

Finite and $\mathcal{O}(\epsilon)$ integrals for minimally divergent integral bases and special relations

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Work with David Kosower, Giulio Gambuti, and Lorenzo Tancredi

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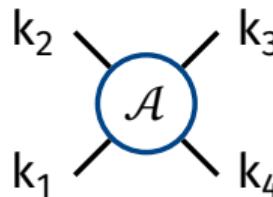
Summary

1. Organizing the basis of master integrals according to their degree of divergence in ϵ may simplify amplitude computations
2. For a given diagram, we can find sets of (locally) finite and $\mathcal{O}(\epsilon)$ integrals systematically
3. These integrals can be compactly described in terms of generating Gram determinants

Motivation

$$\begin{array}{c} k_2 \quad \quad \quad k_3 \\ \diagdown \quad \diagup \\ \text{\textcircled{A}} \\ \diagup \quad \diagdown \\ k_1 \quad \quad \quad k_4 \end{array} = c_1 \text{Master}_1 + \cdots + c_N \text{Master}_N$$

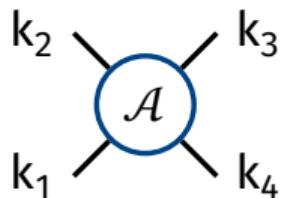
Motivation


$$= c_1 \text{Master}_1 + \dots + c_N \text{Master}_N$$

1. Minimally divergent bases: reduce the number of divergent masters

Q: Which integrals are finite?

Motivation


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1. Minimally divergent bases: reduce the number of divergent masters

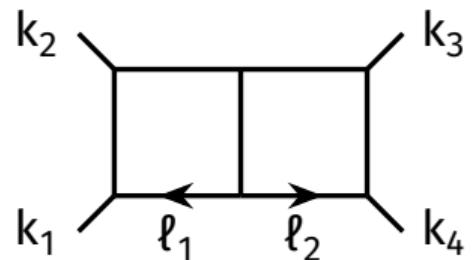
Q: Which integrals are finite?
2. Special relations: remove masters which are redundant to $\mathcal{O}(\epsilon)$

Q: Which integrals are $\mathcal{O}(\epsilon)$?

Problem statement

$\text{Num} = \text{Poly}(\ell_i \cdot \ell_j, \ell_i \cdot k_j)$ with coefficients being Rational($k_i \cdot k_j, m_i$)

$$\int d\ell \frac{\text{Num}}{\text{Den}_1 \cdots \text{Den}_E}$$

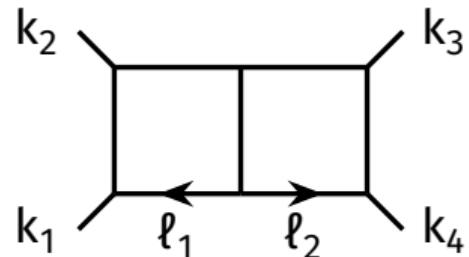


$$\text{Den} = \left(\sum \pm \ell_i \pm k_j \right)^2 - m^2 + i\epsilon$$

Problem statement

$\text{Num} = \text{Poly}(\ell_i \cdot \ell_j, \ell_i \cdot k_j)$ with coefficients being Rational($k_i \cdot k_j, m_i$)

Find Num such that $\int d\ell \frac{\text{Num}}{\text{Den}_1 \cdots \text{Den}_E} = \mathcal{O}(\epsilon^r)$



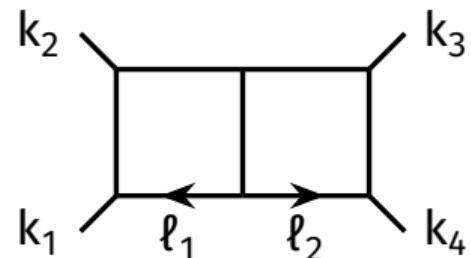
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This talk:
 $r = 0$ (finite)
 $r = 1$ (vanishing in 4d)

UV divergences

Weinberg's theorem

An integral is UV-finite, if:

- it converges superficially
- all **subintegrations** converge superficially

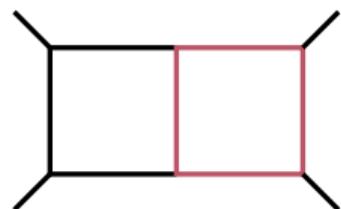
[Dyson 1949; Nakanishi 1957; Weinberg 1960; Hahn, Zimmermann 1968]

Weinberg's theorem

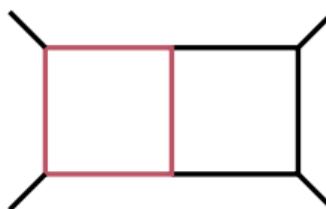
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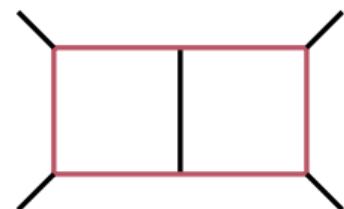
[Dyson 1949; Nakanishi 1957; Weinberg 1960; Hahn, Zimmermann 1968]



$$\ell_1 = C, \quad \ell_2 = \lambda\ell$$



$$\ell_1 = \lambda\ell, \quad \ell_2 = C$$



$$\ell_1 = \lambda\ell, \quad \ell_2 = C - \lambda\ell$$

UV-finite numerators

- form a linear space:

$\text{coef}_1 \cdot \text{Num}_1 + \text{coef}_2 \cdot \text{Num}_2$ is UV-finite

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$$\text{Num}(\ell_1, \ell_2) = \text{coef}_1 + \text{coef}_2 (\ell_1 \cdot k_1) + \cdots + \text{coef}_N (\ell_2^2)^2 (\ell_2 \cdot k_3)$$

UV-finite numerators

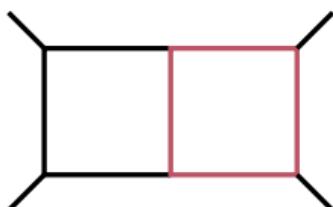
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- subdivergences \rightarrow linear constraints on coef_i :



$$\text{Num}(C, \lambda\ell) = \underbrace{(\text{coef}_N \dots)}_{=0} \lambda^5 + (\dots) \lambda^4 + \dots$$

IR divergences

UV vs IR

UV

$$\left\{ \begin{array}{l} \ell_1 = \infty, \\ \ell_1 + \ell_2 = C \end{array} \right\}$$

divergent surface

$$\left\{ \begin{array}{l} \ell_1 \rightarrow \lambda \ell, \\ \ell_2 \rightarrow C - \lambda \ell \end{array} \right\}$$

power-counting rule

UV vs IR

UV

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$$\left\{ \begin{array}{l} \ell_1 \rightarrow \lambda \ell, \\ \ell_2 \rightarrow C - \lambda \ell \end{array} \right\}$$

power-counting rule

IR

$$\left\{ \begin{array}{l} \ell_1 = k_1, \\ \ell_2 = x k_4 \end{array} \right\}$$

$$\left\{ \begin{array}{l} \ell_1 \rightarrow k_1 + \lambda^2 \ell_s, \\ \ell_2 \rightarrow x k_4 + \lambda^2 \eta_4 + \lambda \ell_\perp \end{array} \right\}$$

[see, e.g., Agarwal, Magnea et al. 2021; Collins 2011; Anastasiou, Sterman 2018]

IR-divergent surfaces

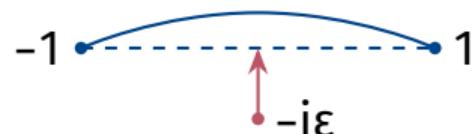
$$\int_{-1}^1 \frac{dx}{x}$$

IR-divergent surfaces

$$\lim_{\varepsilon \rightarrow +0} \int_{-1}^1 \frac{dx}{x + i\varepsilon}$$

IR-divergent surfaces

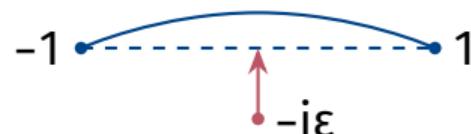
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IR-divergent surfaces

no divergence

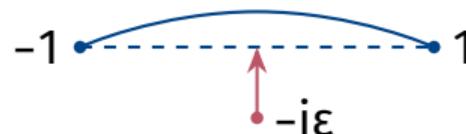
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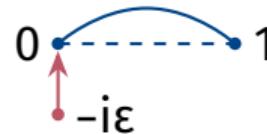
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end-point
divergence

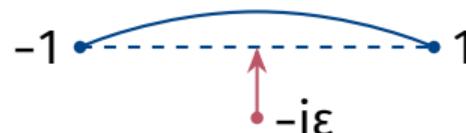
$$\lim_{\varepsilon \rightarrow +0} \int_0^1 \frac{dx}{x + i\varepsilon}$$



IR-divergent surfaces

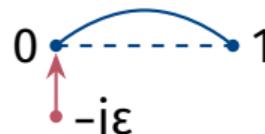
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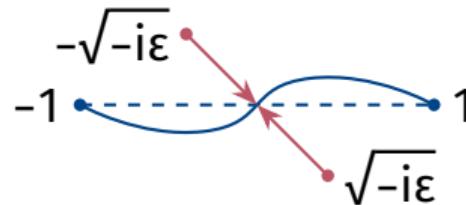
end-point
divergence

$$\lim_{\varepsilon \rightarrow +0} \int_0^1 \frac{dx}{x + i\varepsilon}$$



pinch
divergence

$$\lim_{\varepsilon \rightarrow +0} \int_{-1}^1 \frac{dx}{x^2 + i\varepsilon}$$



Landau equations

$$\forall i: \frac{\partial}{\partial \ell_i} \sum_e \alpha_e D e_n = 0$$

$$\forall e: \alpha_e D e_n = 0$$

[Bjorken 1959; Landau 1959; Nakanishi 1959;
see also Collins 2020]

→ Sebastian's talk on Wednesday

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$$\mathcal{U}(\alpha) \neq 0 \rightarrow \ell_1 = \frac{\alpha_1}{\alpha_1 + \alpha_2} k_1 + \dots$$

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$$\alpha_2 = \alpha_5 = 0$$

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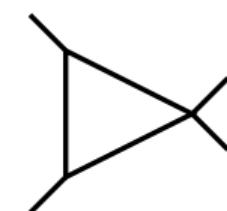
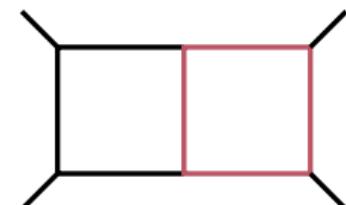
$$U(\alpha) \neq 0 \rightarrow \ell_1 = \frac{\alpha_1}{\alpha_1 + \alpha_2} k_1 + \dots$$

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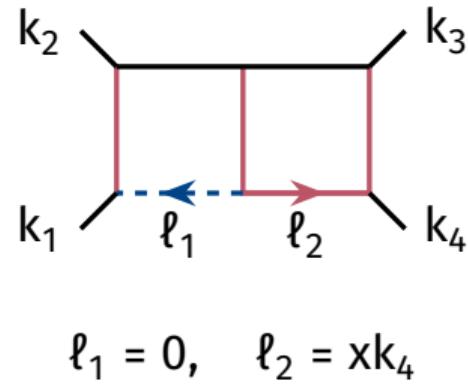
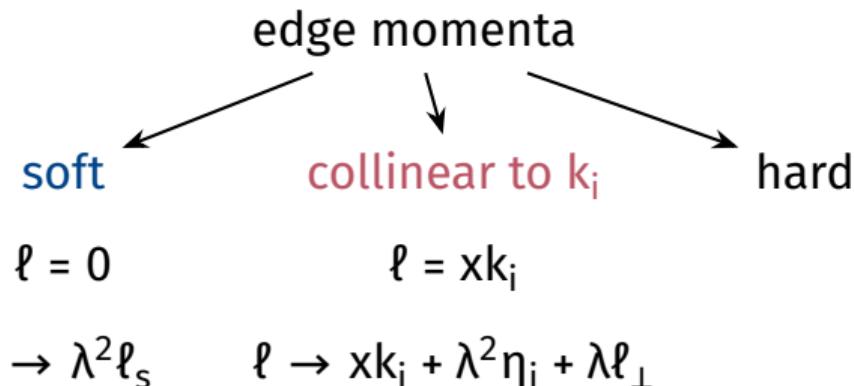
$$\ell_1 = k_1 + \dots$$

$$U(\alpha) = 0$$



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General structure of solutions



[Coleman, Norton 1965; Sterman 1978; Libby, Sterman 1978]

IR-finite numerators

- linear constraints on coef_i as in the UV case:

$$\text{Num}(\lambda^2 \ell_s, xk_4 + \lambda^2 \eta_4 + \lambda \ell_{\perp}) = \underbrace{(\text{coef}_i \dots)}_{=0} + (\dots) \lambda + \dots$$

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- form a polynomial ideal:

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- form a polynomial ideal:

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Conjecture: IR-finite ideal
can be built using Gram determinants

$$G \begin{pmatrix} p_1 & \dots & p_n \\ q_1 & \dots & q_n \end{pmatrix} = \det(2p_i \cdot q_j)$$

$\mathcal{O}(\epsilon)$ numerators

1. Start with the most general finite numerator:

$$\text{Num}(\ell) = \text{coef}_1 \text{Num}_1 + \dots + \text{coef}_N \text{Num}_N$$

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2. Restrict loop momenta to 4d:

$$\ell = b_1 k_1 + b_2 k_2 + b_3 k_3 + b_4 k_{\perp} \quad k_{\perp} \cdot k_i = 0$$

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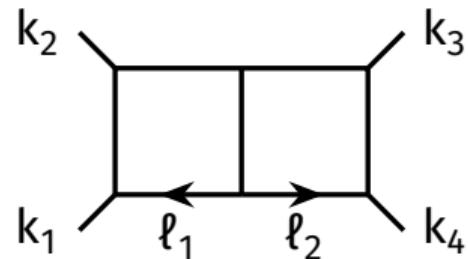
3. Require that **Num** vanish \rightarrow linear constraints on coef_i

Conjecture: $\mathcal{O}(\epsilon)$ numerators
can be built using **Gram determinants**

Results

Result: planar double box

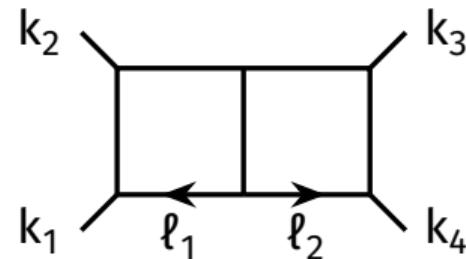
max. rank	1	2	3	4	5	
$\mathcal{O}(\epsilon^0)$	lin. dim.	0	2	18	89	247
$\mathcal{O}(\epsilon^1)$	lin. dim.	0	0	0	1	7



31 IR-div. surfaces

Result: planar double box

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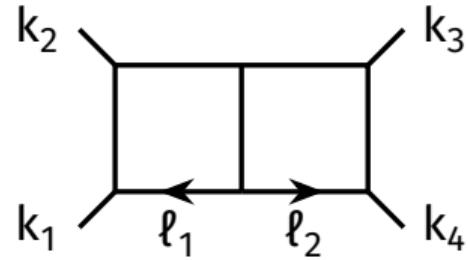


31 IR-div. surfaces

$$\begin{aligned}
 \text{Num} = & \frac{1}{2} (s_{12} + s_{23}) (\ell_1 \cdot \ell_2) + (\ell_1 \cdot k_3) (\ell_2 \cdot k_1) + (\ell_1 \cdot k_3) (\ell_2 \cdot k_3) \\
 & - \frac{s_{23}}{s_{12}} (\ell_1 \cdot k_1) (\ell_2 \cdot k_1) - \frac{s_{23}}{s_{12}} (\ell_1 \cdot k_1) (\ell_2 \cdot k_3) \\
 & - \left(1 + \frac{s_{23}}{s_{12}}\right) (\ell_1 \cdot k_1) (\ell_2 \cdot k_2) + \left(1 + \frac{s_{23}}{s_{12}}\right) (\ell_1 \cdot k_2) (\ell_2 \cdot k_3)
 \end{aligned}$$

Result: planar double box

max. rank	1	2	3	4	5	
$\mathcal{O}(\epsilon^0)$	lin. dim.	0	2	18	89	247
	Grams	0	2	6	10	10
$\mathcal{O}(\epsilon^1)$	lin. dim.	0	0	0	1	7
	Grams	0	0	0	1	1



31 IR-div. surfaces

$$\begin{aligned}
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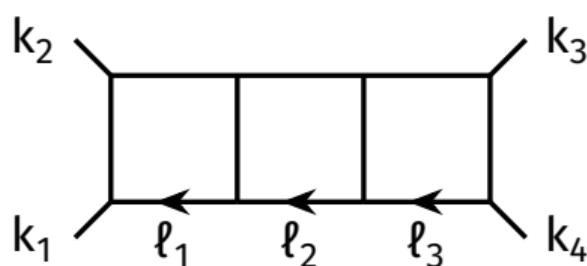
rank 2	$G\begin{pmatrix} \ell_1 & k_1 & k_2 \\ \ell_2 & k_3 & k_4 \end{pmatrix}$	$G\begin{pmatrix} \ell_1 & k_1 & k_2 \\ k_1 & k_2 & k_4 \end{pmatrix} G\begin{pmatrix} \ell_2 & k_3 & k_4 \\ k_1 & k_2 & k_4 \end{pmatrix}$
rank 3	$(\ell_1 - k_1)^2 G\begin{pmatrix} \ell_2 & k_3 & k_4 \\ k_1 & k_2 & k_4 \end{pmatrix}$	$G\begin{pmatrix} \ell_1 & k_1 & k_2 \\ k_1 & k_2 & k_4 \end{pmatrix} G(\ell_2, k_3, k_4)$
	$(\ell_2 - k_4)^2 G\begin{pmatrix} \ell_1 & k_1 & k_2 \\ k_1 & k_2 & k_4 \end{pmatrix}$	$G\begin{pmatrix} \ell_2 & k_3 & k_4 \\ k_1 & k_2 & k_4 \end{pmatrix} G(\ell_1, k_1, k_2)$
rank 4	$(\ell_1 - k_1)^2 G(\ell_2, k_3, k_4)$	$(\ell_1 - k_1)^2 (\ell_2 - k_4)^2$
	$(\ell_2 - k_4)^2 G(\ell_1, k_1, k_2)$	$G(\ell_1, \ell_2, k_1, k_2, k_4)$

Result: planar double box

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rank 3	$(\ell_1 - k_1)^2 G \begin{pmatrix} \ell_2 & k_3 & k_4 \\ k_1 & k_2 & k_4 \end{pmatrix}$	$G \begin{pmatrix} \ell_1 & k_1 & k_2 \\ k_1 & k_2 & k_4 \end{pmatrix} G (\ell_2 \quad k_3 \quad k_4)$
	$(\ell_2 - k_4)^2 G \begin{pmatrix} \ell_1 & k_1 & k_2 \\ k_1 & k_2 & k_4 \end{pmatrix}$	$G \begin{pmatrix} \ell_2 & k_3 & k_4 \\ k_1 & k_2 & k_4 \end{pmatrix} G (\ell_1 \quad k_1 \quad k_2)$
rank 4	$(\ell_1 - k_1)^2 G (\ell_2 \quad k_3 \quad k_4)$	$(\ell_1 - k_1)^2 (\ell_2 - k_4)^2$
	$(\ell_2 - k_4)^2 G (\ell_1 \quad k_1 \quad k_2)$	$G (\ell_1 \quad \ell_2 \quad k_1 \quad k_2 \quad k_4) \quad \mathcal{O}(\epsilon)$

Result: 3-loop ladder

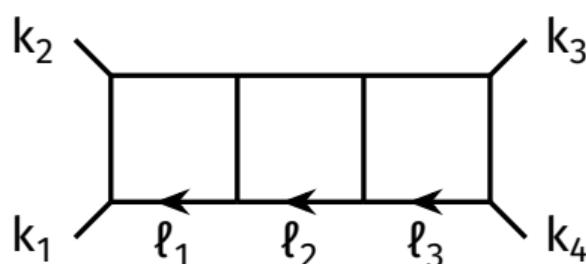
max. rank		1	2	3	4	5	6	7
$\mathcal{O}(\epsilon^0)$	lin. dim.	0	2	26	184	850	2807	6044
	Grams	0	2	8	17	17	17	17
$\mathcal{O}(\epsilon^1)$	lin. dim.	0	0	0	4	42	229	678
	Grams	0	0	0	4	6	8	8



71 IR-div. surfaces

Result: 3-loop ladder

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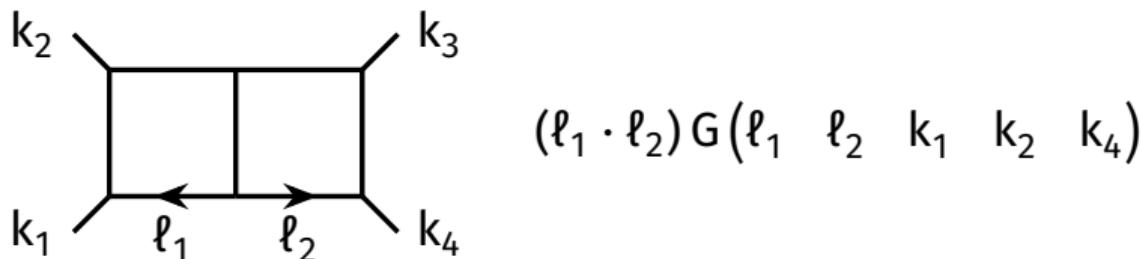


$$G \begin{pmatrix} \ell_1 & \ell_2 & k_1 & k_2 & k_4 \\ \ell_1 & \ell_3 & k_1 & k_2 & k_4 \end{pmatrix}, \dots$$

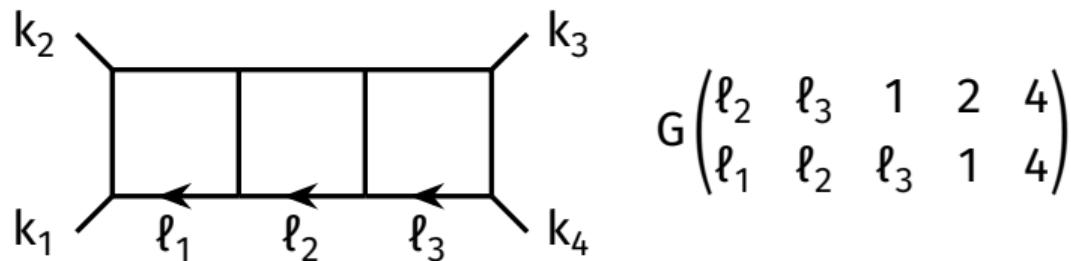
71 IR-div. surfaces

Beyond locally finite integrals

- $\epsilon_{\text{UV}}^{-1} \cdot \epsilon_{4d}^0 \sim \epsilon^0$



- $\epsilon_{\text{IR}}^{-1} \cdot \epsilon_{4d}^0 \sim \epsilon^0$



→ Ben's talk tomorrow

Summary

1. Organizing the basis of master integrals according to their degree of divergence in ϵ may simplify amplitude computations
2. For a given diagram, we can find sets of (locally) finite and $\mathcal{O}(\epsilon)$ integrals systematically
3. These integrals can be compactly described in terms of generating Gram determinants