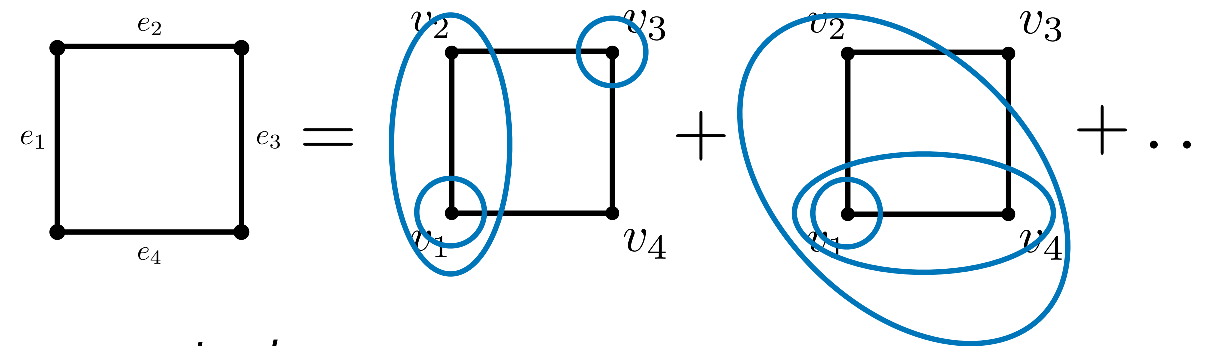
Exposing the threshold structure of loop integrals

Zeno Capatti
ETH Zürich

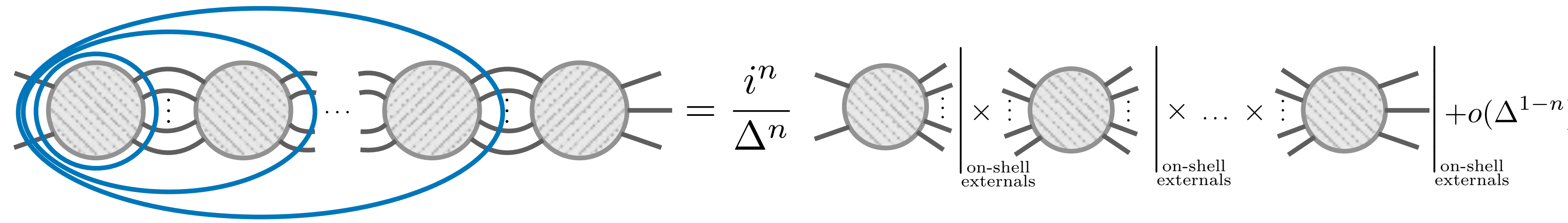
RADCOR 2023
30/05/2023, Crieff, Scotland

Summary:

1. The Cross-Free Family representation



2. Iterated connectedness

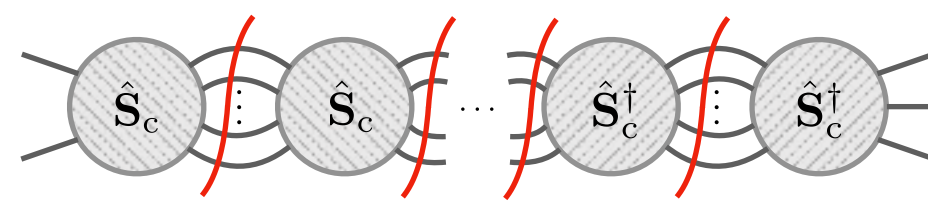


3. Steinmann relations

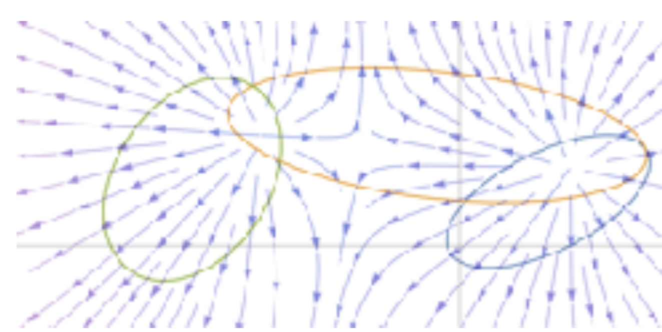
$$\mathrm{disc}_s \mathrm{disc}_t\, I(s,t) = 0$$

4.Cluster decomposition, Unitarity and Infrared Finiteness

$$\sum_{\boldsymbol{\alpha}} \sum_{\boldsymbol{\beta}} \langle \boldsymbol{\alpha} | (\hat{\mathbf{S}}_{\mathrm{c}})^n | \boldsymbol{\beta} \rangle \langle \boldsymbol{\beta} | (\hat{\mathbf{S}}_{\mathrm{c}}^\dagger)^m | \boldsymbol{\alpha} \rangle =$$



5. How to compute discontinuities



Backup slides!

The Cross-Free Family representation

G. F. Sterman, An Introduction to QFT

Borinsky, ZC, Salas-Bernadez, Laenen
arXiv: 2210.05532 (2022)

ZC, arXiv: 2211.09653 (2022)

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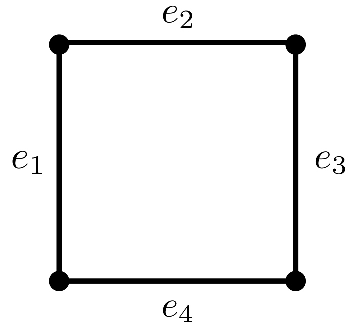

$$= i \int \frac{\mathrm{d}k^0}{2\pi} \frac{N(\{k+p_i\}_{i=1}^4)}{\prod_{i=1}^4 ((k+p_i)^2 + m_i^2 + i\varepsilon)}$$

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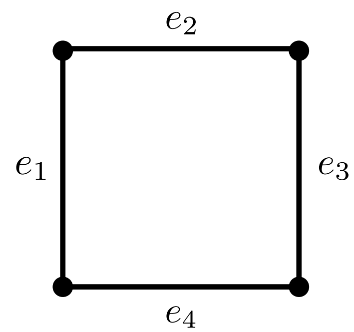

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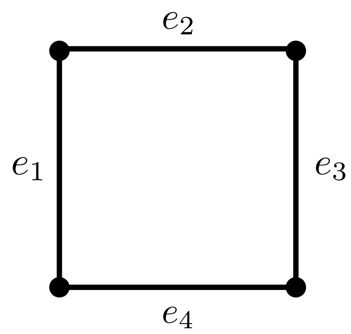

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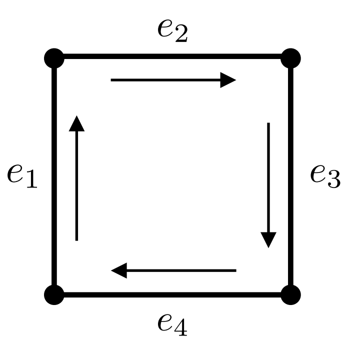

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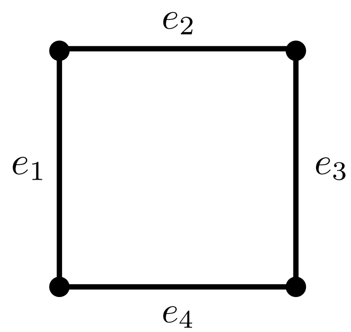

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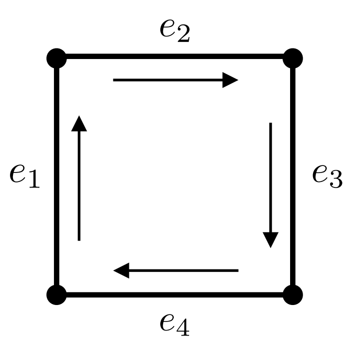

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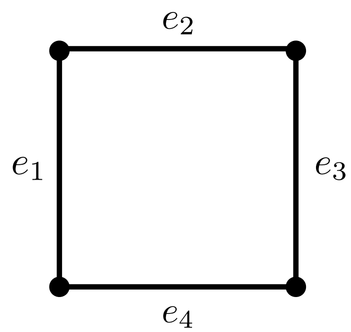

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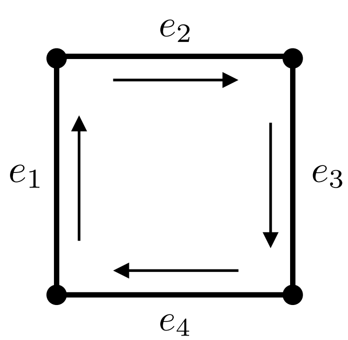

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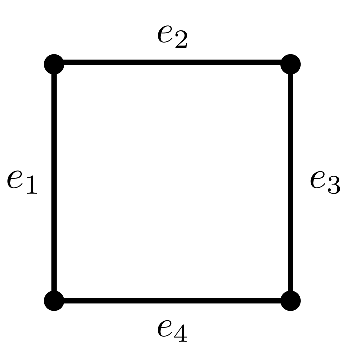

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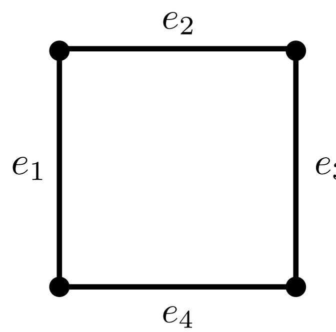

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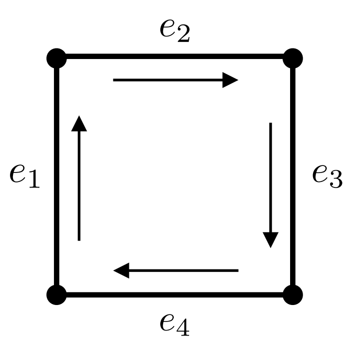

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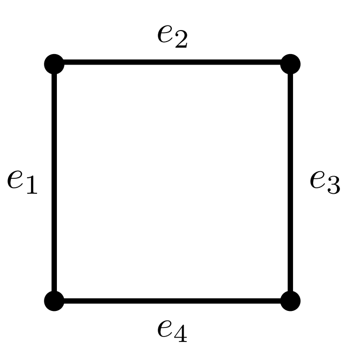

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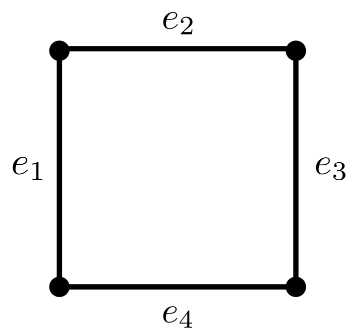

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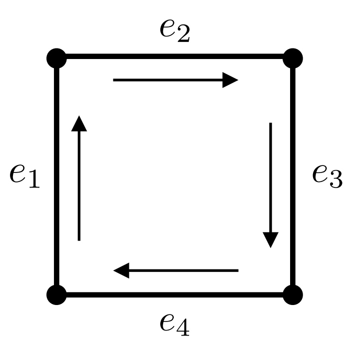

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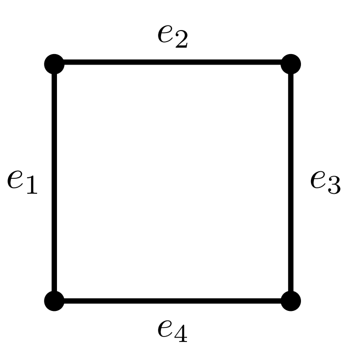

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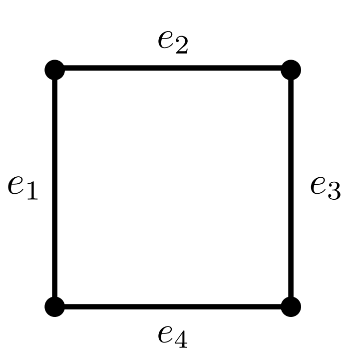

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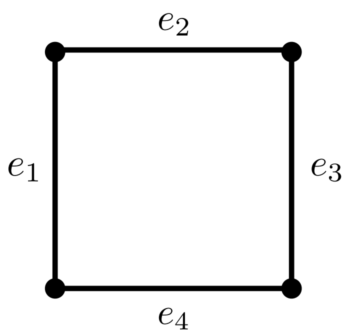

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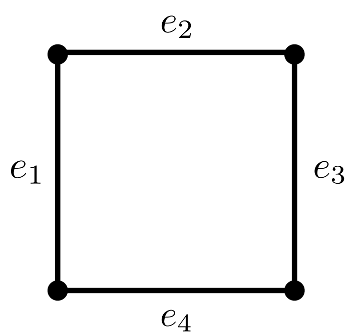

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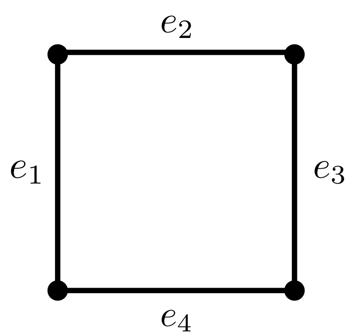

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The signs switch the orientation of the edge! Interpret sum over signs as sum over orientations

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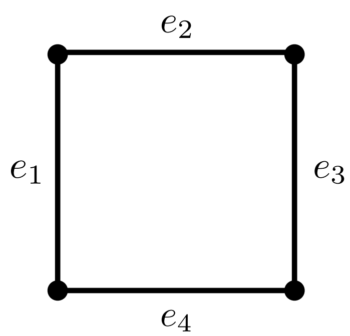

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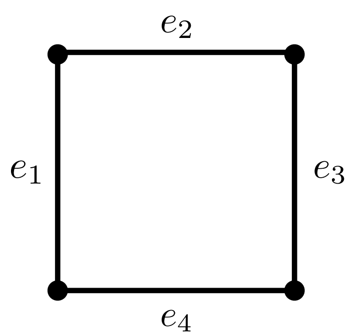

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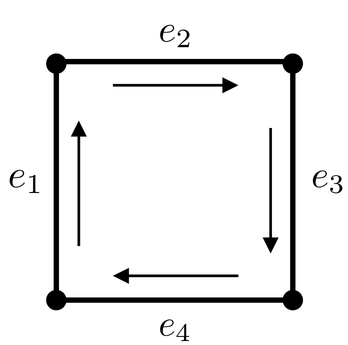

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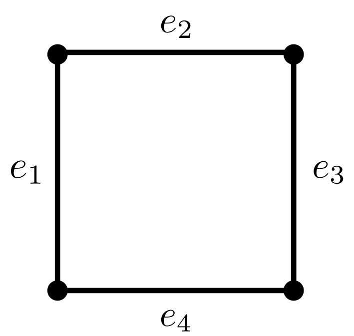

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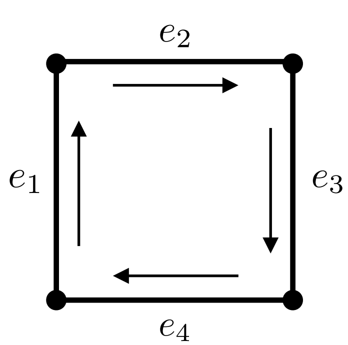
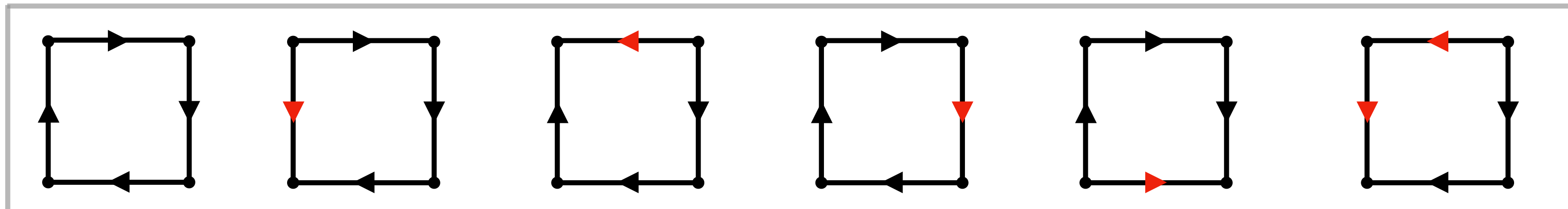
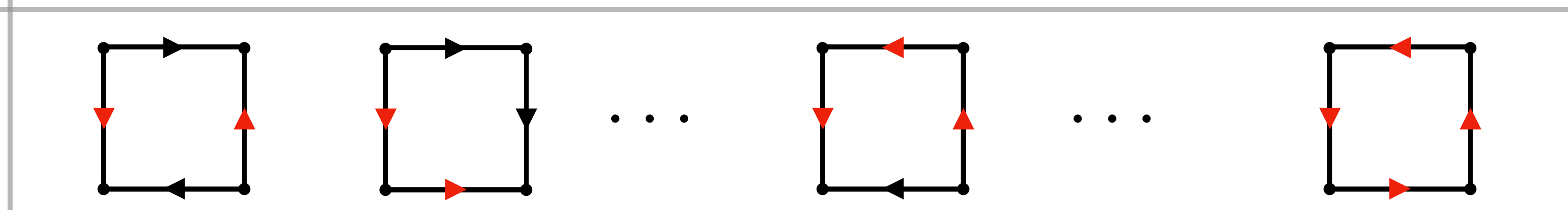




$\vec{\sigma}$	$(1,1,1,1)$	$(-1,1,1,1)$	$(1,-1,1,1)$	$(1,1,-1,1)$	$(1,1,1,-1)$	$(-1,-1,1,1)$



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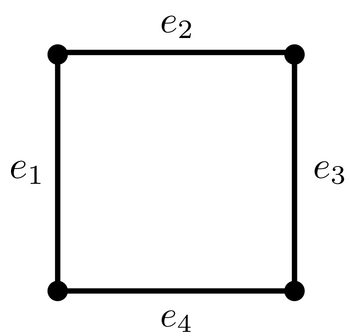

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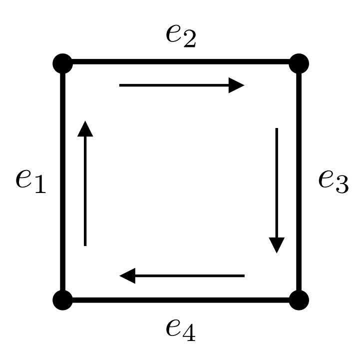
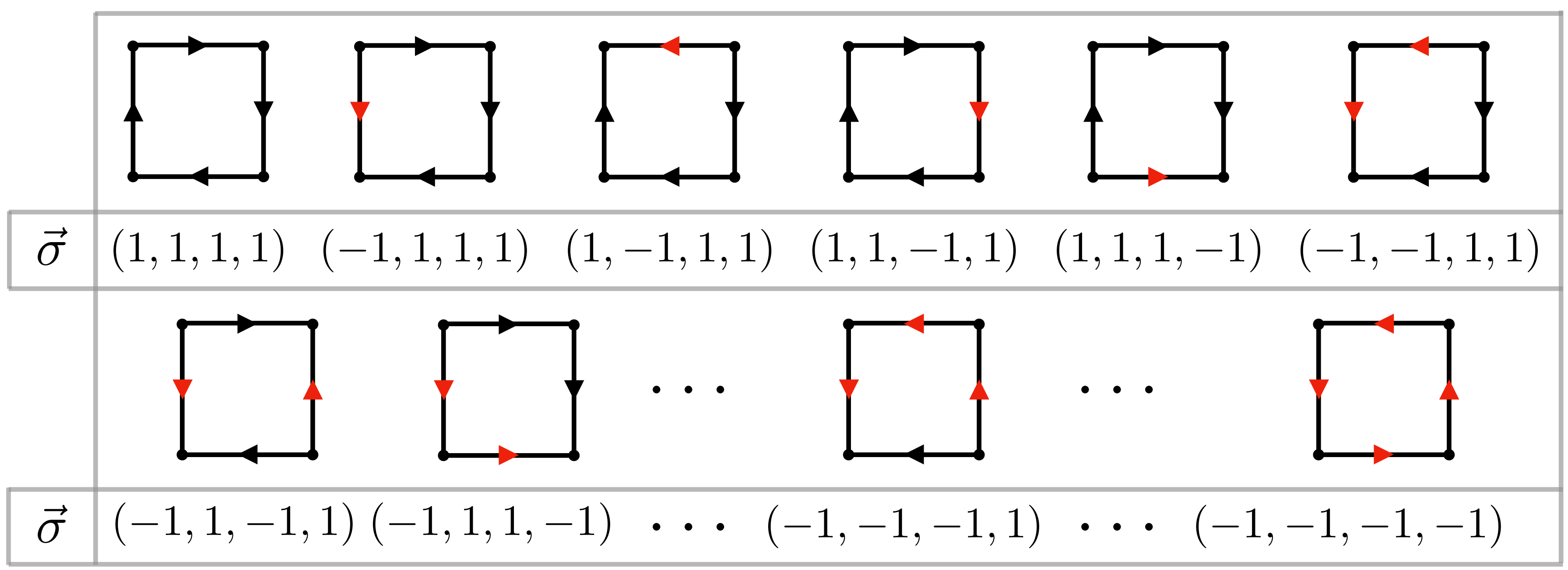
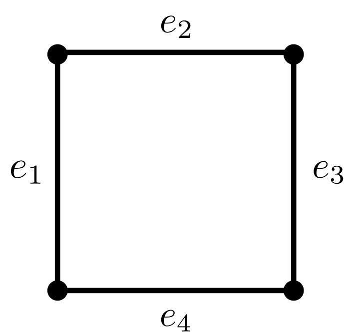




To each orientation corresponds an integral, e.g.

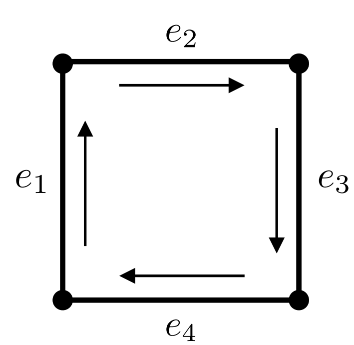
Finally, we perform the energy loop integration

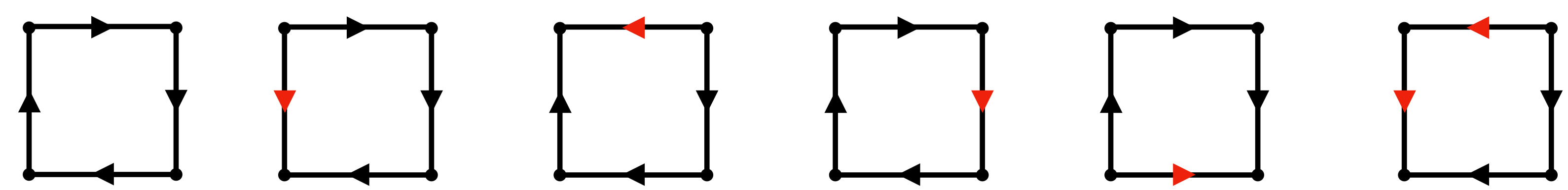


$$= i \sum_{\vec{\sigma}\in\{\pm 1\}^4} N(\{\sigma_j E_j\}_{i=1}^4) \int \frac{\mathrm{d}k^0}{2\pi} \int \left[\prod_{j=1}^{4} \frac{\mathrm{d}\tau_j}{2E_j} e^{i\tau_j(E_j^0 - \sigma_j(k^0+p_j^0))}\Theta(\tau_j)\right]$$

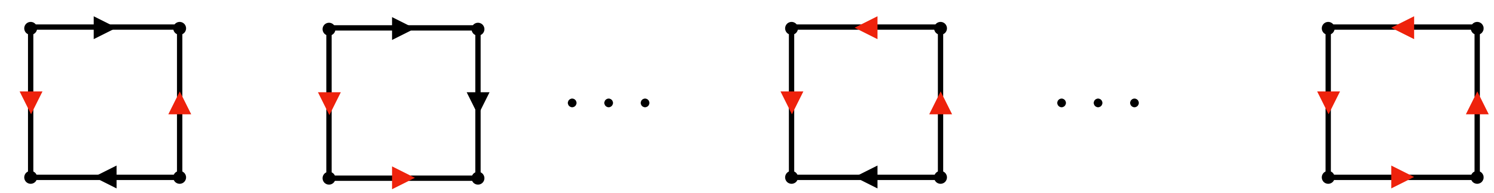
$$= \sum_{\vec{\sigma}\in\{\pm 1\}^4} N(\{\sigma_j E_j\}_{i=1}^4) \int \left[\prod_{j=1}^{4} \frac{\mathrm{d}\tau_j}{2E_j} e^{i\tau_j(E_j^0 - \sigma_j p_j^0)}\Theta(\tau_j)\right] \delta\left(\sum_{i=1}^{4}\sigma_i\tau_i\right)$$

The signs switch the orientation of the edge! Interpret sum over signs as sum over orientations





$\vec{\sigma}$	$(1,1,1,1)$	$(-1,1,1,1)$	$(1,-1,1,1)$	$(1,1,-1,1)$	$(1,1,1,-1)$	$(-1,-1,1,1)$



$\vec{\sigma}$	$(-1,1,-1,1)$	$(-1,1,1,-1)$	$\cdots$	$(-1,-1,-1,1)$	$\cdots$	$(-1,-1,-1,-1)$

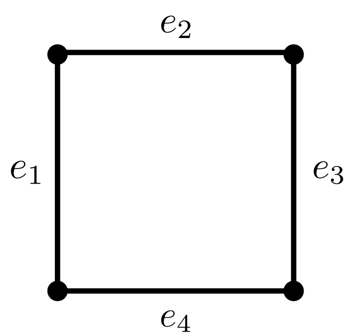
To each orientation corresponds an integral, e.g.

Integration domain:

$$\tau_j > 0,\ \ j = 1, ..., 4$$

$$\tau_1 + \tau_2 + \tau_3 + \tau_4 = 0$$

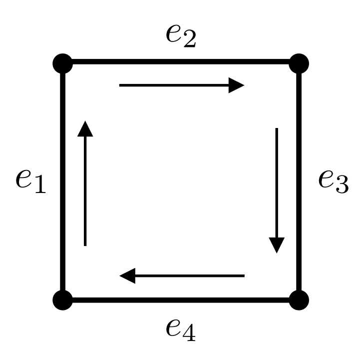
Finally, we perform the energy loop integration

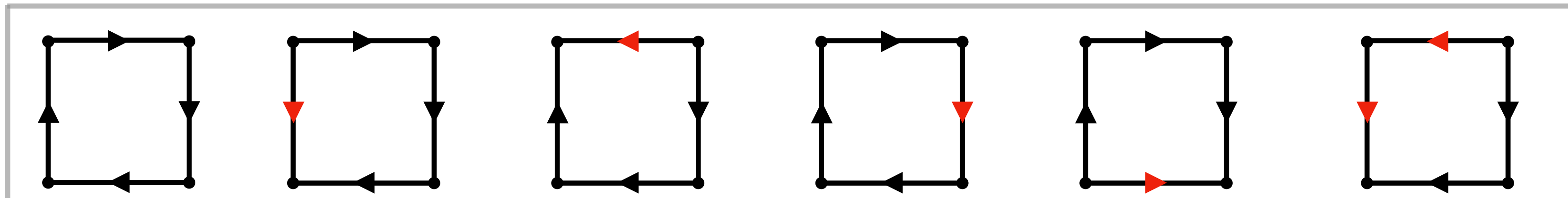


$$= i \sum_{\vec{\sigma}\in\{\pm 1\}^4} N(\{\sigma_j E_j\}_{i=1}^4) \int \frac{\mathrm{d}k^0}{2\pi} \int \left[\prod_{j=1}^{4} \frac{\mathrm{d}\tau_j}{2E_j} e^{i\tau_j (E_j^0 - \sigma_j (k^0 + p_j^0))} \Theta(\tau_j) \right]$$

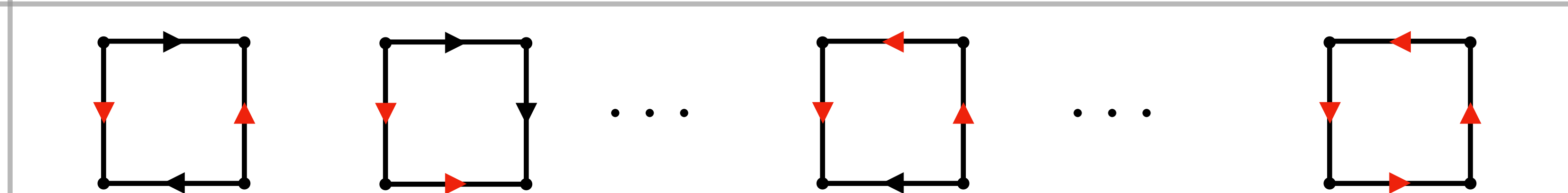
$$= \sum_{\vec{\sigma}\in\{\pm 1\}^4} N(\{\sigma_j E_j\}_{i=1}^4) \int \left[\prod_{j=1}^{4} \frac{\mathrm{d}\tau_j}{2E_j} e^{i\tau_j (E_j^0 - \sigma_j p_j^0)} \Theta(\tau_j) \right] \delta\left(\sum_{i=1}^{4} \sigma_i \tau_i \right)$$

The signs switch the orientation of the edge! Interpret sum over signs as sum over orientations





$\vec{\sigma}$	$(1,1,1,1)$	$(-1,1,1,1)$	$(1,-1,1,1)$	$(1,1,-1,1)$	$(1,1,1,-1)$	$(-1,-1,1,1)$



$\vec{\sigma}$	$(-1,1,-1,1)$	$(-1,1,1,-1)$	$\cdots$	$(-1,-1,-1,1)$	$\cdots$	$(-1,-1,-1,-1)$

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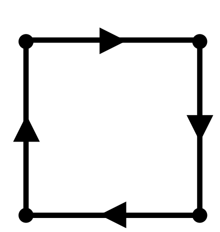


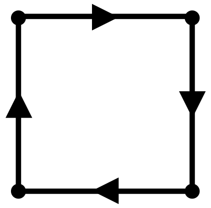
$$= \int \left[\prod_{j=1}^{4} \frac{\mathrm{d}\tau_j}{2E_j} e^{i\tau_j (E_j^0 - \sigma_j p_j^0)} \Theta(\tau_j) \right] \delta\left(\tau_1 + \tau_2 + \tau_3 + \tau_4\right)$$

Integration domain:

$\tau_j > 0, \ \ j = 1, \ldots, 4$

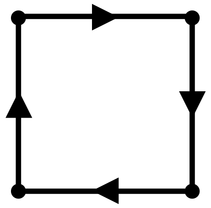
$\tau_1 + \tau_2 + \tau_3 + \tau_4 = 0$

Sborlini, arXiv:2102.05062 (2021) Borinsky, ZC, Salas-Bernadez, Laenen, arXiv: 2210.05532 (2022) ZC, arXiv: 2211.09653 (2022)



$$= \int \left[\prod_{j=1}^{4} \frac{\mathrm{d}\tau_j}{2E_j} e^{i\tau_j (E_j^0 - \sigma_j p_j^0)} \Theta(\tau_j) \right] \delta\left(\tau_1 + \tau_2 + \tau_3 + \tau_4\right)$$

Sborlini, arXiv:2102.05062 (2021) Borinsky, ZC, Salas-Bernadez, Laenen, arXiv: 2210.05532 (2022) ZC, arXiv: 2211.09653 (2022)



$$= \int \left[\prod_{j=1}^{4} \frac{\mathrm{d}\tau_j}{2E_j} e^{i\tau_j (E_j^0 - \sigma_j p_j^0)} \Theta(\tau_j) \right] \delta\left(\tau_1 + \tau_2 + \tau_3 + \tau_4\right)$$

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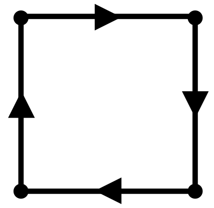

$$= \int \left[\prod_{j=1}^{4} \frac{\mathrm{d}\tau_j}{2E_j} e^{i\tau_j (E_j^0 - \sigma_j p_j^0)} \Theta(\tau_j) \right] \delta\left(\tau_1 + \tau_2 + \tau_3 + \tau_4\right)$$

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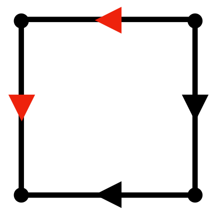

$$= \int \left[\prod_{j=1}^{4} \frac{\mathrm{d}\tau_j}{2E_j} e^{i\tau_j (E_j^0 - \sigma_j p_j^0)} \Theta(\tau_j) \right] \delta\left(-\tau_1 - \tau_2 + \tau_3 + \tau_4\right)$$

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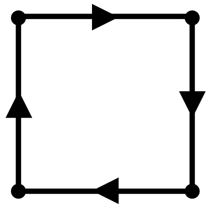
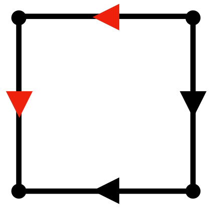


$$= \int \left[\prod_{j=1}^{4} \frac{\mathrm{d}\tau_j}{2E_j} e^{i\tau_j (E_j^0 - \sigma_j p_j^0)} \Theta(\tau_j) \right] \delta\left(\tau_1 + \tau_2 + \tau_3 + \tau_4\right)$$

$\tau_j > 0, \;\; j = 1, ..., 4$

$\tau_1 + \tau_2 + \tau_3 + \tau_4 = 0$



$$= \int \left[\prod_{j=1}^{4} \frac{\mathrm{d}\tau_j}{2E_j} e^{i\tau_j (E_j^0 - \sigma_j p_j^0)} \Theta(\tau_j) \right] \delta\left(-\tau_1 - \tau_2 + \tau_3 + \tau_4\right)$$

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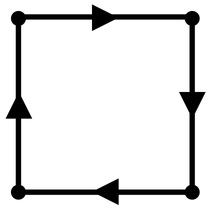

$$= \int \left[\prod_{j=1}^{4} \frac{\mathrm{d}\tau_j}{2E_j} e^{i\tau_j (E_j^0 - \sigma_j p_j^0)} \Theta(\tau_j) \right] \delta\left(\tau_1 + \tau_2 + \tau_3 + \tau_4\right)$$

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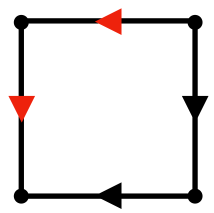

$$= \int \left[\prod_{j=1}^{4} \frac{\mathrm{d}\tau_j}{2E_j} e^{i\tau_j (E_j^0 - \sigma_j p_j^0)} \Theta(\tau_j) \right] \delta\left(-\tau_1 - \tau_2 + \tau_3 + \tau_4\right)$$

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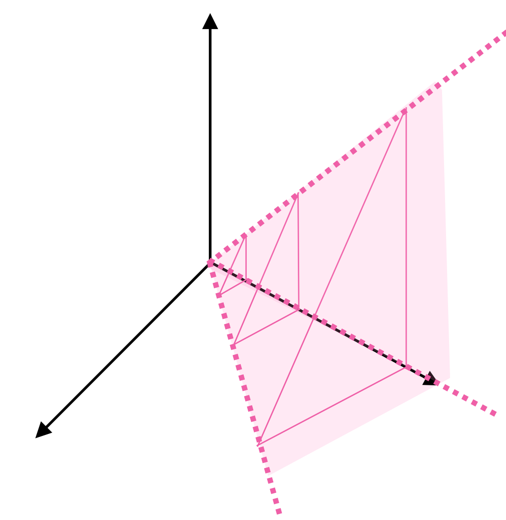

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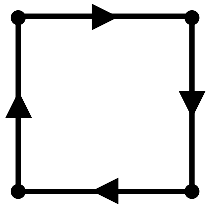

$$= \int \left[\prod_{j=1}^{4} \frac{\mathrm{d}\tau_j}{2E_j} e^{i\tau_j (E_j^0 - \sigma_j p_j^0)} \Theta(\tau_j) \right] \delta\left(\tau_1 + \tau_2 + \tau_3 + \tau_4\right)$$

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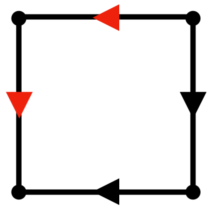

$$= \int \left[\prod_{j=1}^{4} \frac{\mathrm{d}\tau_j}{2E_j} e^{i\tau_j (E_j^0 - \sigma_j p_j^0)} \Theta(\tau_j) \right] \delta\left(-\tau_1 - \tau_2 + \tau_3 + \tau_4\right)$$

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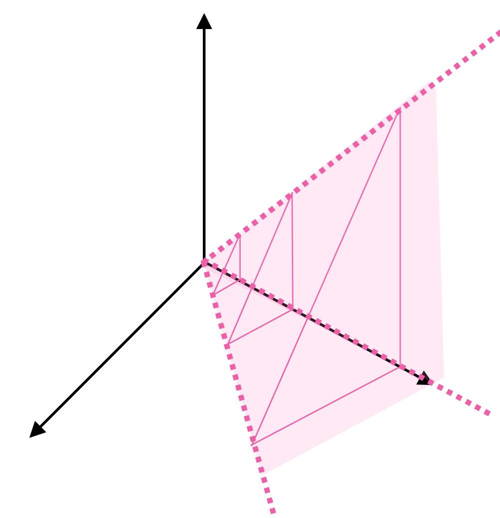
Observation: first integration domain is empty, second is a convex cone



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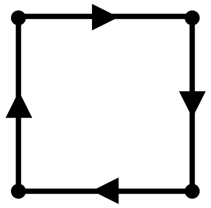

$$= \int \left[\prod_{j=1}^{4} \frac{\mathrm{d}\tau_j}{2E_j} e^{i\tau_j (E_j^0 - \sigma_j p_j^0)} \Theta(\tau_j) \right] \delta\left(\tau_1 + \tau_2 + \tau_3 + \tau_4\right)$$

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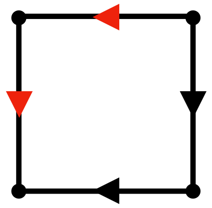

$$= \int \left[\prod_{j=1}^{4} \frac{\mathrm{d}\tau_j}{2E_j} e^{i\tau_j (E_j^0 - \sigma_j p_j^0)} \Theta(\tau_j) \right] \delta\left(-\tau_1 - \tau_2 + \tau_3 + \tau_4\right)$$

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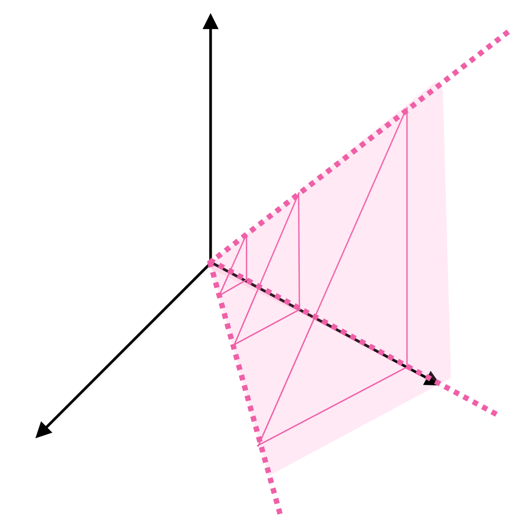
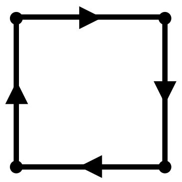
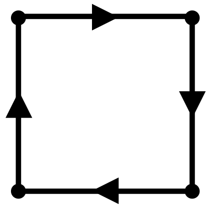




$$= 0$$



$$= \int \left[\prod_{j=1}^{4} \frac{\mathrm{d}\tau_j}{2E_j} e^{i\tau_j(E_j^0 - \sigma_j p_j^0)} \Theta(\tau_j) \right] \delta\left(\tau_1 + \tau_2 + \tau_3 + \tau_4\right)$$

$$\tau_j > 0,\ \ j = 1, ..., 4$$
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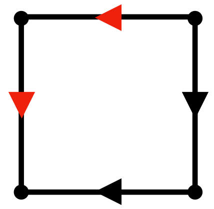

$$= \int \left[\prod_{j=1}^{4} \frac{\mathrm{d}\tau_j}{2E_j} e^{i\tau_j(E_j^0 - \sigma_j p_j^0)} \Theta(\tau_j) \right] \delta\left(-\tau_1 - \tau_2 + \tau_3 + \tau_4\right)$$

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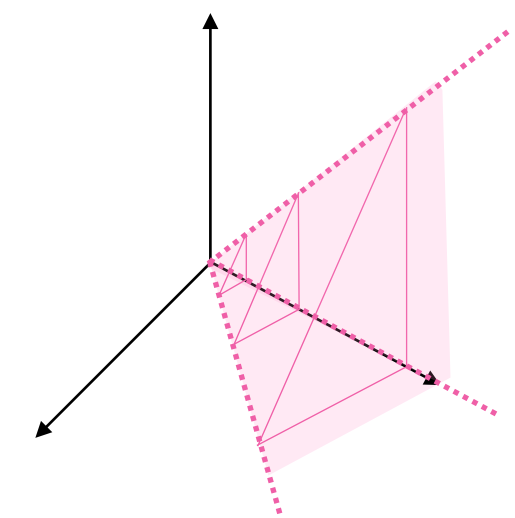
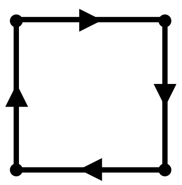
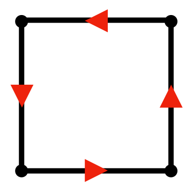
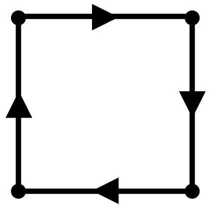
Observation: first integration domain is empty, second is a convex cone



 $= 0$  $= 0$

Sborlini, arXiv:2102.05062 (2021) Borinsky, ZC, Salas-Bernadez, Laenen, arXiv: 2210.05532 (2022) ZC, arXiv: 2211.09653 (2022)



$$= \int \left[\prod_{j=1}^{4} \frac{\mathrm{d}\tau_j}{2E_j} e^{i\tau_j (E_j^0 - \sigma_j p_j^0)} \Theta(\tau_j) \right] \delta\left(\tau_1 + \tau_2 + \tau_3 + \tau_4\right)$$

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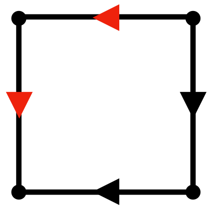

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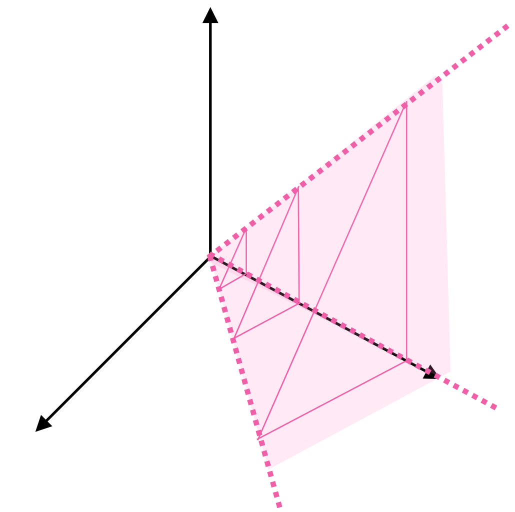
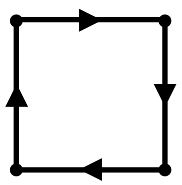
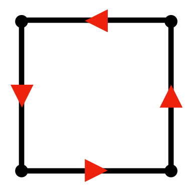
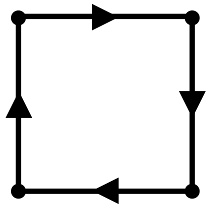


 $= 0$  $= 0$ The graphs have a directed cycle!

Sborlini, arXiv:2102.05062 (2021) Borinsky, ZC, Salas-Bernadez, Laenen, arXiv: 2210.05532 (2022) ZC, arXiv: 2211.09653 (2022)



$$= \int \left[\prod_{j=1}^{4} \frac{\mathrm{d}\tau_j}{2E_j} e^{i\tau_j(E_j^0-\sigma_j p_j^0)}\Theta(\tau_j)\right] \delta\left(\tau_1+\tau_2+\tau_3+\tau_4\right)$$

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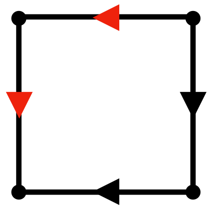

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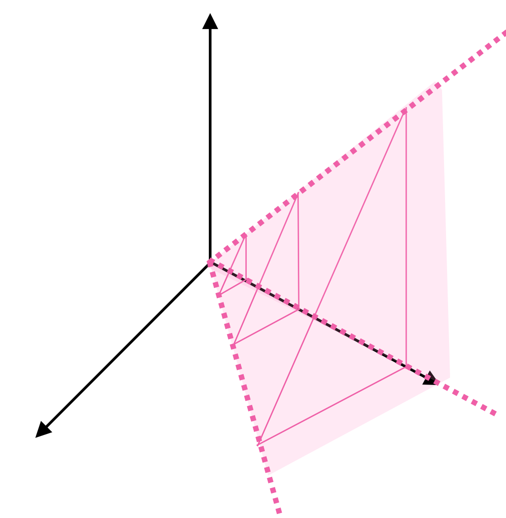
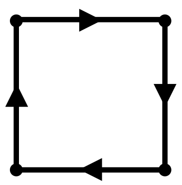
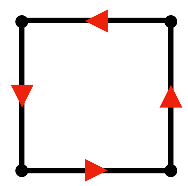
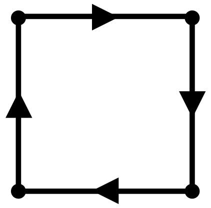


Observation: first integration domain is empty, second is a convex cone

 $= 0$  $= 0$ The graphs have a directed cycle!

Only **acyclic** graphs give non zero contribution!

Sborlini, arXiv:2102.05062 (2021) Borinsky, ZC, Salas-Bernadez, Laenen, arXiv: 2210.05532 (2022) ZC, arXiv: 2211.09653 (2022)



$$= \int \left[\prod_{j=1}^{4} \frac{\mathrm{d}\tau_j}{2E_j} e^{i\tau_j (E_j^0 - \sigma_j p_j^0)} \Theta(\tau_j) \right] \delta\left(\tau_1 + \tau_2 + \tau_3 + \tau_4\right)$$

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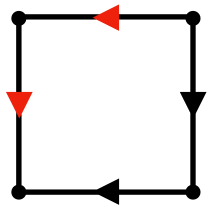

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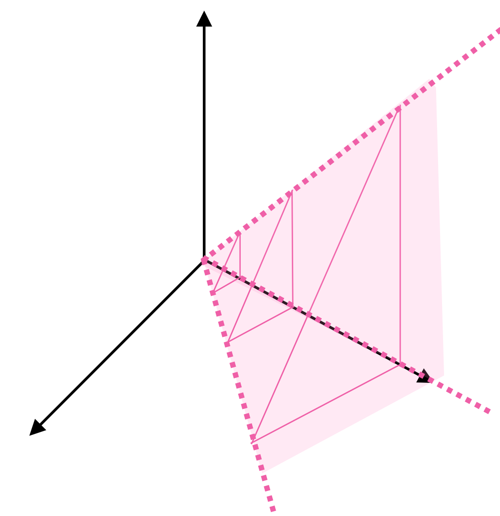
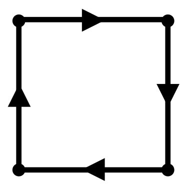
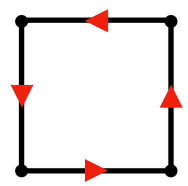


 $= 0$  $= 0$ The graphs have a directed cycle!

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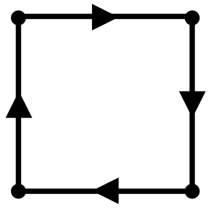

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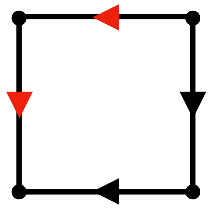

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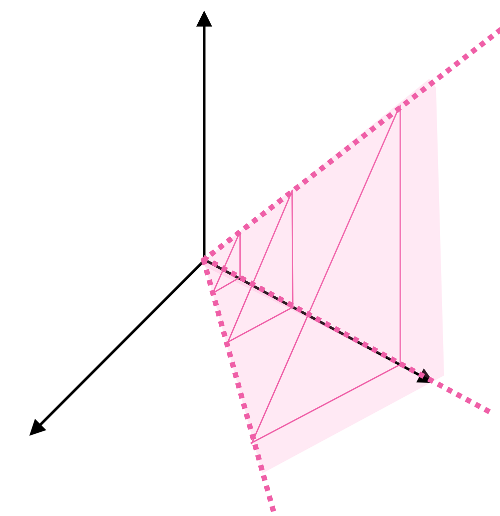
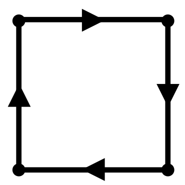
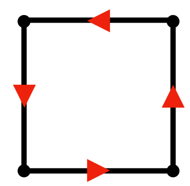


 $= 0$  $= 0$ The graphs have a directed cycle!

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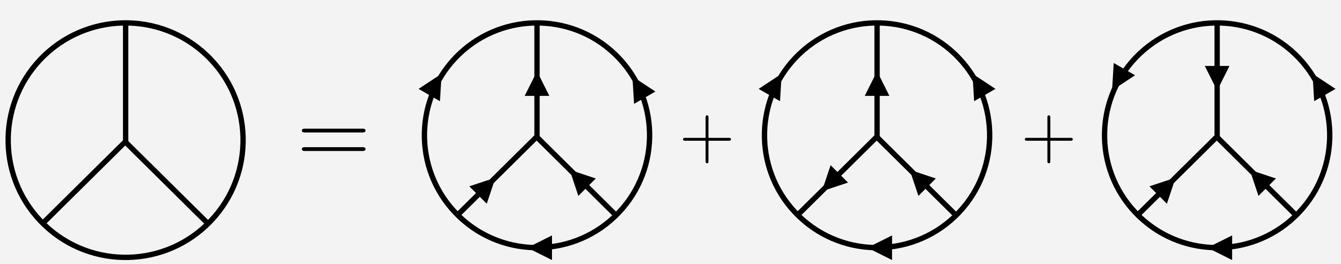

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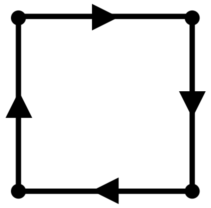

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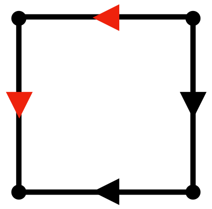

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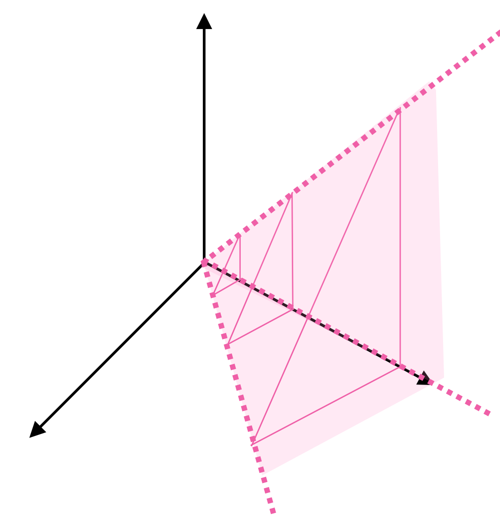
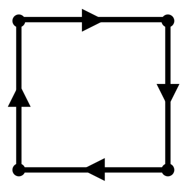
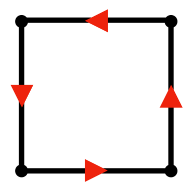


 $= 0$  $= 0$ The graphs have a directed cycle!

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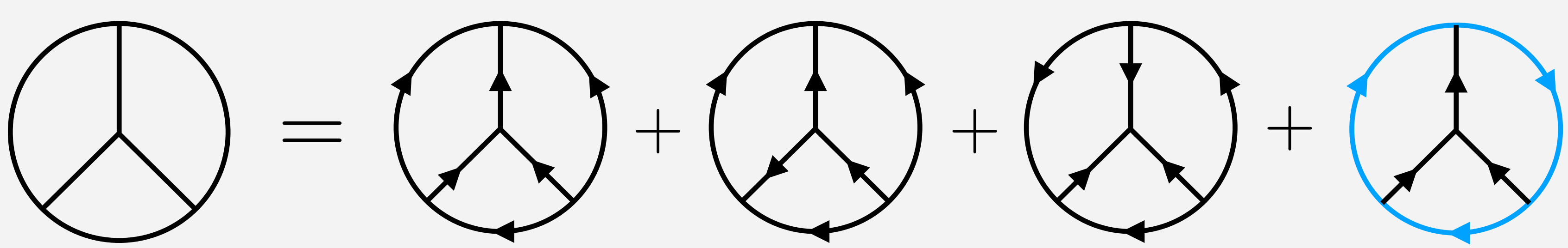

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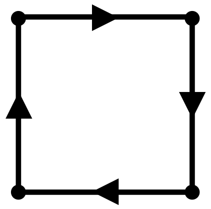

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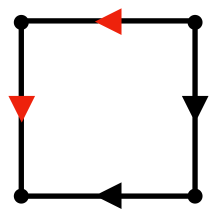

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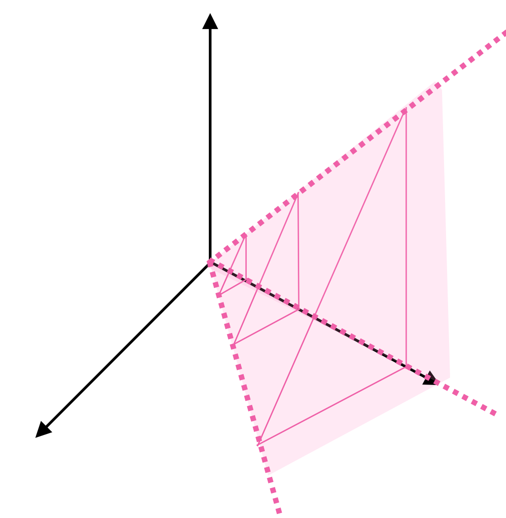
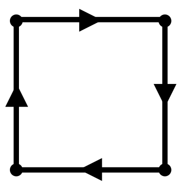
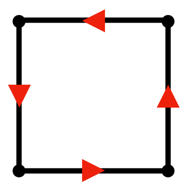


 $= 0$  $= 0$ The graphs have a directed cycle!

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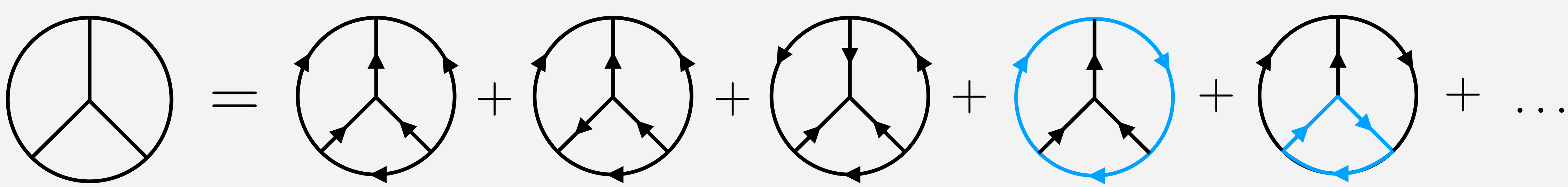

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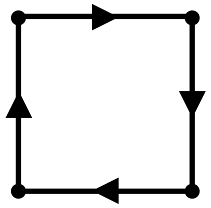
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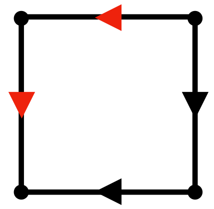
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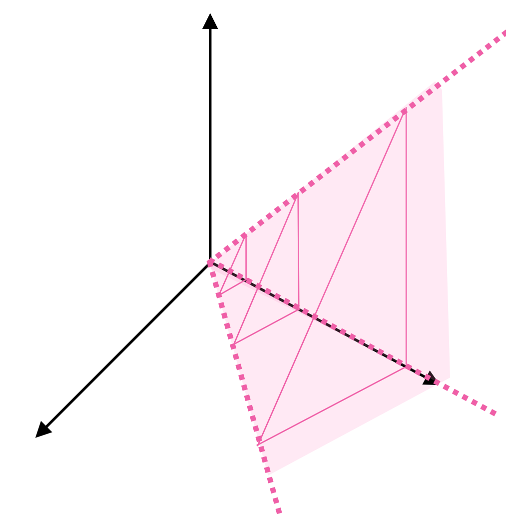
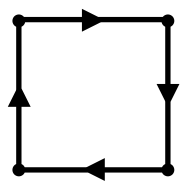
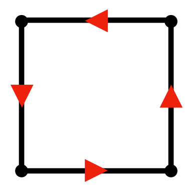


 $= 0$  $= 0$ The graphs have a directed cycle!

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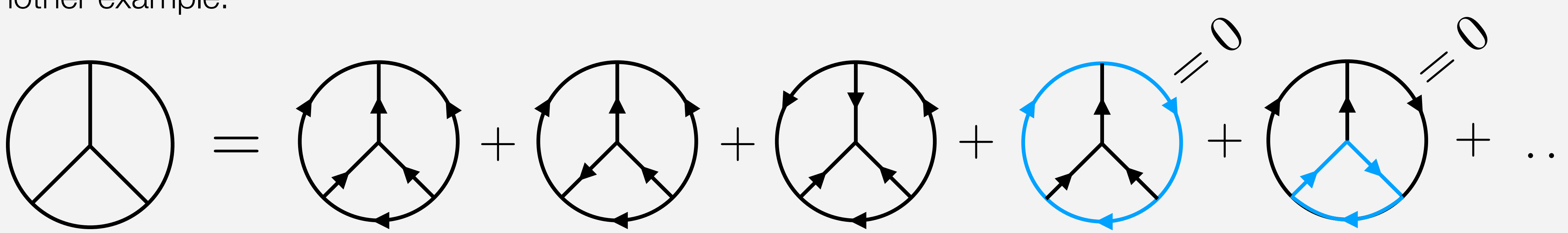

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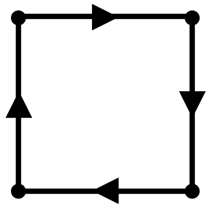

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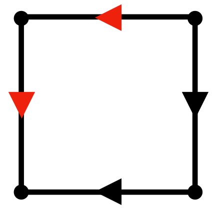

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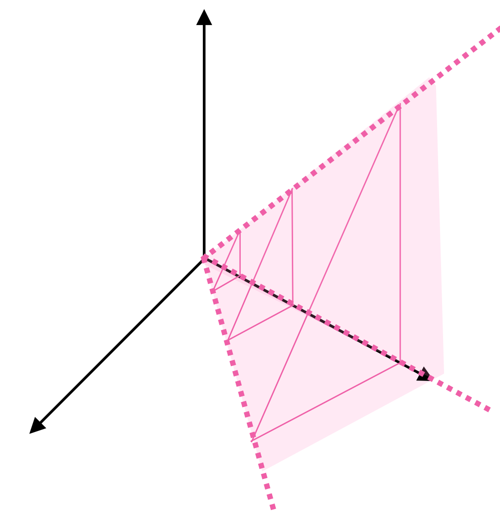
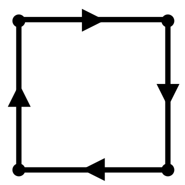
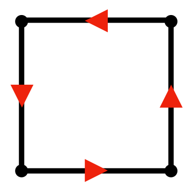


 $= 0$  $= 0$ The graphs have a directed cycle!

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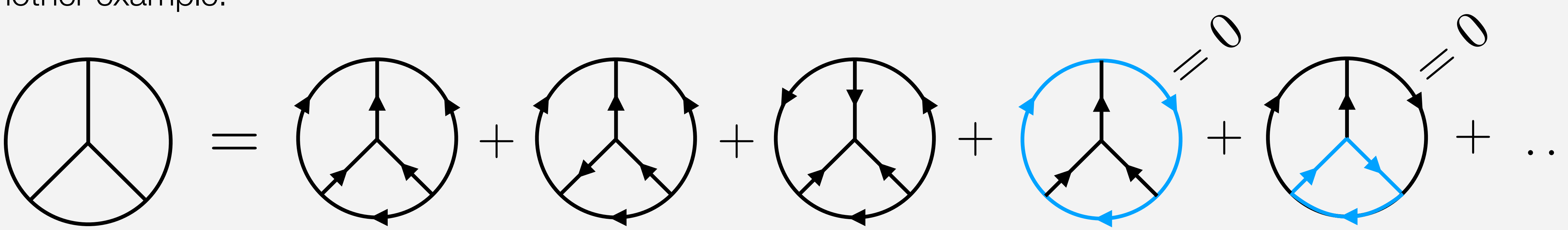

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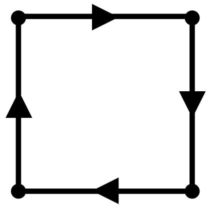

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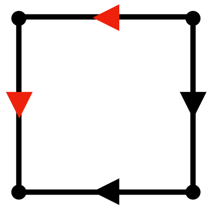

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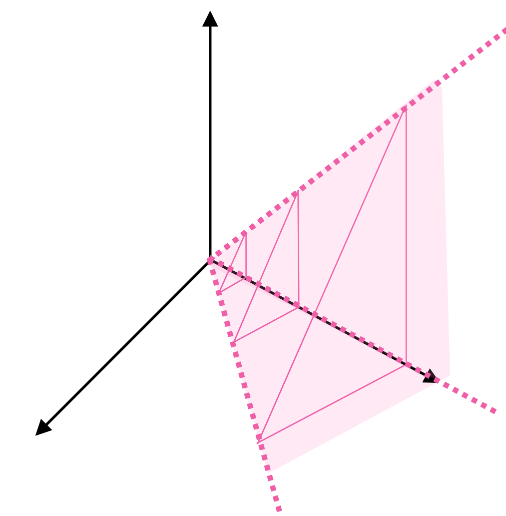
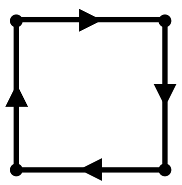
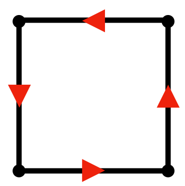


 $= 0$  $= 0$ The graphs have a directed cycle!

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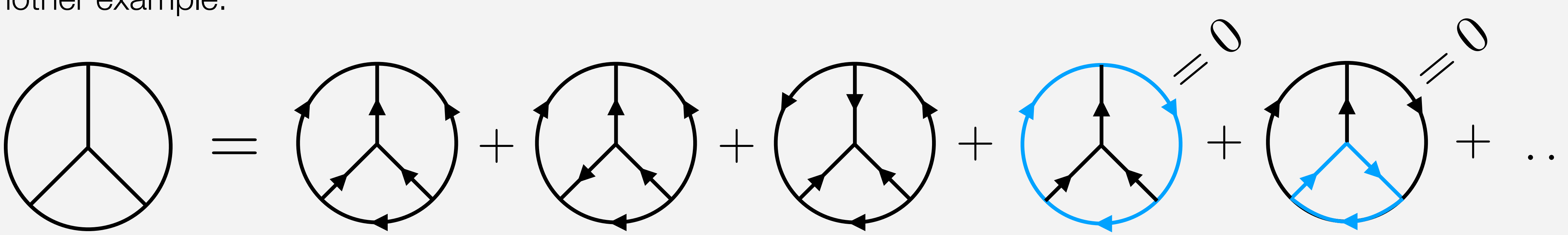

Sum one integral for each acyclic graph

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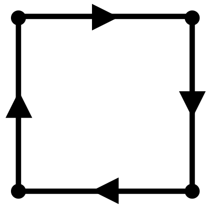

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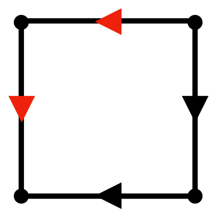

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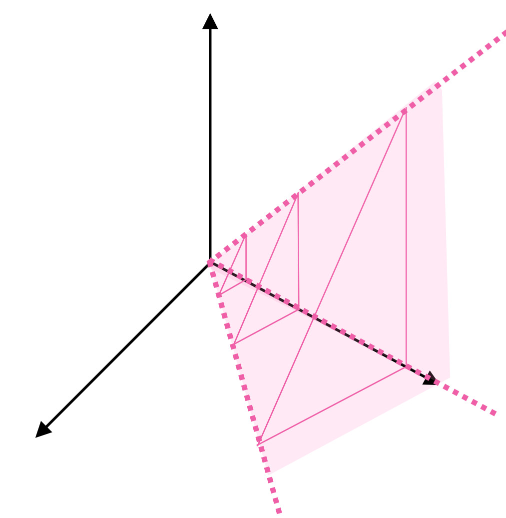
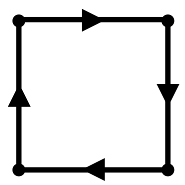
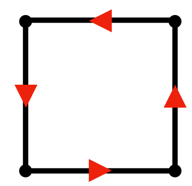


 $= 0$  $= 0$ The graphs have a directed cycle!

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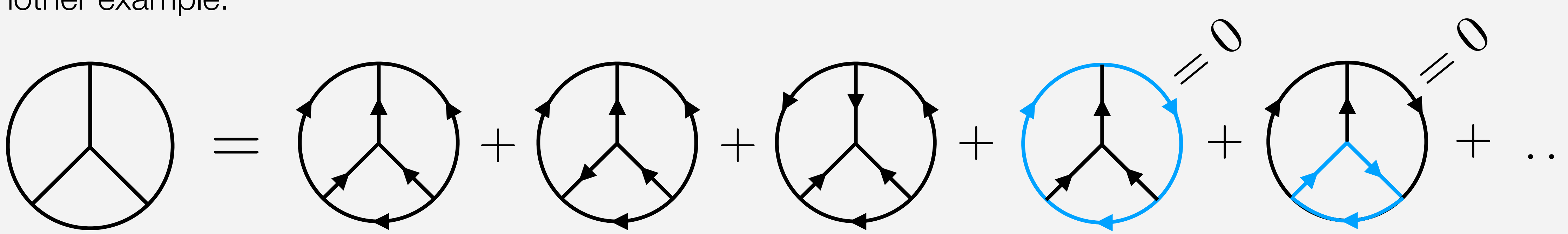
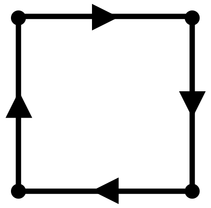


Sum one integral for each acyclic graph

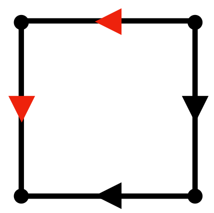
$$= \sum_{\text{acyclic}\, G} N_G \int \left[\prod_{e \in E} \frac{\mathrm{d}\tau_e}{2E_e} e^{i\tau_e (E_e - (p_e^G)^0)} \Theta(\tau_e) \right] \prod_{i=1}^{L} \delta\left(\sum_{e \in c_i} \tau_e s_{ei} \right)$$

Sborlini, arXiv:2102.05062 (2021) Borinsky, ZC, Salas-Bernadez, Laenen, arXiv: 2210.05532 (2022) ZC, arXiv: 2211.09653 (2022)



$$= \int \left[\prod_{j=1}^{4} \frac{\mathrm{d}\tau_j}{2E_j} e^{i\tau_j (E_j^0 - \sigma_j p_j^0)} \Theta(\tau_j) \right] \delta\left(\tau_1 + \tau_2 + \tau_3 + \tau_4\right)$$

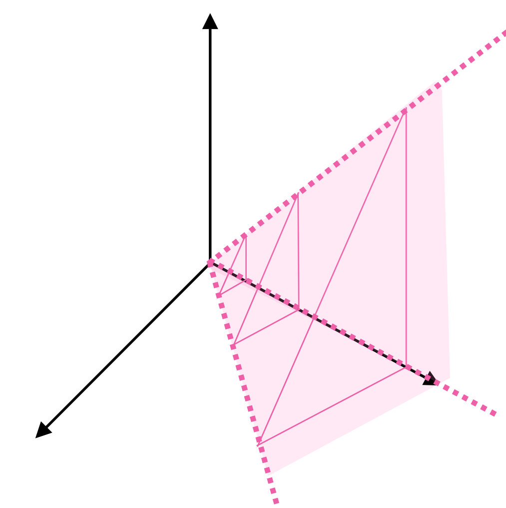
$$\tau_j > 0, \;\; j = 1, ..., 4$$
$$\tau_1 + \tau_2 + \tau_3 + \tau_4 = 0$$

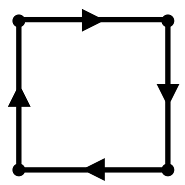
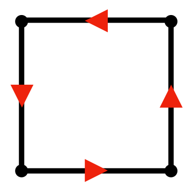


$$= \int \left[\prod_{j=1}^{4} \frac{\mathrm{d}\tau_j}{2E_j} e^{i\tau_j (E_j^0 - \sigma_j p_j^0)} \Theta(\tau_j) \right] \delta\left(-\tau_1 - \tau_2 + \tau_3 + \tau_4\right)$$

$$\tau_j > 0, \;\; j = 1, ..., 4$$
$$-\tau_1 - \tau_2 + \tau_3 + \tau_4 = 0$$

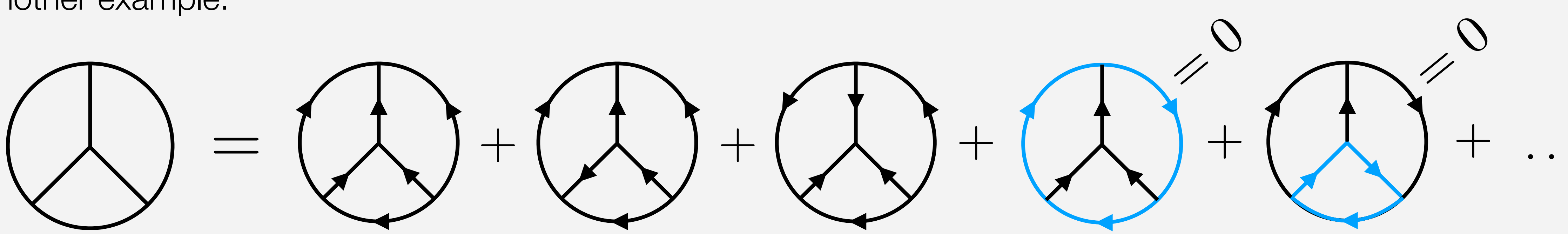
Observation: first integration domain is empty, second is a convex cone



 $= 0$  $= 0$ The graphs have a directed cycle!

Only **acyclic** graphs give non zero contribution!

Another example:

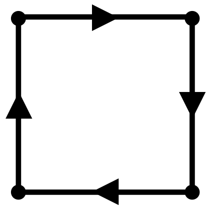


Sum one integral for each acyclic graph

$$= \sum_{\text{acyclic } G} N_G \int \left[\prod_{e \in E} \frac{\mathrm{d}\tau_e}{2E_e} e^{i\tau_e (E_e - (p_e^G)^0)} \Theta(\tau_e) \right] \prod_{i=1}^{L} \delta\left(\sum_{e \in c_i} \tau_e s_{ei} \right)$$

One time integration per edge

Sborlini, arXiv:2102.05062 (2021) Borinsky, ZC, Salas-Bernadez, Laenen, arXiv: 2210.05532 (2022) ZC, arXiv: 2211.09653 (2022)



$$= \int \left[\prod_{j=1}^{4} \frac{\mathrm{d}\tau_j}{2E_j} e^{i\tau_j (E_j^0 - \sigma_j p_j^0)} \Theta(\tau_j) \right] \delta\left(\tau_1 + \tau_2 + \tau_3 + \tau_4\right)$$

$$\tau_j > 0, \; j = 1, \ldots, 4$$
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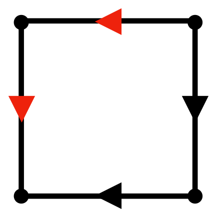

$$= \int \left[\prod_{j=1}^{4} \frac{\mathrm{d}\tau_j}{2E_j} e^{i\tau_j (E_j^0 - \sigma_j p_j^0)} \Theta(\tau_j) \right] \delta\left(-\tau_1 - \tau_2 + \tau_3 + \tau_4\right)$$

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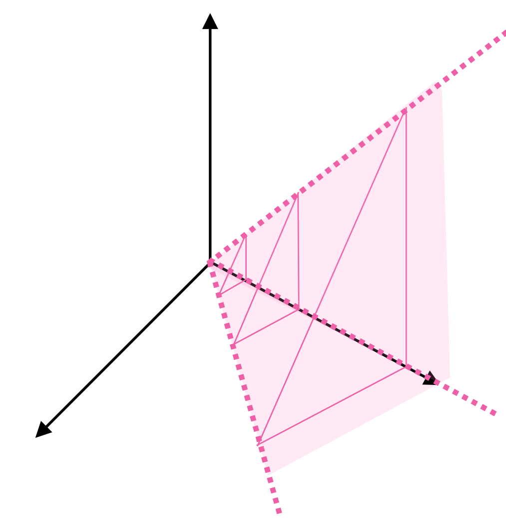
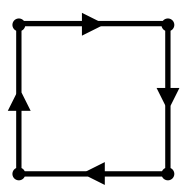
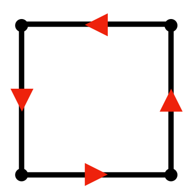
Observation: first integration domain is empty, second is a convex cone



 $= 0$  $= 0$ The graphs have a directed cycle!

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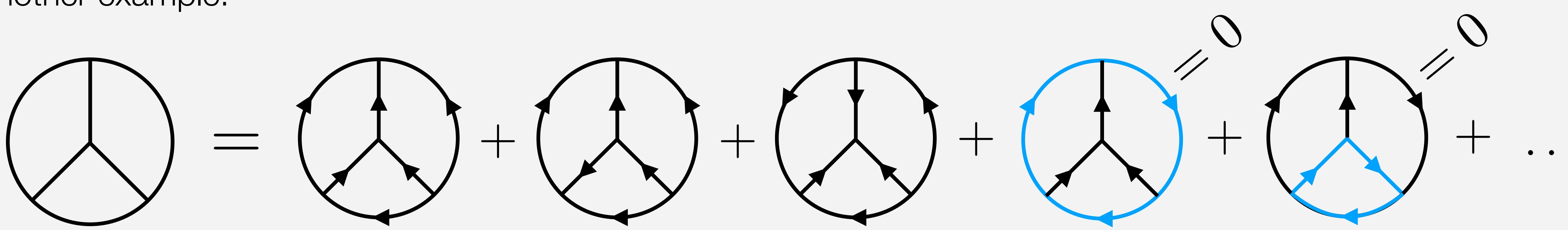


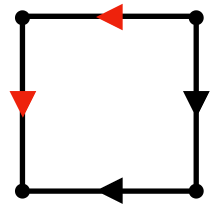
Sum one integral for each acyclic graph

$$= \sum_{\text{acyclic } G} N_G \int \left[\prod_{e \in E} \frac{\mathrm{d}\tau_e}{2E_e} e^{i\tau_e (E_e - (p_e^G)^0)} \Theta(\tau_e) \right] \prod_{i=1}^{L} \delta\left(\sum_{e \in c_i} \tau_e s_{ei} \right)$$

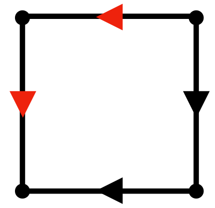
Times along cycles must sum to zero

One time integration per edge

Sborlini, arXiv:2102.05062 (2021) Borinsky, ZC, Salas-Bernadez, Laenen, arXiv: 2210.05532 (2022) ZC, arXiv: 2211.09653 (2022)

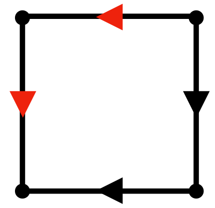


$$= \int \left[\prod_{j=1}^{4} \frac{\mathrm{d}\tau_j}{2E_j} e^{i\tau_j (E_j^0 - \sigma_j p_j^0)} \Theta(\tau_j) \right] \delta\left(-\tau_1 - \tau_2 + \tau_3 + \tau_4\right)$$



$$= \int \left[\prod_{j=1}^{4} \frac{\mathrm{d}\tau_j}{2E_j} e^{i\tau_j (E_j^0 - \sigma_j p_j^0)} \Theta(\tau_j) \right] \delta\left(-\tau_1 - \tau_2 + \tau_3 + \tau_4\right)$$

How do we perform the remaining integrations (**one for each edge**)? Edge-contraction



$$= \int \left[\prod_{j=1}^{4} \frac{\mathrm{d}\tau_j}{2E_j} e^{i\tau_j (E_j^0 - \sigma_j p_j^0)} \Theta(\tau_j) \right] \delta\left(-\tau_1 - \tau_2 + \tau_3 + \tau_4\right)$$

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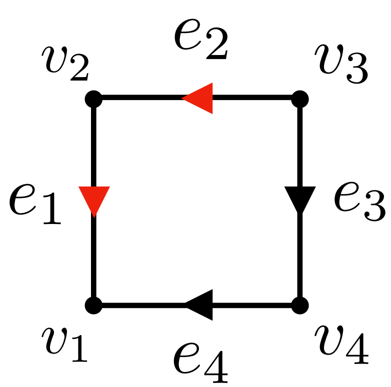

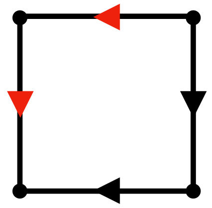

$$= \int \left[\prod_{j=1}^{4} \frac{\mathrm{d}\tau_j}{2E_j} e^{i\tau_j (E_j^0 - \sigma_j p_j^0)} \Theta(\tau_j) \right] \delta\left(-\tau_1 - \tau_2 + \tau_3 + \tau_4\right)$$

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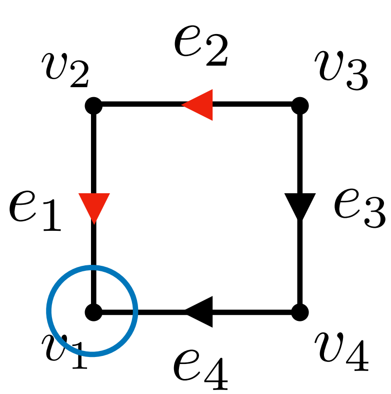
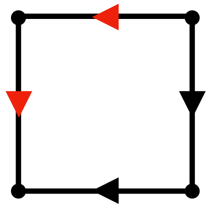


1. Choose sink/source with connected complement



$$= \int \left[\prod_{j=1}^{4} \frac{\mathrm{d}\tau_j}{2E_j} e^{i\tau_j(E_j^0-\sigma_j p_j^0)}\Theta(\tau_j)\right]\delta\left(-\tau_1-\tau_2+\tau_3+\tau_4\right)$$

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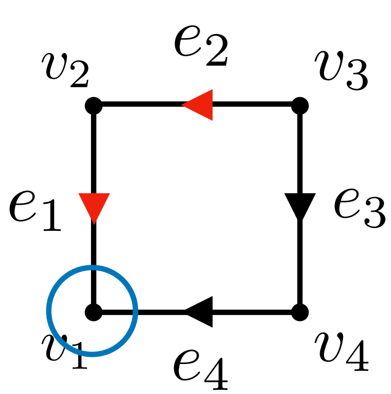
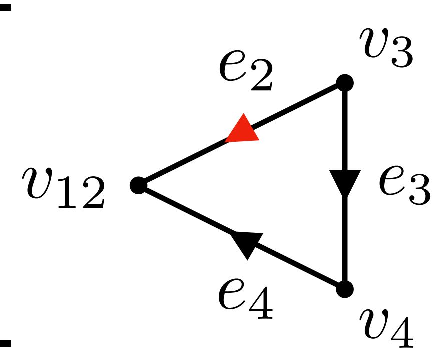


$$= \frac{i}{E_1+E_4-p_1^0}\Bigg[$$



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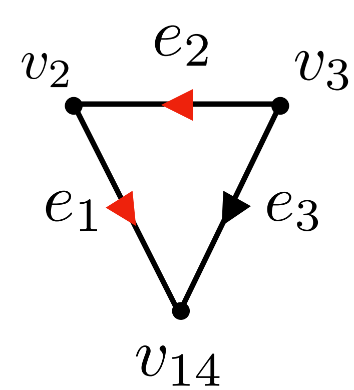
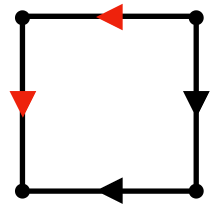


$$\Bigg]$$

1. Choose sink/source with connected complement
2. Contract one-by-one adjacent edges
3. Multiply by inverse sum of energies of adjacent edges



$$= \int \left[\prod_{j=1}^{4} \frac{\mathrm{d}\tau_j}{2E_j} e^{i\tau_j (E_j^0 - \sigma_j p_j^0)} \Theta(\tau_j) \right] \delta\left(-\tau_1 - \tau_2 + \tau_3 + \tau_4\right)$$

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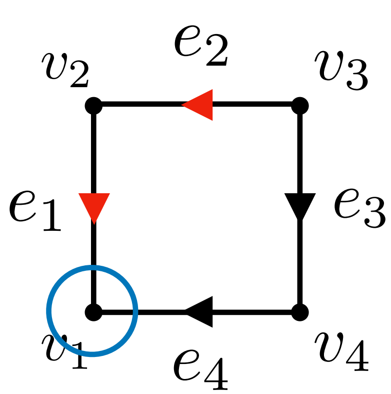
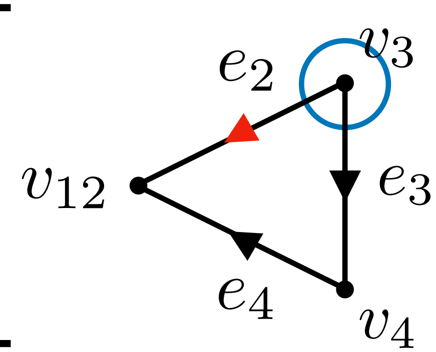


$$= \frac{i}{E_1 + E_4 - p_1^0} \Bigg[$$



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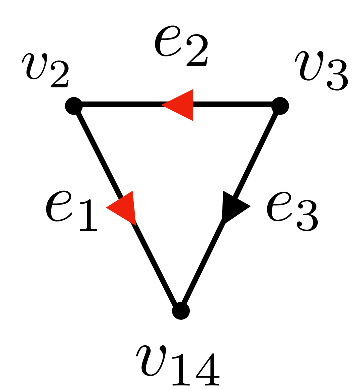

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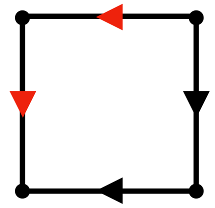

$$= \int \left[\prod_{j=1}^{4} \frac{\mathrm{d}\tau_j}{2E_j} e^{i\tau_j (E_j^0 - \sigma_j p_j^0)} \Theta(\tau_j) \right] \delta\left(-\tau_1 - \tau_2 + \tau_3 + \tau_4\right)$$

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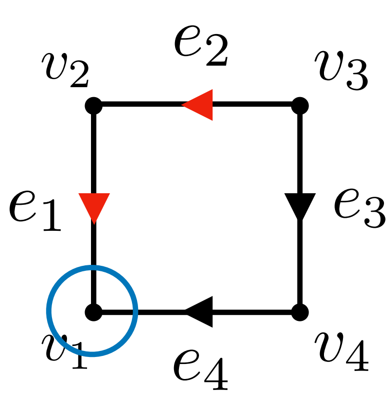

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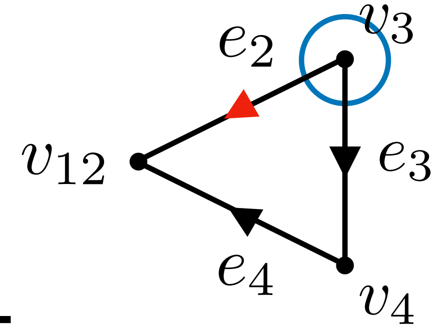

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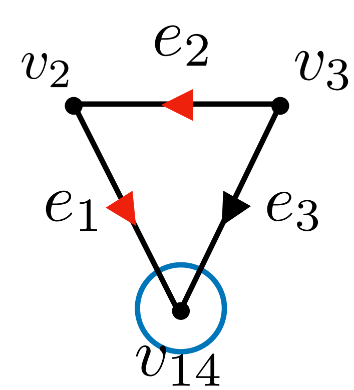

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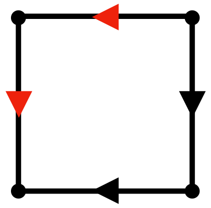

$$= \int \left[\prod_{j=1}^{4} \frac{\mathrm{d}\tau_j}{2E_j} e^{i\tau_j (E_j^0 - \sigma_j p_j^0)} \Theta(\tau_j) \right] \delta\left(-\tau_1 - \tau_2 + \tau_3 + \tau_4\right)$$

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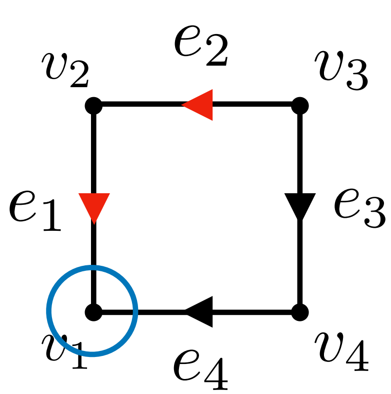
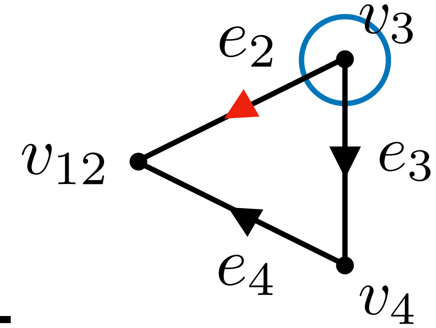


$$= \frac{i}{E_1 + E_4 - p_1^0} \Bigg[$$



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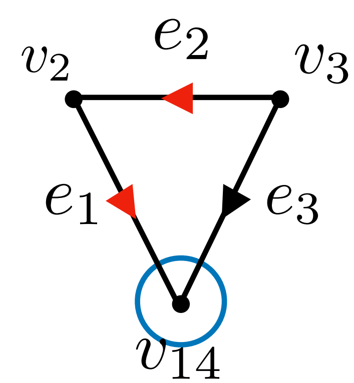

$\Bigg]$

1. Choose sink/source with connected complement
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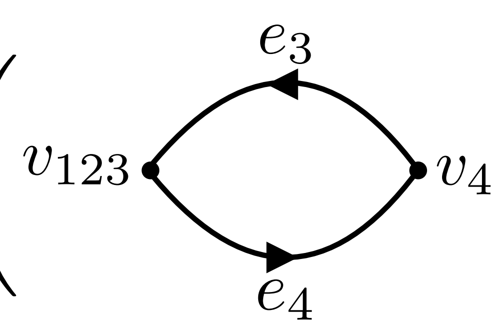
$$= \frac{i}{E_1 + E_4 - p_1^0} \Bigg[\frac{i}{E_2 + E_3 + p_3^0} \Bigg($$



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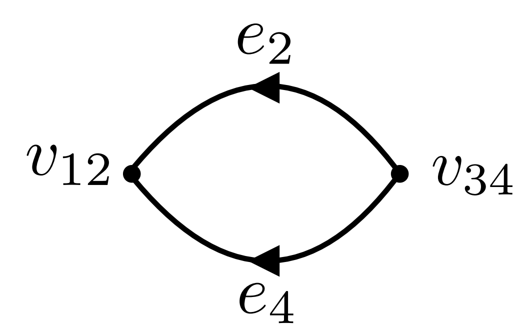
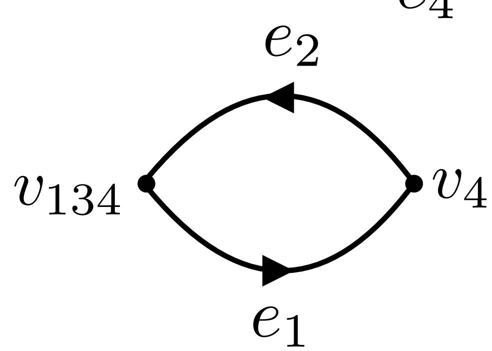


$\Bigg)$

$$+ \frac{i}{E_1 + E_3 - p_1^0 - p_4^0} \Bigg($$



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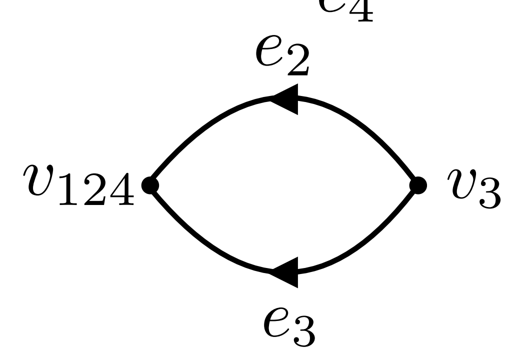
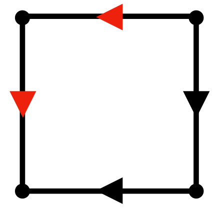
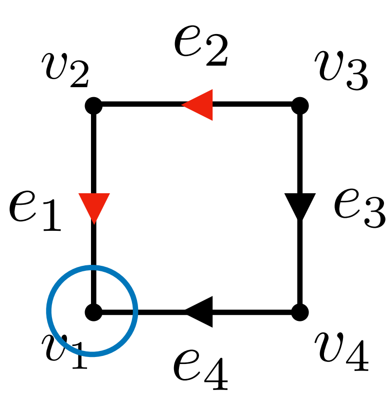


$\Bigg) \Bigg]$

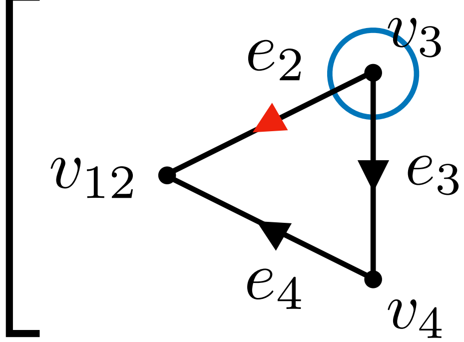


$$= \int \left[\prod_{j=1}^{4} \frac{\mathrm{d}\tau_j}{2E_j} e^{i\tau_j(E_j^0-\sigma_j p_j^0)}\Theta(\tau_j)\right]\delta\left(-\tau_1-\tau_2+\tau_3+\tau_4\right)$$

How do we perform the remaining integrations (**one for each edge**)? Edge-contraction



$$= \frac{i}{E_1+E_4-p_1^0}\Bigg[$$



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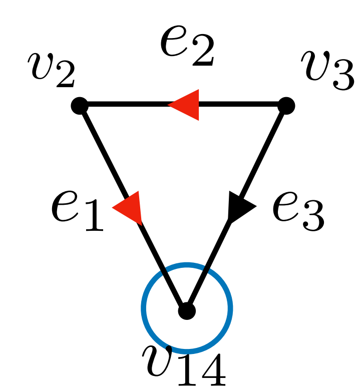

$$\Bigg]$$

1. Choose sink/source with connected complement
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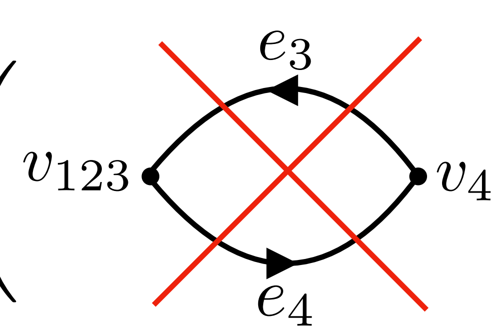
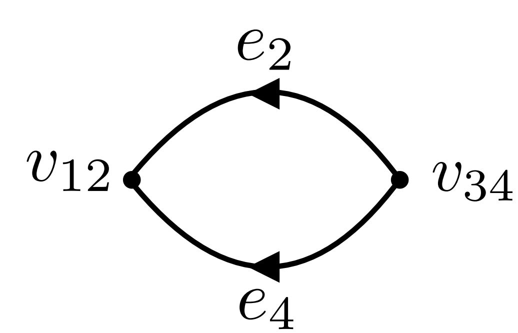
$$= \frac{i}{E_1+E_4-p_1^0}\Bigg[\frac{i}{E_2+E_3+p_3^0}\Bigg($$



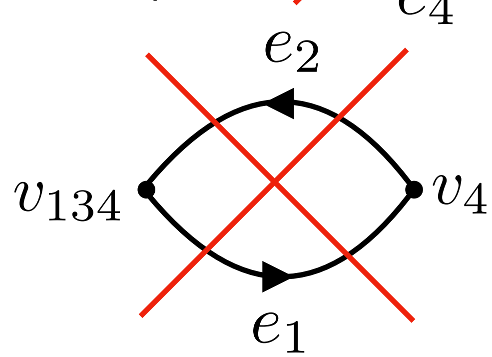
$$+$$



$$\Bigg)$$

4. Throw out non-acyclic graphs

$$+\frac{i}{E_1+E_3-p_1^0-p_4^0}\Bigg($$



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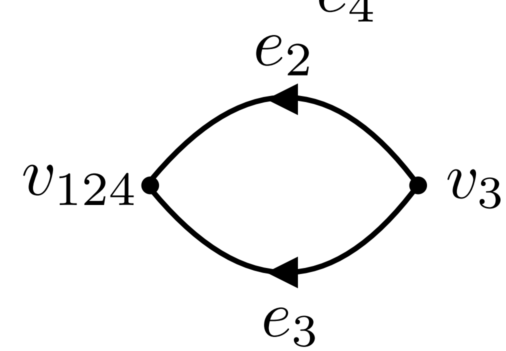
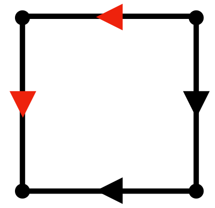
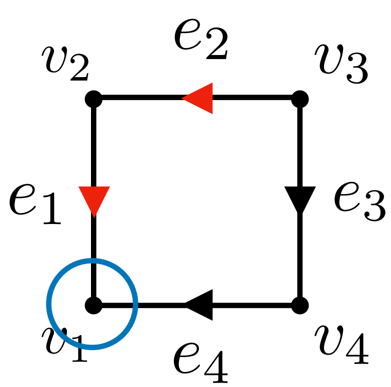


$$\Bigg)\Bigg]$$

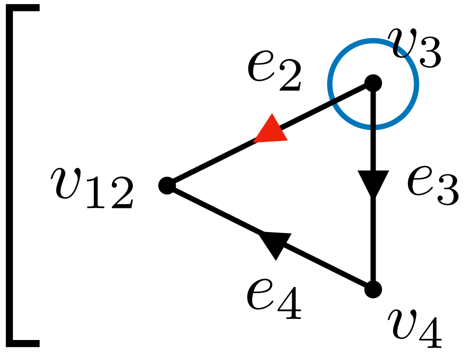


$$= \int \left[\prod_{j=1}^{4} \frac{\mathrm{d}\tau_j}{2E_j} e^{i\tau_j (E_j^0 - \sigma_j p_j^0)} \Theta(\tau_j) \right] \delta\left(-\tau_1 - \tau_2 + \tau_3 + \tau_4\right)$$

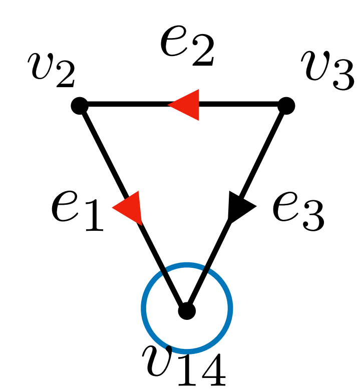
How do we perform the remaining integrations (**one for each edge**)? Edge-contraction



$$= \frac{i}{E_1 + E_4 - p_1^0} \Bigg[$$



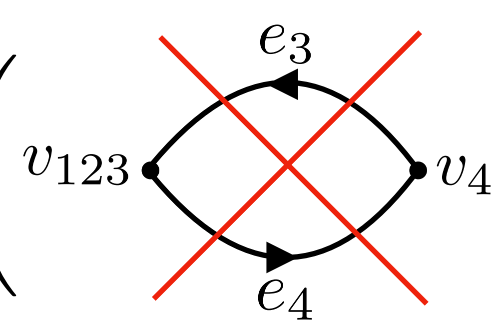
$$+$$



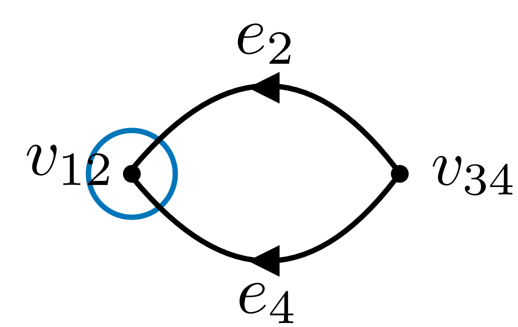
$$\Bigg]$$

1. Choose sink/source with connected complement
2. Contract one-by-one adjacent edges
3. Multiply by inverse sum of energies of adjacent edges

$$= \frac{i}{E_1 + E_4 - p_1^0} \Bigg[\frac{i}{E_2 + E_3 + p_3^0} \Bigg($$



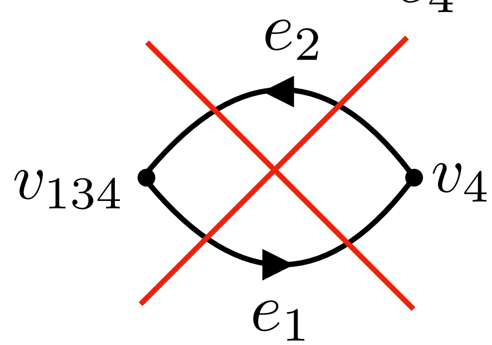
$$+$$



$$\Bigg)$$

4. Throw out non-acyclic graphs

$$+ \frac{i}{E_1 + E_3 - p_1^0 - p_4^0} \Bigg($$



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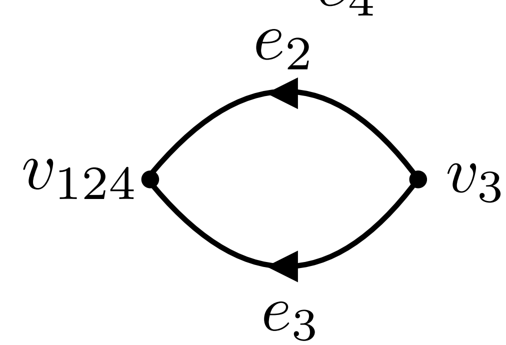

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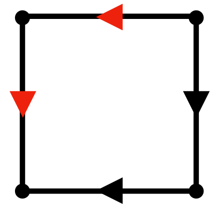
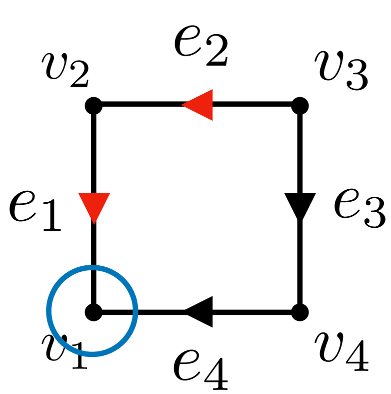


$$= \int \left[\prod_{j=1}^{4} \frac{\mathrm{d}\tau_j}{2E_j} e^{i\tau_j (E_j^0 - \sigma_j p_j^0)} \Theta(\tau_j) \right] \delta\left(-\tau_1 - \tau_2 + \tau_3 + \tau_4\right)$$

How do we perform the remaining integrations (**one for each edge**)? Edge-contraction



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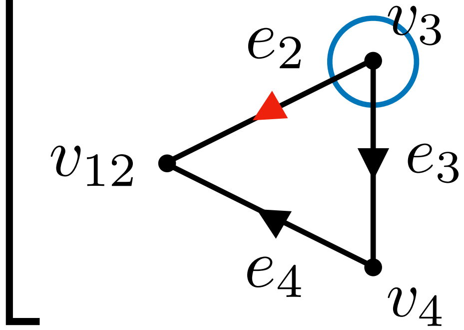
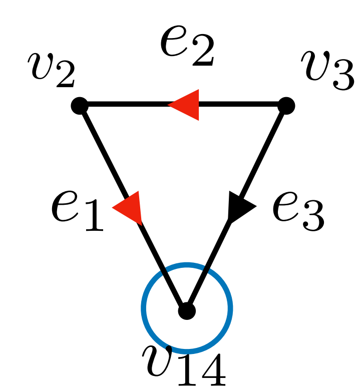


$$+$$



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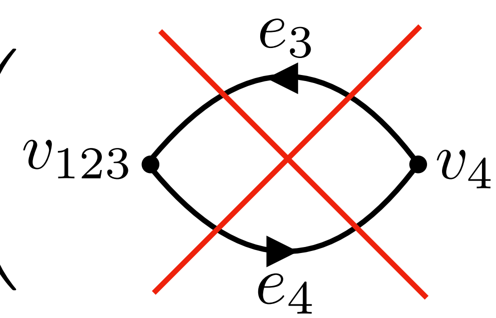

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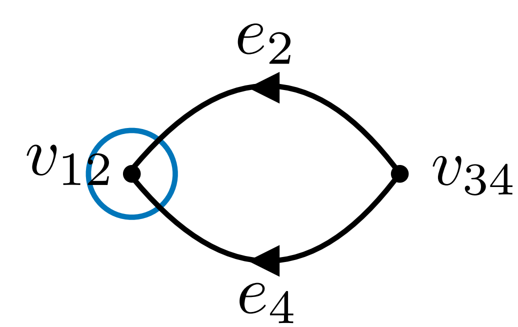

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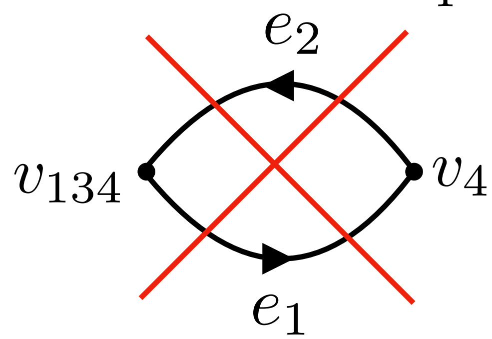

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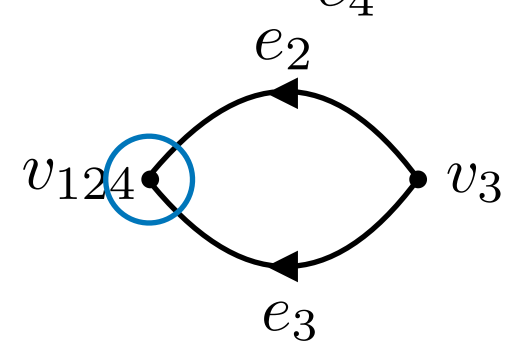
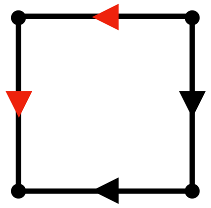


$$\Bigg) \Bigg]$$



$$= \int \left[\prod_{j=1}^{4} \frac{\mathrm{d}\tau_j}{2E_j} e^{i\tau_j (E_j^0 - \sigma_j p_j^0)} \Theta(\tau_j) \right] \delta\left(-\tau_1 - \tau_2 + \tau_3 + \tau_4\right)$$

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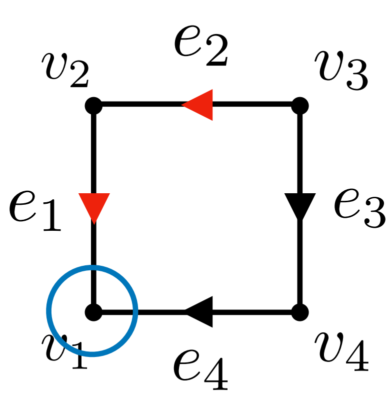

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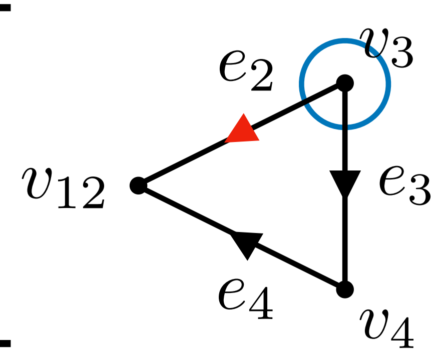

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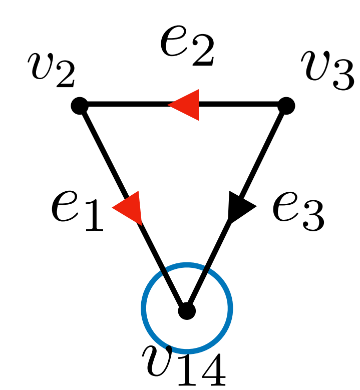

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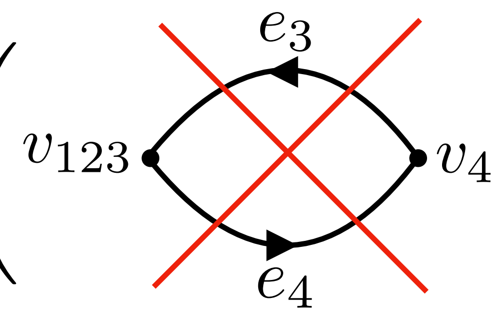
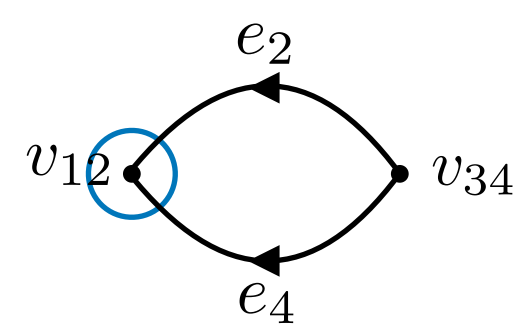


$+$



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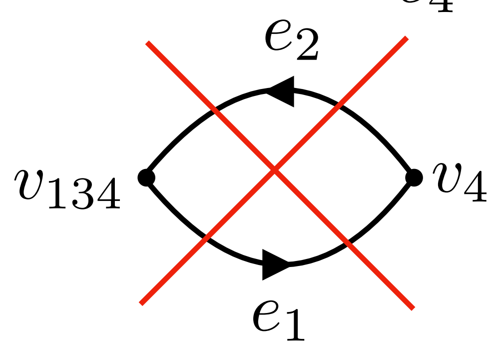

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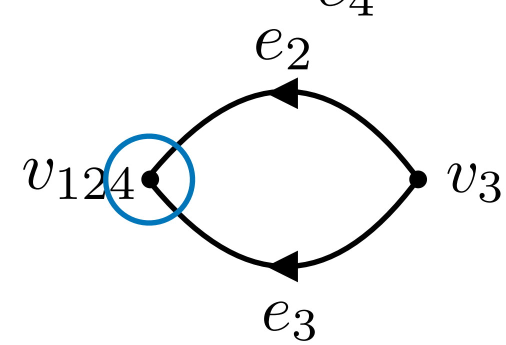

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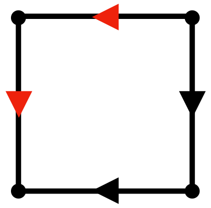

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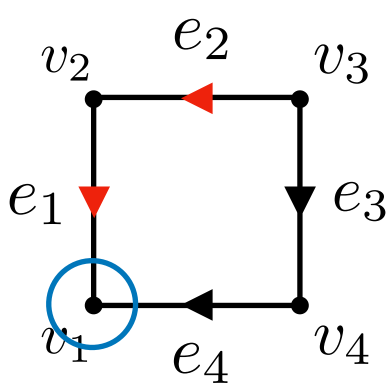

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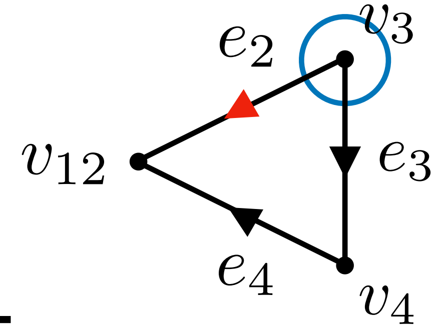
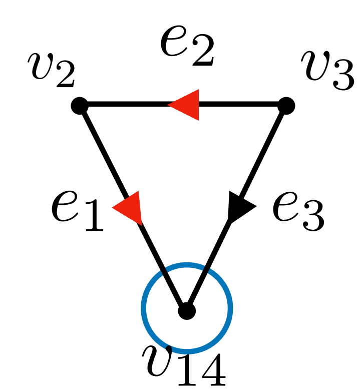


$$+$$



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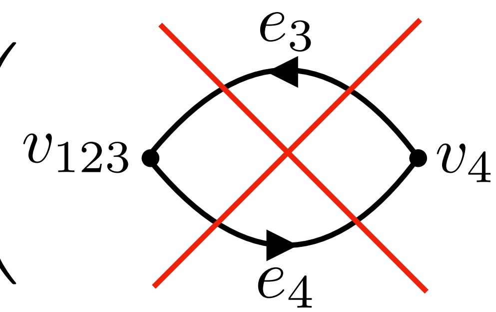
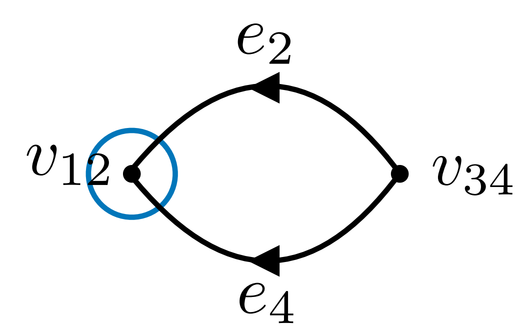


$$+$$



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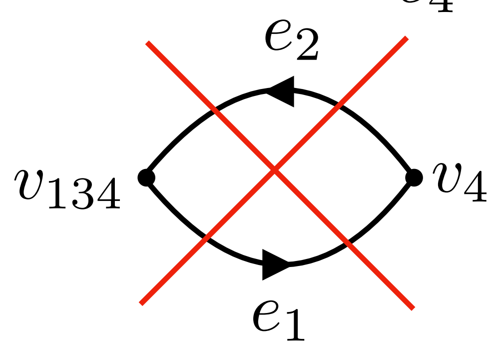
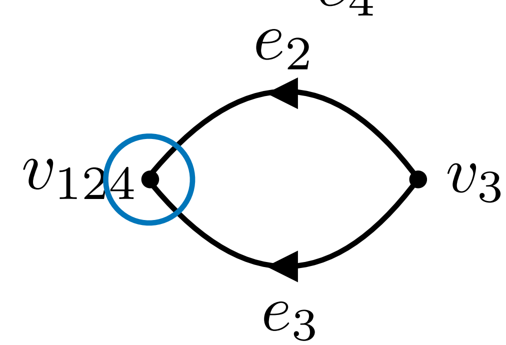


$$+$$

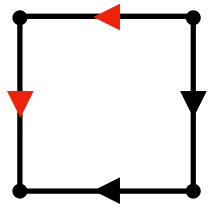


$$\Bigg) \Bigg]$$

4. Throw out non-acyclic graphs

$$= \frac{i}{E_1 + E_4 - p_1^0} \left[\frac{i}{E_2 + E_3 + p_3^0} \frac{i}{E_2 + E_4 - p_1^0 - p_2^0} + \frac{i}{E_1 + E_3 - p_1^0 - p_4^0} \frac{i}{E_2 + E_3 + p_3^0} \right]$$

5. Contract parallel edges



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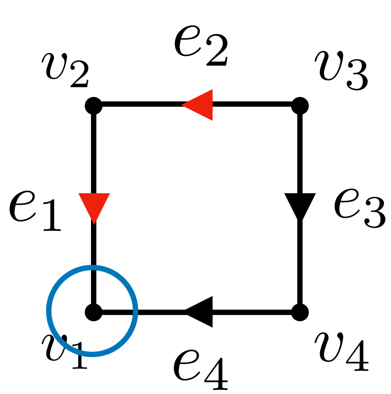

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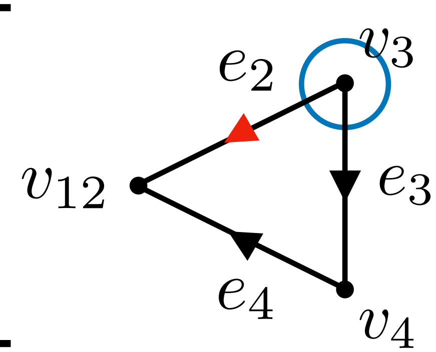
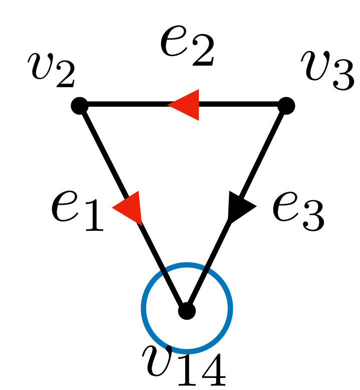


$$+$$



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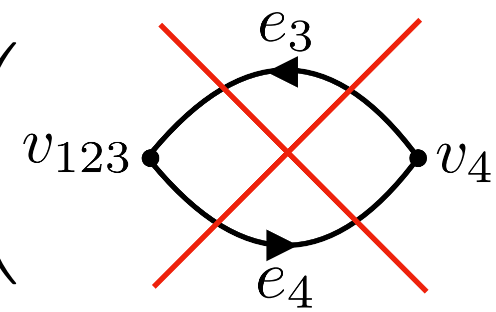
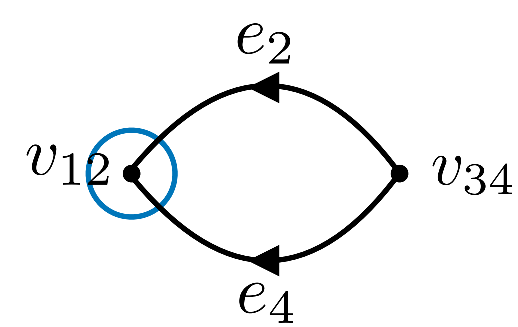


$$+$$



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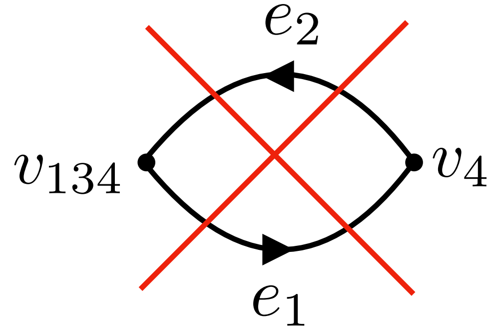

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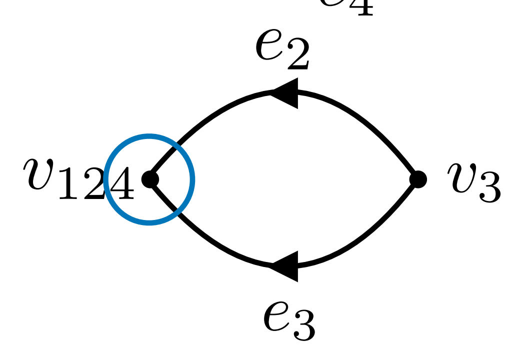

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All time integrations are performed diagrammatically!

Now collect, for each term, the choice of sink and source that led to it

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Term: $$\frac{i}{E_1+E_4-p_1^0}\,\frac{i}{E_2+E_3+p_3^0}\,\frac{i}{E_2+E_4-p_1^0-p_2^0}$$

Now collect, for each term, the choice of sink and source that led to it

Term: $\frac{i}{E_1 + E_4 - p_1^0} \frac{i}{E_2 + E_3 + p_3^0} \frac{i}{E_2 + E_4 - p_1^0 - p_2^0}$

Vertex sequence: $v_1 \longrightarrow v_3 \longrightarrow v_{12}$

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Family of cuts: $F_1 = \{\{v_1\}, \{v_3\}, \{v_1, v_2\}\}$

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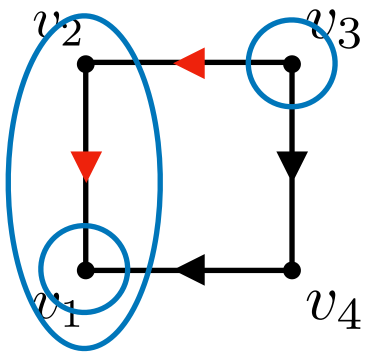

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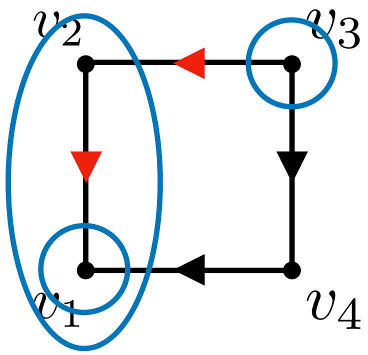

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e.g. $\partial(\{v_1,v_2\}) = \{e_2,e_4\}$

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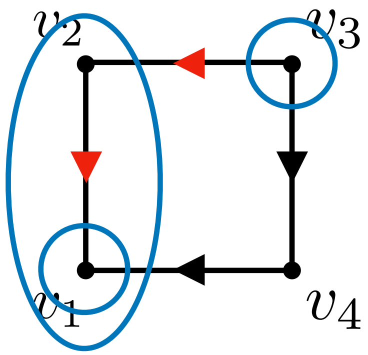

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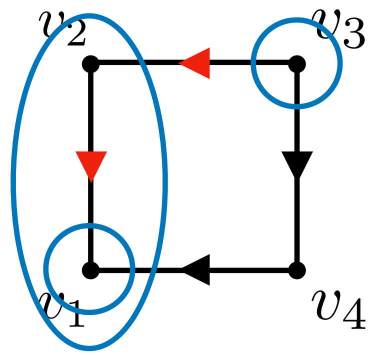

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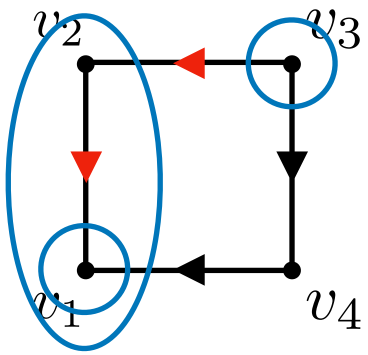

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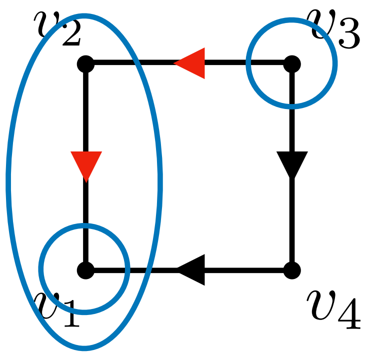

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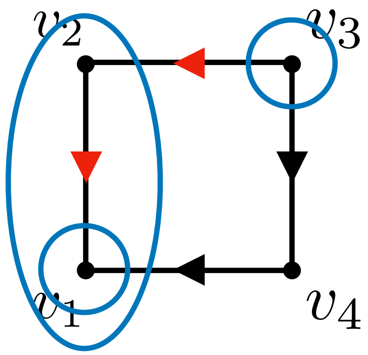

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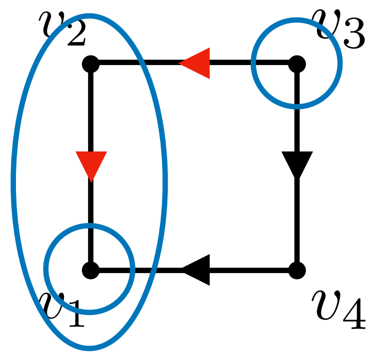
Term: $\frac{i}{E_1+E_4-p_1^0}\,\frac{i}{E_1+E_3-p_1^0-p_4^0}\,\frac{i}{E_2+E_3+p_3^0}$

Vertex sequence: $v_1 \longrightarrow v_{14} \longrightarrow v_{124}$

Now collect, for each term, the choice of sink and source that led to it



Term: $\dfrac{i}{E_1 + E_4 - p_1^0} \dfrac{i}{E_2 + E_3 + p_3^0} \dfrac{i}{E_2 + E_4 - p_1^0 - p_2^0}$

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Vertex sequence: $v_1 \longrightarrow v_{14} \longrightarrow v_{124}$

Family of cuts: $F_2 = \{\{v_1\}, \{v_1, v_4\}, \{v_1, v_2, v_4\}\}$

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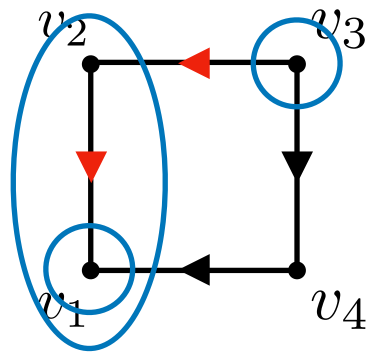

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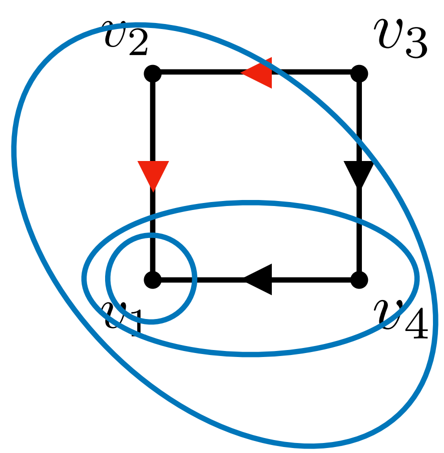

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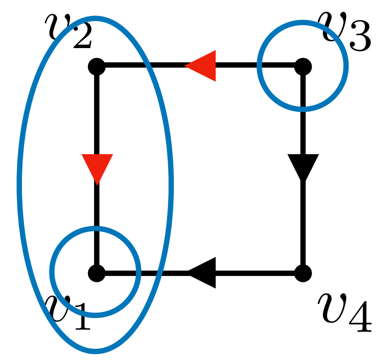

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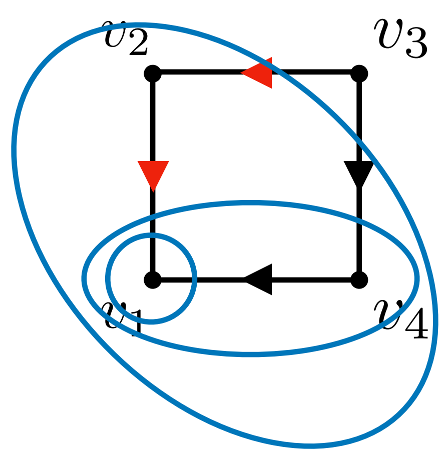

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$$S \in F \quad \Rightarrow \quad S,\ V \setminus S \text{ are connected}$$

$$S_1, S_2 \in F \quad \Rightarrow \quad S_1 \subset S_2 \text{ or } S_2 \subset S_1 \text{ or } S_1 \cap S_2 = \emptyset$$

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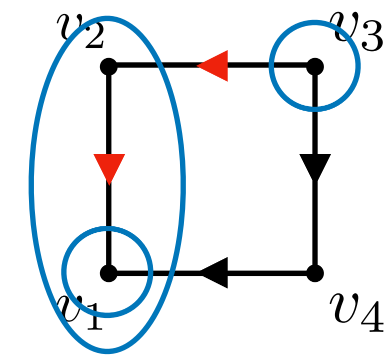

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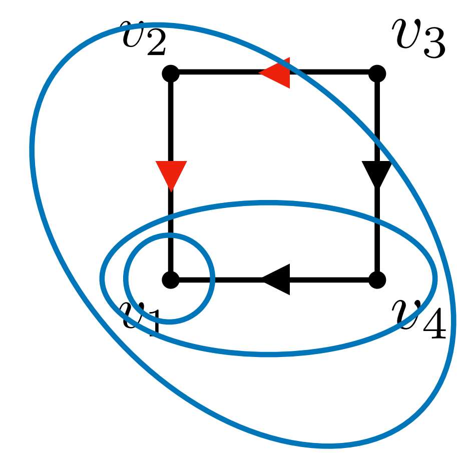

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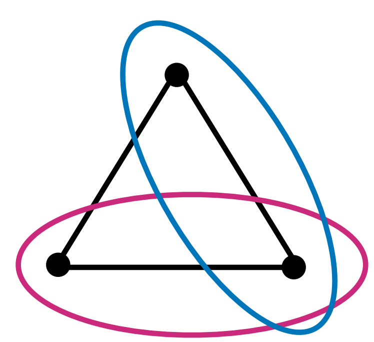
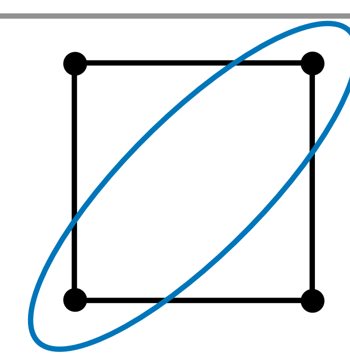
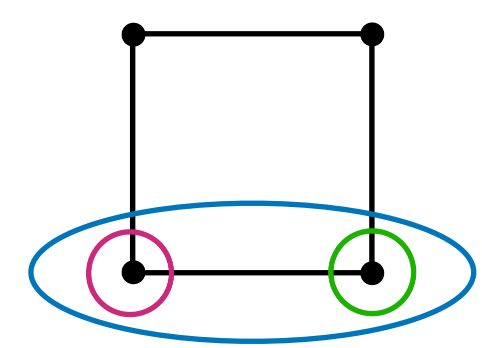
Forbidden configurations:



crossing



disconnected



obstruction

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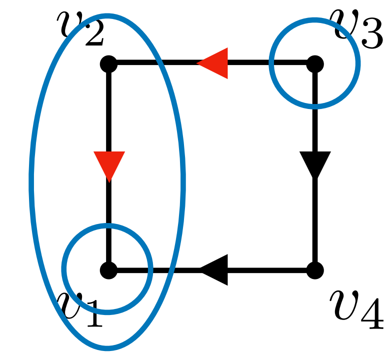

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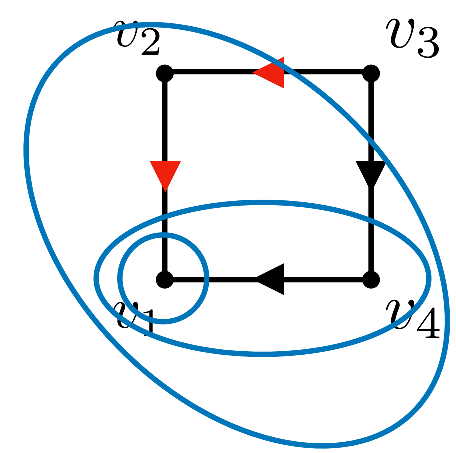

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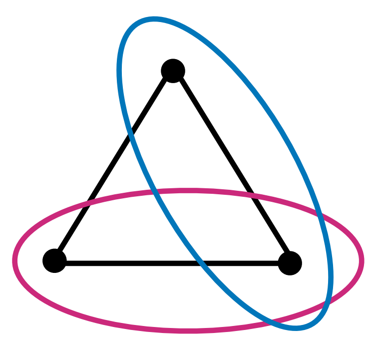

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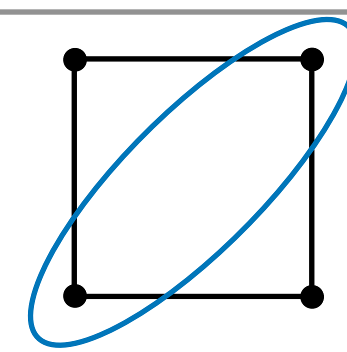
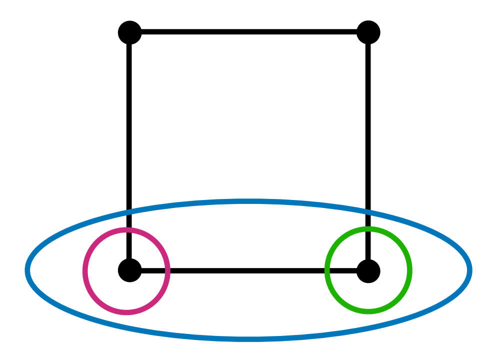


disconnected



obstruction

These properties are completely general!

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Cross-Free Family representation of loop integrals

$$\int \left[\prod_{i=1}^{L} \mathrm{d}^4 k_i\right] \frac{N}{\prod_e (q_e^2 - m_e^2)} = \int \left[\prod_{i=1}^{L} \mathrm{d}^3 \vec{k}_i\right] \sum_{\substack{\text{acyclic} \\ \text{graph } G}} \frac{N_G}{\prod_e 2E_e} \sum_{F \in \mathcal{F}_G} \frac{1}{\prod_{S \in F} [\sum_{e \in \delta(S)} E_e]}$$

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Difference with TOPT: **TOPT violates the connectedness property**

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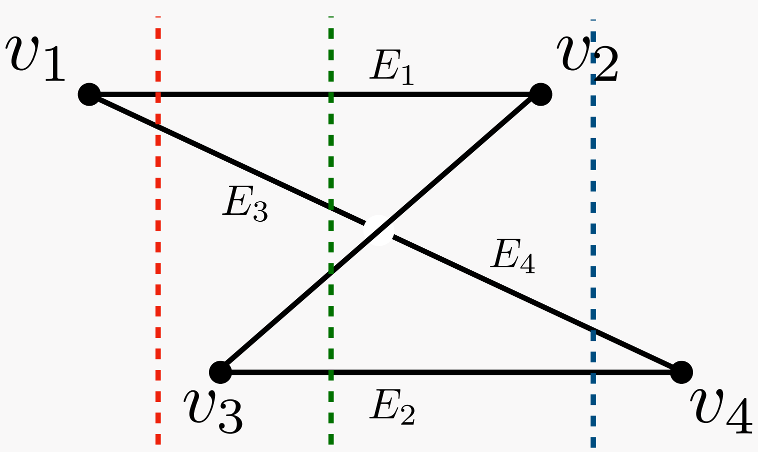
Focus on the TOPT term



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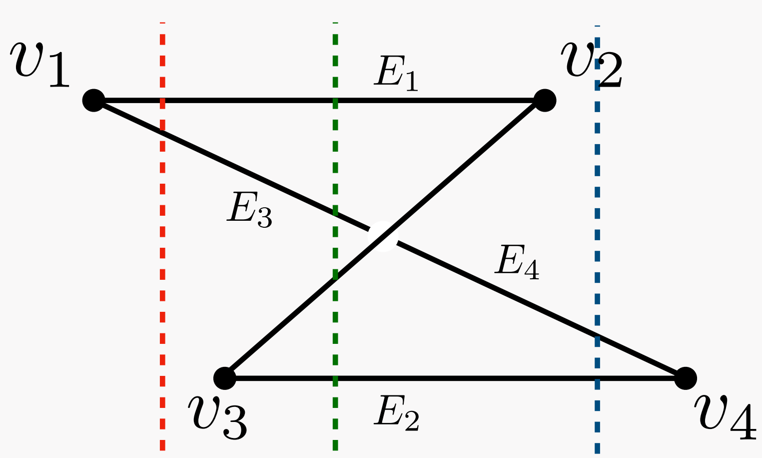
Difference with TOPT: **TOPT violates the connectedness property**

Focus on the TOPT term



Consider the threshold

$$E_1 + E_2 + E_3 + E_4 - p_1^0 - p_3^0 = 0$$

Cross-Free Family representation of loop integrals

Check out cLTD.m!
github.com/apelloni/cLTD

Sum over cross-free families

$$\int \left[\prod_{i=1}^{L} \mathrm{d}^4 k_i\right] \frac{N}{\prod_e (q_e^2 - m_e^2)} = \int \left[\prod_{i=1}^{L} \mathrm{d}^3 \vec{k}_i\right] \sum_{\substack{\text{acyclic}\\ \text{graph } G}} \frac{N_G}{\prod_e 2E_e} \sum_{F \in \mathcal{F}_G} \frac{1}{\prod_{S \in F}[\sum_{e \in \delta(S)} E_e]}$$

Product over cuts in family

Sum over acyclic graphs

$$F = \{S_1, \ldots, S_n\}$$

- S_i and $V \setminus S_i$ are connected
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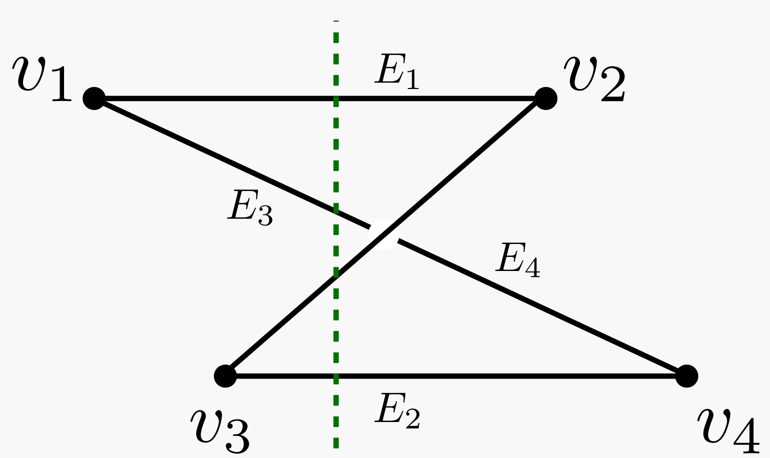

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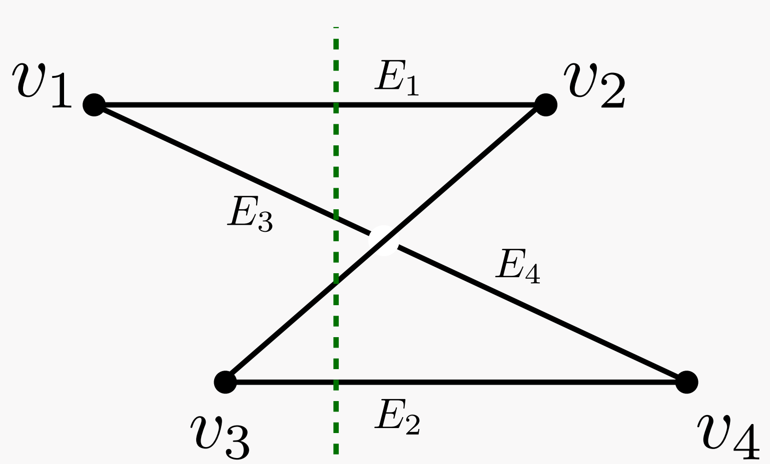

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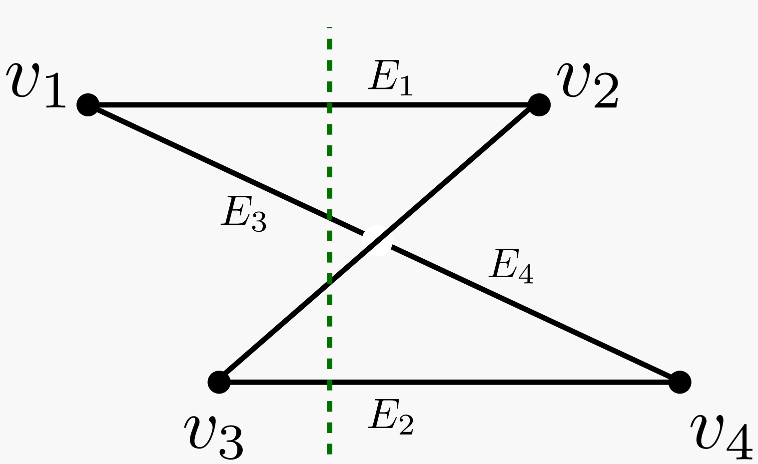

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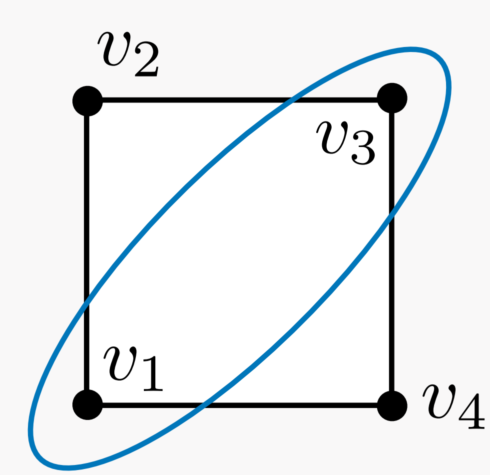

Iterated connectedness

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But how does the integrand behave at the location of a threshold?

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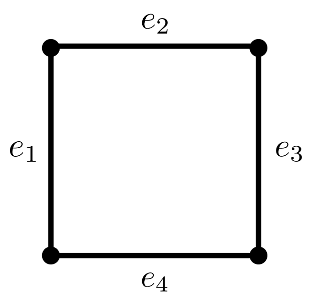

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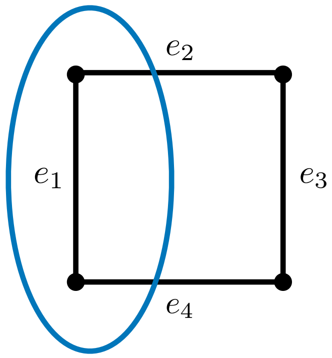

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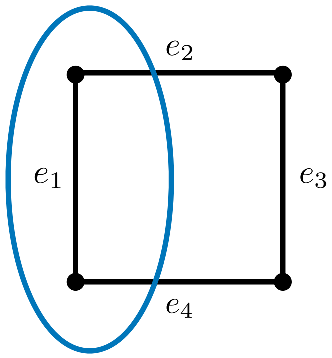

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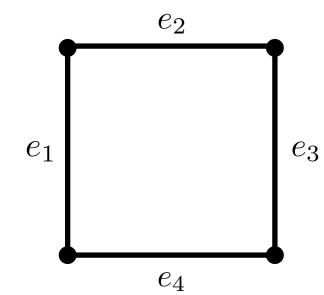

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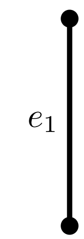

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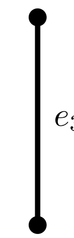

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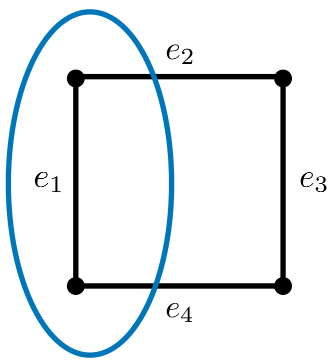

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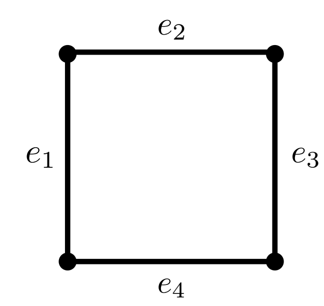

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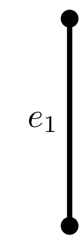

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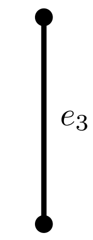

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In general:

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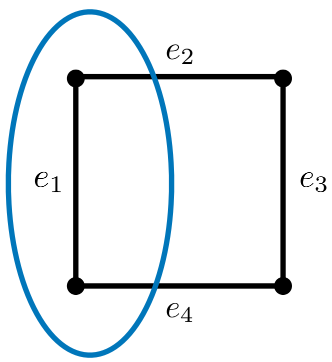

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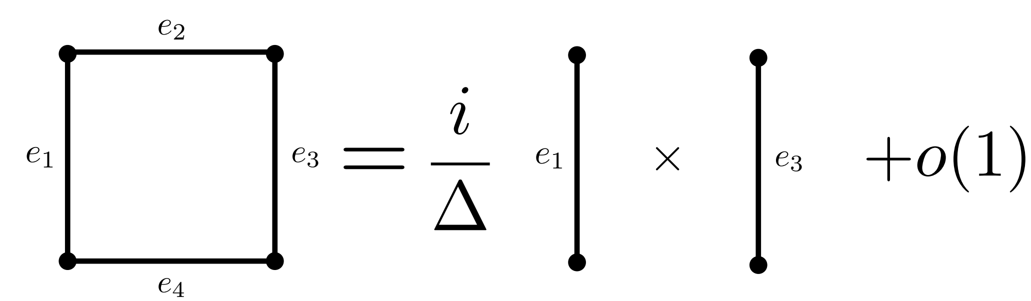

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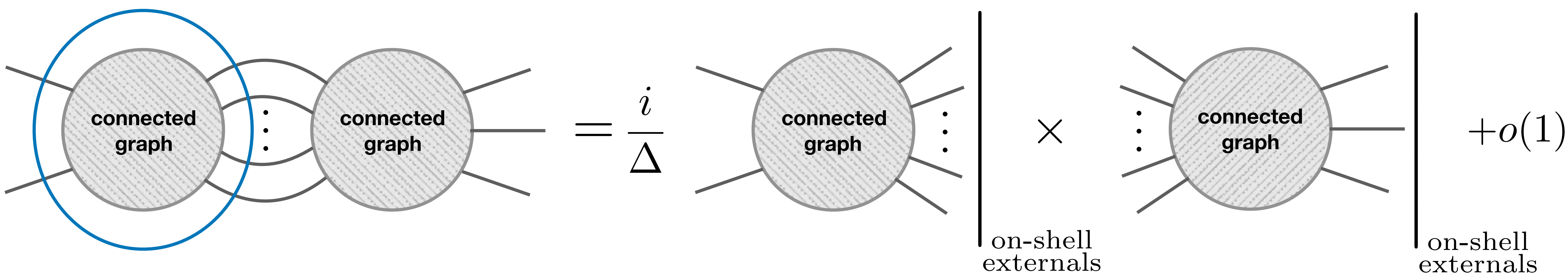

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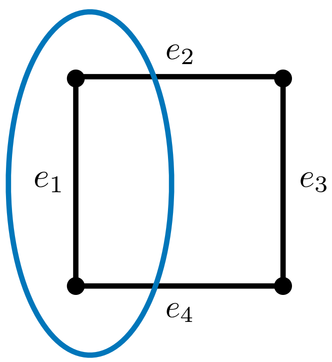

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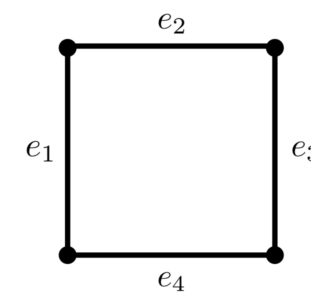

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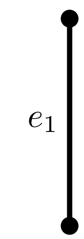

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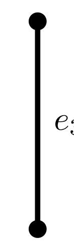

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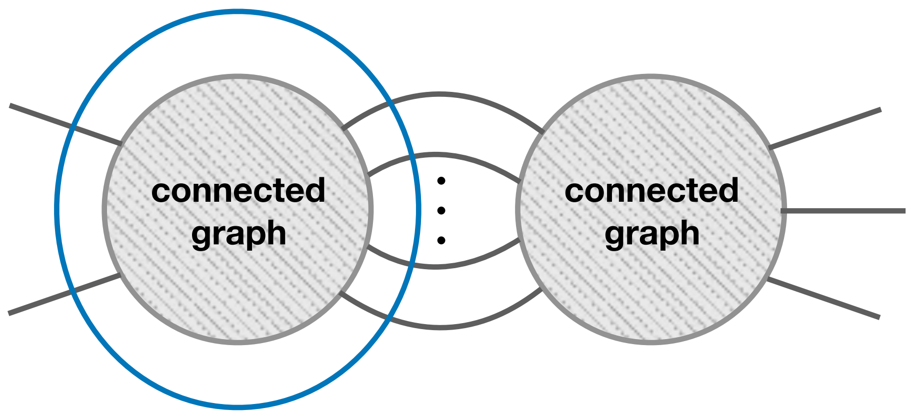
This relation also holds at the amplitude level with underlying tensor contraction!



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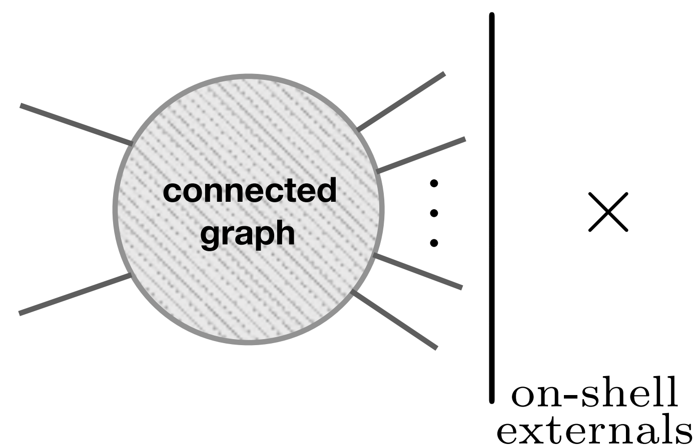

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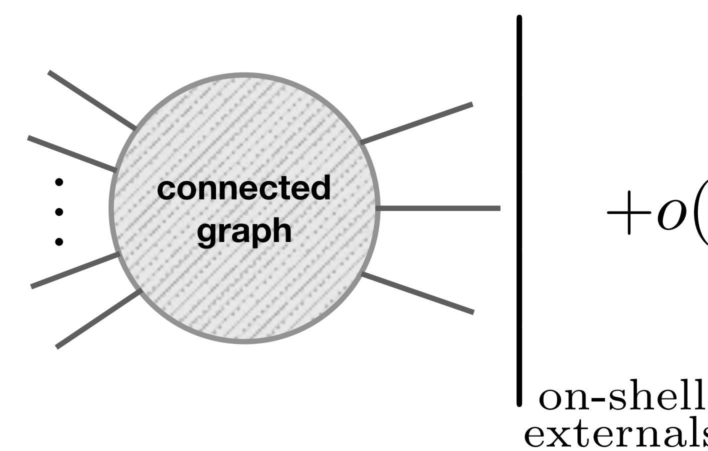

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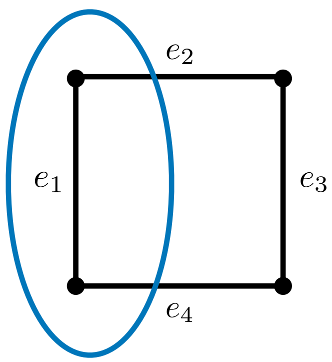

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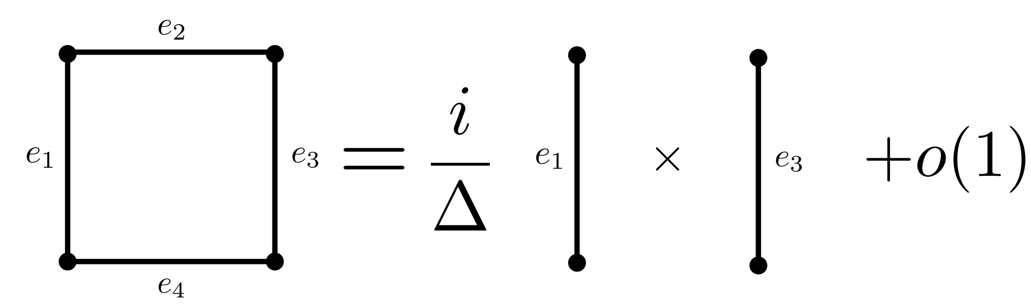

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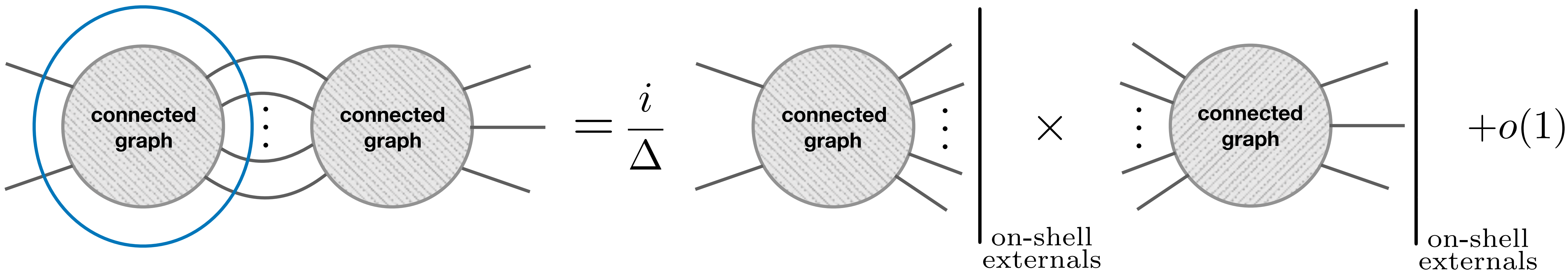

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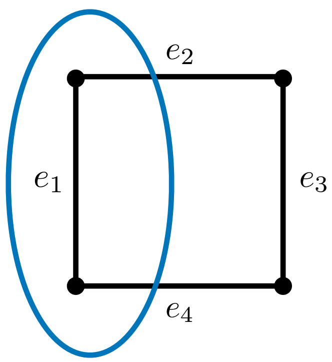

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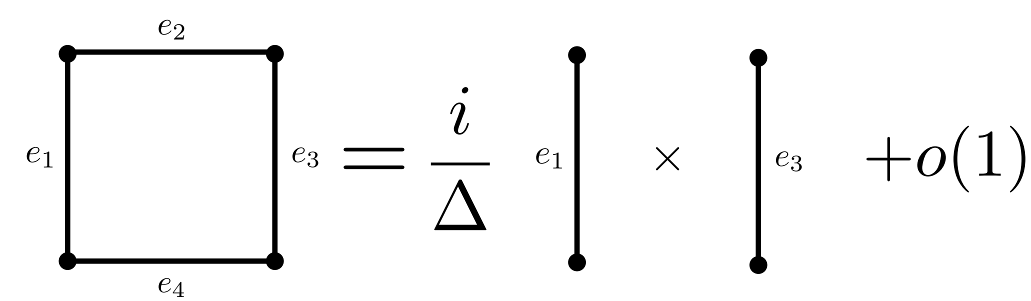

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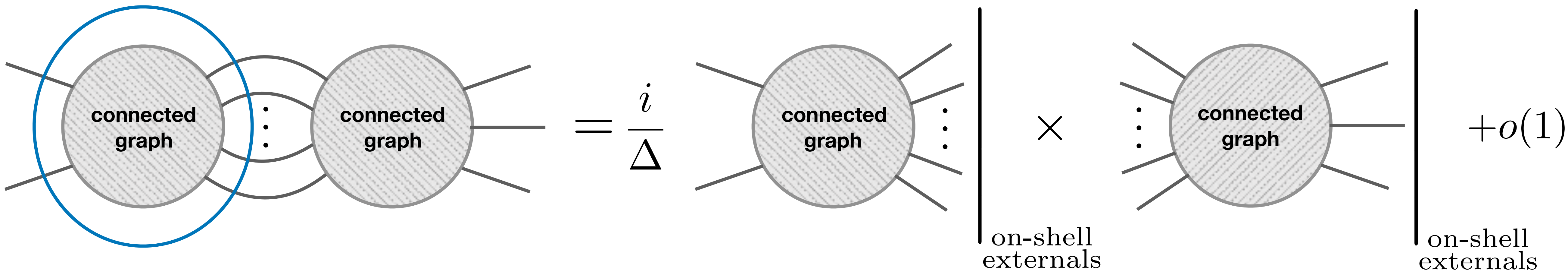

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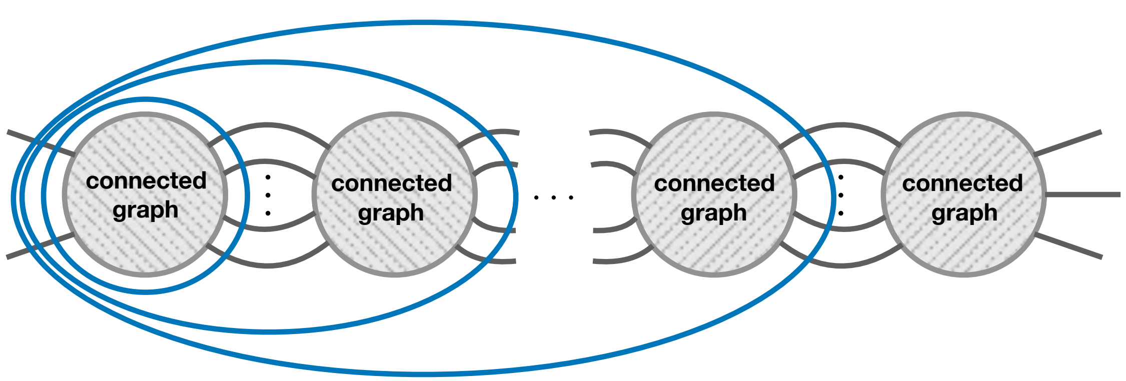

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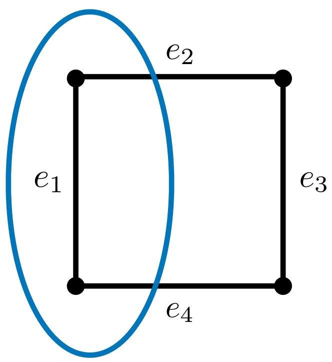

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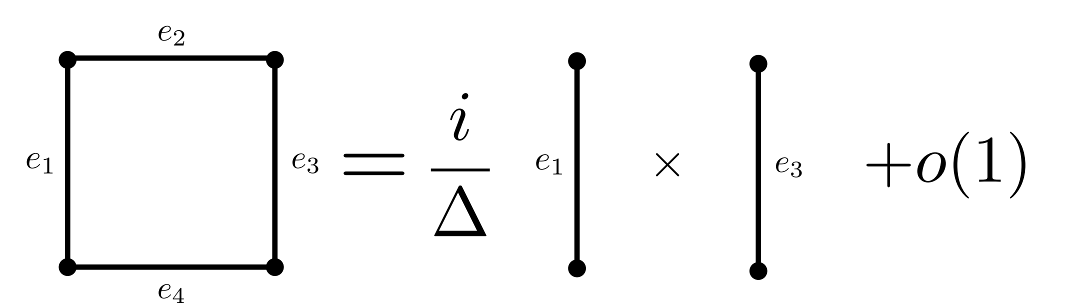

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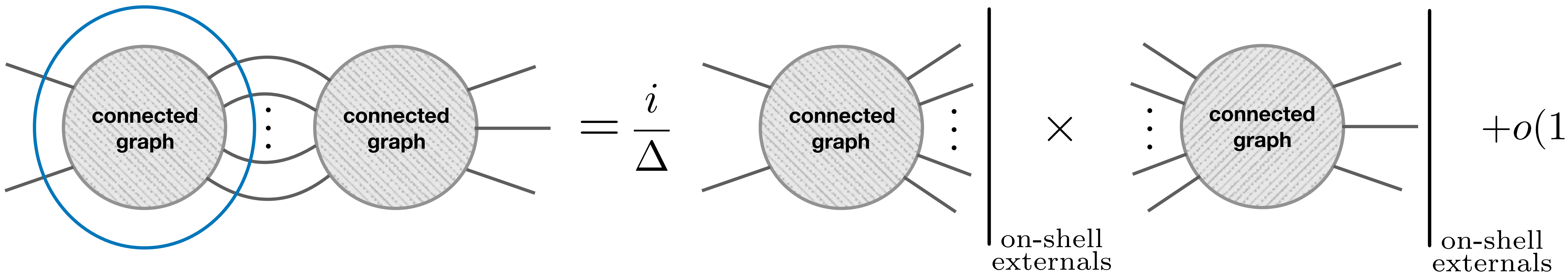

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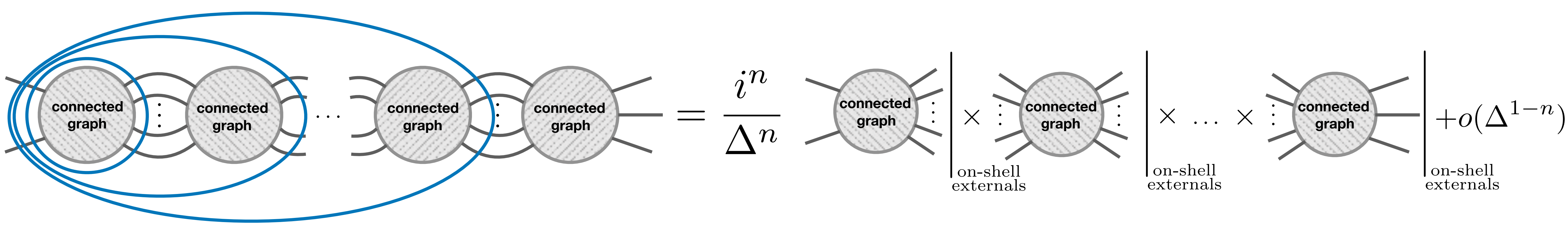

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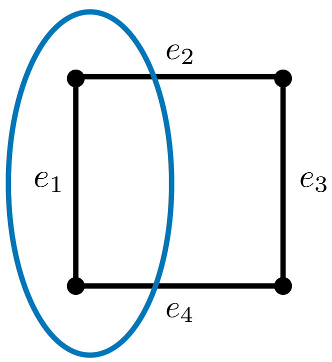

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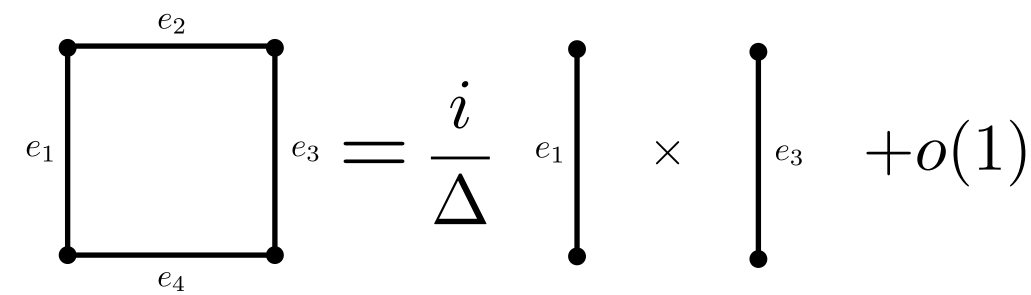

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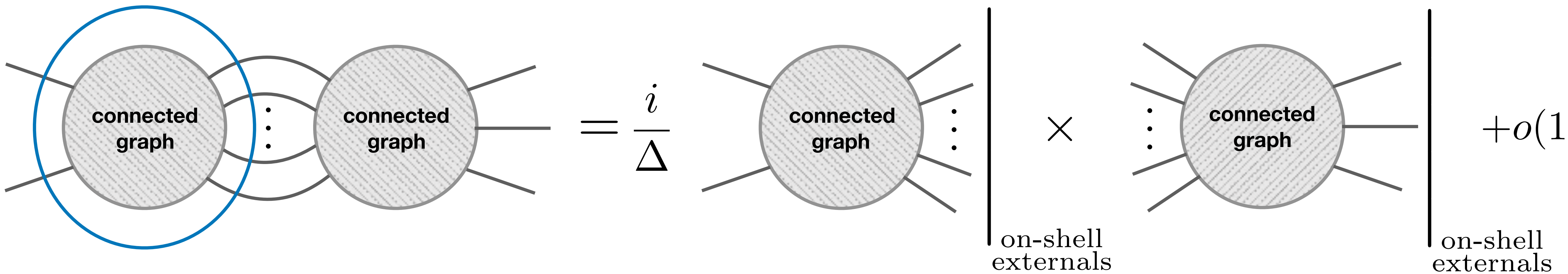

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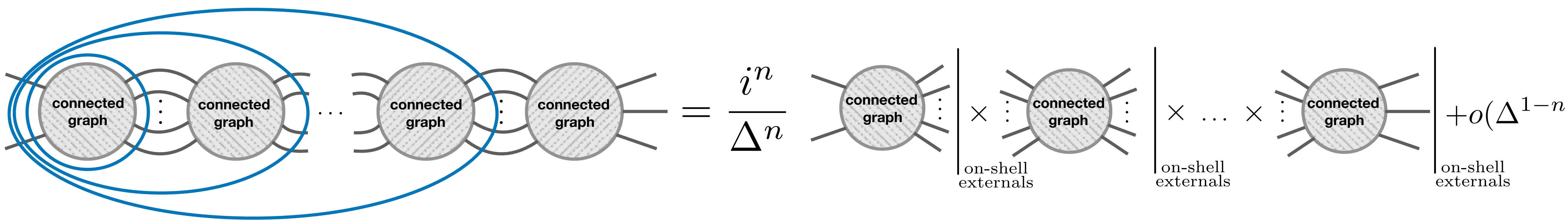

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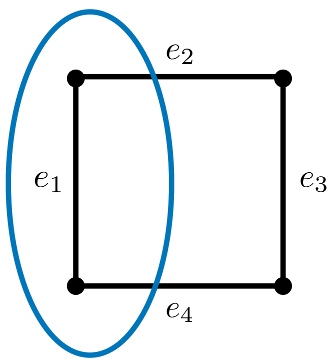

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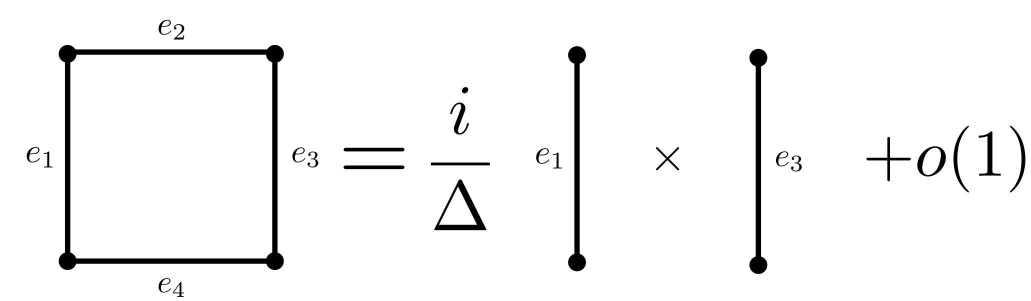

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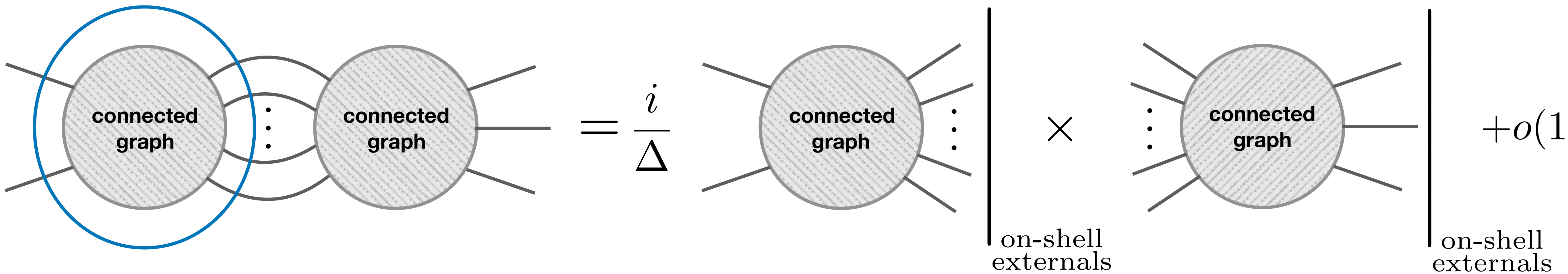

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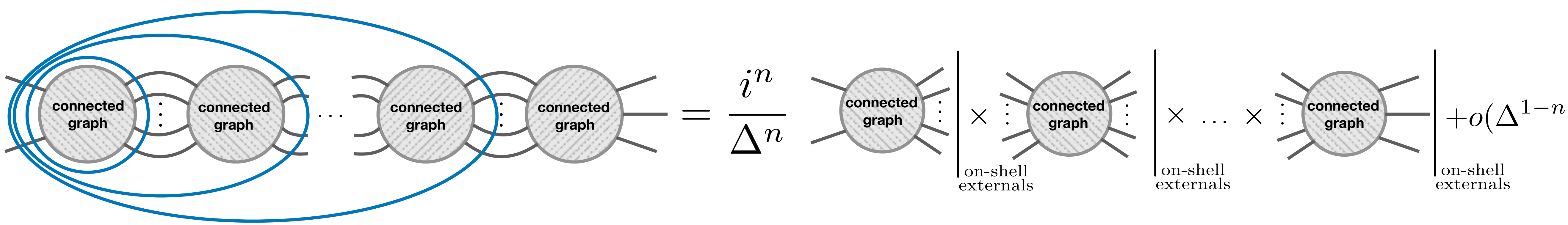

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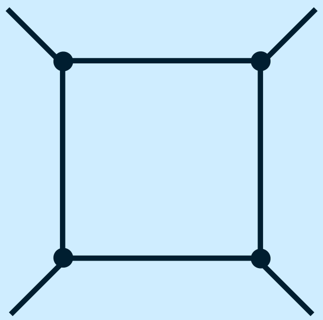
A set of n thresholds leads to a Δ^{-n} scaling only if the corresponding cuts divide the graph in exactly $n+1$ connected components and not more

Steinmann relations and unitarity cuts: consider the box diagram

Steinmann relations

$$I(s,t) =$$



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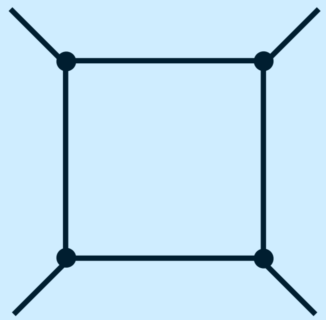
$I(s,t) =$ 

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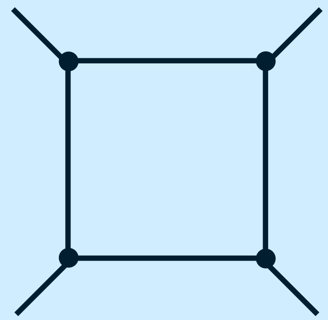

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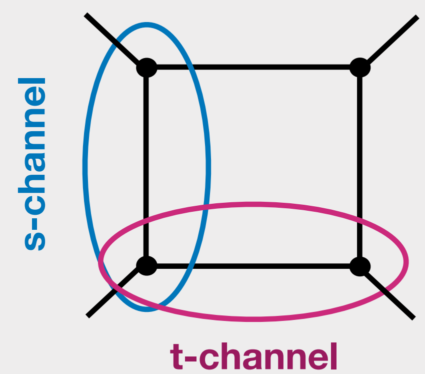

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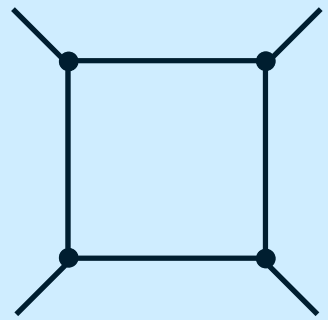
$I(s,t) =$ 

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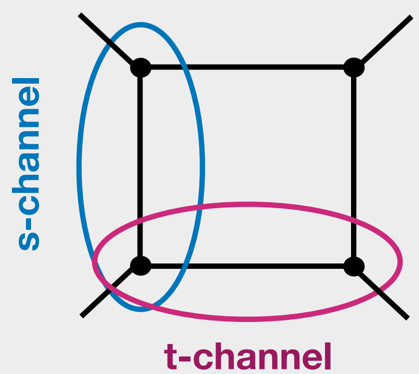

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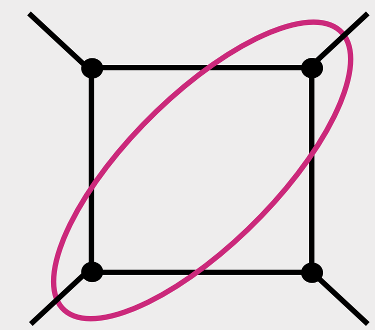

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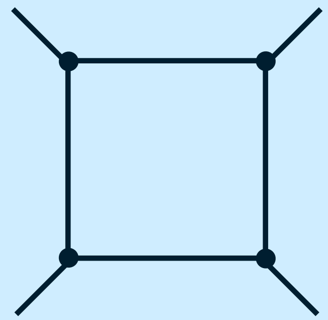
$I(s,t) =$ 

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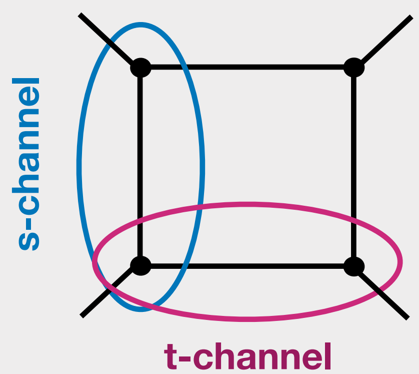

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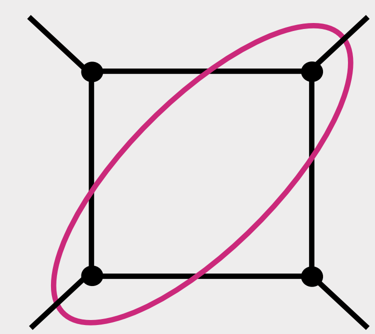

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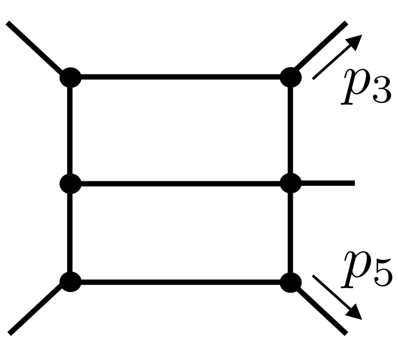

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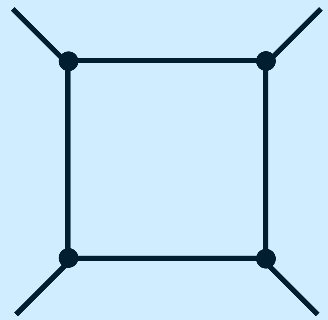
$I(s,t) =$ 

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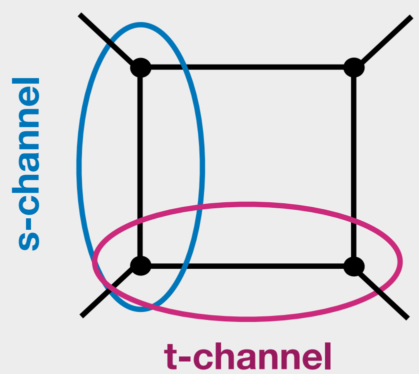

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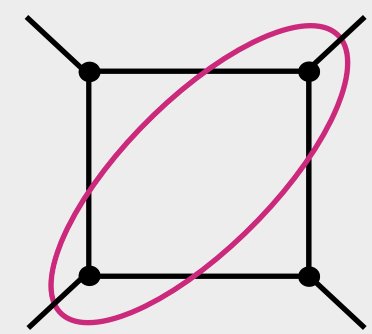

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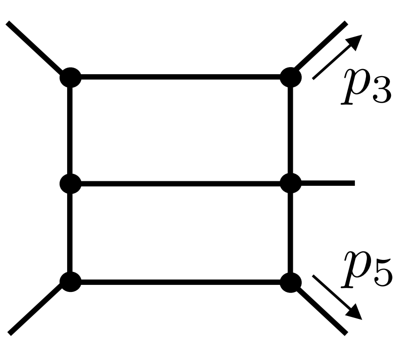

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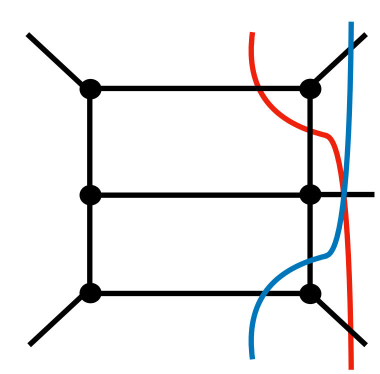

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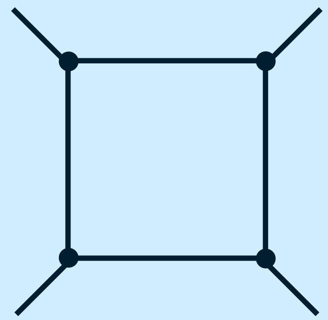
$I(s,t) =$ 

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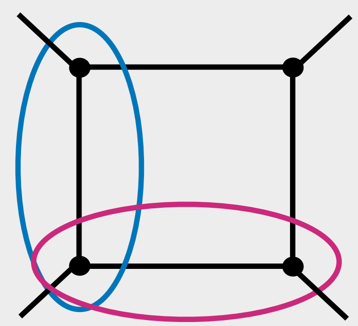

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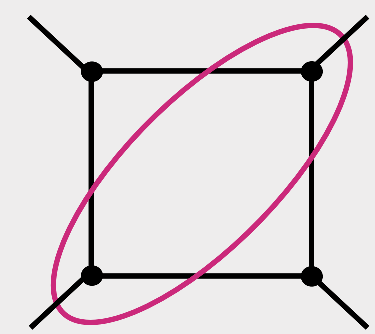

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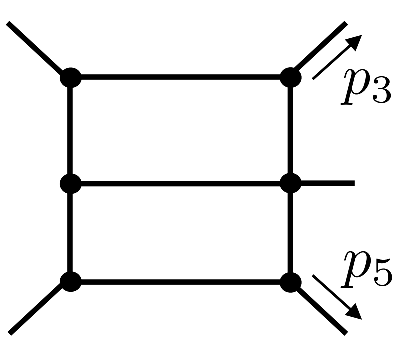

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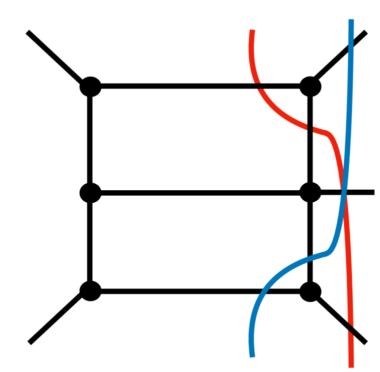

But these two cuts are simultaneously present in a few terms of the CFF expression!

Cluster decomposition principle, Unitarity and Infrared Finiteness

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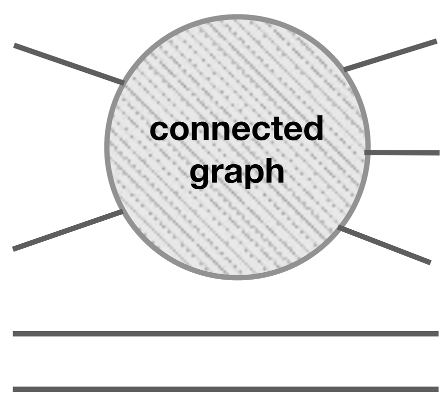

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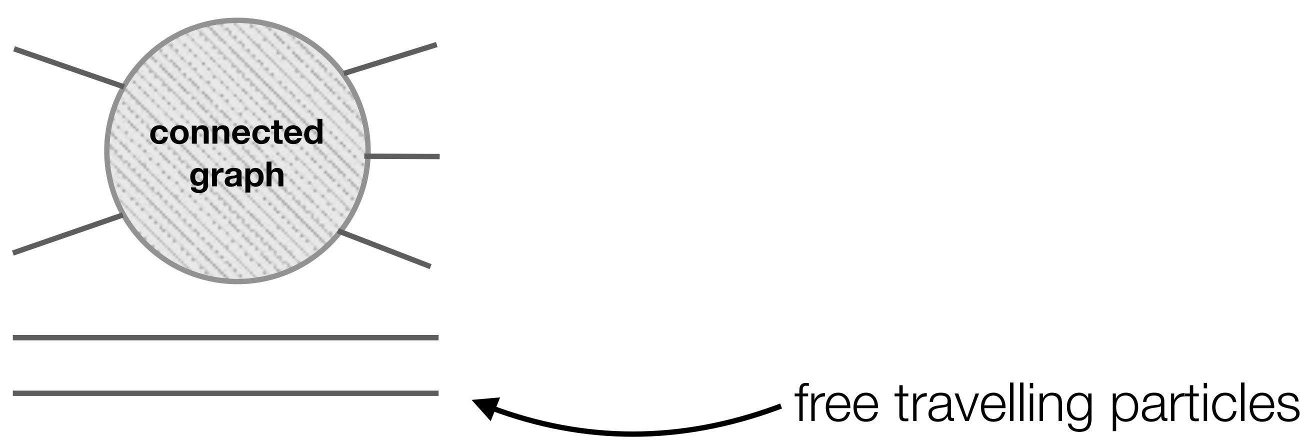

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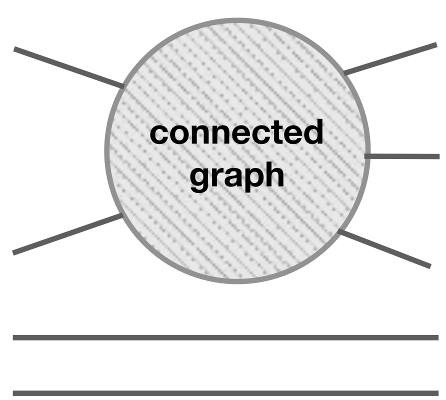

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free travelling particles

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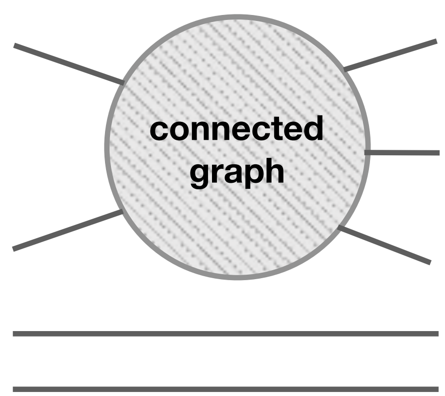

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The cluster decomposition principle and unitarity constraints

$$\hat{\mathbf{S}} = \hat{\mathbf{S}}^\dagger$$

$$\lim_{\substack{d_{12}^2 \text{large} \\ \text{and negative}}} \langle \boldsymbol{\alpha}_1 \boldsymbol{\alpha}_2 | \hat{\mathbf{S}} | \boldsymbol{\beta}_1 \boldsymbol{\beta}_2 \rangle = \langle \boldsymbol{\alpha}_1 | \hat{\mathbf{S}} | \boldsymbol{\beta}_1 \rangle \langle \boldsymbol{\alpha}_2 | \hat{\mathbf{S}} | \boldsymbol{\beta}_2 \rangle$$

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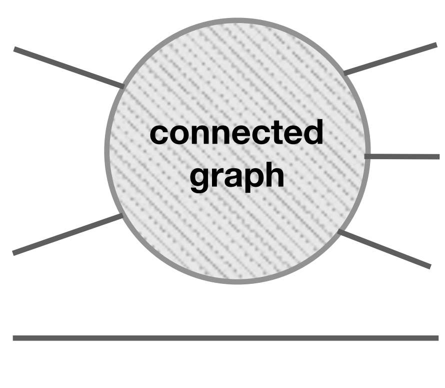

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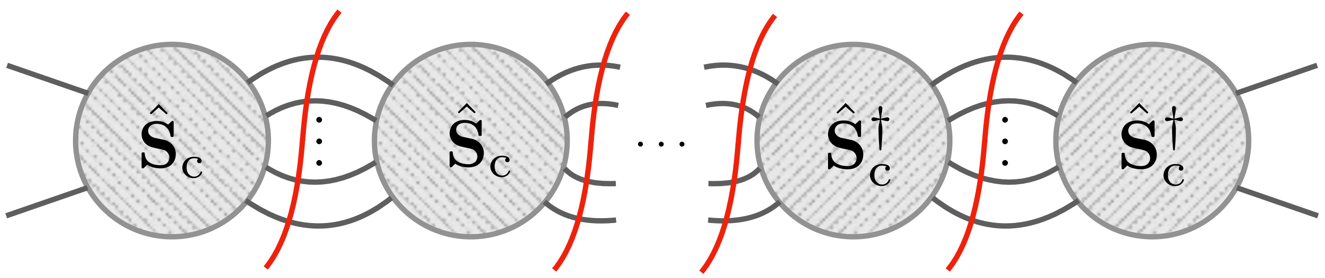
fix the coefficients of the series

$$a_n = \frac{i^n}{n!}$$

Scattering cross-sections then correspond to sums of terms

$$\sum_{\boldsymbol{\alpha}} \sum_{\boldsymbol{\beta}} \langle \boldsymbol{\alpha} | (\hat{\mathbf{S}}_{\rm c})^n | \boldsymbol{\beta} \rangle \langle \boldsymbol{\beta} | (\hat{\mathbf{S}}_{\rm c}^\dagger)^m | \boldsymbol{\alpha} \rangle =$$



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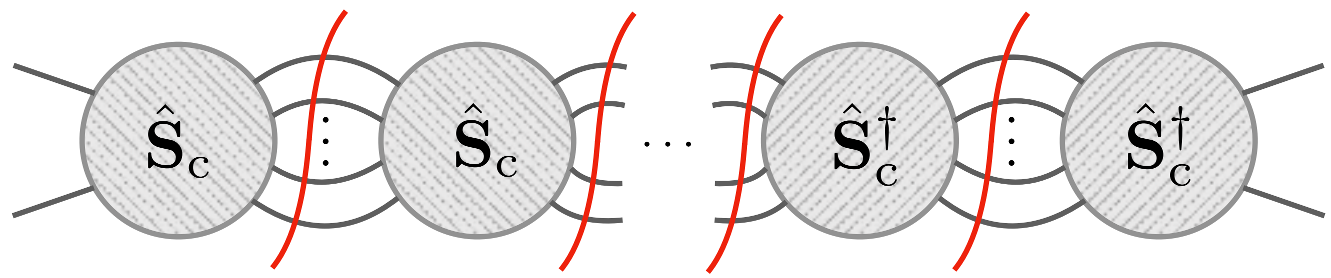

Notice the multiple Cutkosky cuts, as opposed to the usual definition of the connected S-matrix

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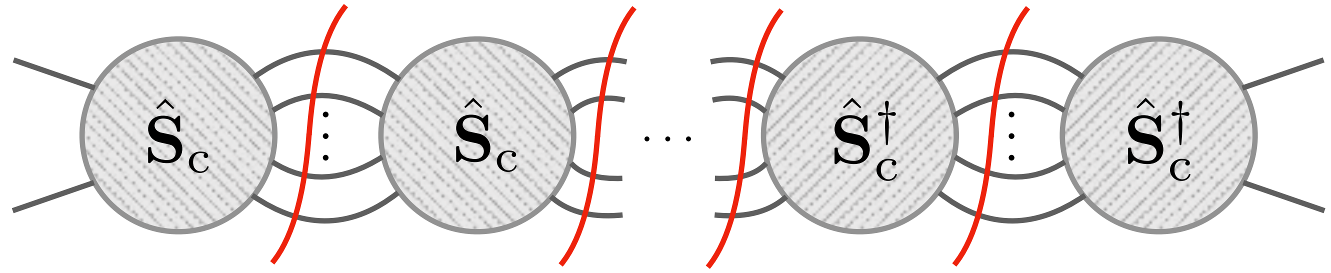

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Work for fixed **n** and **m**, e.g.

$n = 1,\, m = 1$

$+$

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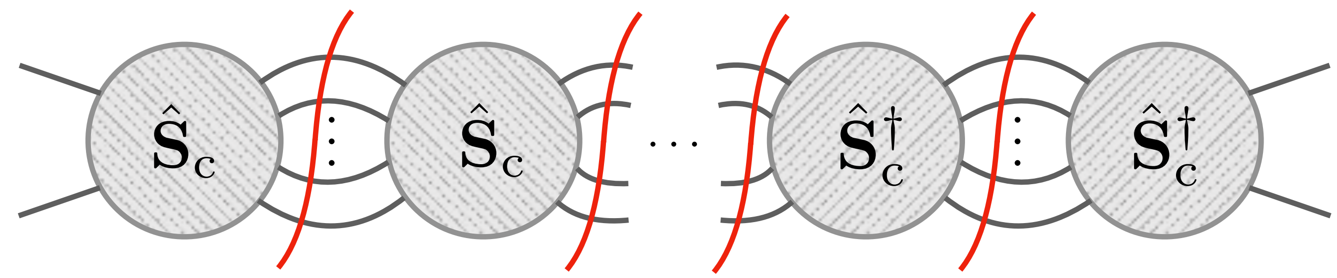

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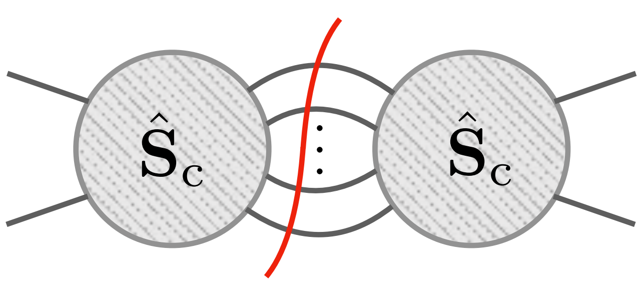

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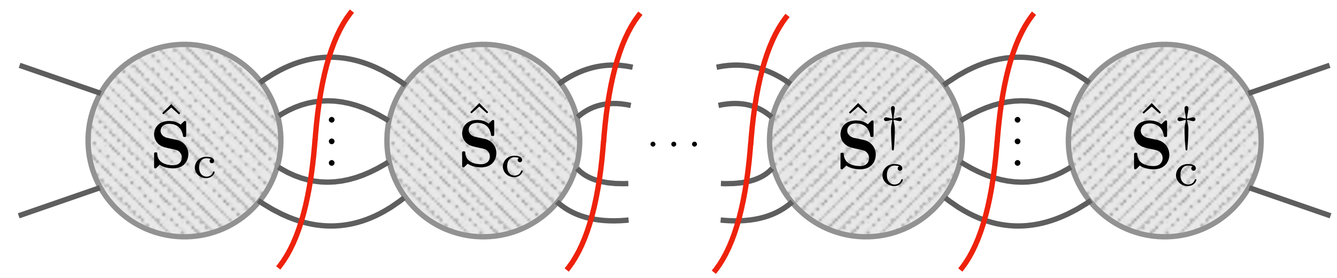

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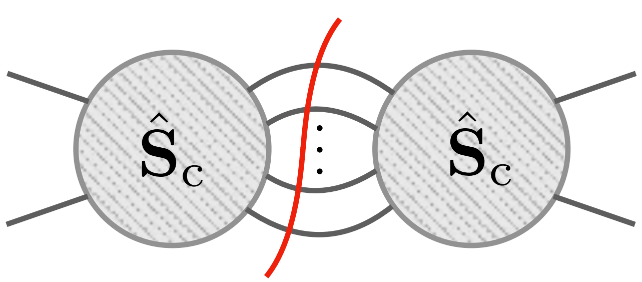

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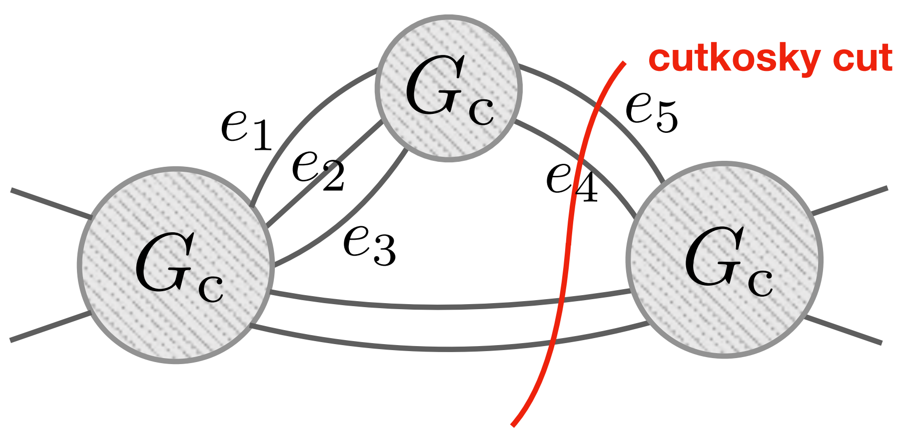

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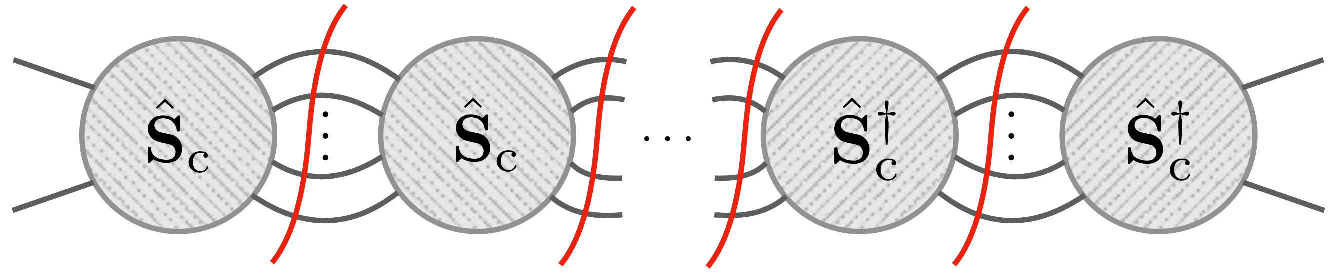

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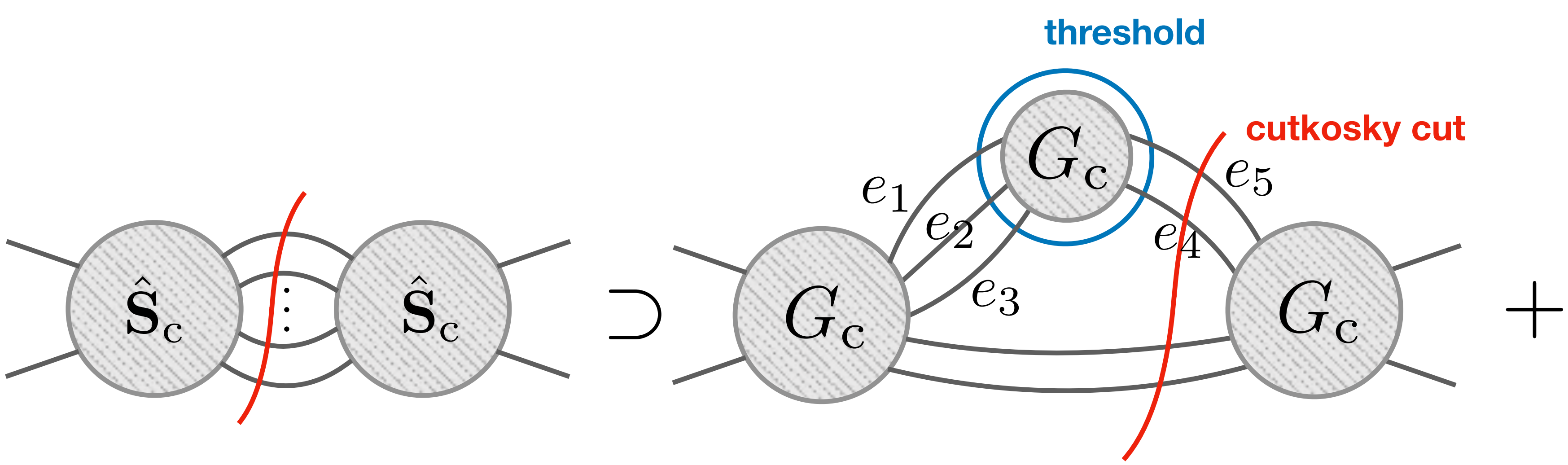

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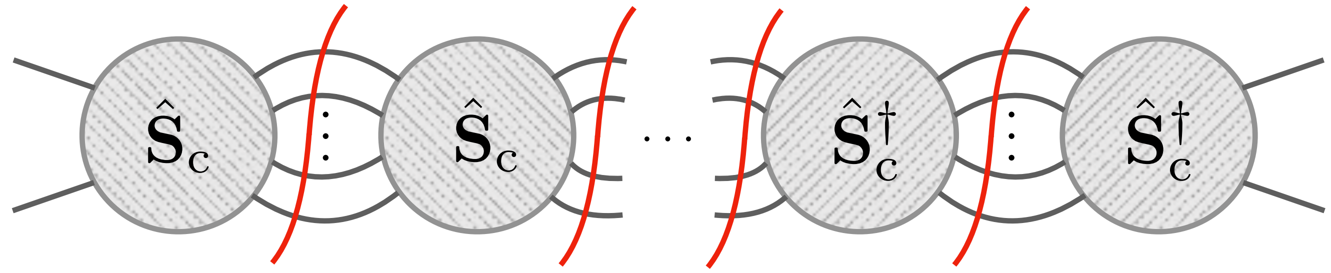

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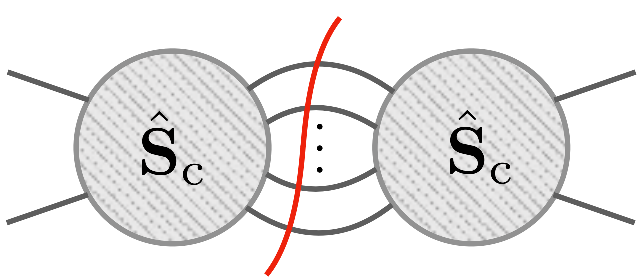

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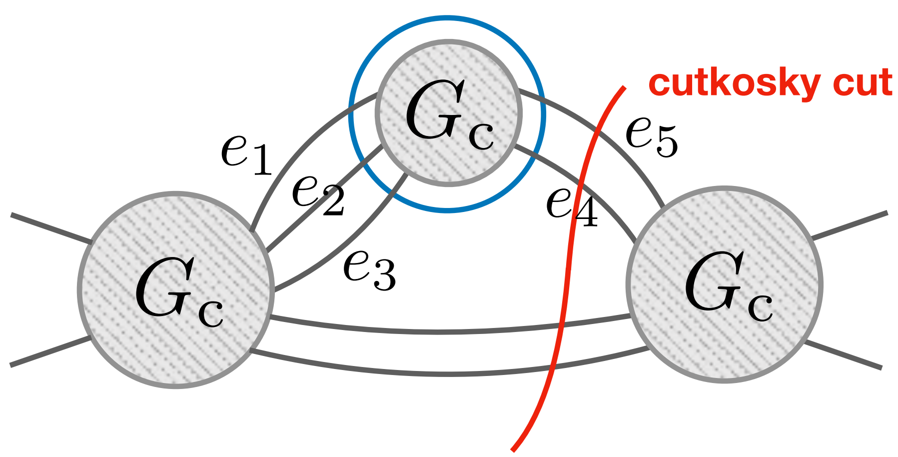

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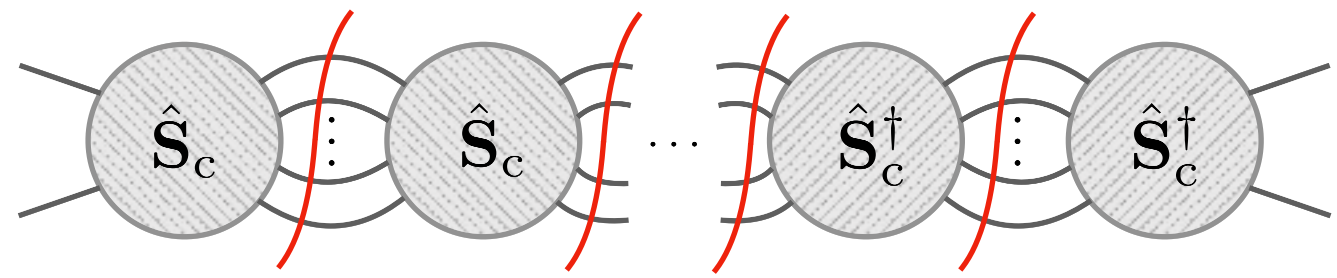

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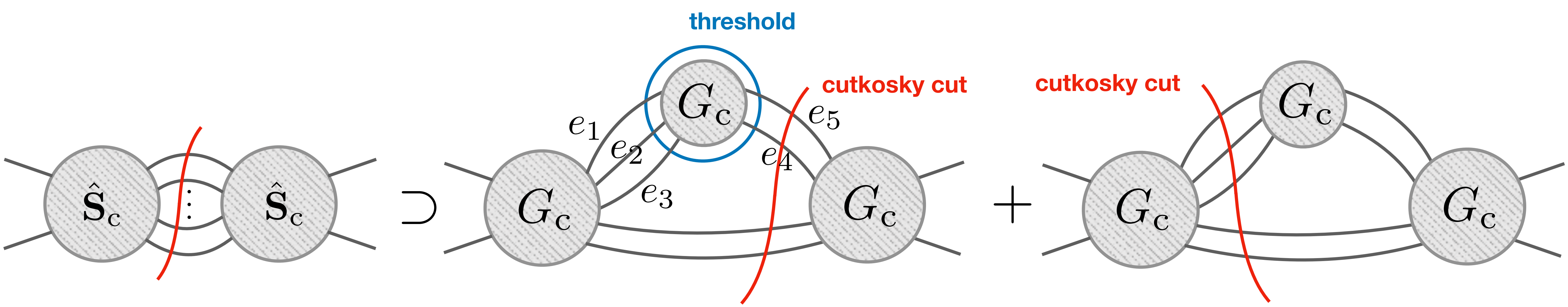

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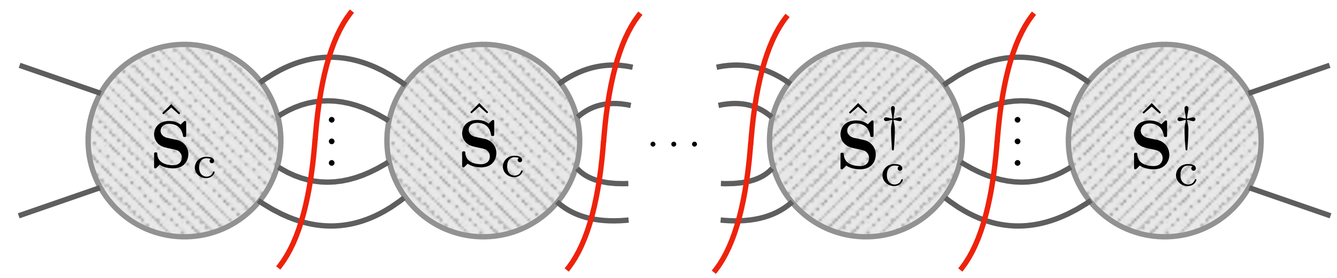

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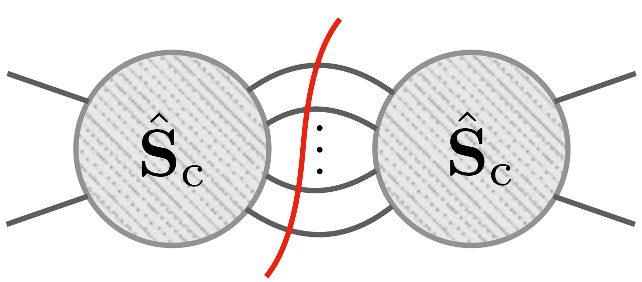

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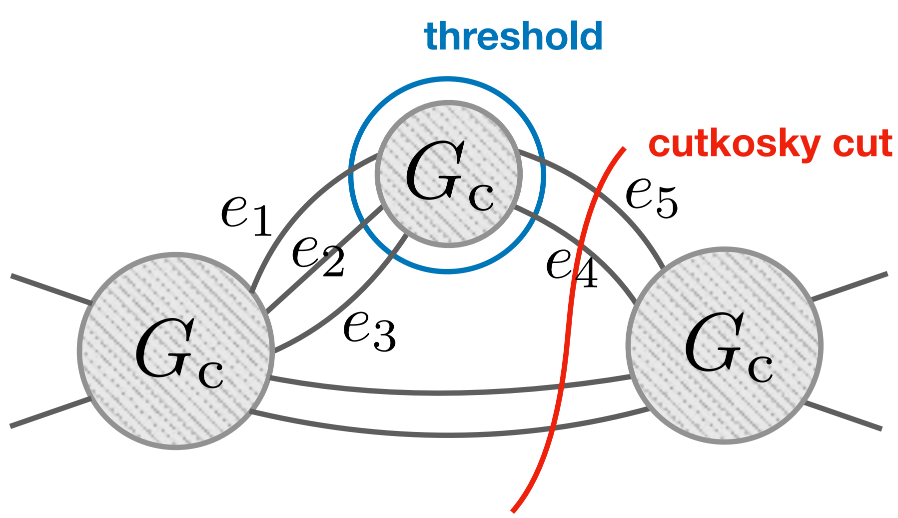

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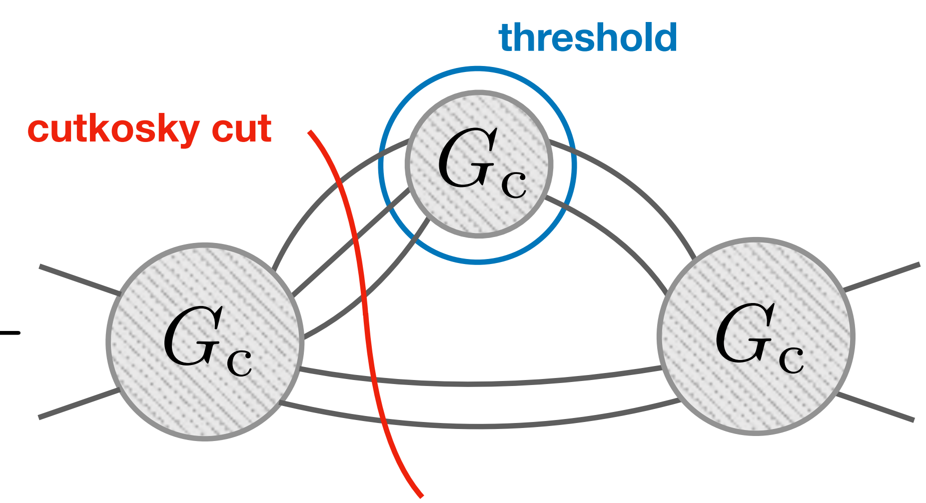

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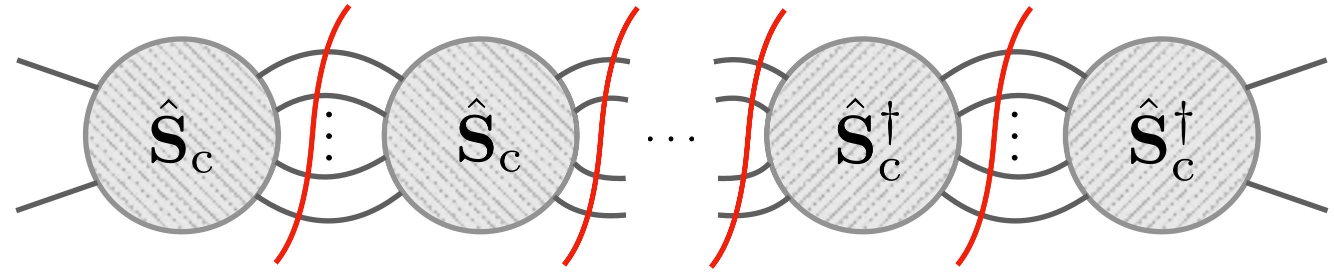

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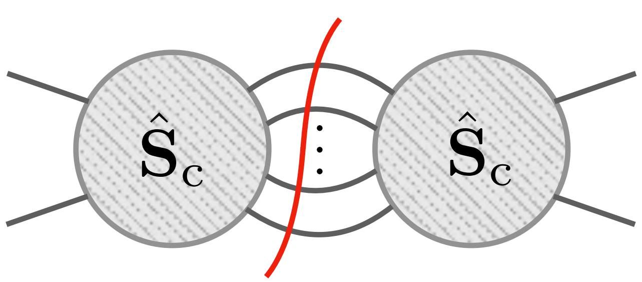

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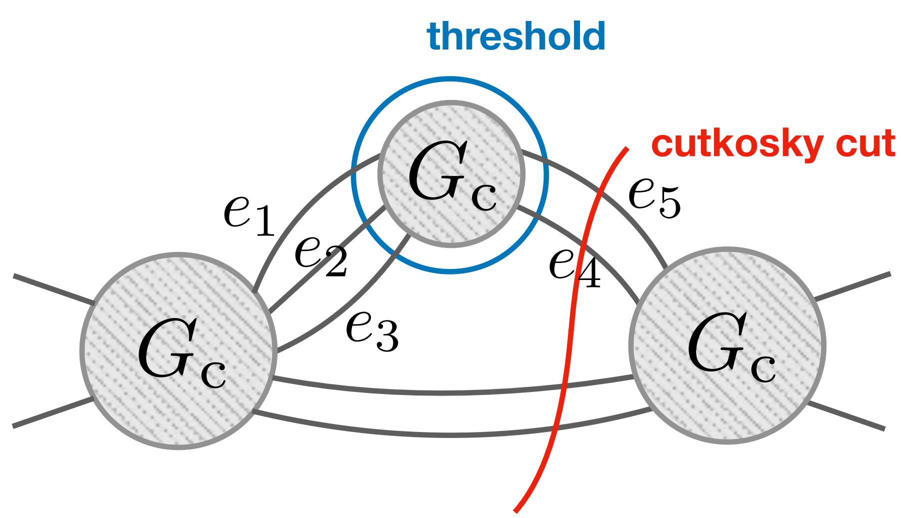

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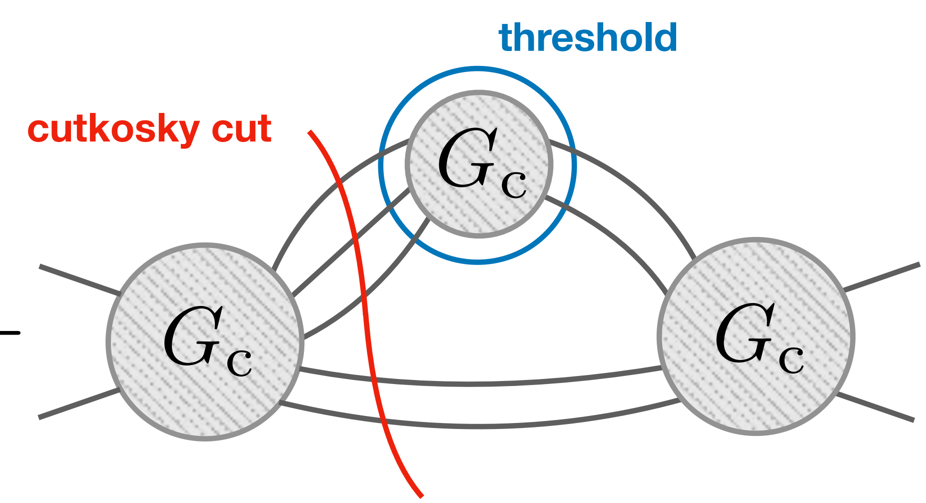

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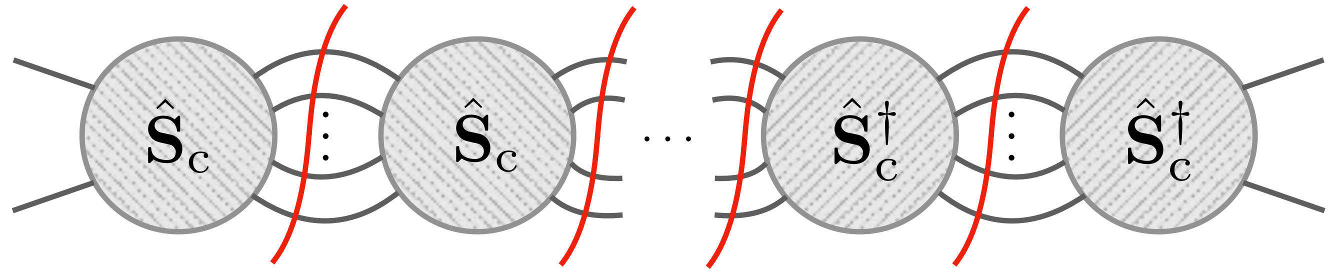

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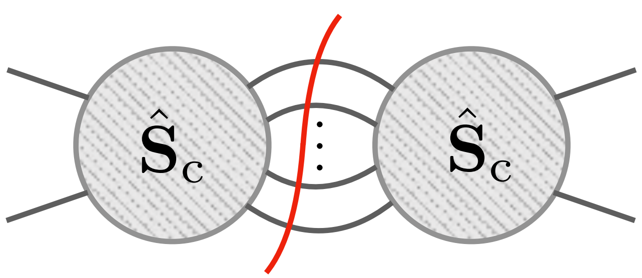

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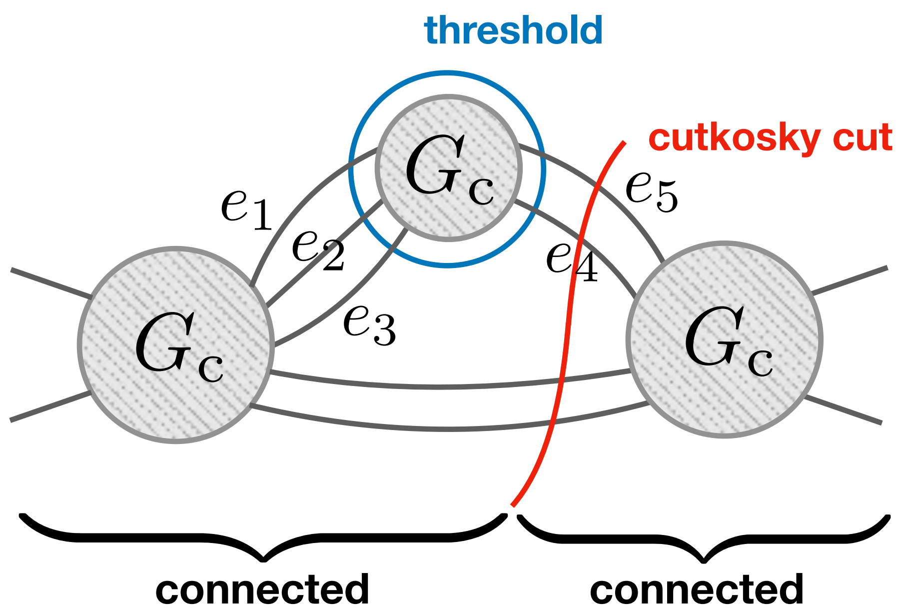

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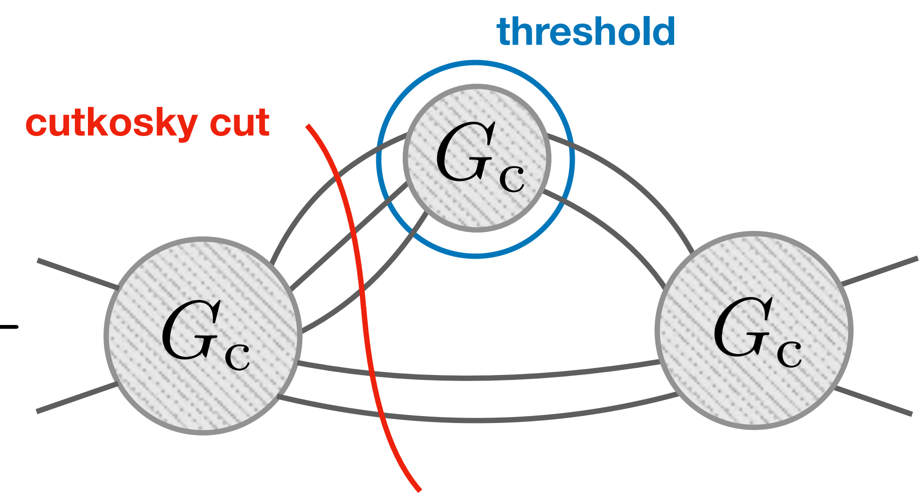

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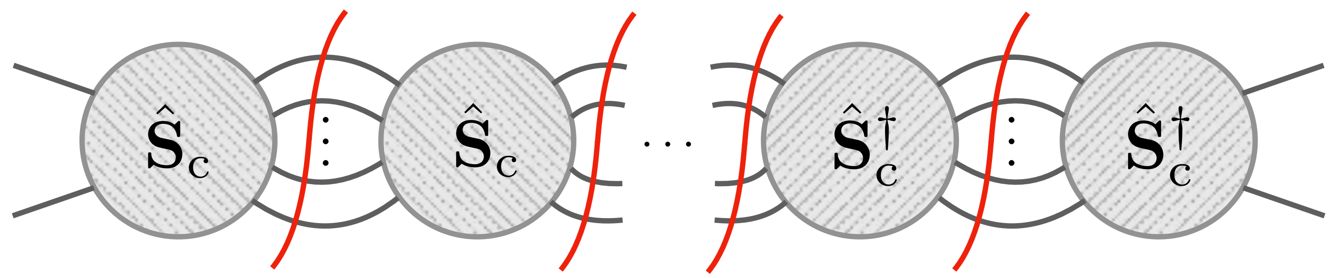

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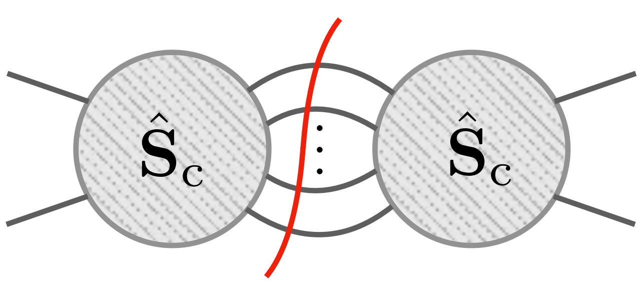

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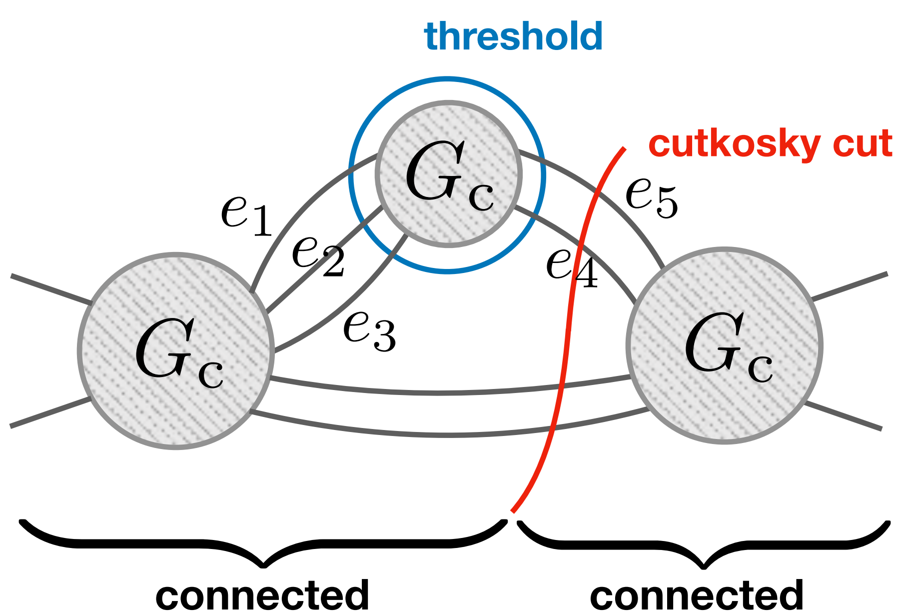

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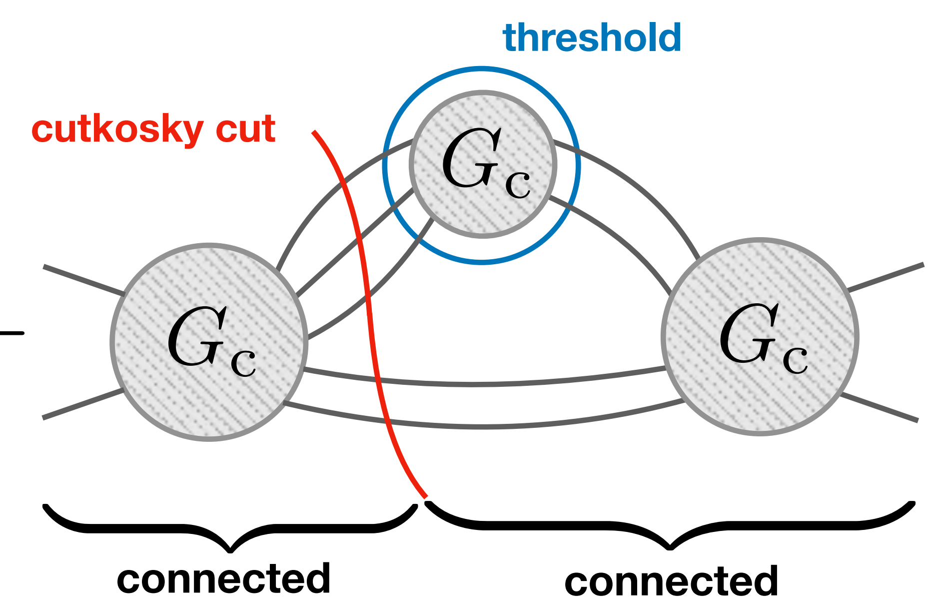

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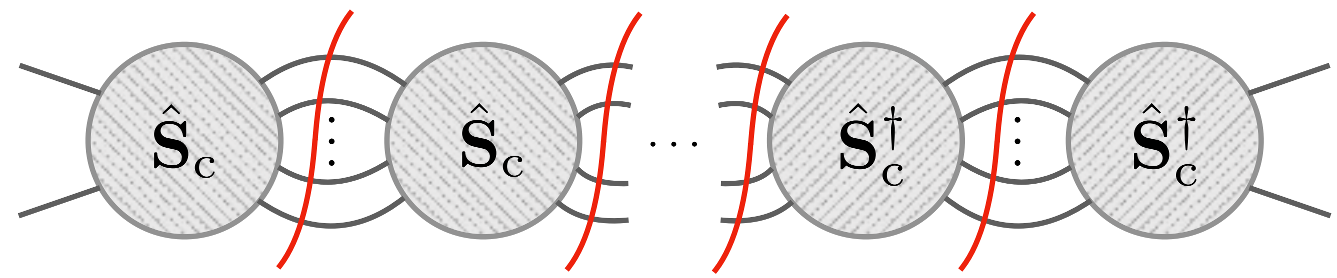

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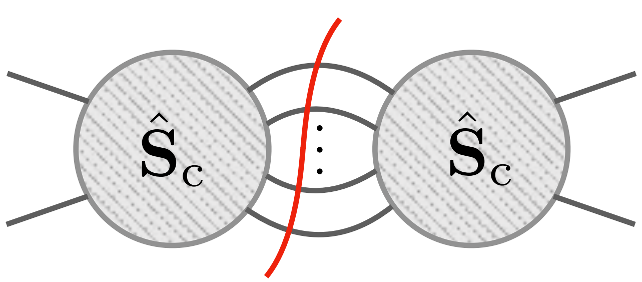

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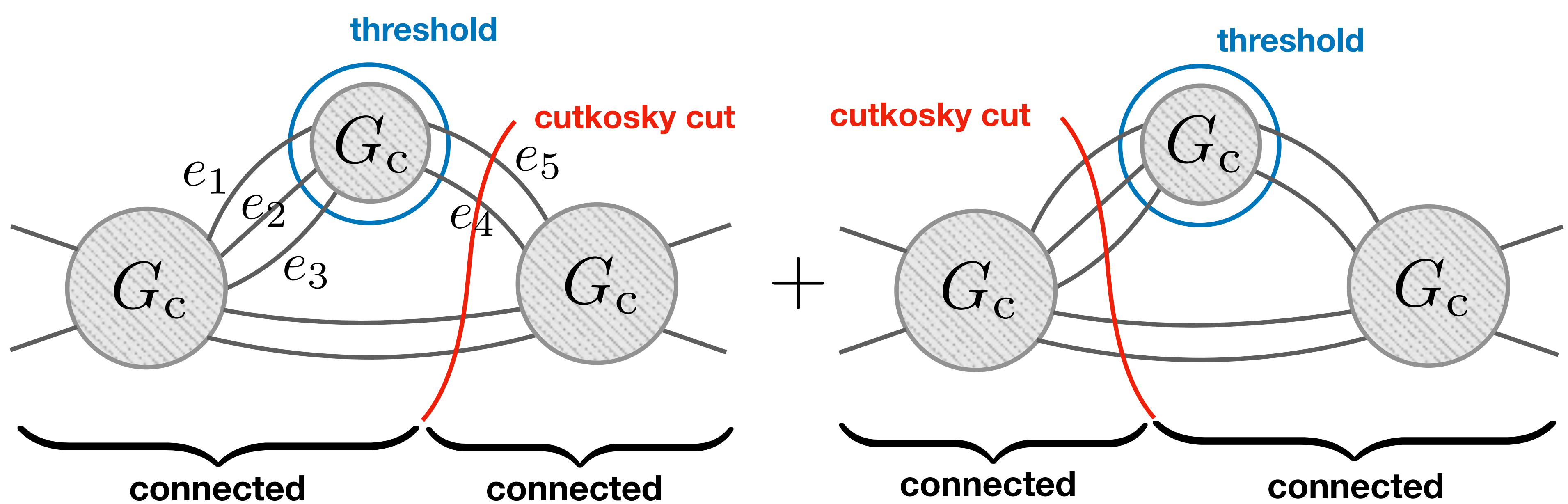

It is finite when $E_4 + E_5 - E_1 - E_2 - E_3 \to 0$ (delta argument becomes same). In general:

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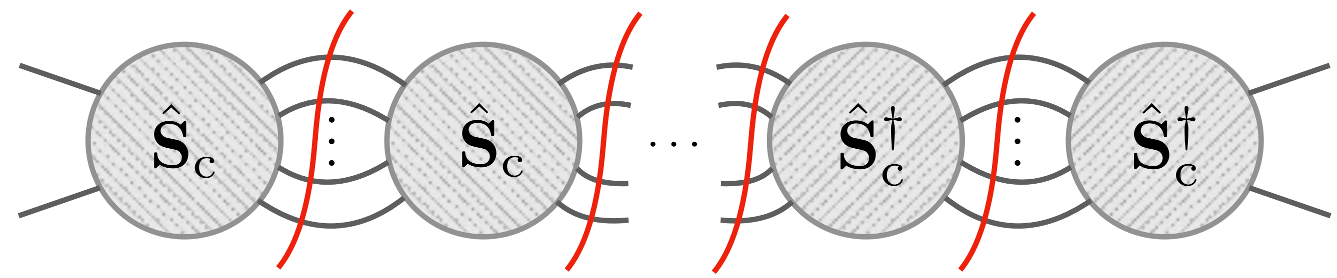

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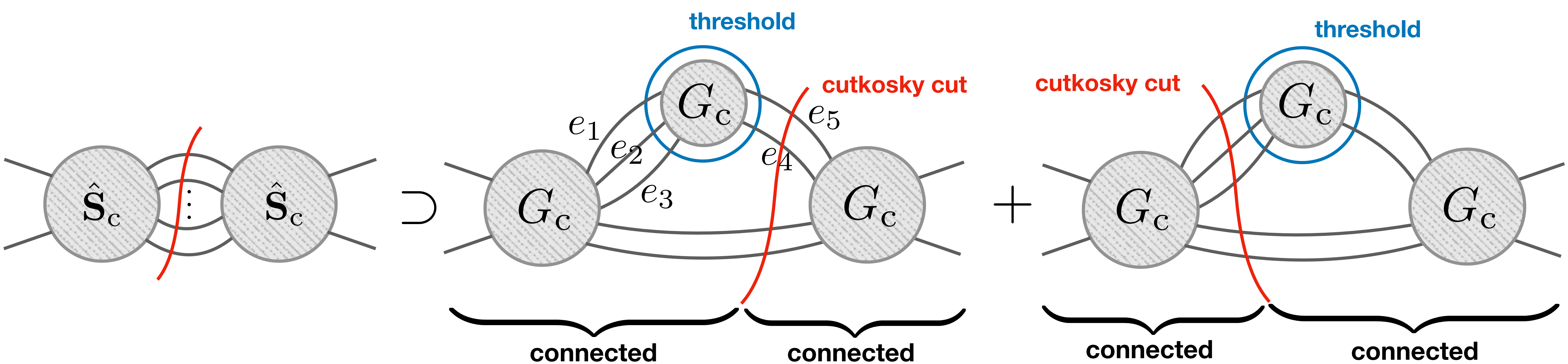

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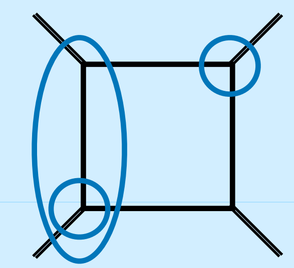
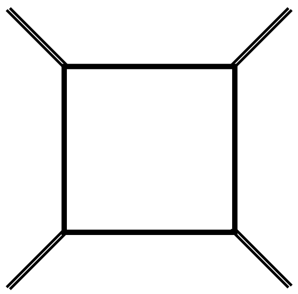
Is IR finite for fixed n and m, that is *for fixed number of connected components*!

Backup slides

Thresholds in three-dimensional space

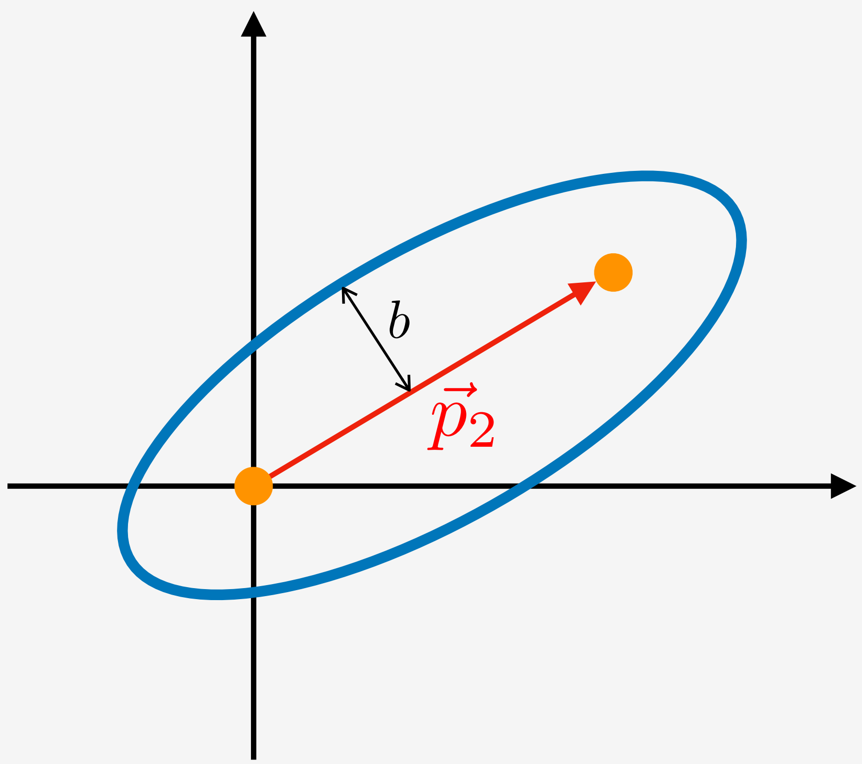
We kept on-shell energy dependence implicit. Let's re-instate it





$$= \ldots + \int \mathrm{d}^3\vec{k} \frac{1}{\prod_{i=1}^{4} 2|\vec{k}+\vec{p}_i|} \frac{i^3}{(|\vec{k}|+|\vec{k}-\vec{p}_2|-p_2^0)(|\vec{k}+\vec{p}_1|+|\vec{k}+\vec{p}_1-\vec{p}_3|-p_3^0)(|\vec{k}+\vec{p}_1|+|\vec{k}-\vec{p}_2|-p_{12}^0)} + \ldots$$

We can now draw the threshold locations in the space of spatial loop momenta



$$|\vec{k}|+|\vec{k}-\vec{p}_2|-p_2^0=0 \quad \Rightarrow \quad \text{ellipse!}$$

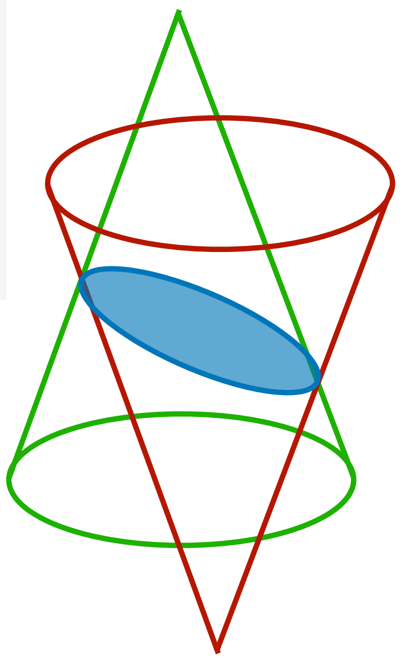
Foci: $\vec{k}=0,\ \vec{k}=\vec{p}_2$

Minor/major axis lengths: $a=p_2^0,\ b=\sqrt{(p_2^0)^2-|\vec{p}_2|^2}$

It exists if $p_2^0 \geq 0, p_2^2 \geq 0$

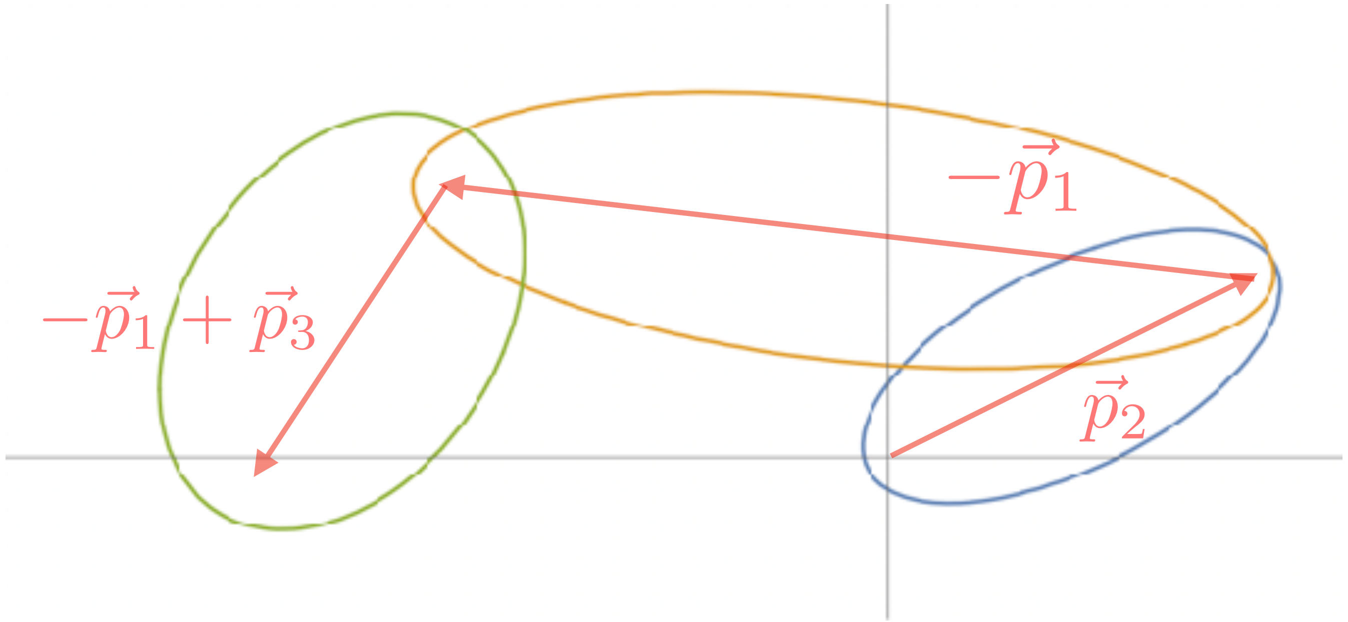
Describes a segment connecting the foci when $p_2^2=0$

Conic obtained by intersecting a forward light cone with a backward one



$$\{k^2=0,\ k^0>0\} \cap \{(k-p_2)^2=0,\ k^0-p_2^0<0\}$$

Now we can draw all the remaining thresholds



$$\eta_1 = |\vec{k}| + |\vec{k} - \vec{p}_2| - p_2^0 = 0$$

$$\eta_2 = |\vec{k} + \vec{p}_1| + |\vec{k} - \vec{p}_2| - p_{12}^0 = 0$$

$$\eta_3 = |\vec{k} + \vec{p}_1| + |\vec{k} + \vec{p}_1 - \vec{p}_3| - p_3^0 = 0$$

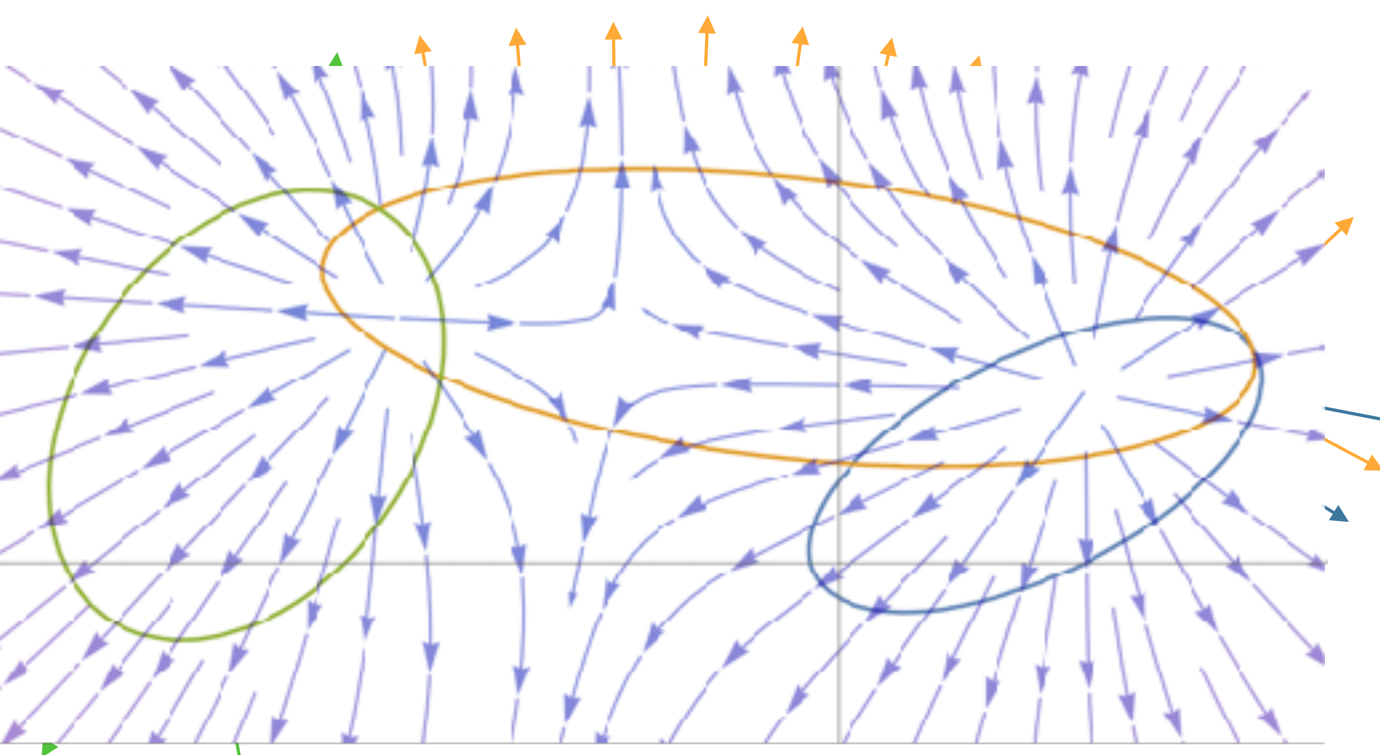
We now want to regulate the thresholds through a contour deformation $\vec{k} \to \vec{k} - i\kappa$

κ has to satisfy constraints

- No crossing of branch-cuts
- Stay on well-defined Riemann sheet
- Satisfy causal prescription $\kappa \cdot \nabla \eta_i > 0, \quad \text{when } \eta_i = 0$

(In Feynman parameter space, $\kappa \propto \nabla \mathcal{F}$)

Focus on the last one



The vector field

$$\kappa = \lambda_1(\vec{k})(\vec{k} - \vec{s}_1) + \lambda_2(\vec{k})(\vec{k} - \vec{s}_2)$$

satisfies the causal constraint for all thresholds!

1. Find out whether thresholds overlap

combinatorial problem

2. Find a point inside each overlap

convex problem

ZC, Hirschi, Kermanshah, Pelloni, Ruijl, arXiv: 2009.05509 (2020)
Kermanshah, arXiv: 2110.06869 (2020)

In general, the problem of finding the correct contour deformation decomposes

1. **Combinatorial problem:** enumerate overlaps
2. **Convex problem:** find point inside each overlap

When all internal propagators are massless, these problems can be solved in closed form, thanks to the factorisation formula!

This principle can be translated into a constraint on overlaps, establishing that there is one overlap for each spanning tree of the graph

For each of these overlaps, the point inside it, when internal propagators are massless, corresponds to the origin in the loop momentum basis identified by the spanning tree

$$\kappa = \sum_{\substack{\text{spanning} \\ \text{tree } T}} \lambda_T(\vec{k})(\vec{k} - \vec{s}_T)$$

This in turn shows that a deformation for massless internals always exists, and thus the Feynman diagram is well-defined!

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A set of n thresholds leads to a Δ^{-n} scaling only if the corresponding cuts divide the graph in exactly $n+1$ connected components and not more

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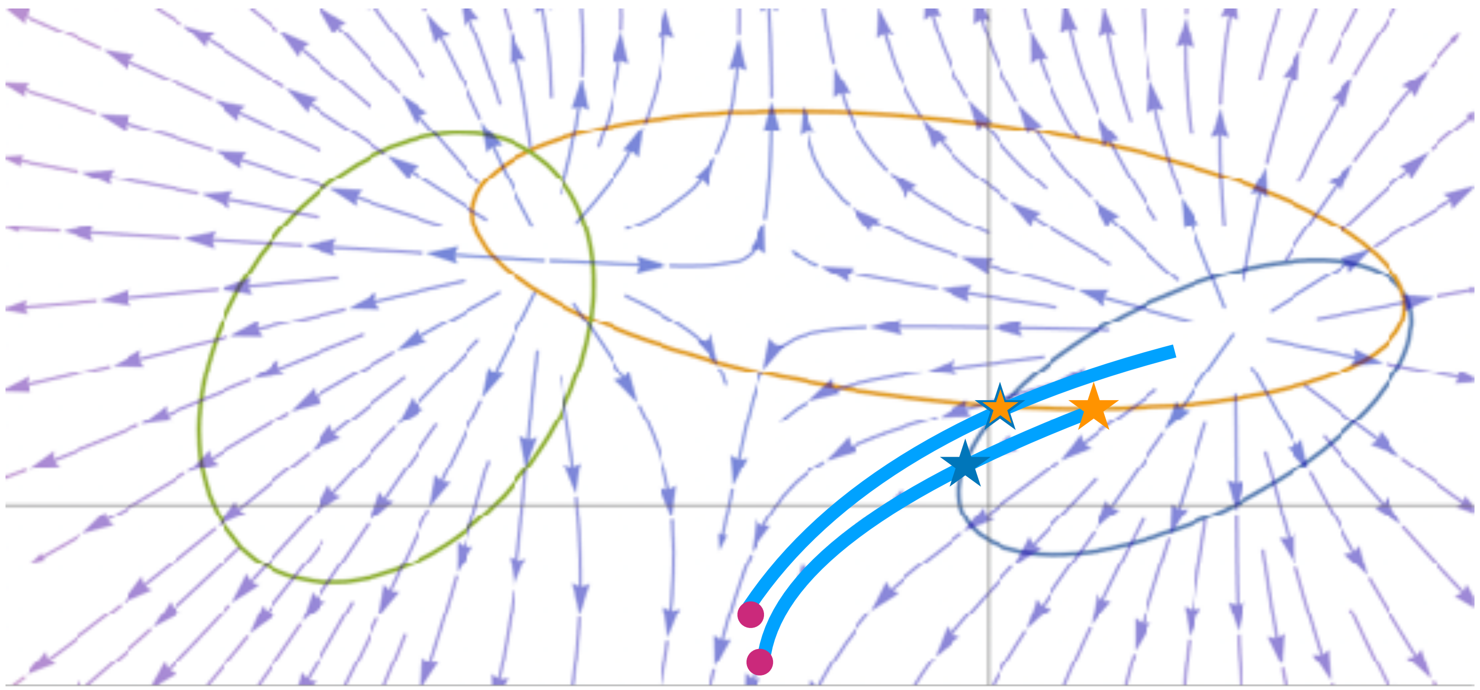
Local Unitarity

We can then use the deformation field to construct a parametrisation of the surfaces

Causal flow
$$\begin{cases} \partial_t \phi(t,\vec{k}) = \kappa(\phi(t,\vec{k})) \\ \phi(0,\vec{k}) = \vec{k} \end{cases}$$

Capatti, Hirschi, Pelloni, Ruijl, arXiv:2010.01068 (2020)
Capatti, Hirschi, Ruijl, arXiv:2203.11038 (2022)

We use the flow to transport us on the thresholds



$$\eta_i(\phi(t,\vec{k})) = 0, \quad \Rightarrow \quad t = t_i^\star(\vec{k})$$

The function

$$\phi(t_i^\star(\vec{k}),\vec{k})$$

assigns to each point a threshold point

$$\vec{k} \to \phi(t_i^\star(\vec{k}),\vec{k}) \in \{\vec{k} \,|\, \eta_i = 0\}$$

The parametrisation are correlated so that they coincide at intersections!

$$\mathrm{disc}\left[\quad\right] = \sum_{i=1}^{3} \int d^3\vec{k} \frac{\delta(\eta_i(\vec{k}))}{2E_1 2E_2 2E_3} \frac{\mathcal{O}_i}{\prod_{\substack{j=1\\ j\neq i}}^3 \eta_j} = \int d^3\vec{k} \sum_{i=1}^{3} \frac{(\kappa \cdot \nabla \eta_i)^{-1}}{2E_1 2E_2 2E_3} \frac{\mathcal{O}_i}{\prod_{\substack{j=1\\ j\neq i}}^3 \eta_j}\Bigg|_{\phi(t_i^\star(\vec{k}),\vec{k})}$$

The right-hand side is integrable, and corresponds to a **differential cross-section**!

The causal flow aligns the phase-space integration measure of interference diagrams