Relative Cohomology and Feynman Integrals

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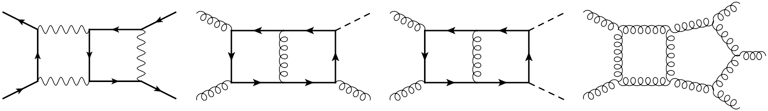

For state-of-the art two-loop scattering amplitude calculations
$\mathcal{O}(1000)$ Feynman diagrams $\rightarrow \mathcal{O}(10000)$ Feynman integrals

$$I_{a_1,\ldots,a_n} = \int \frac{N(k)}{D_1^{a_1}(k)\cdots D_P^{a_P}(k)} \prod_i \frac{\mathrm{d}^d k_i}{\pi^{d/2}}$$

Linear relations bring this down to $\mathcal{O}(100)$ *master integrals*

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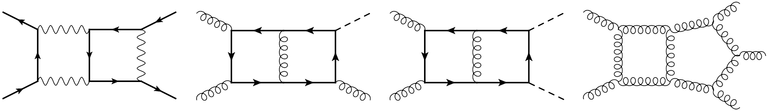

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Linear relations may be derived using IBP (integration by part) identities

$$\int \frac{\mathrm{d}^d k}{\pi^{d/2}} \frac{\partial}{\partial k^\mu} \frac{q^\mu N(k)}{D_1^{a_1}(k)\cdots D_P^{a_P}(k)} = 0$$

Systematic by Laporta's algorithm $\Rightarrow$ Solve a huge linear system.

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The linear relations form a vector space

$$I = \sum_{i \in \text{masters}} c_i I_i$$

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$$\langle v| = \sum_i \langle v v_j^* \rangle \big(\boldsymbol{C}^{-1}\big)_{ji} \langle v_i| \qquad \text{with} \qquad \boldsymbol{C}_{ij} = \langle v_i v_j^* \rangle$$

$$= \sum_i c_i \, \langle v_i| \qquad\qquad \big(c_i = \langle v v_i^* \rangle \;\text{ if }\; \boldsymbol{C}_{ij} = \delta_{ij}\big)$$

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If only there were a way to define an inner product for Feynman integrals...

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$$I = \int_{\mathcal{C}} \mathrm{d}^n x \, \frac{\mathcal{B}^\gamma(x) N(x)}{x_1^{a_1} \cdots x_P^{a_P}} = \int_{\mathcal{C}} u \, \phi$$

$u = \mathcal{B}^\gamma$ is a multivalued function in $\{x\}$

$\phi = \frac{N(x)}{x_1^{a_1} \cdots x_P^{a_P}} \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$ is a form

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$\omega = \mathrm{d}\log(u)$ is *the twist*

$\langle \phi | \mathcal{C}]_\omega$ is a pairing of a *twisted cycle* ($\mathcal{C}$) and a *twisted cocycle* (ϕ)
(equivalence classes of contours and integrands respectively)

P. Mastrolia and S. Mizera, *Feynman Integrals and Intersection Theory*, JHEP **1902** (2019) 139

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We also need the *dual* Feynman integral:

$$I_{\text{dual}} = \int_{\mathcal{C}} u^{-1} \xi \; = \; [\mathcal{C} | \xi \rangle_\omega$$

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$$\langle\phi|\xi\rangle \ := \int (u\phi)_{\text{reg}}\,(u^{-1}\xi) \ = \ \ldots$$

$$\boxed{\langle\phi|\xi\rangle \ = \sum \text{Res}(\psi\xi) \qquad \text{with} \qquad (d + d\log(u))\psi = \phi}$$

ψ can be found with a series expansion $\psi = \sum \psi_i z^i$, a recursive formula, or sometimes a closed expression

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Summary:

$$I = \sum_i c_i I_i \quad \Rightarrow \quad c_i = \langle\phi|\xi_j\rangle(\boldsymbol{C}^{-1})_{ji} \quad \text{with} \quad \boldsymbol{C}_{ij} = \langle\phi_i|\xi_j\rangle$$

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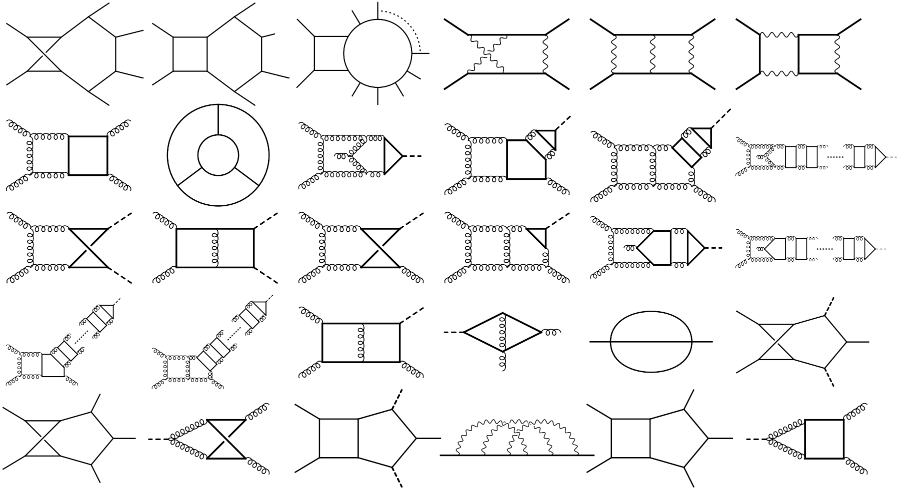
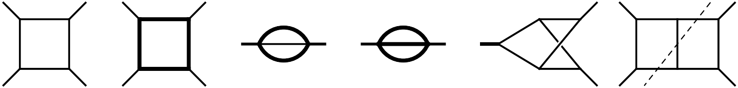
On the maximal cut we did a lot of examples
[HF, Gasparotto, Laporta, Mandal, Mastrolia, Mattiazzi, Mizera (2019)]



Examples of complete reductions:



H. Frellesvig, F. Gasparotto, S. Laporta, M. K. Mandal, P. Mastrolia, L. Mattiazzi, S. Mizera
Decomposition of Feynman Integrals by Multivariate Intersection Numbers
JHEP 03 (2021) **027** arXiv:2008.04823

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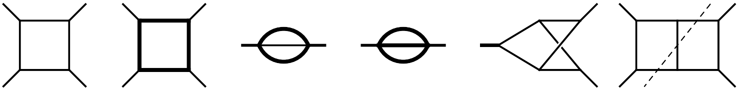

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In *twisted (co)homology* theory

$$I = \int_{\mathcal{C}} u\, \phi \quad \text{with all poles of } \phi \text{ being regulated by } u = \mathcal{B}^{\gamma}$$

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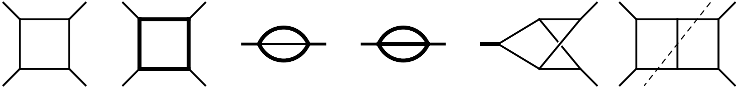

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Solution so far: Introduce *regulators*

$$u \;\to\; u_{\textsf{reg}} = u z_1^{\rho_1} z_2^{\rho_2} \cdots z_m^{\rho_m}$$

and take the limits $\rho_i \to 0$ at the end

I want to get rid of the regulators. One option: *Relative* (twisted) cohomology:
Forms and contours live in a space defined *modulo* a different space

$$I = \int_{\mathcal{C}} u\,\phi \quad \text{with} \quad \phi = \frac{\mathrm{d}^n z}{z_1 \cdots z_m} \; : \quad \text{Work } \textit{relative} \text{ to } \bigcup_i (z_i = 0)$$

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The intersection numbers become easier to compute, involve fewer variables and many are zero.

$$\boldsymbol{C} = \begin{bmatrix} \boldsymbol{C}_{1,1} & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ \boldsymbol{C}_{2,1} & \boldsymbol{C}_{2,2} & \ddots & \boldsymbol{0} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{C}_{n,1} & \boldsymbol{C}_{n,2} & \cdots & \boldsymbol{C}_{n,n} \end{bmatrix}$$

No regulators needed!

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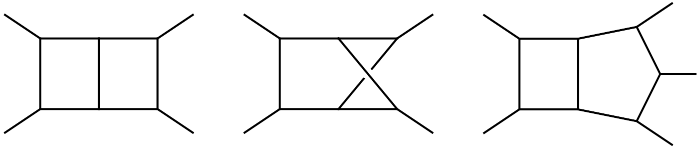
Examples computed using relative cohomologies:



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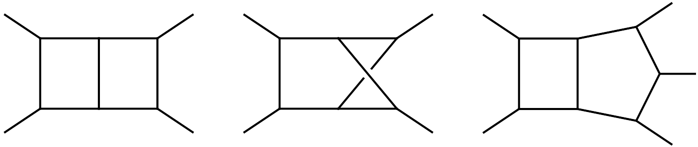

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Other optimizations:

- Get rid of algebraic extensions (e.g. square roots)
  [G. Fontana, T. Peraro (2023)]
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- Different handling of *higher poles*
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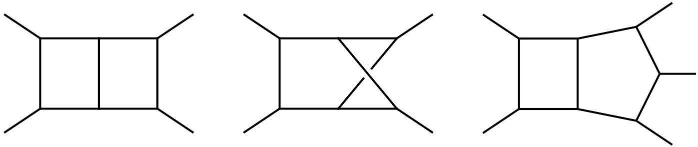

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We are quickly approaching the state of the art

References

Integration-By-Parts (IBP) identities for Feynman Integrals:
[Tkachov (1981)], [Chetyrkin and Tkachov (1981)], [Laporta (2000)], . . .

Public IBP Codes:
[Anastasiou, Lazopoulos (2004)], [Manteuffel, Studerus (2010, 12)], [Lee (2012)],
[Smirnov (2008, 13, 15)], [Smirnov, Chukharev (2019)], [Peraro (2019)],
[Maierhöfer, Usovitsch, Uwer ($2\times$2018)], [Klappert, Lange, Maierhöfer, Usovitsch (2020, 21)]

Intersection Theory and Twisted cohomologies:
[Cho, Matsumoto (1995)], [Matsumoto (1995)], [Matsumoto, Yoshida (1998)], [Matsumoto (2018)], . . .

Intersection Theory and Feynman Integrals:
[Mizera, Mastrolia (2018)], [HF, Gasparotto, Laporta, Mandal, Mastrolia, Mattiazzi, Mizera ($2\times$2019, 20)],
[Mizera, Pokraka (2019)], [Weinzierl (2020)], [Caron-Huot, Pokraka ($2\times$2021)], [Giroux, Pokraka (2022)],
[Chen, Jiang, Ma, Xu, Yang (2021, 22)], [Chestnov, HF, Gasparotto, Mandal, Mastrolia (2022)],
[Chestnov, Gasparotto, Mandal, Mastrolia, Matsubara-Heo, Munch, Takayama (2022, 23)]*
[Cacciatori, Conti, Trevisan (2021)], [Fontana, Peraro (2023)], . . .

Intersection Theory elsewhere in Physics:
[Mizera (2017)], [Mizera (2019)], [Cacciatori, Mastrolia (2022)], [Weinzierl (2021)],
[Gasparotto, Rapakoulias, Weinzierl (2022)], [Gasparotto, Weinzierl, Xu (2023)]**, . . .

Upcoming on Relative Cohomology:
[Brunello, Chestnov, Crisanti, HF, Gasparotto, Mandal, Mastrolia (2023?)]

* See talk by Henrik Munch

** See talk by Federico Gasparotto

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Thank you for listening!

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