-P(n, k)) \Theta(w-2)}{\operatorname{AGM}\left(2 \sqrt{w},(w-1) \sqrt{w^{2}+2 w}\right)}, n=\frac{w^{2}-2 w}{(w-1)^{2}}, \frac{k^{2}}{n}=\frac{(w+1)^{2}}{w^{2}+2 w} \tag{21}
\end{gather*}
$$

and readily discovered a new reduction of a perfect tadpole to tetralogarithms

$$
\begin{equation*}
\widehat{F}\left(\frac{1}{2}, \frac{1}{2}\right)=30 \zeta_{3} \log (2)-16 \zeta_{4}-32 U_{3,1} . \tag{22}
\end{equation*}
$$

## Relations between tadpoles



0


Figure 2: The perfect tadpoles $\widehat{F}\left(\frac{1}{2}, \frac{1}{2}\right), \widehat{F}\left(1, \frac{1}{2}\right)$ and $\widehat{G}\left(\frac{1}{2}\right)$ in relation (24)
In addition to the two-parameter family $\widehat{F}(x, y)$, there is a one-parameter family $\widehat{G}(x)=F_{(x, 1-x, 1)}^{(x, 1, x, 1)}$ of perfect tadpoles, with $x \in\left[0, \frac{1}{2}\right]$ and $\widehat{G}(0)=17 \zeta_{4}+16 U_{3,1}$.
I used the efficient AGM of Gauss to obtain 200 digits of

$$
\begin{equation*}
\widehat{G}\left(\frac{1}{2}\right)=F_{\left(\frac{1}{2}, \frac{1}{2}, 1\right)}^{\left(\frac{1}{2}, \frac{1}{2}, 1\right)}=-\int_{1}^{\infty} \mathrm{d} s\left(\widehat{\sigma}_{\mathrm{N}}^{\prime}(s)+\widehat{\sigma}_{\mathrm{E}}^{\prime}(s)\right) \operatorname{Li}_{2}(1-s) \tag{23}
\end{equation*}
$$

to which all routes are elliptic. This revealed the intriguing empirical relation

$$
\begin{equation*}
2 \widehat{F}\left(\frac{1}{2}, \frac{1}{2}\right)+2 \widehat{F}\left(1, \frac{1}{2}\right)+\widehat{G}\left(\frac{1}{2}\right)=42 \zeta_{4}+24 \zeta_{3} \log (2) \tag{24}
\end{equation*}
$$

A non-elliptic route to $\widehat{F}\left(1, \frac{1}{2}\right)$ led to multiple polylogarithms in an alphabet of forms, $\mathrm{d} x /\left(x-a_{i}\right)$, with $a_{i} \in\{0,1,-1,-2\}$. After help with these, from Steven Charlton, I then found the integer relation

$$
\begin{equation*}
\widehat{G}\left(\frac{1}{2}\right)=6\left(2 \zeta_{4}-3 \operatorname{Li}_{4}\left(\frac{1}{4}\right)\right)+8\left(2 \zeta_{3}-3 \operatorname{Li}_{3}\left(\frac{1}{4}\right)\right) L-12 \operatorname{Li}_{2}\left(\frac{1}{4}\right) L^{2}-4 L^{4} \tag{25}
\end{equation*}
$$

with $L=\log (2)$ and classical polylogs giving 10000 digits in less than a second.
Binary tadpoles, with $m_{k} \in\{0,1\}$, evaluate to multiple polylogarithms in an alphabet containing sixth roots of unity, with $\lambda=(1+\sqrt{-3}) / 2$ appearing if three massive edges meet at a vertex, where $\Delta_{i, j, k}=\sqrt{-3}$. For example, with 5 unit edges

$$
\begin{equation*}
F_{(1,1,1)}^{(1,1,0)}=\frac{109}{6}\left(\frac{\pi}{3}\right)^{4}+16 \Re\left(\frac{\operatorname{Li}_{2}^{2}(\lambda)}{6}+\sum_{m>n>0} \frac{\lambda^{3 m+2 n}}{m^{3} n}\right) . \tag{26}
\end{equation*}
$$

There are linear relations between binary tadpoles, as here:

$$
\begin{align*}
3 F_{(0,0,0)}^{(1,1,1)} & =F_{(1,1,1)}^{(0,0,0)}+2 F_{(1,1,0)}^{(1,0,0)}  \tag{27}\\
3 F_{(1,1,0)}^{(0,0,0)} & =F_{(1,0,0)}^{(0,0)}+2 F_{(1,0,1)}^{(0,0)}  \tag{28}\\
F_{(\mathbf{1}, \mathbf{1}, \mathbf{1})}^{(\mathbf{1}, \mathbf{1})}+F_{(1,0,0)}^{(1,0,0)} & =F_{(1,1,0)}^{(1,1,0)}+F_{(0,0,0)}^{(1,1,1)} \tag{29}
\end{align*}
$$

## Number fields of the alphabets of tadpoles

So far, one might guess that a tadpole with rational masses evaluates to multiple tetralogarithms in an alphabet whose number field is no larger than the compositum of the quadratic number fields associated by Gunnar Källén to the vertices of the tetrahedron, namely the field $Q\left(\Delta_{1,3,4}, \Delta_{2,3,5}, \Delta_{1,2,6}, \Delta_{4,5,6}\right)$.
Yet that is not the case. The imperfect binary tadpole $F_{(1,1,0)}^{(1,0)}$ involves $\Re \operatorname{Li}_{2}^{2}(\lambda)$, but the Källén field is rational.

Faced with this rather limited, yet potent, evidence, I arrive at three suggestions, each too weak to be dignified as a well-tested conjecture.

1. Every tetrahedral tadpole with rational masses reduces to multiple or single tetralogarithms whose alphabet lies in an algebraic number field.
2. If the tadpole is perfect, the alphabet is rational.
3. If the tadpole is imperfect, the alphabetic field may include the Källén field.

Experimentum crucis: I found an empirical relation between the totallly massive imperfect tadpole $F_{\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)}^{(1,1,1)}$ with Källén field $Q(\sqrt{-3})$ and the perfect tadpole $\widehat{G}\left(\frac{1}{2}\right)$, already evaluated in terms of classical polylogs:

$$
\begin{equation*}
F_{\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)}^{(1,1,1)}=3 \zeta_{3} \log (2)-4 U_{3,1}+10 \zeta_{4}+10 \mathrm{Cl}_{2}^{2}(\pi / 3)-\frac{1}{2} \widehat{G}\left(\frac{1}{2}\right) \tag{30}
\end{equation*}
$$

with a Clausen value $\mathrm{Cl}_{2}(\pi / 3)=\Im \operatorname{Li}_{2}(\lambda)$, from the Källén field. It took less than a minute to validate (30) at 600-digit precision.

## Tests and benchmarks for kites and tadpoles

1. Elliptic terms do not depend on the order of phase-space integrations.
2. The derivative of the discontinuity of a kite satisfies the sum rule

$$
\begin{equation*}
\int_{s_{\mathrm{L}}}^{\infty} \mathrm{d} s \sigma^{\prime}(s) \log \left(\frac{s}{s_{\mathrm{L}}}\right)=6 \zeta_{3} \tag{31}
\end{equation*}
$$

3. The high energy behaviour of $s^{2} \sigma^{\prime}(s)$ holds irrespective of anomalous thresholds.
4. The same tadpole is obtained by integrating over 6 distinct kites.

These tests were invariably passed, at high precision, in a plethora of cases.
Benchmark 1: A useful benchmark was established by Stefan Bauberger and Manfred Böhm, who gave 6 decimal digits of $B_{1}=I(50+\mathrm{i} \epsilon ; 1,2,3,4,5) / 50$, with all 4 cuts opened. For $B_{1}$, I obtain the value +0. 173901219069555460362391997806756419040779085211744093645075
-0.118080028202009293890731446888246675922194086181504660940640*I
Benchmark 2: Stephen Martin computed 8 digits of $B_{2}=-I(10+\mathrm{i} \epsilon ; 1,3,5,2,4) / 10$, in a non-anomalous case with only one open cut. For $B_{2}$, I obtain the value $+0.718335353533534129653528554796276560425262176802655670356407$ +0.390162199972762321424365961074218884677858368327292408622989*I


Figure 3: Tadpoles for benchmarks $B_{3}, B_{4}$ and $B_{5}$

The benchmarks of Figure 3 are ambitious targets for adept users of HyperInt.
Benchmark 3: The first example in Figure 3 is the simplest perfect tadpole with 6 distinct non-zero rational masses. I suggest that its alphabet may be rational. For its finite part $B_{3}=\widehat{F}\left(\frac{5}{6}, \frac{1}{3}\right)$, I obtain
13.3861455348739022697615450327228552185248654855497464708212

Benchmark 4: The second example is an imperfect tadpole The benchmark for $B_{4}=F_{\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)}^{(1,1,1)}$ is
16.6059542811980228081648880073141697347243824321176643541089

Benchmark 5: The third example is doubly imperfect. I suggest that its alphabetic field may include $Q(\sqrt{-3}, \sqrt{5})$. The benchmark for $B_{5}=F_{(1,1,1)}^{\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)}$ is 16.5896999071871022548891317280131669711968061643643361121466

## Comments and summary

1. Elliptic substructure of 2-loop kites and 3-loop tadpoles is not a problem. The time taken to evaluate a complete elliptic integral, of whatever kind, is commensurate with the time for a logarithm and less than the time for a dilogarithm. Thanks to Gauss, elliptic integrals should be embraced, not feared.
2. Anomalous terms are not problematic. They submit to Gauss, at high energy.
3. The number theory of tadpoles is subtle. They may be polylogarithmic, even in totally massive cases to which every route is elliptic.
4. I have given far-reaching suggestions on the number theory of tadpoles and benchmarks for users of HyperInt to investigate those suggestions analytically.

Appendix with zero-mass limits: As $m_{3} \rightarrow 0$ with $m_{1} \neq m_{4}$ and $m_{2} \neq m_{5}$,

$$
\begin{gathered}
\Delta_{1,2}(s) \sigma_{1,2}^{\prime}(s) \rightarrow \Re\left(\left(2 s-s_{3}\right) D_{4,5}(s)+\widehat{L}_{4,5}+\sum_{i=0,3} C_{i} \frac{D_{4,5}(s)-D_{4,5}\left(s_{i}\right)}{s-s_{i}}\right) \\
s_{3}=-\frac{\left(m_{1}^{2} m_{5}^{2}-m_{2}^{2} m_{4}^{2}\right)\left(m_{1}^{2}-m_{2}^{2}-m_{4}^{2}+m_{5}^{2}\right)}{M}, \quad M=\left(m_{1}^{2}-m_{4}^{2}\right)\left(m_{2}^{2}-m_{5}^{2}\right) \\
\widehat{L}_{4,5}=\log \left(\frac{m_{4}^{2} m_{5}^{2}}{M}\right), \quad C_{3}=-\left(\frac{m_{1}^{2}}{u}-m_{2}^{2} u\right)\left(\frac{m_{4}^{2}}{u}-m_{5}^{2} u\right), \quad u=\frac{m_{1}^{2}-m_{4}^{2}}{m_{2}^{2}-m_{5}^{2}}
\end{gathered}
$$

As $m_{3} \rightarrow 0$ with $m_{1}=m_{4}$ and $m_{2} \neq m_{5}$

$$
\begin{aligned}
\Delta_{2,4}(s) \sigma_{1,2}^{\prime}(s) & \rightarrow \Re\left(\left(3 s-m_{2}^{2}-2 m_{4}^{2}-m_{5}^{2}\right) D_{4,5}(s)+\log \left(\frac{m_{4} m_{5}^{3}}{\left(m_{2}^{2}-m_{5}^{2}\right)^{2}}\right)\right. \\
& \left.+\left(m_{2}^{2}-m_{4}^{2}\right)\left(m_{4}^{2}-m_{5}^{2}\right) \frac{D_{4,5}(s)-D_{4,5}(0)}{s}\right)
\end{aligned}
$$

The degenerate case with $m_{1}=m_{4}$ and $m_{2}=m_{5}$ will be considered after adding contributions from three-particle cuts.

As $m_{3} \rightarrow 0$, the three-particle cuts yield logarithms:

$$
\begin{gathered}
\sigma_{2,3,4}^{\prime}\left(w^{2}\right) \rightarrow \Re\left(\sum_{i= \pm} E_{i} \frac{\widehat{P}_{2,4}\left(t_{i}, w\right)-\widehat{P}_{2,4}\left(m_{1}^{2}, w\right)}{t_{i}-m_{1}^{2}}\right) \\
\widehat{P}_{j, k}(t, w)=\frac{\left(m_{k}^{2}-t\right) v(t)}{\left(w-m_{j}\right)^{2}-t} \log \left(\frac{v(t)+v\left(m_{k}^{2}\right)}{v(t)-v\left(m_{k}^{2}\right)}\right), v(t)=\left(\frac{\left(w-m_{j}\right)^{2}-t}{\left(w+m_{j}\right)^{2}-t}\right)^{1 / 2} .
\end{gathered}
$$

With $m_{1}=m_{4}$ and $m_{2}=m_{5}$ all four thresholds collide as $m_{3} \rightarrow 0$, giving

$$
\begin{gathered}
\sigma^{\prime}(s) \rightarrow \Theta\left(s-s_{4,5}\right) \frac{2 \mu\left(y_{4}\right)+2 \mu\left(y_{5}\right)-8 \mu\left(y_{4} y_{5}\right)}{\Delta_{4,5}(s)} \\
\mu(y)=\log |1-y|+\frac{y \log |y|}{1-y}, \quad y_{k}=\frac{-2 m_{k}^{2}}{s-m_{4}^{2}-m_{5}^{2}+\Delta_{4,5}(s)}
\end{gathered}
$$

Next, consider cases with $m_{3}>0$ and one of the other masses vanishing. Without loss of generality, take it to be $m_{4}$. As $m_{4} \rightarrow 0$, logarithms from appear in

$$
\sigma_{2,3,4}^{\prime}\left(w^{2}\right) \rightarrow \Re\left(\sum_{i= \pm} E_{i} \frac{\widehat{P}_{2,3}\left(t_{i}, w\right)-\widehat{P}_{2,3}\left(m_{1}^{2}, w\right)}{t_{i}-m_{1}^{2}}\right)
$$

The logarithms for two-particle cuts are modified, as $m_{4} \rightarrow 0$, to give

$$
\begin{aligned}
\Delta_{1,2}(s) \sigma_{1,2}^{\prime}(s) & \rightarrow \Re\left((s+\alpha) \widehat{D}_{5}(s)+\widehat{L}_{5}+\sum_{i=0,+,-} C_{i} \frac{\widehat{D}_{5}(s)-\widehat{D}_{5}\left(s_{i}\right)}{s-s_{i}}\right) \\
\widehat{D}_{5}(s) & =\frac{1}{s-m_{5}^{2}} \log \left(1-\frac{s}{m_{5}^{2}}\right), \quad \widehat{L}_{5}=\log \left(\frac{m_{5}^{2}}{m_{3}^{2}}\right)
\end{aligned}
$$

An elliptic contribution persists if two non-adjacent edges have vanishing mass. As $m_{1} \rightarrow 0$ and $m_{5} \rightarrow 0$,

$$
\begin{gathered}
\left(w^{2}-m_{4}^{2}\right) \sigma_{2,3,4}^{\prime}\left(w^{2}\right) \rightarrow-\frac{4 \pi m_{3} m_{4} \Re R\left(w^{2}, m_{2}^{2}, m_{3}^{2}, m_{4}^{2}\right)}{\operatorname{AGM}\left(\sqrt{16 m_{2} m_{3} m_{4} w}, \sqrt{W}\right)}, \\
R(s, b, c, d)=P(\widehat{n}, k)-\rho P\left(n_{0}, k\right)+(\rho-1) P\left(n_{3}, k\right) \\
\widehat{n}=\frac{w_{-}^{2}-m_{+}^{2}}{w_{-}^{2}-m_{-}^{2}}, \quad \frac{n_{0}}{\widehat{n}}=\frac{m_{-}^{2}}{m_{+}^{2}}, \quad \frac{n_{3}}{\widehat{n}}=\frac{t_{3}-m_{-}^{2}}{t_{3}-m_{+}^{2}}, \quad t_{3}=\frac{(b d-c s)(b-c+d-s)}{(b-c)(d-s)}, \\
\rho=\left(\frac{d-s}{b-c+d-s}\right)\left(\frac{(b+c)(d-s)+(b-c)(b+d)}{b d-c s}\right) \\
R(s, c, c, d)=2 P(\widehat{n}, k)-2 P\left(n_{0}, k\right), \quad R(s, d, d, d)=\frac{s-9 d}{6 d}
\end{gathered}
$$

with a rational result for $R$ in the QED case $m_{2}=m_{3}=m_{4}$.

