Tropical Feynman Integration in the Physical Region

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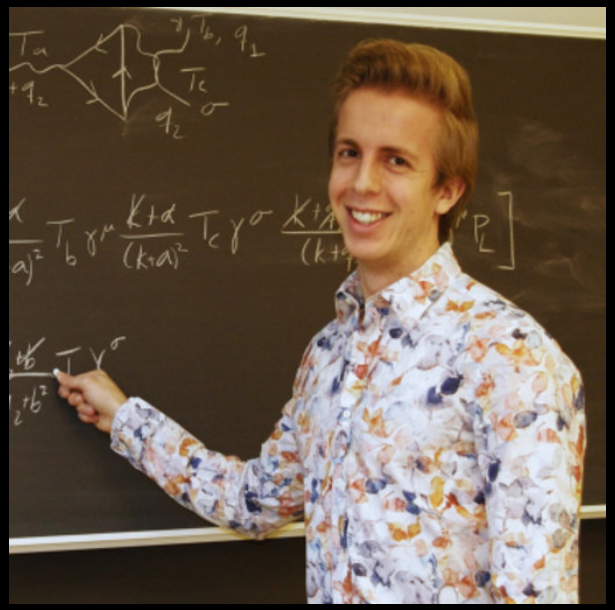
Università degli Studi di Padova

Work in collaboration with

Michael Borinsky



Felix Tellander

- This talk is based on *Tropical Feynman integration in the Minkowski regime [2302.08955]*
- Continuation of research program started by M. Borinsky in *Tropical Monte Carlo quadrature for Feynman integrals [2008.12310]*

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Motivation

From model to prediction

Quantum Field Theory model: $\mathcal{L} \sim -\frac{1}{2}(\partial\varphi)^2 + \frac{\lambda}{4!}\varphi^4$

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Cross section: $\sigma \sim \int |\mathcal{A}|^2$

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Bottleneck: Diagrams with several loops and many scales

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Computing loop diagrams

Analytic approach:

- Express via special functions (MPLs, elliptics, modular forms...)
- Preferably know:
  - Algebraic relations
  - Analytic continuation
  - Rapidly converging power series
- Issue: "Good" space of functions unknown for 2 loops and beyond!

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- Input: Diagram + numerical phase space point
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  - `pySecDec` [Borowka et al.]
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Projective Feynman integrals

Loop diagram data

- Feynman loop diagram $G = (V, E)$:
  - Vertices: $V = \{1, 2, \ldots, |V|\}$
  - Edges: $E = \{(u_1, v_1), \ldots, (u_{|E|}, v_{|E|})\}$ with $u, v \in V$
- Kinematic data:

Can then build the two Symanzik polynomials associated to G

$$\mathcal{U}(x)\,, \quad \mathcal{F}(x)$$

in terms of Schwinger parameters $x = (x_1, \ldots, x_{|E|})$

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$$\mathcal{I} = \Gamma(\omega) \int_{\mathbb{P}_+} \phi$$

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in the physical (not only Euclidean) region of kinematic space.

- In particular, using dimensional regularization

$$D = D_0 - 2\epsilon$$

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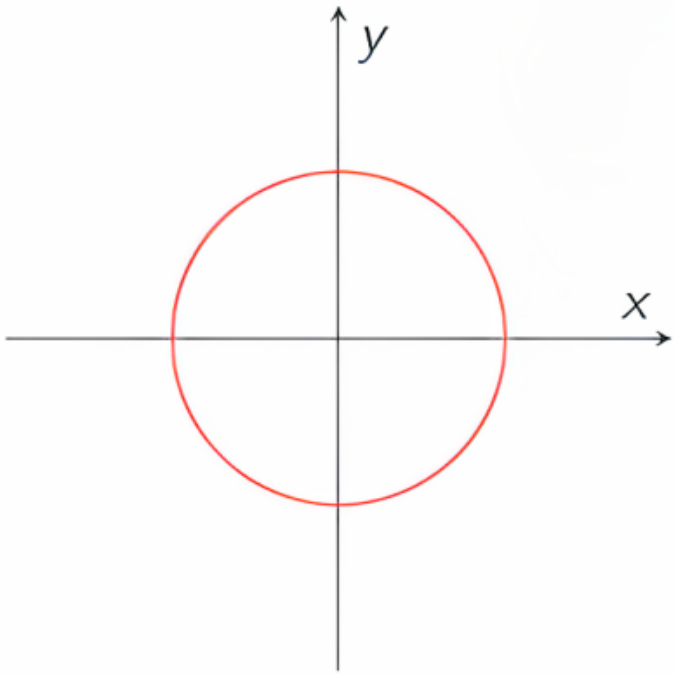
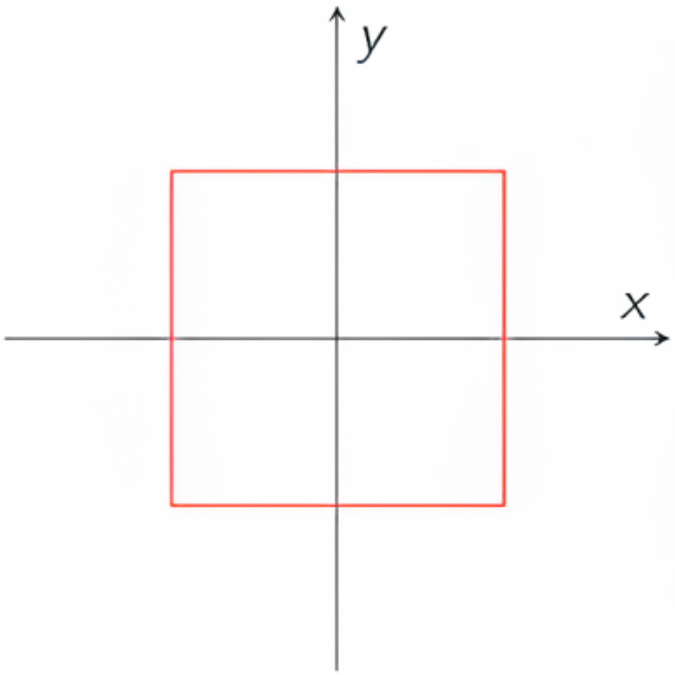
Tropical integration

Tropical geometry

Idea: Deform "smooth" geometries into "flat" pieces

$$1 = x^2 + y^2 \quad \rightarrow \quad 1 = (x^2 + y^2)^{\mathrm{tr}} = \max\{x^2, y^2\}$$





Tropicalized polynomials

- Polynomial

$$p(x) = \sum_{a \in \mathrm{support}(p)} c_a \, x^a \,, \quad x^a = x_1^{a_1} \cdots x_n^{a_n}$$

- Tropicalized version ignores coefficients and only cares about the largest monomial:

$$p^{\mathrm{tr}}(x) = \max_{a \in \mathrm{support}(p)} \{x^a\}$$

- Theorem [Borinsky]: There exist constants $C_1, C_2 > 0$ such that

$$C_1 \leq \frac{|p(x)|}{p^{\mathrm{tr}}(x)} \leq C_2 \quad \text{for all} \quad x \in \mathbb{P}_+$$

assuming a technical condition ($p(x)$ is completely non-vanishing)

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Tropical Monte Carlo

- Tropical Symanzik polynomials (recall $\mathcal{V} = \mathcal{F}/\mathcal{U}$):

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- Monte Carlo sampling from the tropical probability measure

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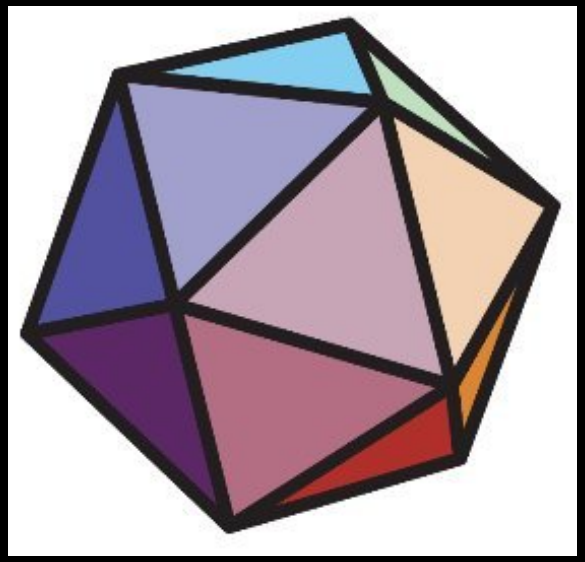
Speeding up tropical Monte Carlo sampling



- Consider the Newton polytope of $\mathcal{F}(x)$, $\mathrm{Newt}[\mathcal{F}]$
- Tropical Monte Carlo is vastly sped up when $\mathrm{Newt}[\mathcal{F}]$ is a Generalized Permutahedron
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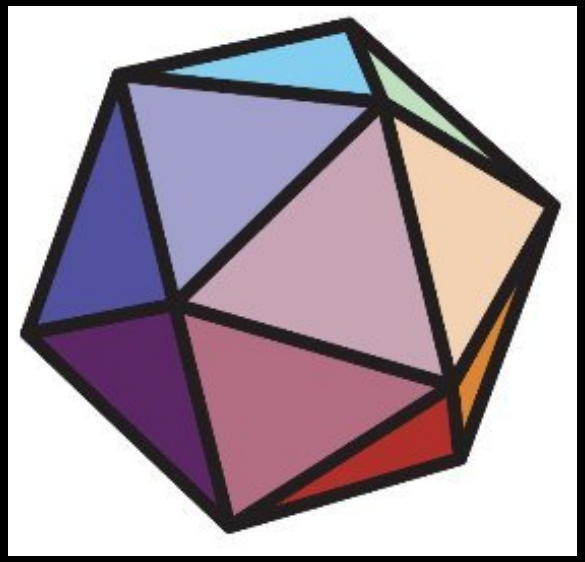

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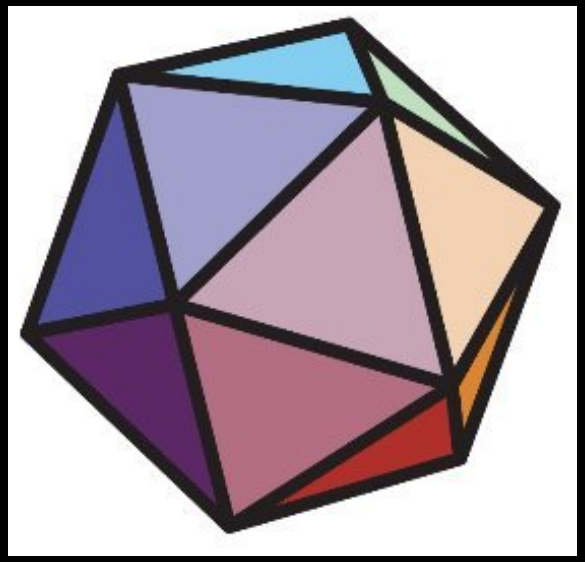

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$i\varepsilon$ implementation

- Required for integration in the Minskowski/physical region
- Infinitesimal $i\varepsilon$ not suited for numerics

- Consider the Schwinger representation:

$$\mathcal{I} = \int_0^\infty \ldots \exp\left[i\left(-\mathcal{V}(x) + i\varepsilon \sum_{e\in E} x_e\right)\right]$$

- Can drop $i\varepsilon$ and still have convergence if

$$\mathrm{Im}(-\mathcal{V}(x)) > 0$$

- Seek a "nice" deformation of x_e such that the above holds true

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$i\varepsilon$ via analytic continuation

- Analytic continuation [Mizera, Telen] [Hannesdottir, Mizera]

$$X_e = x_e \exp\left[-i\lambda \frac{\partial \mathcal{V}(x)}{\partial x_e}\right]$$

- Taylor expand

$$-\mathcal{V}(X) = -\mathcal{V}(x) + i\lambda \sum_{e\in E} x_e \left(\frac{\partial \mathcal{V}(x)}{\partial x_e}\right)^2 + O(\lambda^2)$$

- Then $\mathrm{Im}(-\mathcal{V}(X)) > 0$ for λ small enough if

$$\exists\, e \in E: \quad x_e \frac{\partial \mathcal{V}(x)}{\partial x_e} \neq 0$$

- X_e also preserves projectivity of $\mathcal{I}$

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ϵ-expansion

- Assume that the Feynman integral is quasi-finite [von Manteuffel, Panzer, Schabinger]:

$$\mathcal{I} = (\epsilon - \text{divergent prefactor}) \quad \times \quad (\epsilon - \text{finite integral})$$

- Can always find such a representation [Berkesch, Forsgaard, Passare]

- Let $D = D_0 - 2\epsilon$ and $\omega_0 = \sum_{e \in E} \nu_e - D_0 L/2$
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$$\mathcal{I} = \frac{\Gamma(\omega_0 + \epsilon L)}{\prod_{e \in E} \Gamma(\nu_e)} \sum_{k=0}^{\infty} \mathcal{I}_k \frac{\epsilon^k}{k!}$$

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$$\mathcal{I}_k = \int_{\mathbb{P}_+} \frac{\det \mathcal{J}_\lambda(x) \ \times \ \prod_{e \in E} X_e^{\nu_e}}{\mathcal{U}(X)^{D_0/2} \ \times \ \mathcal{V}(X)^{\omega_0}} \ \times \ \log^k \left(\frac{\mathcal{U}(X)}{\mathcal{V}(X)^L} \right) \ \times \ \Omega$$

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`feyntrop` package

feyntrop

- Tropical Monte Carlo algorithm implemented in `feyntrop`

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- C++ code with python interface
- Includes ✓
- Limitations ×

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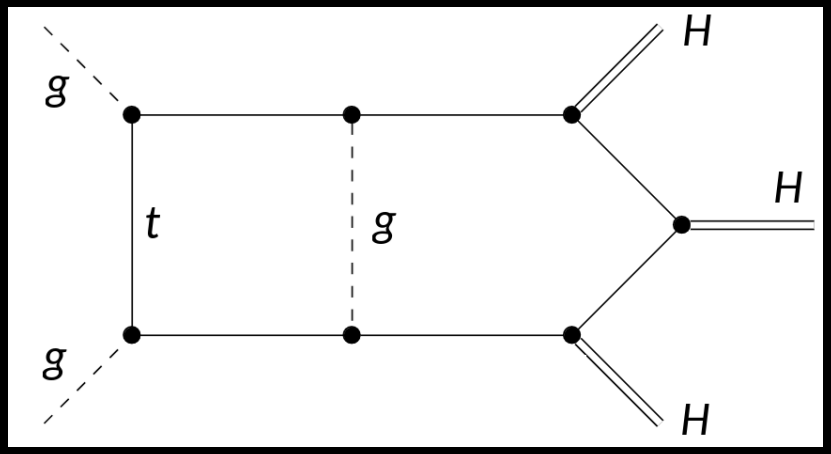
$gg \to HHH$ with internal top quark loop

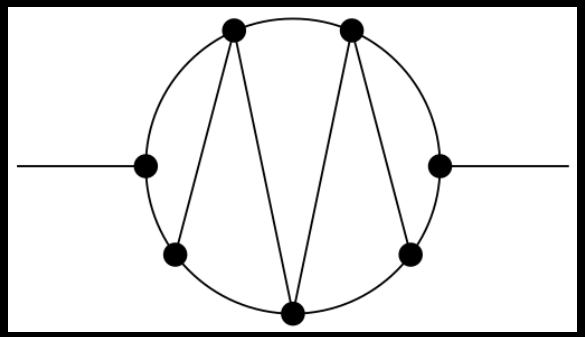


$$(m_H^2,\, m_t^2) = (1,\, 1.8995)$$

$$(s_{12}, s_{13}, s_{14}, s_{23}, s_{24}, s_{34}) = (17.5, -2.3, -2.4, -2.5, -2.6, -2.7)$$

```
* N = 10^10.
* Lambda = 0.29.
* Finished in 10.2 minutes.
-- eps^0: [0.000469553 +/- 0.000000063]  +  i * [-0.000721636 +/- 0.000000062]
-- eps^1: [0.00056055  +/- 0.00000025 ]  +  i * [ 0.00385763  +/- 0.00000024 ]
-- eps^2: [-0.00680350 +/- 0.00000054 ]  +  i * [-0.00516286  +/- 0.00000055 ]
-- eps^3: [0.01194325  +/- 0.00000089 ]  +  i * [-0.00211739  +/- 0.00000089 ]
-- eps^4: [-0.0064124  +/-  0.0000012 ]  +  i * [ 0.0109338   +/-  0.0000012 ]
```

5-loop 2-point graph with 11 different masses



$$p_1^2 = 100, \quad m_1^2 = 1,\, m_1^2 = 2, \ldots, m_{11}^2 = 11$$

```
* N = 10^12.
* Lambda = 0.02.
* Finished in 20 hours.
-- eps^0: [0.000196885 +/- 0.000000032]  +  i * [0.000140824 +/- 0.000000034]
-- eps^1: [-0.00493791 +/- 0.00000040 ]  +  i * [-0.00079691 +/- 0.00000038 ]
-- eps^2: [ 0.0491933  +/-  0.0000025 ]  +  i * [-0.0154647  +/-  0.0000025 ]
-- eps^3: [ -0.253458  +/-  0.000012  ]  +  i * [ 0.246827   +/-  0.000012  ]
-- eps^4: [ 0.587258   +/-  0.000046  ]  +  i * [ -1.720213  +/-  0.000046  ]
-- eps^5: [  1.05452   +/-   0.00015  ]  +  i * [  7.38725   +/-   0.00015  ]
-- eps^6: [ -14.66144  +/-   0.00047  ]  +  i * [ -20.86779  +/-   0.00046  ]
-- eps^7: [  65.8924   +/-   0.0013   ]  +  i * [  35.0793   +/-   0.0013   ]
-- eps^8: [ -190.9702  +/-   0.0036   ]  +  i * [  -4.4620   +/-   0.0034   ]
-- eps^9: [ 393.2522   +/-   0.0092   ]  +  i * [ -183.7431  +/-   0.0087   ]
-- eps^10:[ -558.202   +/-    0.023   ]  +  i * [  688.556   +/-    0.021   ]
```

Conclusion and outlook

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- Efficient numerical evaluation of Feynman integrals via tropical Monte Carlo
- $i\varepsilon$ prescription handled via analytic continuation $x_e \exp\left[-i\lambda\partial_e \mathcal{V}(x)\right]$
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Outlook:

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- Remove quasi-finite condition via the analytic continuation protocol of [Berkesch, Forsgaard, Passare]
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