



Exposing the threshold structure of loop integrals

Zeno Capatti

ETH Zürich

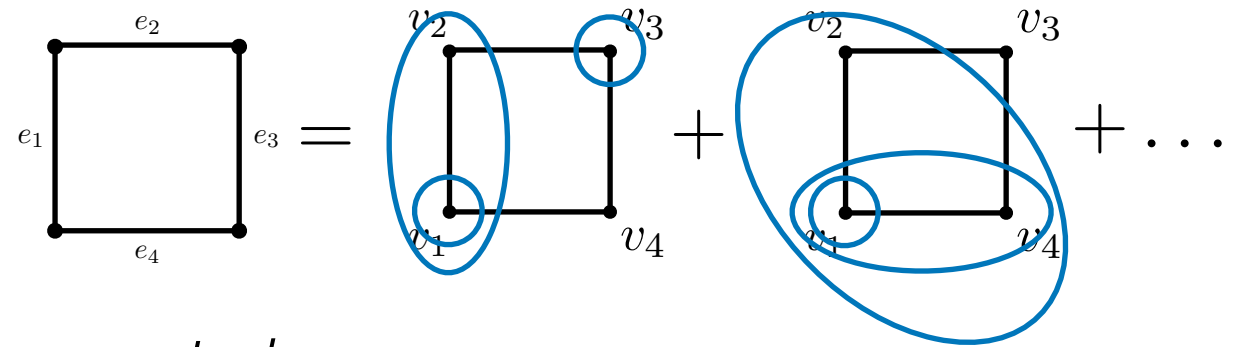
RADCOR 2023

30/05/2023, Crieff, Scotland

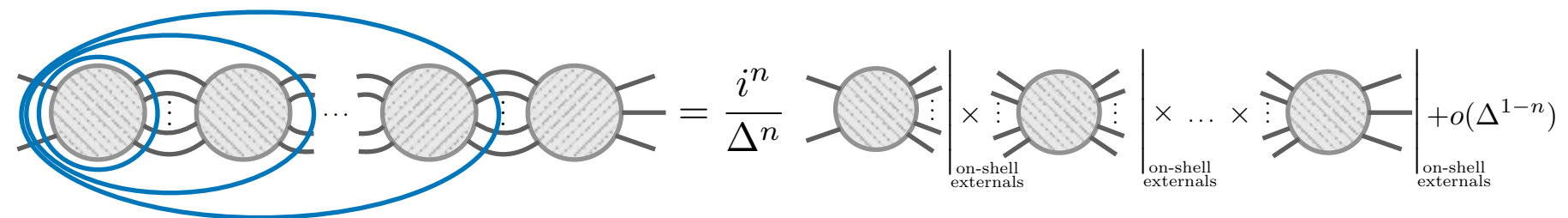


Summary:

1. The Cross-Free Family representation



2. Iterated connectedness



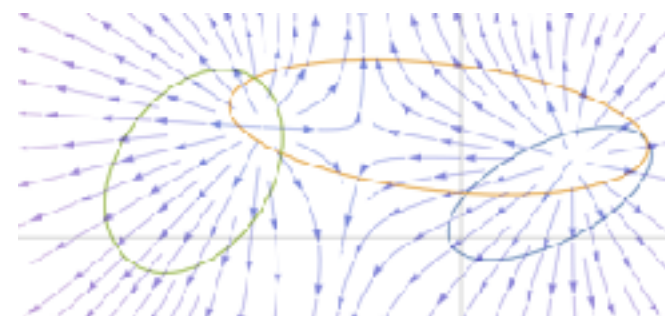
3. Steinmann relations

$$\text{disc}_s \text{disc}_t I(s, t) = 0$$

4. Cluster decomposition, Unitarity and Infrared Finiteness

$$\sum_{\alpha} \sum_{\beta} \langle \alpha | (\hat{S}_c)^n | \beta \rangle \langle \beta | (\hat{S}_c^\dagger)^m | \alpha \rangle =$$

5. How to compute discontinuities



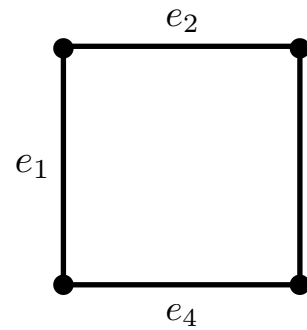
Backup slides!

The Cross-Free Family representation

G. F. Sterman, An Introduction to QFT
Borinsky, ZC, Salas-Bernadez, Laenen
arXiv: 2210.05532 (2022)
ZC, arXiv: [2211.09653](#) (2022)

The Cross-Free Family representation

Consider a box diagram

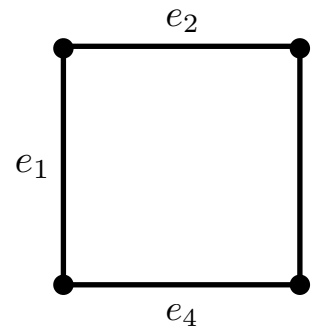


$$e_3 = i \int \frac{dk^0}{2\pi} \frac{N(\{k + p_i\}_{i=1}^4)}{\prod_{i=1}^4 ((k + p_i)^2 + m_i^2 + i\varepsilon)}$$

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$$= i \int \frac{d^4 k}{(2\pi)^4} \frac{N(\{k + p_i\}_{i=1}^4)}{\prod_{i=1}^4 ((k + p_i)^2 + m_i^2 + i\varepsilon)}$$

Introduce one Dirac delta for each edge (as opposed to each vertex, as in TOPT)

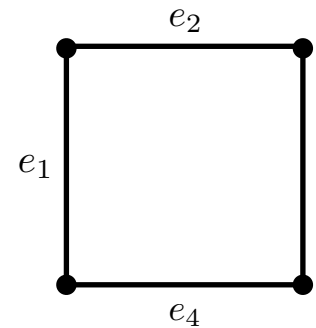
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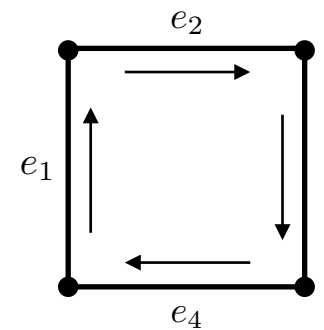
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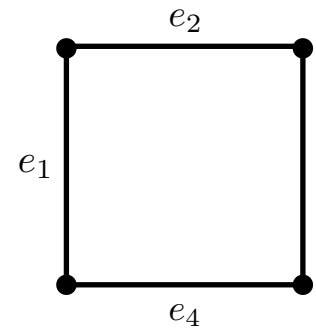
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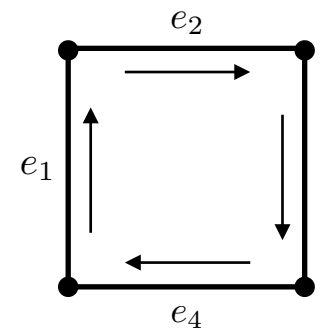
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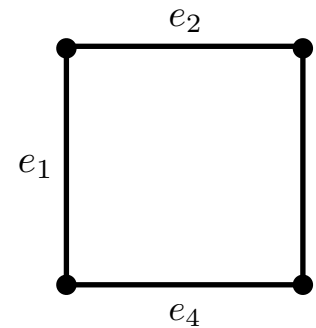
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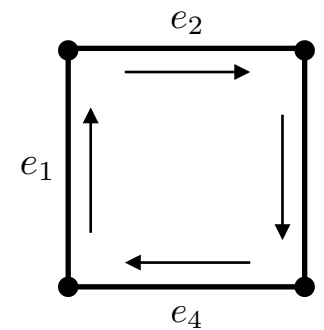
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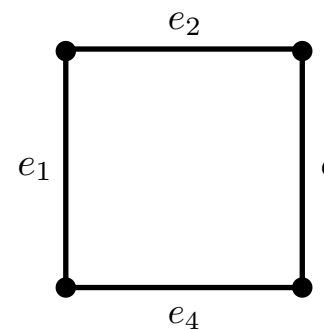
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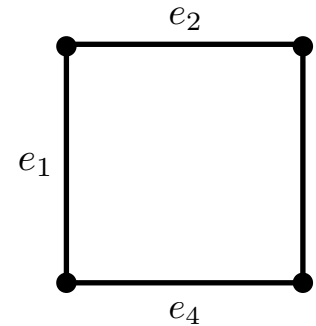


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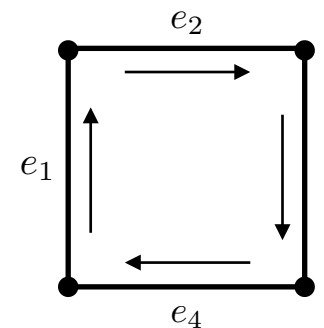
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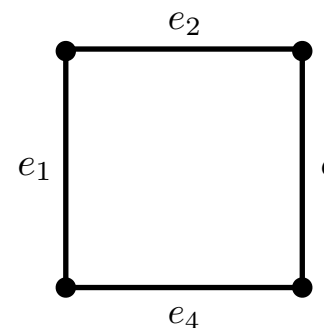
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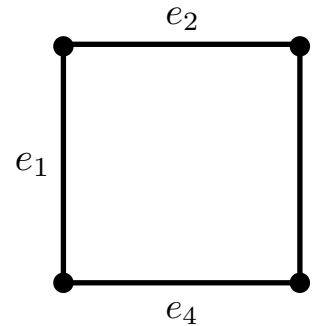


$$= i \int \frac{dk^0}{2\pi} \int \left[\prod_{j=1}^4 dq_j^0 d\tau_j e^{i\tau_j (q_j^0 - k^0 - p_j^0)} \right] \frac{N(\{q_j^0\}_{i=1}^4)}{\prod_{i=1}^4 ((q_i^0)^2 - E_i^2)}$$

Integrate using residue theorem: $\int dq^0 \frac{e^{i\tau q^0}}{(q^0)^2 - E^2} = \sum_{\sigma \in \{\pm 1\}} \frac{e^{i\tau \sigma E}}{2E} \Theta(\sigma \tau)$

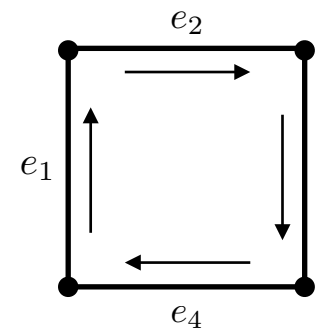
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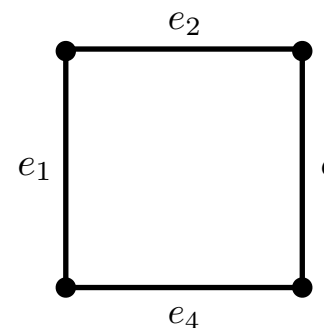
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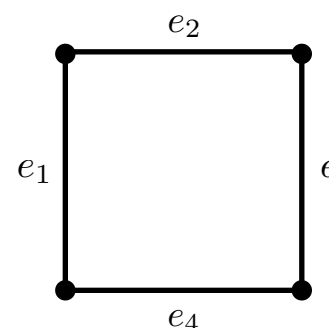
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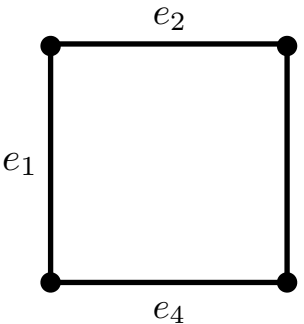
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$$= i \sum_{\vec{\sigma} \in \{\pm 1\}^4} N(\{\sigma_j E_j\}_{i=1}^4) \int \frac{dk^0}{2\pi} \int \left[\prod_{j=1}^4 \frac{d\tau_j}{2E_j} e^{i\tau_j (\sigma_j E_j^0 - k^0 - p_j^0)} \Theta(\sigma_j \tau_j) \right]$$

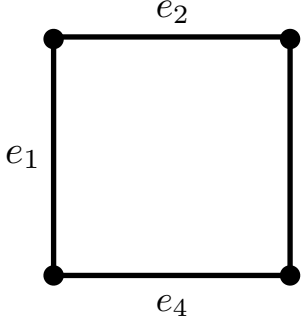
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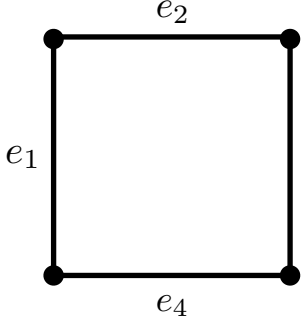
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 &= i \sum_{\vec{\sigma} \in \{\pm 1\}^4} N(\{\sigma_j E_j\}_{j=1}^4) \int \frac{dk^0}{2\pi} \int \left[\prod_{j=1}^4 \frac{d\tau_j}{2E_j} e^{i\tau_j (E_j^0 - \sigma_j (k^0 + p_j^0))} \Theta(\tau_j) \right] \\
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 \end{aligned}$$

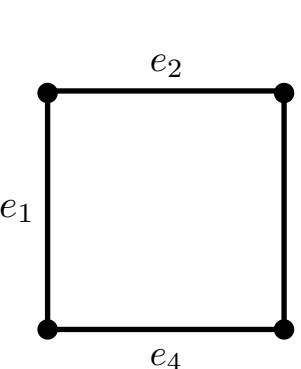
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 \end{aligned}$$

The signs switch the orientation of the edge! Interpret sum over signs as sum over orientations

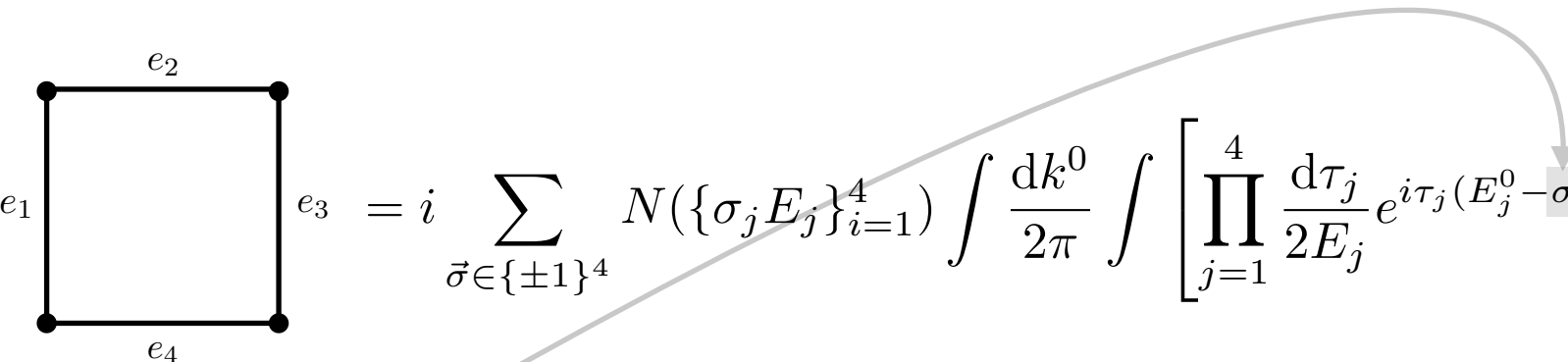
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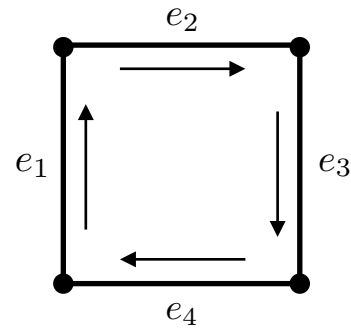
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$\vec{\sigma}$	$(1, 1, 1, 1)$	$(-1, 1, 1, 1)$	$(1, -1, 1, 1)$	$(1, 1, -1, 1)$	$(1, 1, 1, -1)$	$(-1, -1, 1, 1)$
			$\dots$		$\dots$	
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			...		...	
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To each orientation corresponds an integral, e.g.

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Integration domain:

$$\tau_j > 0, \quad j = 1, \dots, 4$$

$$\tau_1 + \tau_2 + \tau_3 + \tau_4 = 0$$

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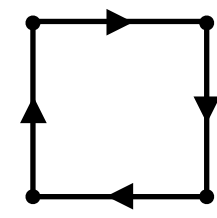
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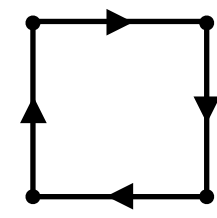
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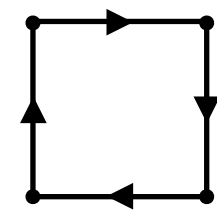
$$= \int \left[\prod_{j=1}^4 \frac{d\tau_j}{2E_j} e^{i\tau_j(E_j^0 - \sigma_j p_j^0)} \Theta(\tau_j) \right] \delta(\tau_1 + \tau_2 + \tau_3 + \tau_4)$$



$$= \int \left[\prod_{j=1}^4 \frac{d\tau_j}{2E_j} e^{i\tau_j(E_j^0 - \sigma_j p_j^0)} \Theta(\tau_j) \right] \delta(\tau_1 + \tau_2 + \tau_3 + \tau_4)$$

$$\tau_j > 0, \quad j = 1, \dots, 4$$

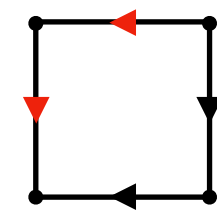
$$\tau_1 + \tau_2 + \tau_3 + \tau_4 = 0$$



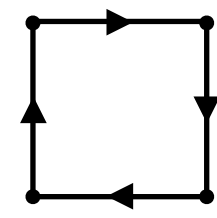
$$= \int \left[\prod_{j=1}^4 \frac{d\tau_j}{2E_j} e^{i\tau_j(E_j^0 - \sigma_j p_j^0)} \Theta(\tau_j) \right] \delta(\tau_1 + \tau_2 + \tau_3 + \tau_4)$$

$$\tau_j > 0, \quad j = 1, \dots, 4$$

$$\tau_1 + \tau_2 + \tau_3 + \tau_4 = 0$$



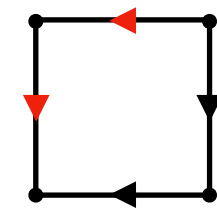
$$= \int \left[\prod_{j=1}^4 \frac{d\tau_j}{2E_j} e^{i\tau_j(E_j^0 - \sigma_j p_j^0)} \Theta(\tau_j) \right] \delta(-\tau_1 - \tau_2 + \tau_3 + \tau_4)$$



$$= \int \left[\prod_{j=1}^4 \frac{d\tau_j}{2E_j} e^{i\tau_j(E_j^0 - \sigma_j p_j^0)} \Theta(\tau_j) \right] \delta(\tau_1 + \tau_2 + \tau_3 + \tau_4)$$

$$\tau_j > 0, \quad j = 1, \dots, 4$$

$$\tau_1 + \tau_2 + \tau_3 + \tau_4 = 0$$



$$= \int \left[\prod_{j=1}^4 \frac{d\tau_j}{2E_j} e^{i\tau_j(E_j^0 - \sigma_j p_j^0)} \Theta(\tau_j) \right] \delta(-\tau_1 - \tau_2 + \tau_3 + \tau_4)$$

$$\tau_j > 0, \quad j = 1, \dots, 4$$

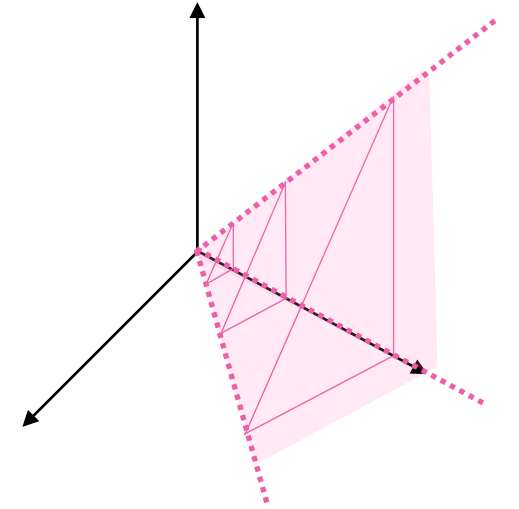
$$-\tau_1 - \tau_2 + \tau_3 + \tau_4 = 0$$

$$\begin{array}{c} \text{Diagram: Square with all edges black and arrows pointing clockwise.} \end{array} = \int \left[\prod_{j=1}^4 \frac{d\tau_j}{2E_j} e^{i\tau_j(E_j^0 - \sigma_j p_j^0)} \Theta(\tau_j) \right] \delta(\tau_1 + \tau_2 + \tau_3 + \tau_4)$$

$$\begin{array}{l} \tau_j > 0, \quad j = 1, \dots, 4 \\ \tau_1 + \tau_2 + \tau_3 + \tau_4 = 0 \end{array}$$

$$\begin{array}{c} \text{Diagram: Square with edges 1 and 2 red and arrows pointing counter-clockwise, edges 3 and 4 black and arrows pointing clockwise.} \end{array} = \int \left[\prod_{j=1}^4 \frac{d\tau_j}{2E_j} e^{i\tau_j(E_j^0 - \sigma_j p_j^0)} \Theta(\tau_j) \right] \delta(-\tau_1 - \tau_2 + \tau_3 + \tau_4)$$

$$\begin{array}{l} \tau_j > 0, \quad j = 1, \dots, 4 \\ -\tau_1 - \tau_2 + \tau_3 + \tau_4 = 0 \end{array}$$



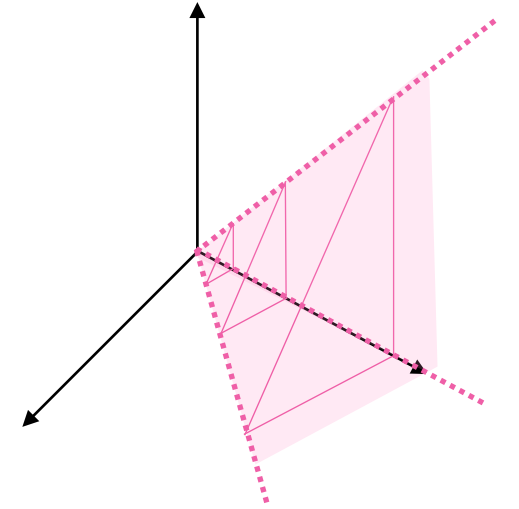
$$\begin{array}{c} \text{Diagram: Square with arrows on all four edges pointing clockwise.} \end{array} = \int \left[\prod_{j=1}^4 \frac{d\tau_j}{2E_j} e^{i\tau_j(E_j^0 - \sigma_j p_j^0)} \Theta(\tau_j) \right] \delta(\tau_1 + \tau_2 + \tau_3 + \tau_4)$$

$$\begin{aligned} \tau_j &> 0, \quad j = 1, \dots, 4 \\ \tau_1 + \tau_2 + \tau_3 + \tau_4 &= 0 \end{aligned}$$

$$\begin{array}{c} \text{Diagram: Square with arrows on edges 1, 3, and 4 pointing clockwise, and edge 2 pointing counter-clockwise.} \end{array} = \int \left[\prod_{j=1}^4 \frac{d\tau_j}{2E_j} e^{i\tau_j(E_j^0 - \sigma_j p_j^0)} \Theta(\tau_j) \right] \delta(-\tau_1 - \tau_2 + \tau_3 + \tau_4)$$

$$\begin{aligned} \tau_j &> 0, \quad j = 1, \dots, 4 \\ -\tau_1 - \tau_2 + \tau_3 + \tau_4 &= 0 \end{aligned}$$

Observation: first integration domain is empty, second is a convex cone



$$\begin{array}{c} \text{Square with all edges pointing clockwise} \end{array} = \int \left[\prod_{j=1}^4 \frac{d\tau_j}{2E_j} e^{i\tau_j(E_j^0 - \sigma_j p_j^0)} \Theta(\tau_j) \right] \delta(\tau_1 + \tau_2 + \tau_3 + \tau_4)$$

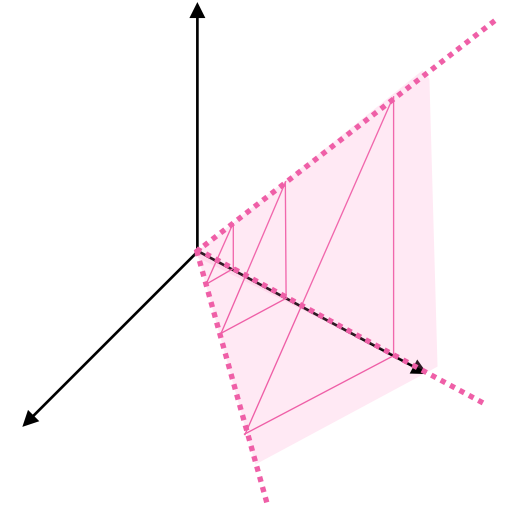
$$\begin{aligned} \tau_j &> 0, \quad j = 1, \dots, 4 \\ \tau_1 + \tau_2 + \tau_3 + \tau_4 &= 0 \end{aligned}$$

$$\begin{array}{c} \text{Square with edges 1 and 2 pointing counter-clockwise, edges 3 and 4 pointing clockwise} \end{array} = \int \left[\prod_{j=1}^4 \frac{d\tau_j}{2E_j} e^{i\tau_j(E_j^0 - \sigma_j p_j^0)} \Theta(\tau_j) \right] \delta(-\tau_1 - \tau_2 + \tau_3 + \tau_4)$$

$$\begin{aligned} \tau_j &> 0, \quad j = 1, \dots, 4 \\ -\tau_1 - \tau_2 + \tau_3 + \tau_4 &= 0 \end{aligned}$$

Observation: first integration domain is empty, second is a convex cone

$$\begin{array}{c} \text{Square with all edges pointing counter-clockwise} \end{array} = 0$$



$$\begin{array}{c} \text{Square with all edges pointing clockwise} \end{array} = \int \left[\prod_{j=1}^4 \frac{d\tau_j}{2E_j} e^{i\tau_j(E_j^0 - \sigma_j p_j^0)} \Theta(\tau_j) \right] \delta(\tau_1 + \tau_2 + \tau_3 + \tau_4)$$

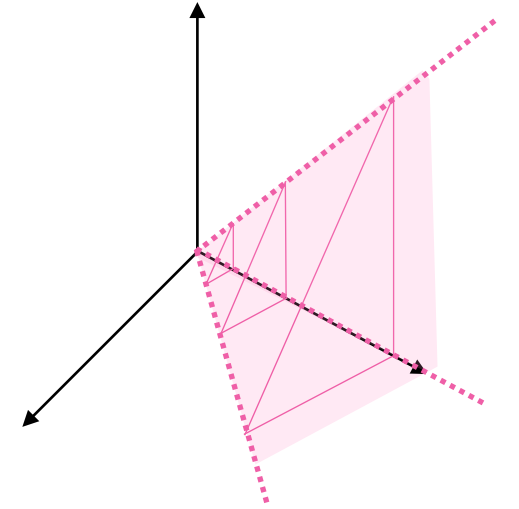
$$\begin{aligned} \tau_j &> 0, \quad j = 1, \dots, 4 \\ \tau_1 + \tau_2 + \tau_3 + \tau_4 &= 0 \end{aligned}$$

$$\begin{array}{c} \text{Square with edges 1 and 2 pointing counter-clockwise, edges 3 and 4 pointing clockwise} \end{array} = \int \left[\prod_{j=1}^4 \frac{d\tau_j}{2E_j} e^{i\tau_j(E_j^0 - \sigma_j p_j^0)} \Theta(\tau_j) \right] \delta(-\tau_1 - \tau_2 + \tau_3 + \tau_4)$$

$$\begin{aligned} \tau_j &> 0, \quad j = 1, \dots, 4 \\ -\tau_1 - \tau_2 + \tau_3 + \tau_4 &= 0 \end{aligned}$$

Observation: first integration domain is empty, second is a convex cone

$$\begin{array}{c} \text{Square with all edges pointing clockwise} \end{array} = 0 \quad \begin{array}{c} \text{Square with edges 1 and 2 pointing counter-clockwise, edges 3 and 4 pointing clockwise} \end{array} = 0$$



$$\begin{array}{c} \text{Square with all edges black and arrows pointing clockwise} \end{array} = \int \left[\prod_{j=1}^4 \frac{d\tau_j}{2E_j} e^{i\tau_j(E_j^0 - \sigma_j p_j^0)} \Theta(\tau_j) \right] \delta(\tau_1 + \tau_2 + \tau_3 + \tau_4)$$

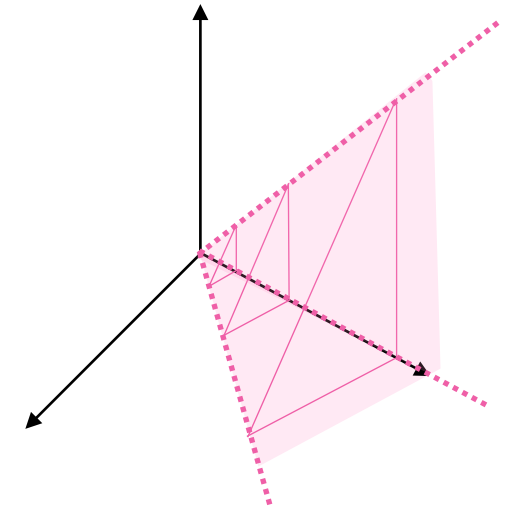
$$\begin{aligned} \tau_j &> 0, \quad j = 1, \dots, 4 \\ \tau_1 + \tau_2 + \tau_3 + \tau_4 &= 0 \end{aligned}$$

$$\begin{array}{c} \text{Square with top and bottom edges black (arrows clockwise), left and right edges red (arrows counter-clockwise)} \end{array} = \int \left[\prod_{j=1}^4 \frac{d\tau_j}{2E_j} e^{i\tau_j(E_j^0 - \sigma_j p_j^0)} \Theta(\tau_j) \right] \delta(-\tau_1 - \tau_2 + \tau_3 + \tau_4)$$

$$\begin{aligned} \tau_j &> 0, \quad j = 1, \dots, 4 \\ -\tau_1 - \tau_2 + \tau_3 + \tau_4 &= 0 \end{aligned}$$

Observation: first integration domain is empty, second is a convex cone

$$\begin{array}{c} \text{Square with all edges black and arrows pointing clockwise} \end{array} = 0 \quad \begin{array}{c} \text{Square with all edges red and arrows pointing counter-clockwise} \end{array} = 0 \quad \text{The graphs have a directed cycle!}$$



$$\begin{array}{c} \text{Square with all edges black and arrows pointing clockwise} \end{array} = \int \left[\prod_{j=1}^4 \frac{d\tau_j}{2E_j} e^{i\tau_j(E_j^0 - \sigma_j p_j^0)} \Theta(\tau_j) \right] \delta(\tau_1 + \tau_2 + \tau_3 + \tau_4)$$

$$\begin{aligned} \tau_j &> 0, \quad j = 1, \dots, 4 \\ \tau_1 + \tau_2 + \tau_3 + \tau_4 &= 0 \end{aligned}$$

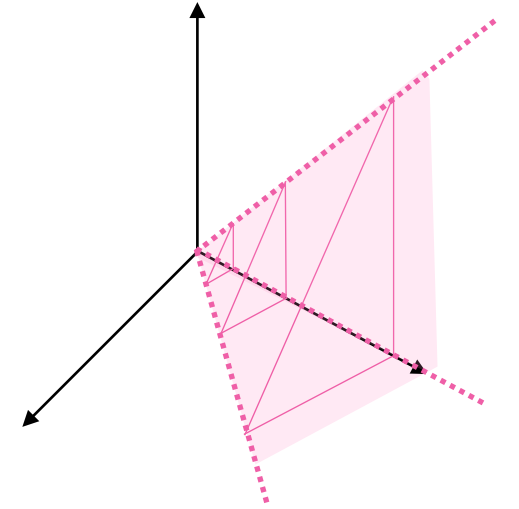
$$\begin{array}{c} \text{Square with top and bottom edges black (arrows clockwise), left and right edges red (arrows counter-clockwise)} \end{array} = \int \left[\prod_{j=1}^4 \frac{d\tau_j}{2E_j} e^{i\tau_j(E_j^0 - \sigma_j p_j^0)} \Theta(\tau_j) \right] \delta(-\tau_1 - \tau_2 + \tau_3 + \tau_4)$$

$$\begin{aligned} \tau_j &> 0, \quad j = 1, \dots, 4 \\ -\tau_1 - \tau_2 + \tau_3 + \tau_4 &= 0 \end{aligned}$$

Observation: first integration domain is empty, second is a convex cone

$$\begin{array}{c} \text{Square with all edges black and arrows pointing clockwise} \end{array} = 0 \quad \begin{array}{c} \text{Square with all edges red and arrows pointing counter-clockwise} \end{array} = 0 \quad \text{The graphs have a directed cycle!}$$

Only **acyclic** graphs give non zero contribution!



$$\begin{array}{c} \text{Square with all edges pointing clockwise} \end{array} = \int \left[\prod_{j=1}^4 \frac{d\tau_j}{2E_j} e^{i\tau_j(E_j^0 - \sigma_j p_j^0)} \Theta(\tau_j) \right] \delta(\tau_1 + \tau_2 + \tau_3 + \tau_4)$$

$$\begin{aligned} \tau_j &> 0, \quad j = 1, \dots, 4 \\ \tau_1 + \tau_2 + \tau_3 + \tau_4 &= 0 \end{aligned}$$

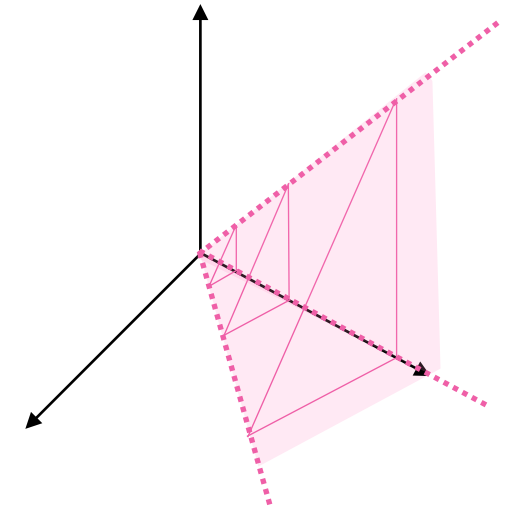
$$\begin{array}{c} \text{Square with edges 1 and 2 pointing counter-clockwise, edges 3 and 4 pointing clockwise} \end{array} = \int \left[\prod_{j=1}^4 \frac{d\tau_j}{2E_j} e^{i\tau_j(E_j^0 - \sigma_j p_j^0)} \Theta(\tau_j) \right] \delta(-\tau_1 - \tau_2 + \tau_3 + \tau_4)$$

$$\begin{aligned} \tau_j &> 0, \quad j = 1, \dots, 4 \\ -\tau_1 - \tau_2 + \tau_3 + \tau_4 &= 0 \end{aligned}$$

Observation: first integration domain is empty, second is a convex cone

$$\begin{array}{c} \text{Square with all edges pointing clockwise} \end{array} = 0 \quad \begin{array}{c} \text{Square with edges 1 and 2 pointing counter-clockwise, edges 3 and 4 pointing clockwise} \end{array} = 0 \quad \text{The graphs have a directed cycle!}$$

Only **acyclic** graphs give non zero contribution!



Another example:

$$\begin{array}{c} \text{Square with all edges clockwise} \end{array} = \int \left[\prod_{j=1}^4 \frac{d\tau_j}{2E_j} e^{i\tau_j(E_j^0 - \sigma_j p_j^0)} \Theta(\tau_j) \right] \delta(\tau_1 + \tau_2 + \tau_3 + \tau_4)$$

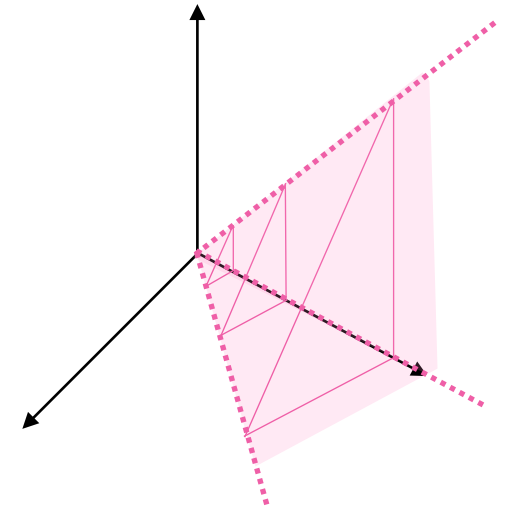
$$\begin{aligned} \tau_j &> 0, \quad j = 1, \dots, 4 \\ \tau_1 + \tau_2 + \tau_3 + \tau_4 &= 0 \end{aligned}$$

$$\begin{array}{c} \text{Square with edges 1, 2, 4 clockwise and edge 3 counter-clockwise} \end{array} = \int \left[\prod_{j=1}^4 \frac{d\tau_j}{2E_j} e^{i\tau_j(E_j^0 - \sigma_j p_j^0)} \Theta(\tau_j) \right] \delta(-\tau_1 - \tau_2 + \tau_3 + \tau_4)$$

$$\begin{aligned} \tau_j &> 0, \quad j = 1, \dots, 4 \\ -\tau_1 - \tau_2 + \tau_3 + \tau_4 &= 0 \end{aligned}$$

Observation: first integration domain is empty, second is a convex cone

$$\begin{array}{c} \text{Square with all edges clockwise} \end{array} = 0 \quad \begin{array}{c} \text{Square with edges 1, 2, 4 clockwise and edge 3 counter-clockwise} \end{array} = 0 \quad \text{The graphs have a directed cycle!}$$



Only **acyclic** graphs give non zero contribution!

Another example:

$$\begin{array}{c} \text{Circle with three internal lines meeting at a central vertex} \end{array} = \begin{array}{c} \text{Circle with clockwise cycle and internal lines} \end{array} + \begin{array}{c} \text{Circle with counter-clockwise cycle and internal lines} \end{array} + \begin{array}{c} \text{Circle with clockwise cycle and internal lines} \end{array}$$

$$\begin{array}{c} \text{Square with all edges clockwise} \end{array} = \int \left[\prod_{j=1}^4 \frac{d\tau_j}{2E_j} e^{i\tau_j(E_j^0 - \sigma_j p_j^0)} \Theta(\tau_j) \right] \delta(\tau_1 + \tau_2 + \tau_3 + \tau_4)$$

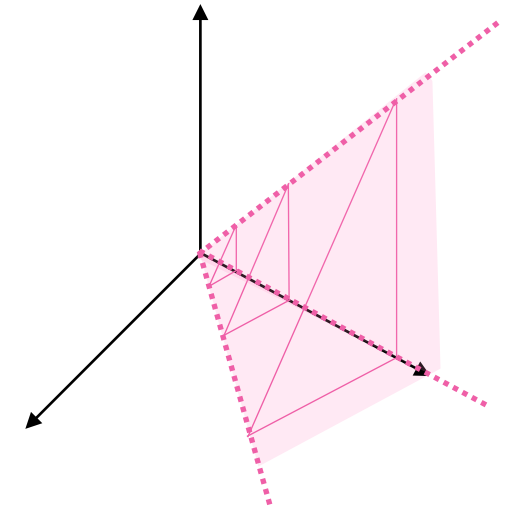
$$\begin{aligned} \tau_j &> 0, \quad j = 1, \dots, 4 \\ \tau_1 + \tau_2 + \tau_3 + \tau_4 &= 0 \end{aligned}$$

$$\begin{array}{c} \text{Square with edges 1, 2, 4 clockwise and edge 3 counter-clockwise} \end{array} = \int \left[\prod_{j=1}^4 \frac{d\tau_j}{2E_j} e^{i\tau_j(E_j^0 - \sigma_j p_j^0)} \Theta(\tau_j) \right] \delta(-\tau_1 - \tau_2 + \tau_3 + \tau_4)$$

$$\begin{aligned} \tau_j &> 0, \quad j = 1, \dots, 4 \\ -\tau_1 - \tau_2 + \tau_3 + \tau_4 &= 0 \end{aligned}$$

Observation: first integration domain is empty, second is a convex cone

$$\begin{array}{c} \text{Square with all edges clockwise} \end{array} = 0 \quad \begin{array}{c} \text{Square with edges 1, 2, 4 clockwise and edge 3 counter-clockwise} \end{array} = 0 \quad \text{The graphs have a directed cycle!}$$



Only **acyclic** graphs give non zero contribution!

Another example:

$$\begin{array}{c} \text{Circle with three internal lines meeting at a central vertex} \end{array} = \begin{array}{c} \text{Circle with clockwise cycle} \end{array} + \begin{array}{c} \text{Circle with counter-clockwise cycle} \end{array} + \begin{array}{c} \text{Circle with clockwise cycle} \end{array} + \begin{array}{c} \text{Circle with counter-clockwise cycle} \end{array}$$

$$\begin{array}{c} \text{Clockwise square} \end{array} = \int \left[\prod_{j=1}^4 \frac{d\tau_j}{2E_j} e^{i\tau_j(E_j^0 - \sigma_j p_j^0)} \Theta(\tau_j) \right] \delta(\tau_1 + \tau_2 + \tau_3 + \tau_4)$$

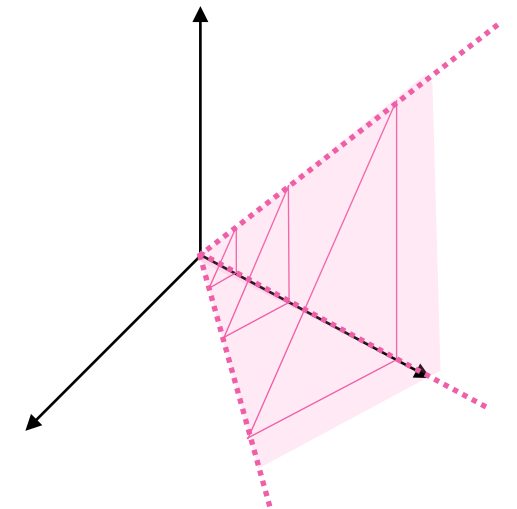
$$\begin{aligned} \tau_j &> 0, \quad j = 1, \dots, 4 \\ \tau_1 + \tau_2 + \tau_3 + \tau_4 &= 0 \end{aligned}$$

$$\begin{array}{c} \text{Counter-clockwise square} \end{array} = \int \left[\prod_{j=1}^4 \frac{d\tau_j}{2E_j} e^{i\tau_j(E_j^0 - \sigma_j p_j^0)} \Theta(\tau_j) \right] \delta(-\tau_1 - \tau_2 + \tau_3 + \tau_4)$$

$$\begin{aligned} \tau_j &> 0, \quad j = 1, \dots, 4 \\ -\tau_1 - \tau_2 + \tau_3 + \tau_4 &= 0 \end{aligned}$$

Observation: first integration domain is empty, second is a convex cone

$$\begin{array}{c} \text{Clockwise square} \end{array} = 0 \quad \begin{array}{c} \text{Counter-clockwise square} \end{array} = 0 \quad \text{The graphs have a directed cycle!}$$



Only **acyclic** graphs give non zero contribution!

Another example:

$$\begin{array}{c} \text{Triangle} \end{array} = \begin{array}{c} \text{Triangle with clockwise cycle} \end{array} + \begin{array}{c} \text{Triangle with counter-clockwise cycle} \end{array} + \begin{array}{c} \text{Triangle with clockwise cycle} \end{array} + \begin{array}{c} \text{Triangle with counter-clockwise cycle} \end{array} + \dots$$

$$\begin{array}{c} \text{Square with all edges clockwise} \end{array} = \int \left[\prod_{j=1}^4 \frac{d\tau_j}{2E_j} e^{i\tau_j(E_j^0 - \sigma_j p_j^0)} \Theta(\tau_j) \right] \delta(\tau_1 + \tau_2 + \tau_3 + \tau_4)$$

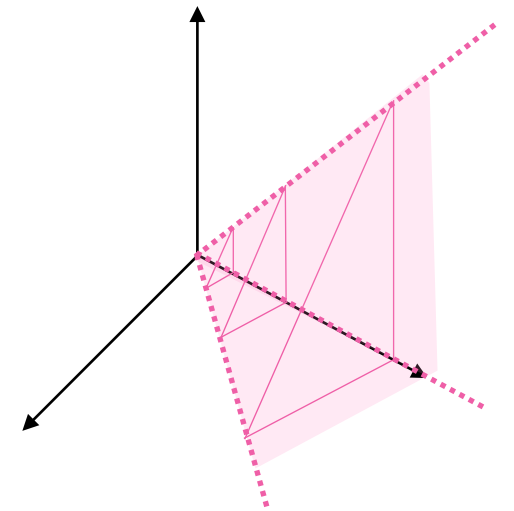
$$\begin{aligned} \tau_j &> 0, \quad j = 1, \dots, 4 \\ \tau_1 + \tau_2 + \tau_3 + \tau_4 &= 0 \end{aligned}$$

$$\begin{array}{c} \text{Square with top and bottom edges clockwise, left and right edges counter-clockwise} \end{array} = \int \left[\prod_{j=1}^4 \frac{d\tau_j}{2E_j} e^{i\tau_j(E_j^0 - \sigma_j p_j^0)} \Theta(\tau_j) \right] \delta(-\tau_1 - \tau_2 + \tau_3 + \tau_4)$$

$$\begin{aligned} \tau_j &> 0, \quad j = 1, \dots, 4 \\ -\tau_1 - \tau_2 + \tau_3 + \tau_4 &= 0 \end{aligned}$$

Observation: first integration domain is empty, second is a convex cone

$$\begin{array}{c} \text{Square with all edges clockwise} \end{array} = 0 \quad \begin{array}{c} \text{Square with top and bottom edges clockwise, left and right edges counter-clockwise} \end{array} = 0 \quad \text{The graphs have a directed cycle!}$$



Only **acyclic** graphs give non zero contribution!

Another example:

$$\begin{array}{c} \text{Circle with three internal lines meeting at a central vertex} \end{array} = \begin{array}{c} \text{Diagram 1} \end{array} + \begin{array}{c} \text{Diagram 2} \end{array} + \begin{array}{c} \text{Diagram 3} \end{array} + \begin{array}{c} \text{Diagram 4} \end{array} + \begin{array}{c} \text{Diagram 5} \end{array} + \dots$$

Diagrams 4 and 5 are marked with $\equiv 0$ above them, indicating they are zero due to the presence of a directed cycle.

$$\begin{array}{c} \text{Square with all edges clockwise} \end{array} = \int \left[\prod_{j=1}^4 \frac{d\tau_j}{2E_j} e^{i\tau_j(E_j^0 - \sigma_j p_j^0)} \Theta(\tau_j) \right] \delta(\tau_1 + \tau_2 + \tau_3 + \tau_4)$$

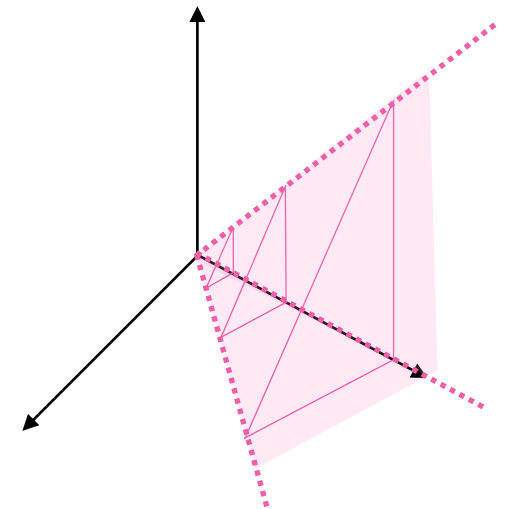
$$\begin{aligned} \tau_j &> 0, \quad j = 1, \dots, 4 \\ \tau_1 + \tau_2 + \tau_3 + \tau_4 &= 0 \end{aligned}$$

$$\begin{array}{c} \text{Square with top and bottom edges clockwise, left and right edges counter-clockwise} \end{array} = \int \left[\prod_{j=1}^4 \frac{d\tau_j}{2E_j} e^{i\tau_j(E_j^0 - \sigma_j p_j^0)} \Theta(\tau_j) \right] \delta(-\tau_1 - \tau_2 + \tau_3 + \tau_4)$$

$$\begin{aligned} \tau_j &> 0, \quad j = 1, \dots, 4 \\ -\tau_1 - \tau_2 + \tau_3 + \tau_4 &= 0 \end{aligned}$$

Observation: first integration domain is empty, second is a convex cone

$$\begin{array}{c} \text{Square with all edges clockwise} \end{array} = 0 \quad \begin{array}{c} \text{Square with top and bottom edges clockwise, left and right edges counter-clockwise} \end{array} = 0 \quad \text{The graphs have a directed cycle!}$$



Only **acyclic** graphs give non zero contribution!

Another example:

$$\begin{array}{c} \text{Circle with three internal lines meeting at a central vertex} \end{array} = \begin{array}{c} \text{Diagram 1} \end{array} + \begin{array}{c} \text{Diagram 2} \end{array} + \begin{array}{c} \text{Diagram 3} \end{array} + \begin{array}{c} \text{Diagram 4} \end{array} + \begin{array}{c} \text{Diagram 5} \end{array} + \dots$$

Diagrams 4 and 5 are marked with $\equiv 0$ above them, indicating they are zero due to the presence of a directed cycle.

$$\begin{array}{c} \text{Clockwise square} \\ \hline \end{array} = \int \left[\prod_{j=1}^4 \frac{d\tau_j}{2E_j} e^{i\tau_j(E_j^0 - \sigma_j p_j^0)} \Theta(\tau_j) \right] \delta(\tau_1 + \tau_2 + \tau_3 + \tau_4)$$

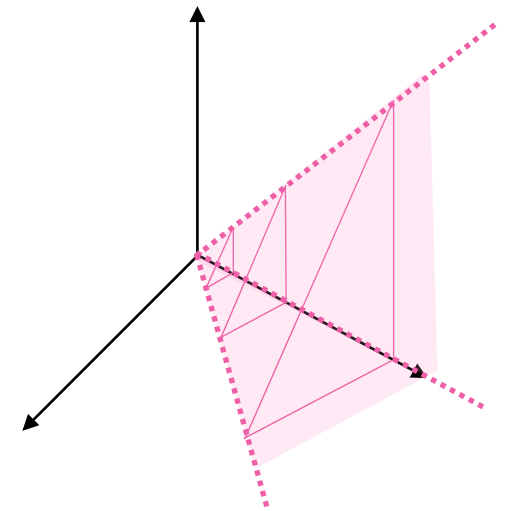
$$\begin{aligned} \tau_j &> 0, \quad j = 1, \dots, 4 \\ \tau_1 + \tau_2 + \tau_3 + \tau_4 &= 0 \end{aligned}$$

$$\begin{array}{c} \text{Counter-clockwise square} \\ \hline \end{array} = \int \left[\prod_{j=1}^4 \frac{d\tau_j}{2E_j} e^{i\tau_j(E_j^0 - \sigma_j p_j^0)} \Theta(\tau_j) \right] \delta(-\tau_1 - \tau_2 + \tau_3 + \tau_4)$$

$$\begin{aligned} \tau_j &> 0, \quad j = 1, \dots, 4 \\ -\tau_1 - \tau_2 + \tau_3 + \tau_4 &= 0 \end{aligned}$$

Observation: first integration domain is empty, second is a convex cone

$$\begin{array}{c} \text{Clockwise square} \\ \hline \end{array} = 0 \quad \begin{array}{c} \text{Counter-clockwise square} \\ \hline \end{array} = 0 \quad \text{The graphs have a directed cycle!}$$



Only **acyclic** graphs give non zero contribution!

Another example:

$$\begin{array}{c} \text{Circle with three spokes} \\ \hline \end{array} = \begin{array}{c} \text{Acyclic graph 1} \\ \hline \end{array} + \begin{array}{c} \text{Acyclic graph 2} \\ \hline \end{array} + \begin{array}{c} \text{Acyclic graph 3} \\ \hline \end{array} + \begin{array}{c} \text{Cyclic graph 1} \\ \hline \end{array} \stackrel{=0}{=} + \begin{array}{c} \text{Cyclic graph 2} \\ \hline \end{array} \stackrel{=0}{=} + \dots$$

Sum one integral for each acyclic graph

$$\begin{array}{c} \text{Clockwise square} \end{array} = \int \left[\prod_{j=1}^4 \frac{d\tau_j}{2E_j} e^{i\tau_j(E_j^0 - \sigma_j p_j^0)} \Theta(\tau_j) \right] \delta(\tau_1 + \tau_2 + \tau_3 + \tau_4)$$

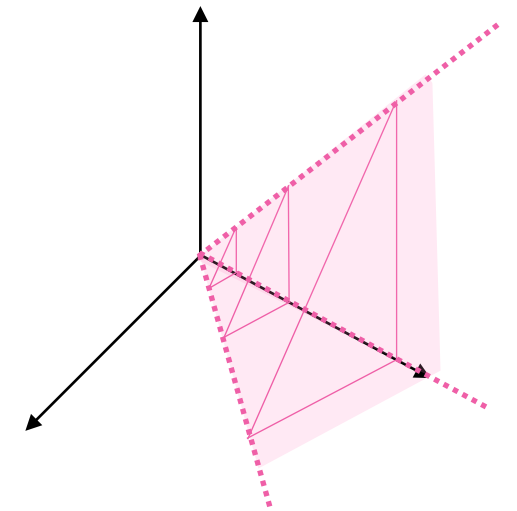
$$\begin{aligned} \tau_j &> 0, \quad j = 1, \dots, 4 \\ \tau_1 + \tau_2 + \tau_3 + \tau_4 &= 0 \end{aligned}$$

$$\begin{array}{c} \text{Counter-clockwise square} \end{array} = \int \left[\prod_{j=1}^4 \frac{d\tau_j}{2E_j} e^{i\tau_j(E_j^0 - \sigma_j p_j^0)} \Theta(\tau_j) \right] \delta(-\tau_1 - \tau_2 + \tau_3 + \tau_4)$$

$$\begin{aligned} \tau_j &> 0, \quad j = 1, \dots, 4 \\ -\tau_1 - \tau_2 + \tau_3 + \tau_4 &= 0 \end{aligned}$$

Observation: first integration domain is empty, second is a convex cone

$$\begin{array}{c} \text{Clockwise square} \end{array} = 0 \quad \begin{array}{c} \text{Counter-clockwise square} \end{array} = 0 \quad \text{The graphs have a directed cycle!}$$



Only **acyclic** graphs give non zero contribution!

Another example:

$$\begin{array}{c} \text{Triangle} \end{array} = \begin{array}{c} \text{Acyclic triangle 1} \end{array} + \begin{array}{c} \text{Acyclic triangle 2} \end{array} + \begin{array}{c} \text{Acyclic triangle 3} \end{array} + \begin{array}{c} \text{Cyclic triangle 1} \end{array} \overset{=0}{=} + \begin{array}{c} \text{Cyclic triangle 2} \end{array} \overset{=0}{=} + \dots$$

Sum one integral for each acyclic graph

$$\begin{array}{c} \text{Triangle} \end{array} = \sum_{\text{acyclic } G} N_G \int \left[\prod_{e \in E} \frac{d\tau_e}{2E_e} e^{i\tau_e(E_e - (p_e^G)^0)} \Theta(\tau_e) \right] \prod_{i=1}^L \delta \left(\sum_{e \in c_i} \tau_e s_{ei} \right)$$

$$\begin{array}{c} \text{Clockwise square} \\ \text{Counter-clockwise square} \end{array} = \int \left[\prod_{j=1}^4 \frac{d\tau_j}{2E_j} e^{i\tau_j(E_j^0 - \sigma_j p_j^0)} \Theta(\tau_j) \right] \delta(\tau_1 + \tau_2 + \tau_3 + \tau_4)$$

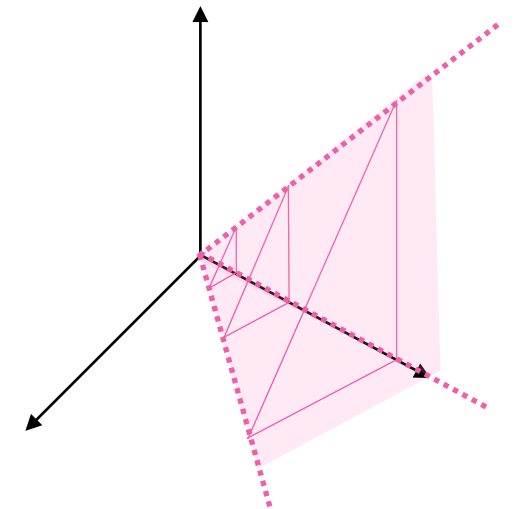
$$\begin{aligned} \tau_j &> 0, \quad j = 1, \dots, 4 \\ \tau_1 + \tau_2 + \tau_3 + \tau_4 &= 0 \end{aligned}$$

$$\begin{array}{c} \text{Clockwise square} \\ \text{Counter-clockwise square} \end{array} = \int \left[\prod_{j=1}^4 \frac{d\tau_j}{2E_j} e^{i\tau_j(E_j^0 - \sigma_j p_j^0)} \Theta(\tau_j) \right] \delta(-\tau_1 - \tau_2 + \tau_3 + \tau_4)$$

$$\begin{aligned} \tau_j &> 0, \quad j = 1, \dots, 4 \\ -\tau_1 - \tau_2 + \tau_3 + \tau_4 &= 0 \end{aligned}$$

Observation: first integration domain is empty, second is a convex cone

$$\begin{array}{c} \text{Clockwise square} \\ \text{Counter-clockwise square} \end{array} = 0 \quad \begin{array}{c} \text{Clockwise square} \\ \text{Counter-clockwise square} \end{array} = 0 \quad \text{The graphs have a directed cycle!}$$



Only **acyclic** graphs give non zero contribution!

Another example:

$$\text{Triangle} = \text{Acyclic 1} + \text{Acyclic 2} + \text{Acyclic 3} + \text{Cyclic 1} + \text{Cyclic 2} + \dots$$

Sum one integral for each acyclic graph

$$\text{Triangle} = \sum_{\text{acyclic } G} N_G \int \left[\prod_{e \in E} \frac{d\tau_e}{2E_e} e^{i\tau_e(E_e - (p_e^G)^0)} \Theta(\tau_e) \right] \prod_{i=1}^L \delta \left(\sum_{e \in c_i} \tau_e s_{ei} \right)$$

One time integration per edge

$$\begin{array}{c} \text{Clockwise square} \end{array} = \int \left[\prod_{j=1}^4 \frac{d\tau_j}{2E_j} e^{i\tau_j(E_j^0 - \sigma_j p_j^0)} \Theta(\tau_j) \right] \delta(\tau_1 + \tau_2 + \tau_3 + \tau_4)$$

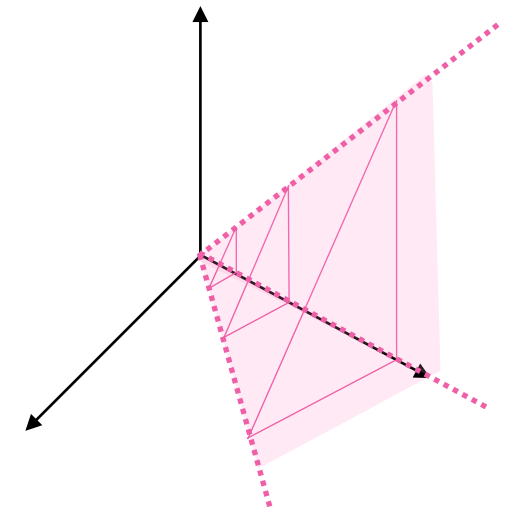
$$\begin{aligned} \tau_j &> 0, \quad j = 1, \dots, 4 \\ \tau_1 + \tau_2 + \tau_3 + \tau_4 &= 0 \end{aligned}$$

$$\begin{array}{c} \text{Counter-clockwise square} \end{array} = \int \left[\prod_{j=1}^4 \frac{d\tau_j}{2E_j} e^{i\tau_j(E_j^0 - \sigma_j p_j^0)} \Theta(\tau_j) \right] \delta(-\tau_1 - \tau_2 + \tau_3 + \tau_4)$$

$$\begin{aligned} \tau_j &> 0, \quad j = 1, \dots, 4 \\ -\tau_1 - \tau_2 + \tau_3 + \tau_4 &= 0 \end{aligned}$$

Observation: first integration domain is empty, second is a convex cone

$$\begin{array}{c} \text{Clockwise square} \end{array} = 0 \quad \begin{array}{c} \text{Counter-clockwise square} \end{array} = 0 \quad \text{The graphs have a directed cycle!}$$



Only **acyclic** graphs give non zero contribution!

Another example:

$$\text{Y-junction graph} = \text{Acyclic graph 1} + \text{Acyclic graph 2} + \text{Acyclic graph 3} + \text{Cyclic graph 1} + \text{Cyclic graph 2} + \dots$$

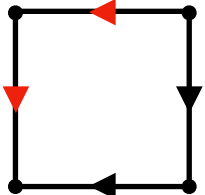
(The cyclic graphs are marked with $= 0$)

Sum one integral for each acyclic graph

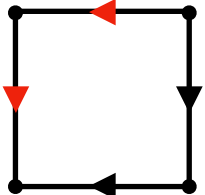
$$\text{Y-junction graph} = \sum_{\text{acyclic } G} N_G \int \left[\prod_{e \in E} \frac{d\tau_e}{2E_e} e^{i\tau_e(E_e - (p_e^G)^0)} \Theta(\tau_e) \right] \prod_{i=1}^L \delta \left(\sum_{e \in c_i} \tau_e s_{ei} \right)$$

Times along cycles must sum to zero

One time integration per edge



$$= \int \left[\prod_{j=1}^4 \frac{d\tau_j}{2E_j} e^{i\tau_j(E_j^0 - \sigma_j p_j^0)} \Theta(\tau_j) \right] \delta(-\tau_1 - \tau_2 + \tau_3 + \tau_4)$$

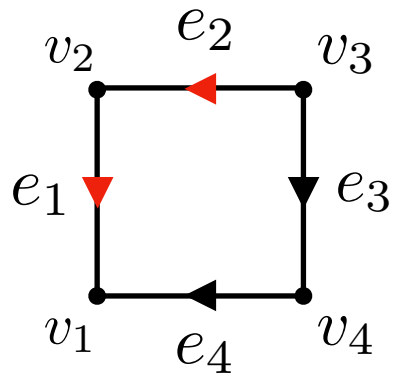


$$= \int \left[\prod_{j=1}^4 \frac{d\tau_j}{2E_j} e^{i\tau_j(E_j^0 - \sigma_j p_j^0)} \Theta(\tau_j) \right] \delta(-\tau_1 - \tau_2 + \tau_3 + \tau_4)$$

How do we perform the remaining integrations (**one for each edge**)? Edge-contraction

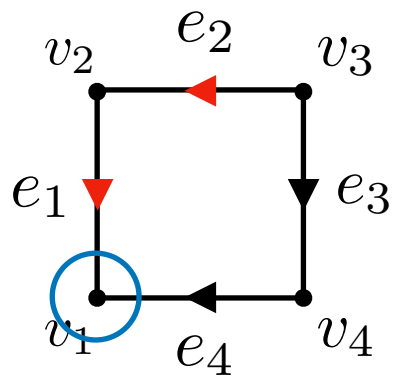
$$\begin{array}{|c|} \hline \text{Diagram: A square loop with four vertices. The top edge has a red arrow pointing right. The left edge has a red arrow pointing down. The bottom edge has a black arrow pointing left. The right edge has a black arrow pointing down.} \\ \hline \end{array} = \int \left[\prod_{j=1}^4 \frac{d\tau_j}{2E_j} e^{i\tau_j(E_j^0 - \sigma_j p_j^0)} \Theta(\tau_j) \right] \delta(-\tau_1 - \tau_2 + \tau_3 + \tau_4)$$

How do we perform the remaining integrations (**one for each edge**)? Edge-contraction



$$\begin{array}{c} \bullet \\ \downarrow \\ \bullet \end{array} \begin{array}{c} \bullet \\ \leftarrow \\ \bullet \end{array} \begin{array}{c} \bullet \\ \uparrow \\ \bullet \end{array} = \int \left[\prod_{j=1}^4 \frac{d\tau_j}{2E_j} e^{i\tau_j(E_j^0 - \sigma_j p_j^0)} \Theta(\tau_j) \right] \delta(-\tau_1 - \tau_2 + \tau_3 + \tau_4)$$

How do we perform the remaining integrations (**one for each edge**)? Edge-contraction



1. Choose sink/source with connected complement

$$\begin{array}{c} \bullet \\ \uparrow \\ \bullet \end{array} \begin{array}{c} \bullet \\ \uparrow \\ \bullet \end{array} = \int \left[\prod_{j=1}^4 \frac{d\tau_j}{2E_j} e^{i\tau_j(E_j^0 - \sigma_j p_j^0)} \Theta(\tau_j) \right] \delta(-\tau_1 - \tau_2 + \tau_3 + \tau_4)$$

How do we perform the remaining integrations (**one for each edge**)? Edge-contraction

$$\begin{array}{c} v_2 \\ \uparrow e_1 \\ v_1 \end{array} \begin{array}{c} v_3 \\ \uparrow e_2 \\ v_4 \end{array} = \frac{i}{E_1 + E_4 - p_1^0} \left[\begin{array}{c} v_{12} \\ \uparrow e_2 \\ v_3 \end{array} \begin{array}{c} v_4 \\ \uparrow e_3 \\ v_{14} \end{array} + \begin{array}{c} v_2 \\ \uparrow e_1 \\ v_{14} \end{array} \begin{array}{c} v_3 \\ \uparrow e_3 \\ v_{14} \end{array} \right]$$

1. Choose sink/source with connected complement
2. Contract one-by-one adjacent edges
3. Multiply by inverse sum of energies of adjacent edges

$$\begin{array}{|c|} \hline \text{Diagram: A square loop with vertices at the corners. The top edge has a red arrow pointing right. The bottom edge has a black arrow pointing left. The left edge has a red arrow pointing down. The right edge has a black arrow pointing down.} \\ \hline \end{array} = \int \left[\prod_{j=1}^4 \frac{d\tau_j}{2E_j} e^{i\tau_j(E_j^0 - \sigma_j p_j^0)} \Theta(\tau_j) \right] \delta(-\tau_1 - \tau_2 + \tau_3 + \tau_4)$$

How do we perform the remaining integrations (**one for each edge**)? Edge-contraction

$$\begin{array}{|c|} \hline \text{Diagram: A square loop with vertices } v_1, v_2, v_3, v_4. \text{ Edges are } e_1 (v_1 \rightarrow v_2, \text{ red}), e_2 (v_2 \rightarrow v_3, \text{ red}), e_3 (v_3 \rightarrow v_4, \text{ black}), e_4 (v_4 \rightarrow v_1, \text{ black}). \text{ Vertex } v_1 \text{ is circled in blue.} \\ \hline \end{array} = \frac{i}{E_1 + E_4 - p_1^0} \left[\begin{array}{|c|} \hline \text{Diagram 1: A triangle with vertices } v_{12}, v_3, v_4. \text{ Edges are } e_2 (v_{12} \rightarrow v_3, \text{ red}), e_3 (v_3 \rightarrow v_4, \text{ black}), e_4 (v_4 \rightarrow v_{12}, \text{ black}). \text{ Vertex } v_3 \text{ is circled in blue.} \\ \hline \end{array} + \begin{array}{|c|} \hline \text{Diagram 2: A triangle with vertices } v_2, v_3, v_{14}. \text{ Edges are } e_2 (v_2 \rightarrow v_3, \text{ red}), e_1 (v_2 \rightarrow v_{14}, \text{ red}), e_3 (v_3 \rightarrow v_{14}, \text{ black}). \\ \hline \end{array} \right]$$

1. Choose sink/source with connected complement
2. Contract one-by-one adjacent edges
3. Multiply by inverse sum of energies of adjacent edges

$$\begin{array}{c} \bullet \\ \uparrow \\ \bullet \end{array} \begin{array}{c} \bullet \\ \uparrow \\ \bullet \end{array} = \int \left[\prod_{j=1}^4 \frac{d\tau_j}{2E_j} e^{i\tau_j(E_j^0 - \sigma_j p_j^0)} \Theta(\tau_j) \right] \delta(-\tau_1 - \tau_2 + \tau_3 + \tau_4)$$

How do we perform the remaining integrations (**one for each edge**)? Edge-contraction

$$\begin{array}{c} v_2 \\ \uparrow e_2 \\ v_3 \\ \uparrow e_3 \\ v_4 \\ \uparrow e_4 \\ v_1 \end{array} = \frac{i}{E_1 + E_4 - p_1^0} \left[\begin{array}{c} v_{12} \\ \uparrow e_2 \\ v_3 \\ \uparrow e_3 \\ v_4 \end{array} + \begin{array}{c} v_2 \\ \uparrow e_2 \\ v_3 \\ \uparrow e_3 \\ v_{14} \end{array} \right]$$

1. Choose sink/source with connected complement
2. Contract one-by-one adjacent edges
3. Multiply by inverse sum of energies of adjacent edges

$$\begin{array}{c} \bullet \\ \uparrow \\ \bullet \end{array} \begin{array}{c} \bullet \\ \uparrow \\ \bullet \end{array} = \int \left[\prod_{j=1}^4 \frac{d\tau_j}{2E_j} e^{i\tau_j(E_j^0 - \sigma_j p_j^0)} \Theta(\tau_j) \right] \delta(-\tau_1 - \tau_2 + \tau_3 + \tau_4)$$

How do we perform the remaining integrations (**one for each edge**)? Edge-contraction

$$\begin{array}{c} v_2 \\ \uparrow e_1 \\ v_1 \end{array} \begin{array}{c} v_3 \\ \uparrow e_2 \\ v_2 \end{array} \begin{array}{c} v_4 \\ \uparrow e_3 \\ v_3 \end{array} \begin{array}{c} v_4 \\ \uparrow e_4 \\ v_1 \end{array} = \frac{i}{E_1 + E_4 - p_1^0} \left[\begin{array}{c} v_{12} \\ \uparrow e_2 \\ v_{14} \end{array} \begin{array}{c} v_3 \\ \uparrow e_3 \\ v_4 \end{array} + \begin{array}{c} v_2 \\ \uparrow e_1 \\ v_{14} \end{array} \begin{array}{c} v_3 \\ \uparrow e_3 \\ v_4 \end{array} \right]$$

1. Choose sink/source with connected complement
2. Contract one-by-one adjacent edges
3. Multiply by inverse sum of energies of adjacent edges

$$= \frac{i}{E_1 + E_4 - p_1^0} \left[\frac{i}{E_2 + E_3 + p_3^0} \left(\begin{array}{c} v_{123} \\ \uparrow e_3 \\ v_4 \end{array} \begin{array}{c} v_4 \\ \uparrow e_4 \\ v_{123} \end{array} + \begin{array}{c} v_{12} \\ \uparrow e_2 \\ v_{34} \end{array} \begin{array}{c} v_{34} \\ \uparrow e_4 \\ v_{12} \end{array} \right) \right. \\ \left. + \frac{i}{E_1 + E_3 - p_1^0 - p_4^0} \left(\begin{array}{c} v_{134} \\ \uparrow e_2 \\ v_4 \end{array} \begin{array}{c} v_4 \\ \uparrow e_1 \\ v_{134} \end{array} + \begin{array}{c} v_{124} \\ \uparrow e_2 \\ v_3 \end{array} \begin{array}{c} v_3 \\ \uparrow e_3 \\ v_{124} \end{array} \right) \right]$$

$$\begin{array}{c} \bullet \\ \uparrow \\ \bullet \end{array} \begin{array}{c} \bullet \\ \uparrow \\ \bullet \end{array} = \int \left[\prod_{j=1}^4 \frac{d\tau_j}{2E_j} e^{i\tau_j(E_j^0 - \sigma_j p_j^0)} \Theta(\tau_j) \right] \delta(-\tau_1 - \tau_2 + \tau_3 + \tau_4)$$

How do we perform the remaining integrations (**one for each edge**)? Edge-contraction

$$\begin{array}{c} v_2 \\ \uparrow e_1 \\ v_1 \end{array} \begin{array}{c} v_3 \\ \uparrow e_2 \\ v_2 \end{array} \begin{array}{c} v_4 \\ \uparrow e_3 \\ v_3 \end{array} \begin{array}{c} v_4 \\ \uparrow e_4 \\ v_1 \end{array} = \frac{i}{E_1 + E_4 - p_1^0} \left[\begin{array}{c} v_{12} \\ \uparrow e_2 \\ v_{14} \end{array} \begin{array}{c} v_3 \\ \uparrow e_3 \\ v_4 \end{array} + \begin{array}{c} v_2 \\ \uparrow e_1 \\ v_{14} \end{array} \begin{array}{c} v_3 \\ \uparrow e_3 \\ v_4 \end{array} \right]$$

1. Choose sink/source with connected complement
2. Contract one-by-one adjacent edges
3. Multiply by inverse sum of energies of adjacent edges

$$= \frac{i}{E_1 + E_4 - p_1^0} \left[\frac{i}{E_2 + E_3 + p_3^0} \left(\begin{array}{c} v_{123} \\ \uparrow e_3 \\ v_4 \end{array} \begin{array}{c} v_4 \\ \uparrow e_4 \\ v_{123} \end{array} + \begin{array}{c} v_{12} \\ \uparrow e_2 \\ v_{34} \end{array} \begin{array}{c} v_{34} \\ \uparrow e_4 \\ v_{12} \end{array} \right) \right. \\ \left. + \frac{i}{E_1 + E_3 - p_1^0 - p_4^0} \left(\begin{array}{c} v_{134} \\ \uparrow e_2 \\ v_4 \end{array} \begin{array}{c} v_4 \\ \uparrow e_1 \\ v_{134} \end{array} + \begin{array}{c} v_{124} \\ \uparrow e_2 \\ v_3 \end{array} \begin{array}{c} v_3 \\ \uparrow e_3 \\ v_{124} \end{array} \right) \right]$$

4. Throw out non-acyclic graphs

$$\begin{array}{c} \bullet \xrightarrow{\text{red}} \bullet \\ \uparrow \text{red} \quad \downarrow \\ \bullet \xleftarrow{\text{red}} \bullet \end{array} = \int \left[\prod_{j=1}^4 \frac{d\tau_j}{2E_j} e^{i\tau_j(E_j^0 - \sigma_j p_j^0)} \Theta(\tau_j) \right] \delta(-\tau_1 - \tau_2 + \tau_3 + \tau_4)$$

How do we perform the remaining integrations (**one for each edge**)? Edge-contraction

$$\begin{array}{c} \begin{array}{ccc} v_2 & \xrightarrow{e_2} & v_3 \\ \downarrow e_1 & & \downarrow e_3 \\ \bullet & \xleftarrow{e_4} & v_4 \\ v_1 & & \end{array} \end{array} = \frac{i}{E_1 + E_4 - p_1^0} \left[\begin{array}{c} \begin{array}{ccc} & e_2 & v_3 \\ & \uparrow & \downarrow e_3 \\ v_{12} & & v_4 \\ & e_4 & \end{array} \\ + \\ \begin{array}{ccc} v_2 & \xrightarrow{e_2} & v_3 \\ \downarrow e_1 & & \downarrow e_3 \\ v_{14} & & \end{array} \end{array} \right]$$

1. Choose sink/source with connected complement
2. Contract one-by-one adjacent edges
3. Multiply by inverse sum of energies of adjacent edges

$$= \frac{i}{E_1 + E_4 - p_1^0} \left[\frac{i}{E_2 + E_3 + p_3^0} \left(\begin{array}{ccc} & e_3 & \\ v_{123} & \xrightarrow{\quad} & v_4 \\ & e_4 & \end{array} \right) + \begin{array}{ccc} & e_2 & \\ v_{12} & \xrightarrow{\quad} & v_{34} \\ & e_4 & \end{array} \right) \right.$$

4. Throw out non-acyclic graphs

$$\left. + \frac{i}{E_1 + E_3 - p_1^0 - p_4^0} \left(\begin{array}{ccc} & e_2 & \\ v_{134} & \xrightarrow{\quad} & v_4 \\ & e_1 & \end{array} \right) + \begin{array}{ccc} & e_2 & \\ v_{124} & \xrightarrow{\quad} & v_3 \\ & e_3 & \end{array} \right) \right]$$

$$\begin{array}{c} \bullet \xrightarrow{\text{red}} \bullet \\ \uparrow \text{red} \quad \downarrow \\ \bullet \xleftarrow{\text{red}} \bullet \end{array} = \int \left[\prod_{j=1}^4 \frac{d\tau_j}{2E_j} e^{i\tau_j(E_j^0 - \sigma_j p_j^0)} \Theta(\tau_j) \right] \delta(-\tau_1 - \tau_2 + \tau_3 + \tau_4)$$

How do we perform the remaining integrations (**one for each edge**)? Edge-contraction

$$\begin{array}{c} v_2 \xrightarrow{e_2} v_3 \\ \uparrow e_1 \quad \downarrow e_3 \\ v_1 \xleftarrow{e_4} v_4 \end{array} = \frac{i}{E_1 + E_4 - p_1^0} \left[\begin{array}{c} v_{12} \xrightarrow{e_2} v_3 \\ \uparrow e_3 \quad \downarrow e_4 \\ v_{14} \end{array} + \begin{array}{c} v_2 \xrightarrow{e_2} v_3 \\ \uparrow e_1 \quad \downarrow e_3 \\ v_{14} \end{array} \right]$$

1. Choose sink/source with connected complement
2. Contract one-by-one adjacent edges
3. Multiply by inverse sum of energies of adjacent edges

$$= \frac{i}{E_1 + E_4 - p_1^0} \left[\frac{i}{E_2 + E_3 + p_3^0} \left(\begin{array}{c} v_{123} \xrightarrow{e_3} v_4 \\ \uparrow e_4 \quad \downarrow \end{array} + \begin{array}{c} v_{12} \xrightarrow{e_2} v_{34} \\ \uparrow e_4 \quad \downarrow \end{array} \right) \right.$$

4. Throw out non-acyclic graphs

$$\left. + \frac{i}{E_1 + E_3 - p_1^0 - p_4^0} \left(\begin{array}{c} v_{134} \xrightarrow{e_2} v_4 \\ \uparrow e_1 \quad \downarrow \end{array} + \begin{array}{c} v_{124} \xrightarrow{e_2} v_3 \\ \uparrow e_3 \quad \downarrow \end{array} \right) \right]$$

$$\begin{array}{c} \bullet \xrightarrow{\text{red}} \bullet \\ \uparrow \text{red} \quad \downarrow \\ \bullet \xleftarrow{\text{red}} \bullet \end{array} = \int \left[\prod_{j=1}^4 \frac{d\tau_j}{2E_j} e^{i\tau_j(E_j^0 - \sigma_j p_j^0)} \Theta(\tau_j) \right] \delta(-\tau_1 - \tau_2 + \tau_3 + \tau_4)$$

How do we perform the remaining integrations (**one for each edge**)? Edge-contraction

$$\begin{array}{c} \begin{array}{c} v_2 \xrightarrow{e_2} v_3 \\ \uparrow e_1 \quad \downarrow e_3 \\ v_1 \xleftarrow{e_4} v_4 \end{array} = \frac{i}{E_1 + E_4 - p_1^0} \left[\begin{array}{c} \begin{array}{c} v_3 \\ \uparrow e_2 \\ v_{12} \end{array} \quad \begin{array}{c} v_4 \\ \downarrow e_3 \\ v_{14} \end{array} \\ + \end{array} \right] \end{array}$$

1. Choose sink/source with connected complement
2. Contract one-by-one adjacent edges
3. Multiply by inverse sum of energies of adjacent edges

$$\begin{array}{c} = \frac{i}{E_1 + E_4 - p_1^0} \left[\frac{i}{E_2 + E_3 + p_3^0} \left(\begin{array}{c} \text{graph with } v_{123} \text{ and } v_4 \text{ and edges } e_3, e_4 \text{ crossed out} \\ + \end{array} \right) \right. \\ \left. + \frac{i}{E_1 + E_3 - p_1^0 - p_4^0} \left(\begin{array}{c} \text{graph with } v_{134} \text{ and } v_4 \text{ and edges } e_2, e_1 \text{ crossed out} \\ + \end{array} \right) \right] \end{array}$$

4. Throw out non-acyclic graphs

$$= \frac{i}{E_1 + E_4 - p_1^0} \left[\frac{i}{E_2 + E_3 + p_3^0} \frac{i}{E_2 + E_4 - p_1^0 - p_2^0} + \frac{i}{E_1 + E_3 - p_1^0 - p_4^0} \frac{i}{E_2 + E_3 + p_3^0} \right]$$

$$\begin{array}{|c|} \hline \text{Diagram: A square loop with vertices at the corners. The top edge has a red arrow pointing right. The bottom edge has a black arrow pointing left. The left edge has a red arrow pointing down. The right edge has a black arrow pointing down.} \\ \hline \end{array} = \int \left[\prod_{j=1}^4 \frac{d\tau_j}{2E_j} e^{i\tau_j(E_j^0 - \sigma_j p_j^0)} \Theta(\tau_j) \right] \delta(-\tau_1 - \tau_2 + \tau_3 + \tau_4)$$

How do we perform the remaining integrations (**one for each edge**)? Edge-contraction

$$\begin{array}{|c|} \hline \text{Diagram: A square loop with vertices } v_1, v_2, v_3, v_4. \text{ Edges are } e_1, e_2, e_3, e_4. \text{ Red arrows on } e_1, e_2, e_4. \text{ Black arrow on } e_3. \text{ Vertex } v_1 \text{ is circled in blue.} \\ \hline \end{array} = \frac{i}{E_1 + E_4 - p_1^0} \left[\begin{array}{|c|} \hline \text{Diagram: A triangle with vertices } v_{12}, v_3, v_4. \text{ Edges } e_2, e_3, e_4. \text{ Red arrow on } e_2. \text{ Black arrows on } e_3, e_4. \text{ Vertex } v_3 \text{ is circled in blue.} \\ \hline \end{array} + \begin{array}{|c|} \hline \text{Diagram: A triangle with vertices } v_2, v_3, v_{14}. \text{ Edges } e_2, e_3, e_1. \text{ Red arrow on } e_2. \text{ Black arrows on } e_3, e_1. \text{ Vertex } v_{14} \text{ is circled in blue.} \\ \hline \end{array} \right]$$

1. Choose sink/source with connected complement
2. Contract one-by-one adjacent edges
3. Multiply by inverse sum of energies of adjacent edges

$$= \frac{i}{E_1 + E_4 - p_1^0} \left[\frac{i}{E_2 + E_3 + p_3^0} \left(\begin{array}{|c|} \hline \text{Diagram: A diamond shape with vertices } v_{123}, v_4. \text{ Edges } e_3, e_4. \text{ Red arrows on } e_3, e_4. \text{ Vertices } v_{123} \text{ and } v_4 \text{ are crossed out with red X's.} \\ \hline \end{array} + \begin{array}{|c|} \hline \text{Diagram: A diamond shape with vertices } v_{12}, v_{34}. \text{ Edges } e_2, e_4. \text{ Black arrows on } e_2, e_4. \text{ Vertex } v_{12} \text{ is circled in blue.} \\ \hline \end{array} \right) \right.$$

4. Throw out non-acyclic graphs

$$+ \frac{i}{E_1 + E_3 - p_1^0 - p_4^0} \left(\begin{array}{|c|} \hline \text{Diagram: A diamond shape with vertices } v_{134}, v_4. \text{ Edges } e_2, e_1. \text{ Red arrows on } e_2, e_1. \text{ Vertices } v_{134} \text{ and } v_4 \text{ are crossed out with red X's.} \\ \hline \end{array} + \begin{array}{|c|} \hline \text{Diagram: A diamond shape with vertices } v_{124}, v_3. \text{ Edges } e_2, e_3. \text{ Black arrows on } e_2, e_3. \text{ Vertex } v_{124} \text{ is circled in blue.} \\ \hline \end{array} \right) \Big]$$

$$= \frac{i}{E_1 + E_4 - p_1^0} \left[\frac{i}{E_2 + E_3 + p_3^0} \frac{i}{E_2 + E_4 - p_1^0 - p_2^0} + \frac{i}{E_1 + E_3 - p_1^0 - p_4^0} \frac{i}{E_2 + E_3 + p_3^0} \right]$$

5. Contract parallel edges

$$\begin{array}{|c|c|} \hline \bullet & \bullet \\ \hline \uparrow & \downarrow \\ \hline \bullet & \bullet \\ \hline \end{array} = \int \left[\prod_{j=1}^4 \frac{d\tau_j}{2E_j} e^{i\tau_j(E_j^0 - \sigma_j p_j^0)} \Theta(\tau_j) \right] \delta(-\tau_1 - \tau_2 + \tau_3 + \tau_4)$$

How do we perform the remaining integrations (**one for each edge**)? Edge-contraction

$$\begin{array}{|c|c|} \hline \begin{array}{c} v_2 \\ \uparrow e_1 \\ \bullet \\ \downarrow e_4 \\ v_1 \end{array} \begin{array}{c} \bullet \\ \downarrow e_2 \\ v_3 \end{array} \begin{array}{c} \bullet \\ \downarrow e_3 \\ v_4 \end{array} \\ \hline \end{array} = \frac{i}{E_1 + E_4 - p_1^0} \left[\begin{array}{c} \begin{array}{c} v_{12} \\ \uparrow e_2 \\ \bullet \\ \downarrow e_3 \\ v_4 \end{array} \begin{array}{c} \bullet \\ \downarrow e_4 \\ v_4 \end{array} \\ \hline \end{array} + \begin{array}{c} \begin{array}{c} v_2 \\ \uparrow e_1 \\ \bullet \\ \downarrow e_3 \\ v_{14} \end{array} \begin{array}{c} \bullet \\ \downarrow e_2 \\ v_3 \end{array} \end{array} \right]$$

1. Choose sink/source with connected complement
2. Contract one-by-one adjacent edges
3. Multiply by inverse sum of energies of adjacent edges

$$= \frac{i}{E_1 + E_4 - p_1^0} \left[\frac{i}{E_2 + E_3 + p_3^0} \left(\begin{array}{c} \begin{array}{c} v_{123} \\ \uparrow e_3 \\ \bullet \\ \downarrow e_4 \\ v_4 \end{array} \begin{array}{c} \bullet \\ \downarrow e_2 \\ v_4 \end{array} \\ \hline \end{array} + \begin{array}{c} \begin{array}{c} v_{12} \\ \uparrow e_2 \\ \bullet \\ \downarrow e_4 \\ v_{34} \end{array} \begin{array}{c} \bullet \\ \downarrow e_3 \\ v_4 \end{array} \end{array} \right) \right.$$

4. Throw out non-acyclic graphs

$$+ \frac{i}{E_1 + E_3 - p_1^0 - p_4^0} \left(\begin{array}{c} \begin{array}{c} v_{134} \\ \uparrow e_2 \\ \bullet \\ \downarrow e_1 \\ v_4 \end{array} \begin{array}{c} \bullet \\ \downarrow e_3 \\ v_4 \end{array} \\ \hline \end{array} + \begin{array}{c} \begin{array}{c} v_{124} \\ \uparrow e_2 \\ \bullet \\ \downarrow e_3 \\ v_3 \end{array} \begin{array}{c} \bullet \\ \downarrow e_1 \\ v_3 \end{array} \end{array} \right) \Bigg]$$

$$= \frac{i}{E_1 + E_4 - p_1^0} \left[\frac{i}{E_2 + E_3 + p_3^0} \frac{i}{E_2 + E_4 - p_1^0 - p_2^0} + \frac{i}{E_1 + E_3 - p_1^0 - p_4^0} \frac{i}{E_2 + E_3 + p_3^0} \right]$$

5. Contract parallel edges

All time integrations are performed diagrammatically!

Now collect, for each term, the choice of sink and source that led to it

Now collect, for each term, the choice of sink and source that led to it

Term:

$$\begin{array}{ccc} i & i & i \\ \hline E_1 + E_4 - p_1^0 & E_2 + E_3 + p_3^0 & E_2 + E_4 - p_1^0 - p_2^0 \end{array}$$

Now collect, for each term, the choice of sink and source that led to it

Term:

$$\begin{array}{ccc} i & i & i \\ \hline E_1 + E_4 - p_1^0 & E_2 + E_3 + p_3^0 & E_2 + E_4 - p_1^0 - p_2^0 \end{array}$$

Vertex sequence:

$$v_1 \longrightarrow v_3 \longrightarrow v_{12}$$

Now collect, for each term, the choice of sink and source that led to it

Term: $\overbrace{E_1 + E_4 - p_1^0}^i \overbrace{E_2 + E_3 + p_3^0}^i \overbrace{E_2 + E_4 - p_1^0 - p_2^0}^i$

Vertex sequence: $v_1 \longrightarrow v_3 \longrightarrow v_{12}$

Family of cuts: $F_1 = \{\{v_1\}, \{v_3\}, \{v_1, v_2\}\}$

Now collect, for each term, the choice of sink and source that led to it

Term:

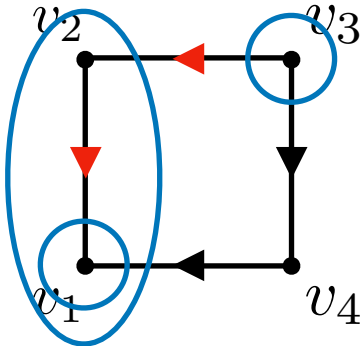
$$\begin{array}{ccc} i & i & i \\ \hline E_1 + E_4 - p_1^0 & E_2 + E_3 + p_3^0 & E_2 + E_4 - p_1^0 - p_2^0 \end{array}$$

Vertex sequence:

$$v_1 \longrightarrow v_3 \longrightarrow v_{12}$$

Family of cuts:

$$F_1 = \{\{v_1\}, \{v_3\}, \{v_1, v_2\}\}$$



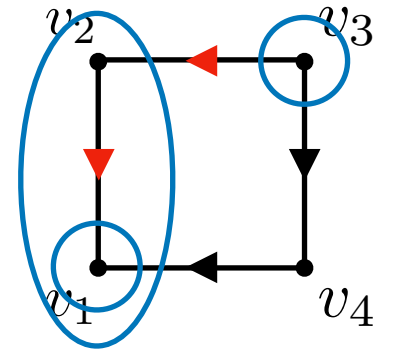
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Term: $\overbrace{E_1 + E_4 - p_1^0}^i \overbrace{E_2 + E_3 + p_3^0}^i \overbrace{E_2 + E_4 - p_1^0 - p_2^0}^i$

Vertex sequence: $v_1 \longrightarrow v_3 \longrightarrow v_{12}$

Family of cuts: $F_1 = \{\{v_1\}, \{v_3\}, \{v_1, v_2\}\}$

e.g. $\partial(\{v_1, v_2\}) = \{e_2, e_4\}$



Now collect, for each term, the choice of sink and source that led to it

Term:

$$\frac{\overset{i}{E_1 + E_4 - p_1^0}}{\quad} \frac{\overset{i}{E_2 + E_3 + p_3^0}}{\quad} \frac{\overset{i}{E_2 + E_4 - p_1^0 - p_2^0}}{\quad}$$

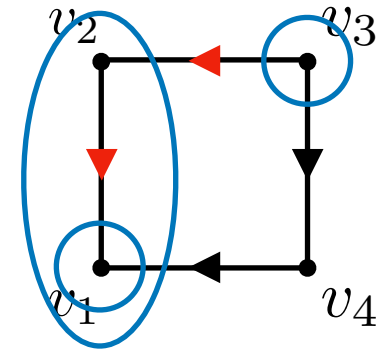
Vertex sequence:

$$v_1 \longrightarrow v_3 \longrightarrow v_{12}$$

Family of cuts:

$$F_1 = \{\{v_1\}, \{v_3\}, \{v_1, v_2\}\}$$

$$\text{e.g. } \partial(\{v_1, v_2\}) = \{e_2, e_4\} \quad \Rightarrow \quad \frac{\overset{i}{E_2 + E_4 - p_1^0 - p_2^0}}{\quad}$$



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Term:

$$\frac{\overset{i}{E_1 + E_4 - p_1^0}}{\quad} \frac{\overset{i}{E_2 + E_3 + p_3^0}}{\quad} \frac{\overset{i}{E_2 + E_4 - p_1^0 - p_2^0}}{\quad}$$

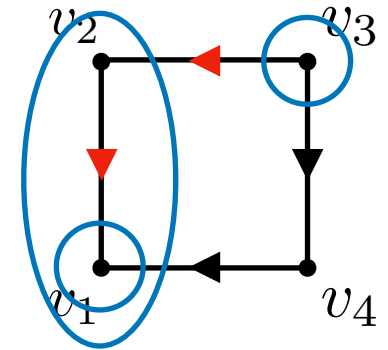
Vertex sequence:

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$$\text{e.g. } \partial(\{v_1, v_2\}) = \{e_2, e_4\} \quad \Rightarrow \quad \frac{\overset{i}{E_2 + E_4 - p_1^0 - p_2^0}}{\quad}$$



Now collect, for each term, the choice of sink and source that led to it

Term:

$$\overbrace{E_1 + E_4 - p_1^0}^i \overbrace{E_2 + E_3 + p_3^0}^i \overbrace{E_2 + E_4 - p_1^0 - p_2^0}^i$$

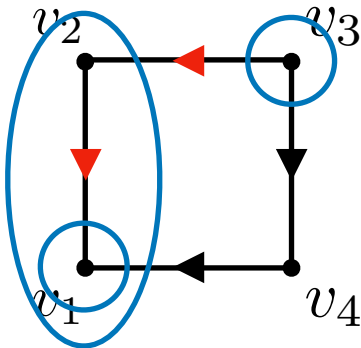
Vertex sequence:

$$v_1 \longrightarrow v_3 \longrightarrow v_{12}$$

Family of cuts:

$$F_1 = \{\{v_1\}, \{v_3\}, \{v_1, v_2\}\}$$

$$\text{e.g. } \partial(\{v_1, v_2\}) = \{e_2, e_4\} \implies \overbrace{E_2 + E_4 - p_1^0 - p_2^0}^i$$



Now collect, for each term, the choice of sink and source that led to it

Term:

$$\frac{\overset{i}{E_1 + E_4 - p_1^0}}{\overset{i}{E_2 + E_3 + p_3^0}} \frac{\overset{i}{E_2 + E_4 - p_1^0 - p_2^0}}$$

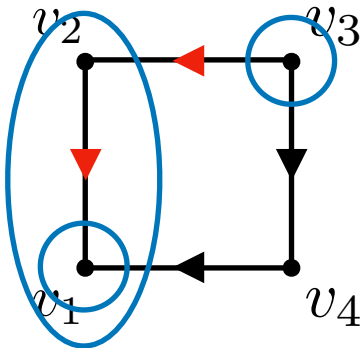
Vertex sequence:

$$v_1 \longrightarrow v_3 \longrightarrow v_{12}$$

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$$F_1 = \{\{v_1\}, \{v_3\}, \{v_1, v_2\}\}$$

$$\text{e.g. } \partial(\{v_1, v_2\}) = \{e_2, e_4\} \implies \frac{\overset{i}{E_2 + E_4 - p_1^0 - p_2^0}}$$



Term:

$$\frac{\overset{i}{E_1 + E_4 - p_1^0}}{\overset{i}{E_1 + E_3 - p_1^0 - p_4^0}} \frac{\overset{i}{E_2 + E_3 + p_3^0}}$$

Now collect, for each term, the choice of sink and source that led to it

Term:

$$\frac{\overset{i}{E_1 + E_4 - p_1^0}}{\frac{\overset{i}{E_2 + E_3 + p_3^0}}{\overset{i}{E_2 + E_4 - p_1^0 - p_2^0}}}$$

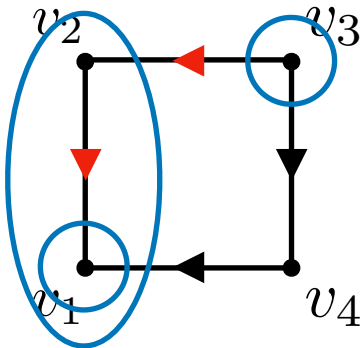
Vertex sequence:

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$$\text{e.g. } \partial(\{v_1, v_2\}) = \{e_2, e_4\} \implies \frac{\overset{i}{E_2 + E_4 - p_1^0 - p_2^0}}{\dots}$$



Term:

$$\frac{\overset{i}{E_1 + E_4 - p_1^0}}{\frac{\overset{i}{E_1 + E_3 - p_1^0 - p_4^0}}{\overset{i}{E_2 + E_3 + p_3^0}}}$$

Vertex sequence:

$$v_1 \longrightarrow v_{14} \longrightarrow v_{124}$$

Now collect, for each term, the choice of sink and source that led to it

Term:

$$\frac{i}{E_1 + E_4 - p_1^0} \frac{i}{E_2 + E_3 + p_3^0} \frac{i}{E_2 + E_4 - p_1^0 - p_2^0}$$

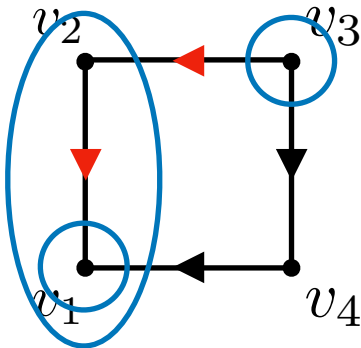
Vertex sequence:

$$v_1 \longrightarrow v_3 \longrightarrow v_{12}$$

Family of cuts:

$$F_1 = \{\{v_1\}, \{v_3\}, \{v_1, v_2\}\}$$

$$\text{e.g. } \partial(\{v_1, v_2\}) = \{e_2, e_4\} \implies \frac{i}{\boxed{E_2} + \boxed{E_4} - \boxed{p_1^0} - \boxed{p_2^0}}$$



Term:

$$\frac{i}{E_1 + E_4 - p_1^0} \frac{i}{E_1 + E_3 - p_1^0 - p_4^0} \frac{i}{E_2 + E_3 + p_3^0}$$

Vertex sequence:

$$v_1 \longrightarrow v_{14} \longrightarrow v_{124}$$

Family of cuts:

$$F_2 = \{\{v_1\}, \{v_1, v_4\}, \{v_1, v_2, v_4\}\}$$

Now collect, for each term, the choice of sink and source that led to it

Term:

$$\frac{\overset{i}{E_1 + E_4 - p_1^0}}{\frac{\overset{i}{E_2 + E_3 + p_3^0}}{\overset{i}{E_2 + E_4 - p_1^0 - p_2^0}}}$$

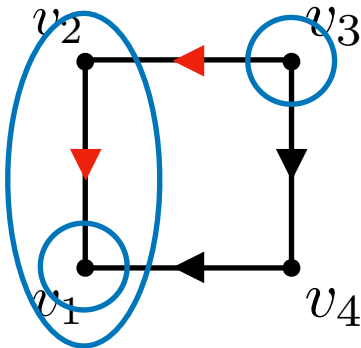
Vertex sequence:

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Family of cuts:

$$F_1 = \{\{v_1\}, \{v_3\}, \{v_1, v_2\}\}$$

$$\text{e.g. } \partial(\{v_1, v_2\}) = \{e_2, e_4\} \implies \frac{\overset{i}{E_2 + E_4 - p_1^0 - p_2^0}}$$



Term:

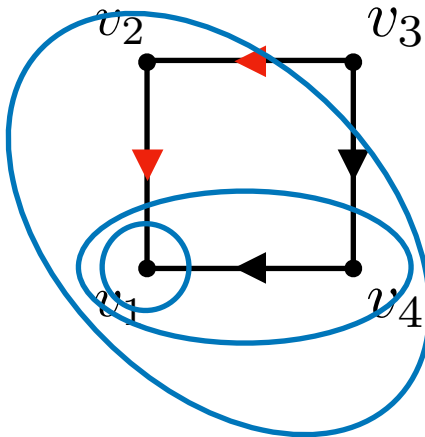
$$\frac{\overset{i}{E_1 + E_4 - p_1^0}}{\frac{\overset{i}{E_1 + E_3 - p_1^0 - p_4^0}}{\overset{i}{E_2 + E_3 + p_3^0}}}$$

Vertex sequence:

$$v_1 \longrightarrow v_{14} \longrightarrow v_{124}$$

Family of cuts:

$$F_2 = \{\{v_1\}, \{v_1, v_4\}, \{v_1, v_2, v_4\}\}$$



Now collect, for each term, the choice of sink and source that led to it

Term:

$$\frac{i}{E_1 + E_4 - p_1^0} \frac{i}{E_2 + E_3 + p_3^0} \frac{i}{E_2 + E_4 - p_1^0 - p_2^0}$$

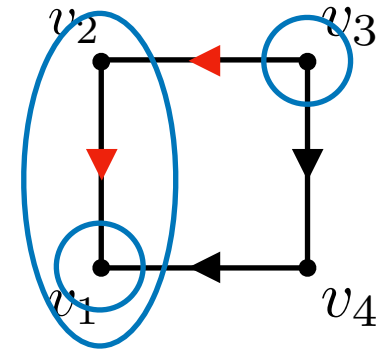
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$$F_1 = \{\{v_1\}, \{v_3\}, \{v_1, v_2\}\}$$

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Term:

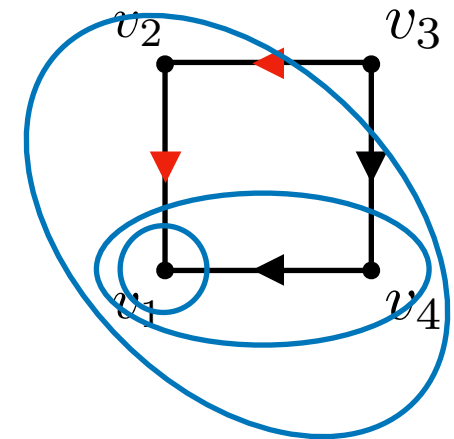
$$\frac{i}{E_1 + E_4 - p_1^0} \frac{i}{E_1 + E_3 - p_1^0 - p_4^0} \frac{i}{E_2 + E_3 + p_3^0}$$

Vertex sequence:

$$v_1 \longrightarrow v_{14} \longrightarrow v_{124}$$

Family of cuts:

$$F_2 = \{\{v_1\}, \{v_1, v_4\}, \{v_1, v_2, v_4\}\}$$



We notice some regularities... these families of cuts satisfy

$$S \in F \quad \Rightarrow \quad S, V \setminus S \text{ are connected}$$

$$S_1, S_2 \in F \quad \Rightarrow \quad S_1 \subset S_2 \text{ or } S_2 \subset S_1 \text{ or } S_1 \cap S_2 = \emptyset$$

$$S \in F \quad \Rightarrow \quad S \text{ cannot be written as union of other sets in } F$$

Now collect, for each term, the choice of sink and source that led to it

Term:

$$\frac{i}{E_1 + E_4 - p_1^0} \frac{i}{E_2 + E_3 + p_3^0} \frac{i}{E_2 + E_4 - p_1^0 - p_2^0}$$

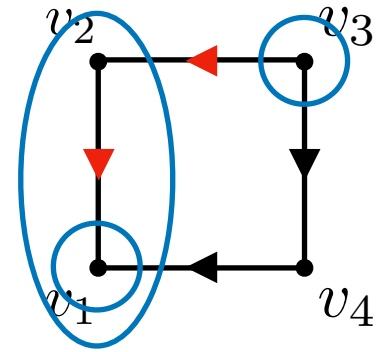
Vertex sequence:

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Family of cuts:

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$$\text{e.g. } \partial(\{v_1, v_2\}) = \{e_2, e_4\} \quad \Rightarrow \quad \frac{i}{E_2 + E_4 - p_1^0 - p_2^0}$$



Term:

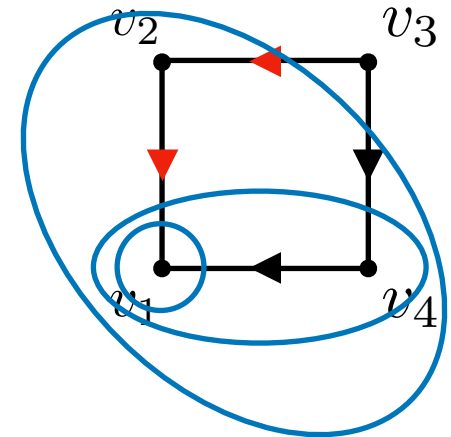
$$\frac{i}{E_1 + E_4 - p_1^0} \frac{i}{E_1 + E_3 - p_1^0 - p_4^0} \frac{i}{E_2 + E_3 + p_3^0}$$

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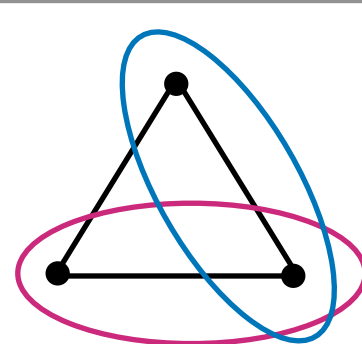
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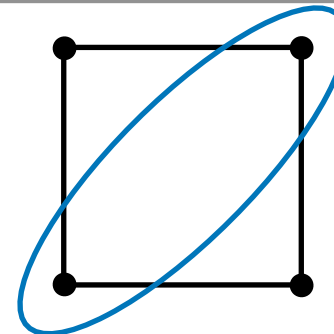
$$S_1, S_2 \in F \quad \Rightarrow \quad S_1 \subset S_2 \text{ or } S_2 \subset S_1 \text{ or } S_1 \cap S_2 = \emptyset$$

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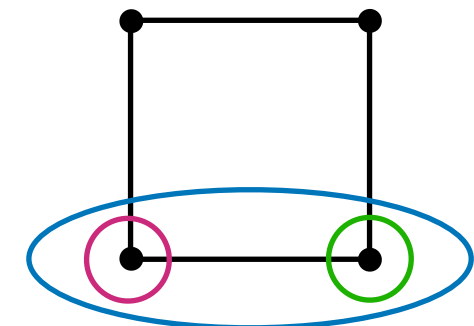
Forbidden configurations:



crossing



disconnected



obstruction

Now collect, for each term, the choice of sink and source that led to it

Term:

$$\frac{i}{E_1 + E_4 - p_1^0} \frac{i}{E_2 + E_3 + p_3^0} \frac{i}{E_2 + E_4 - p_1^0 - p_2^0}$$

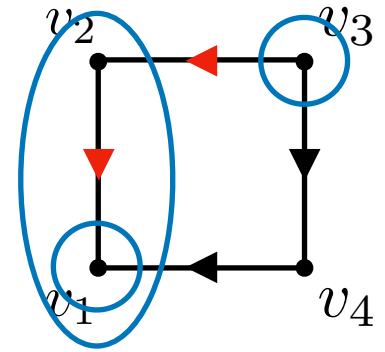
Vertex sequence:

$$v_1 \longrightarrow v_3 \longrightarrow v_{12}$$

Family of cuts:

$$F_1 = \{\{v_1\}, \{v_3\}, \{v_1, v_2\}\}$$

$$\text{e.g. } \partial(\{v_1, v_2\}) = \{e_2, e_4\} \quad \Rightarrow \quad \frac{i}{\overline{E_2} + \overline{E_4} - \overline{p_1^0} - \overline{p_2^0}}$$



Term:

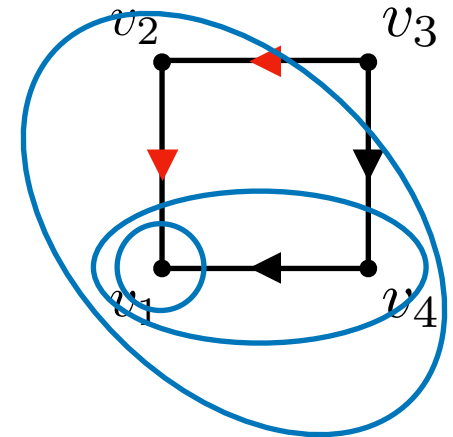
$$\frac{i}{E_1 + E_4 - p_1^0} \frac{i}{E_1 + E_3 - p_1^0 - p_4^0} \frac{i}{E_2 + E_3 + p_3^0}$$

Vertex sequence:

$$v_1 \longrightarrow v_{14} \longrightarrow v_{124}$$

Family of cuts:

$$F_2 = \{\{v_1\}, \{v_1, v_4\}, \{v_1, v_2, v_4\}\}$$



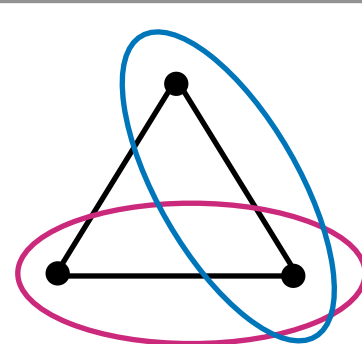
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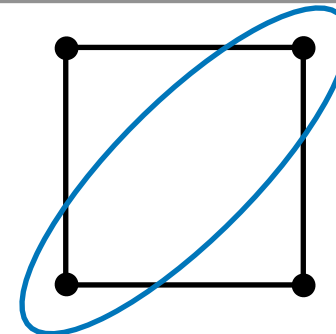
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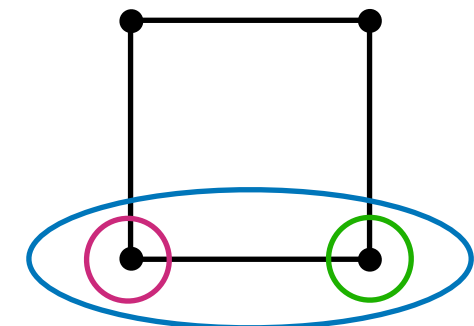
Forbidden configurations:



crossing



disconnected



obstruction

These properties are completely general!

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ZC, arXiv: [2211.09653](#) (2022)

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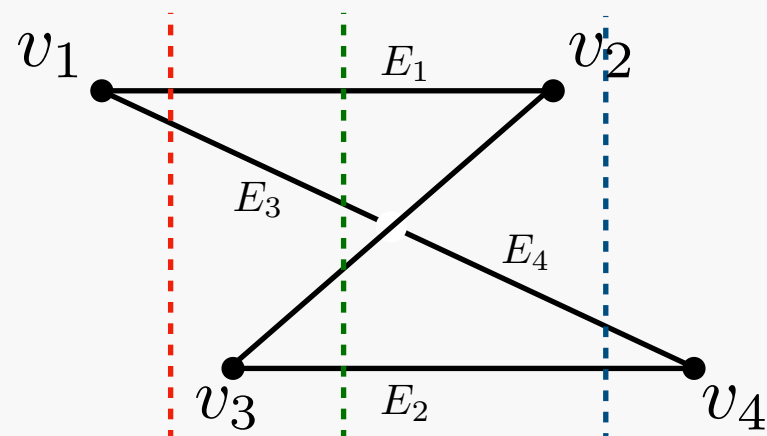
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Focus on the TOPT term



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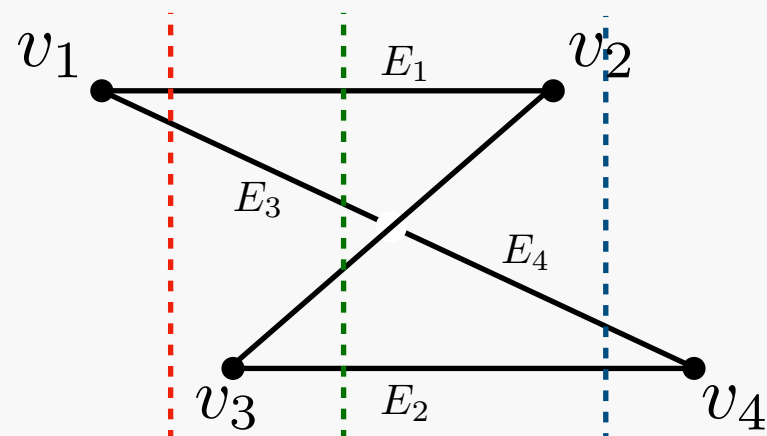
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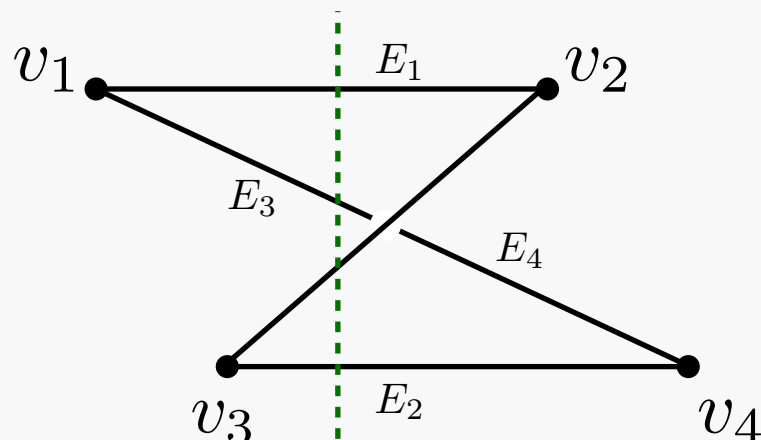
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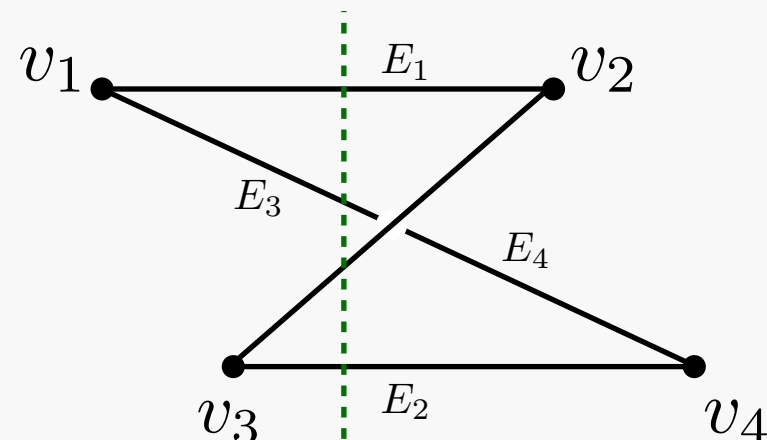
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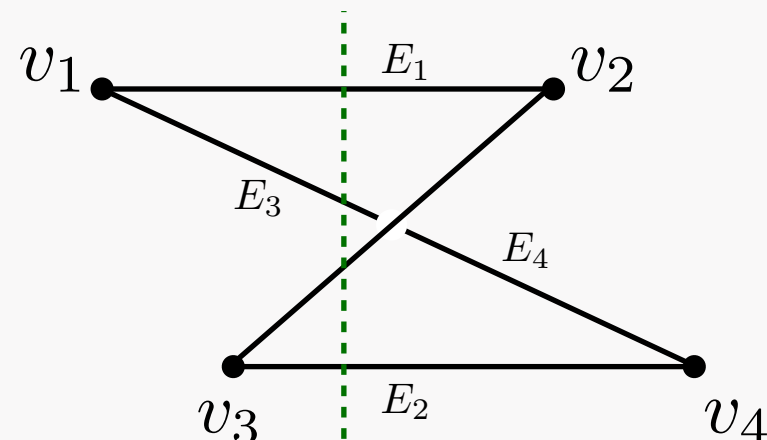
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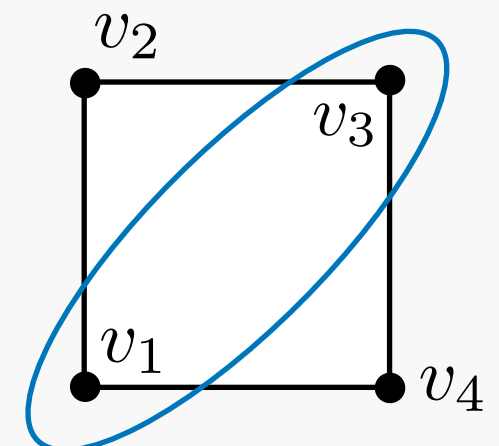
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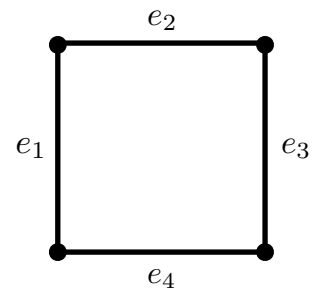
Iterated connectedness

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But how does the integrand behave at the location of a threshold?

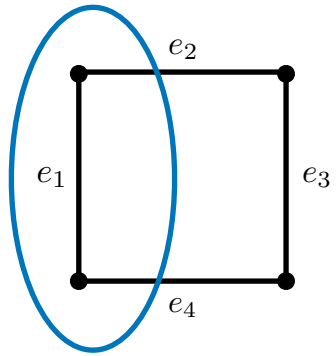
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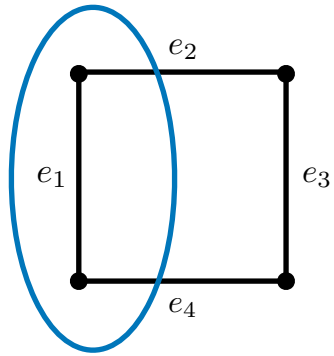
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$$\Delta = E_2 + E_4 - p_1^0 - p_4^0 \rightarrow 0$$

Iterated connectedness

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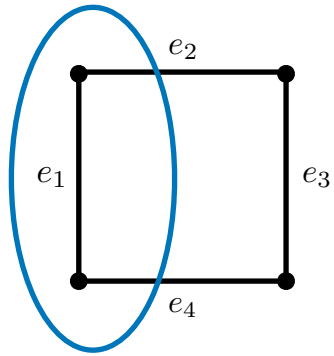
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$$= \frac{i}{\Delta} \begin{array}{c} \bullet \\ | \\ \bullet \end{array}_{e_1} \times \begin{array}{c} \bullet \\ | \\ \bullet \end{array}_{e_3} + o(1)$$

A square diagram with four vertices. The edges are labeled: e_1 (left vertical edge), e_2 (top horizontal edge), e_3 (right vertical edge), and e_4 (bottom horizontal edge).

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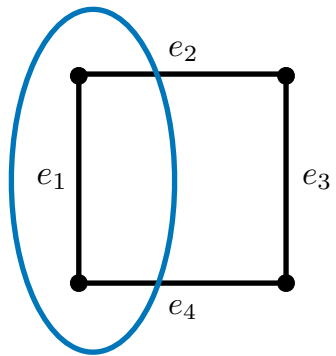
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In general:

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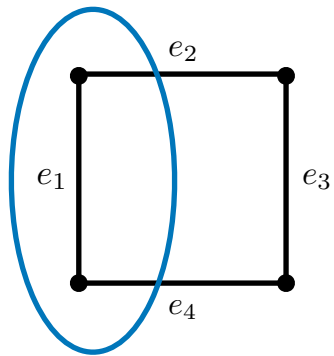
In general:

$$= \frac{i}{\Delta} \left[\begin{array}{c} \text{connected graph} \\ \vdots \end{array} \right] \times \left[\begin{array}{c} \text{connected graph} \\ \vdots \end{array} \right] + o(1)$$

on-shell externals on-shell externals

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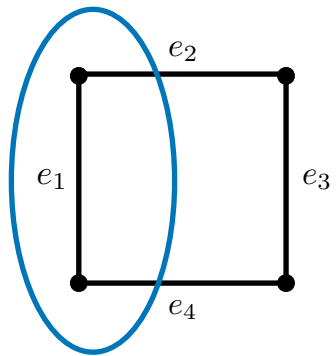
In general:

This relation also holds at the amplitude level with underlying tensor contraction!

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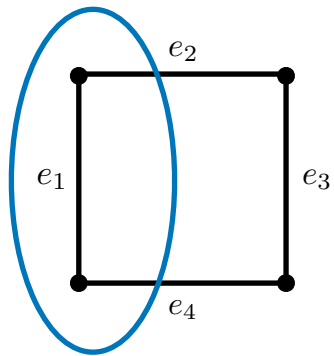
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on-shell externals on-shell externals

We can then iterate the argument for the two smaller graphs...

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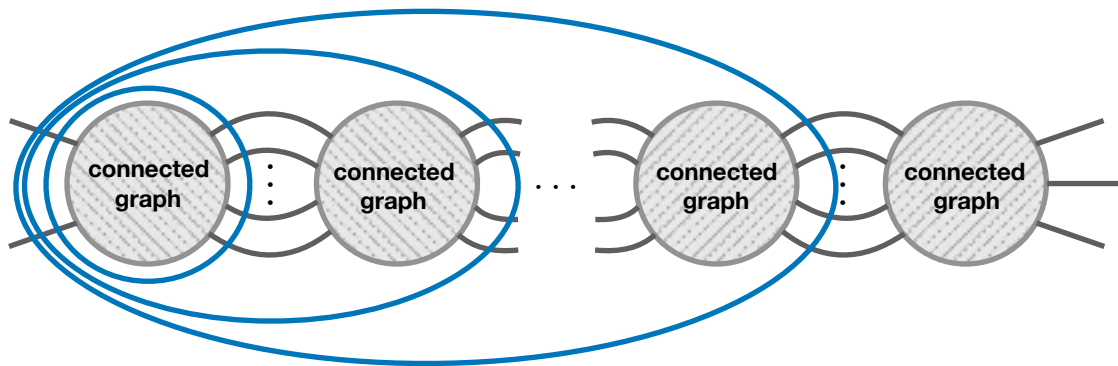
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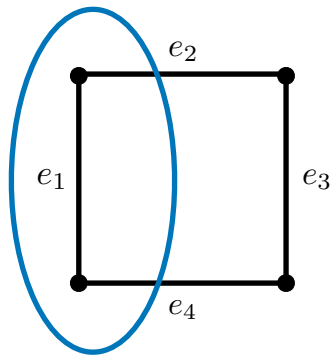
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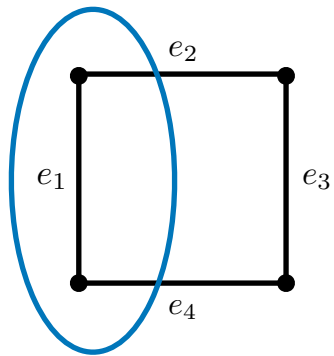
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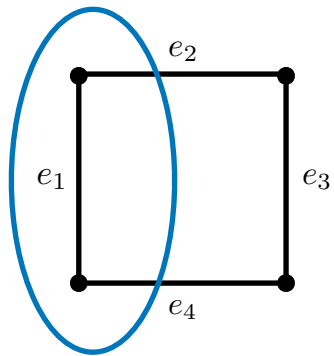
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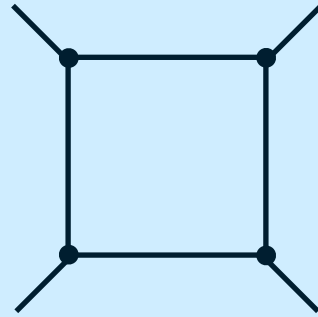
A set of n thresholds leads to a Δ^{-n} scaling only if the corresponding cuts divide the graph in exactly $n + 1$ connected components and not more

Steinmann relations and Unitarity cuts

Steinmann relations and unitarity cuts: consider the box diagram

Steinmann relations

$I(s, t) =$



$$\text{disc}_s \text{disc}_t I(s, t) = 0$$

$$\text{disc}_u I(s, t) = 0$$

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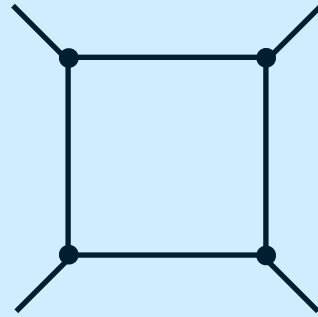
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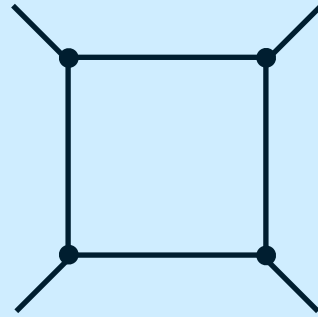
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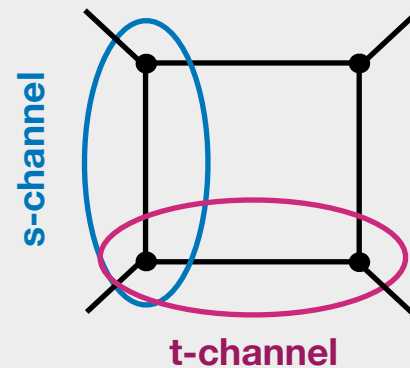
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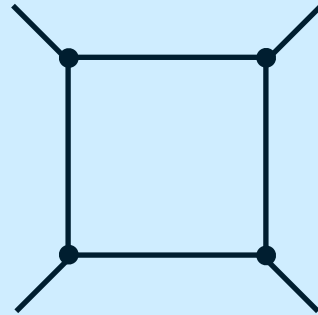


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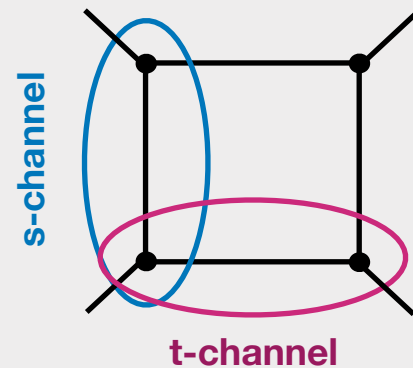
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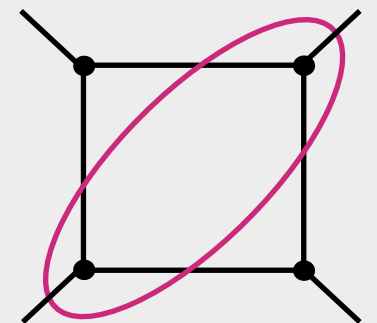
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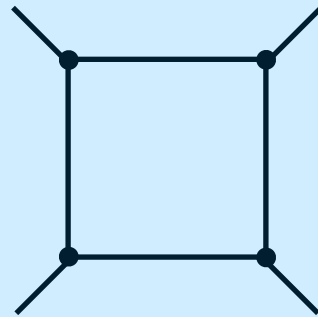


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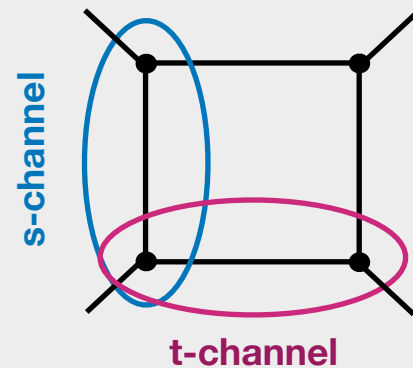
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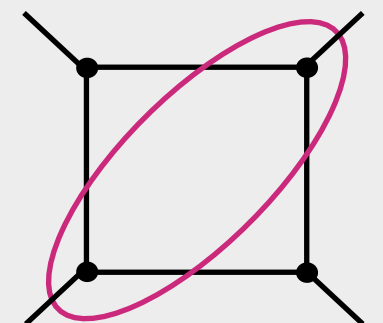
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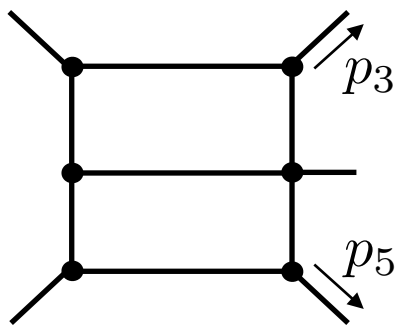
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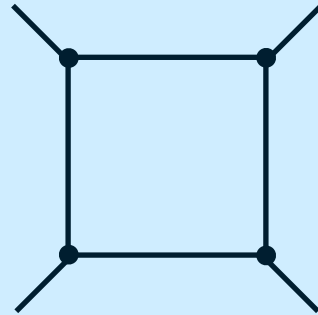
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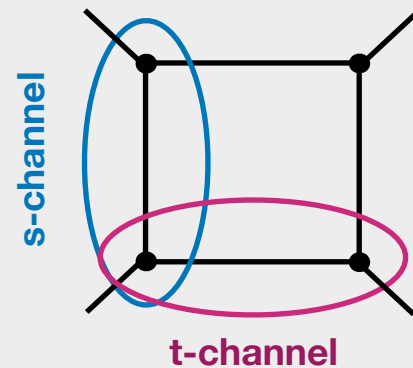
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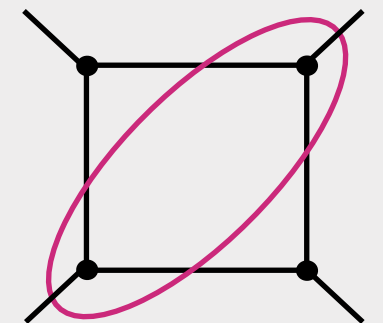
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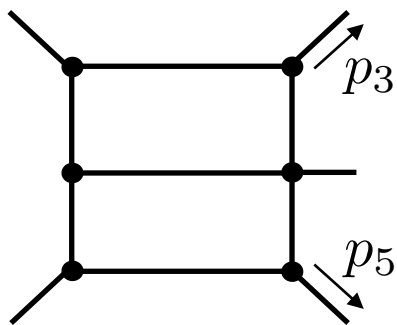
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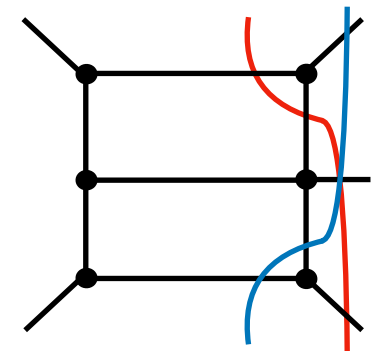
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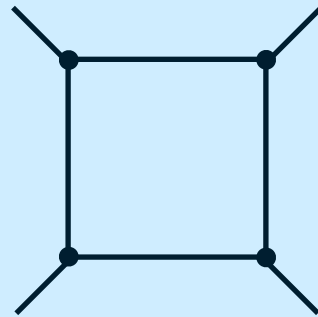


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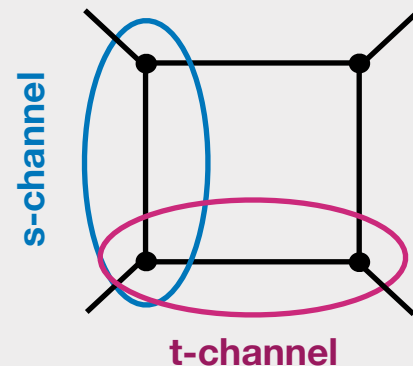
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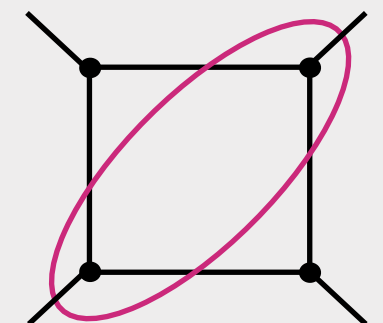
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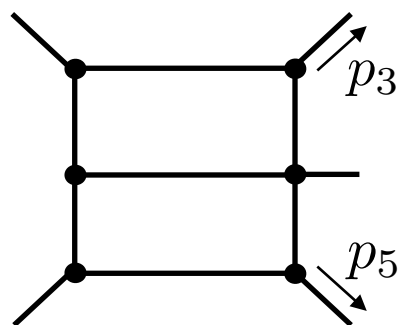
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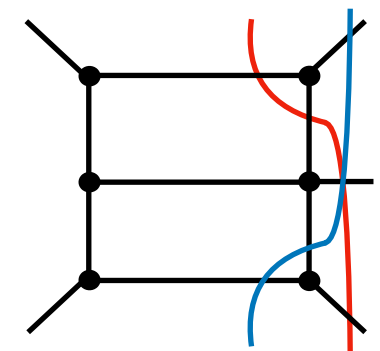
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But these two cuts are simultaneously present in a few terms of the CFF expression!

Cluster decomposition principle, Unitarity and Infrared Finiteness

Recent efforts to construct well defined scattering theory

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We define the connected S-matrix in a slightly different way as usual

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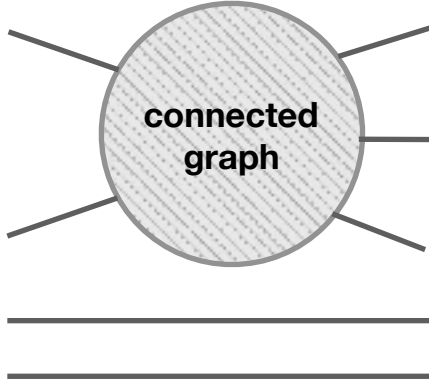
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I would like to focus on the role of **connectedness**

Its role in the Cross-Free Family representation connects the *cluster decomposition principle* with *infrared finiteness*

We define the connected S-matrix in a slightly different way as usual

$$\langle \alpha | \hat{S}_c | \beta \rangle =$$


The diagram shows a central shaded circle labeled "connected graph". Six lines extend from the circle: two from the top-left, two from the top-right, and two from the bottom. The two bottom lines are parallel and horizontal, representing incoming or outgoing particles in a scattering process.

Cluster decomposition principle, Unitarity and Infrared Finiteness

Recent efforts to construct well defined scattering theory

Khalil, Horowitz
arXiv:1701.00763 (2017)

Hannesdottir, Schwartz
arXiv:1906.03271 (2020)

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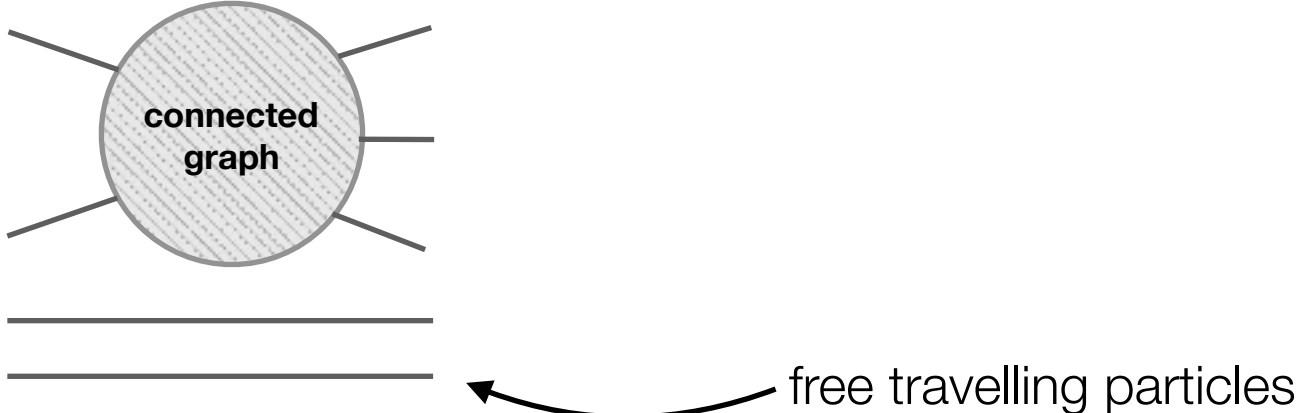
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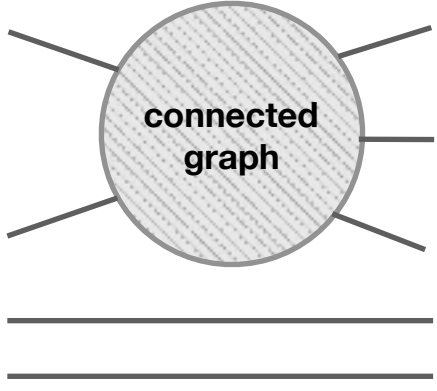
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free travelling particles

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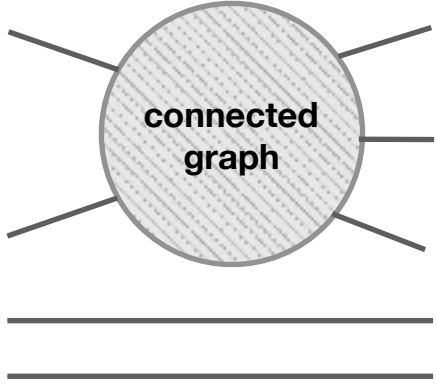
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free travelling particles

The cluster decomposition principle and unitarity constraints

$$\hat{S} = \hat{S}^\dagger$$

$$\lim_{\substack{d_{12}^2 \text{ large} \\ \text{and negative}}} \langle \alpha_1 \alpha_2 | \hat{S} | \beta_1 \beta_2 \rangle = \langle \alpha_1 | \hat{S} | \beta_1 \rangle \langle \alpha_2 | \hat{S} | \beta_2 \rangle$$

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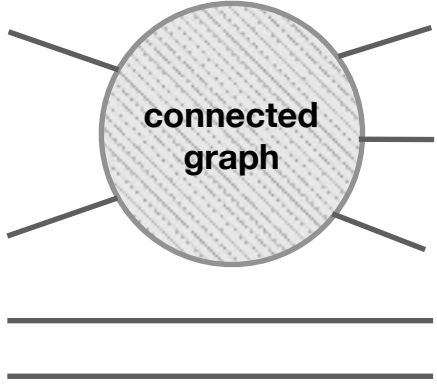
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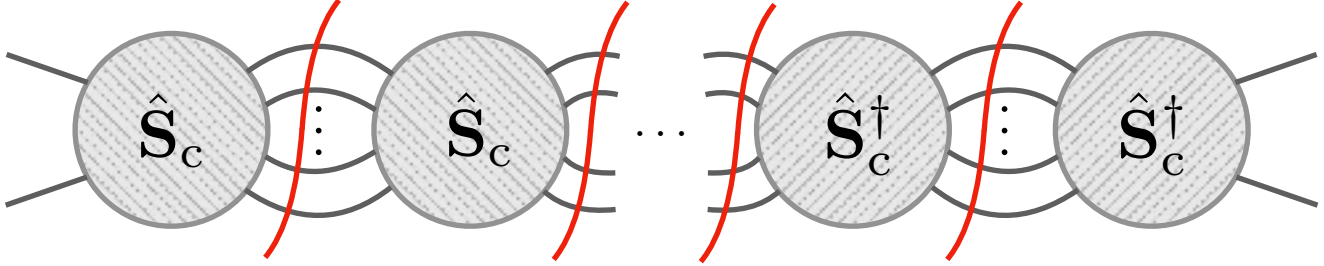
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fix the coefficients of the series

$$a_n = \frac{i^n}{n!}$$

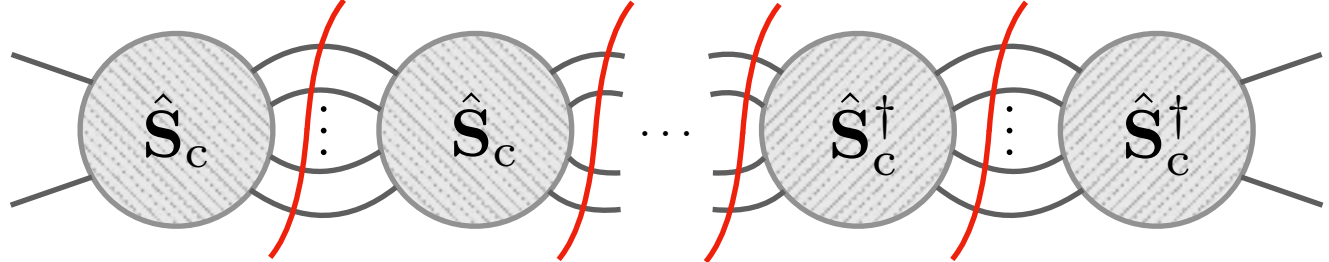
+

Scattering cross-sections then correspond to sums of terms

$$\sum_{\alpha} \sum_{\beta} \langle \alpha | (\hat{S}_c)^n | \beta \rangle \langle \beta | (\hat{S}_c^\dagger)^m | \alpha \rangle =$$


+

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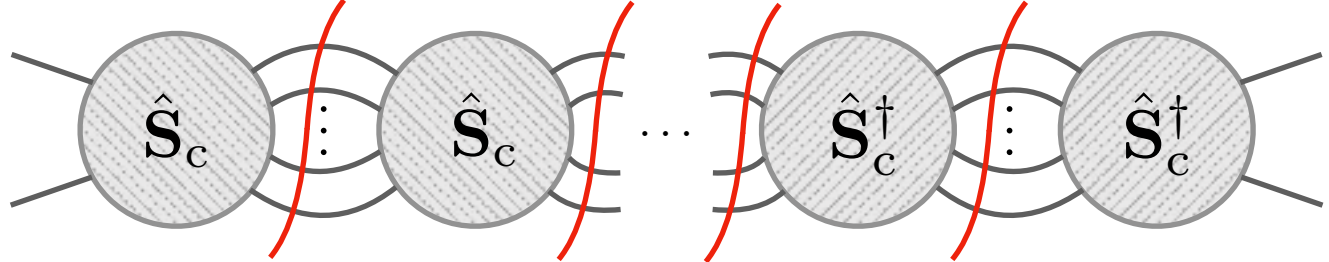
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The diagram illustrates a sequence of scattering processes. It consists of a chain of circular nodes, each labeled with $\hat{S}_c$ or $\hat{S}_c^\dagger$. The first two nodes are labeled $\hat{S}_c$, followed by an ellipsis, and then two nodes labeled $\hat{S}_c^\dagger$. Each node is connected to the next by two horizontal lines. Vertical lines connect the nodes, and red curved lines (Cutkosky cuts) are drawn through these vertical lines. The diagram is followed by an ellipsis and then another set of elements.

Notice the multiple Cutkosky cuts, as opposed to the usual definition of the connected S-matrix

+

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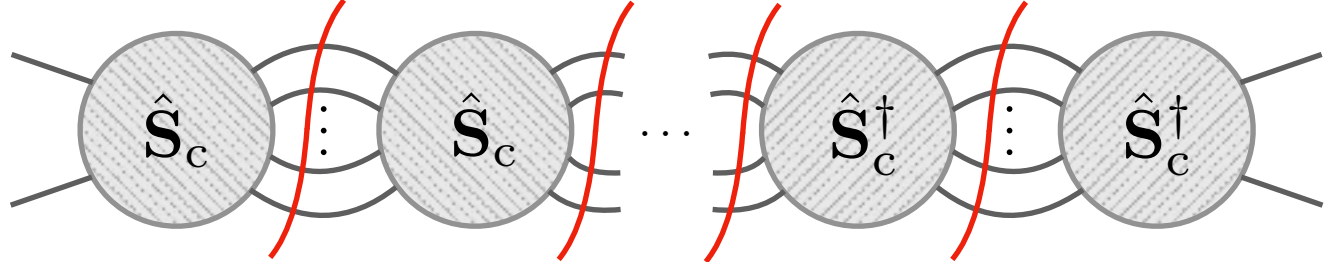
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Work for fixed **n** and **m**, e.g.

$$n = 1, m = 1$$

+

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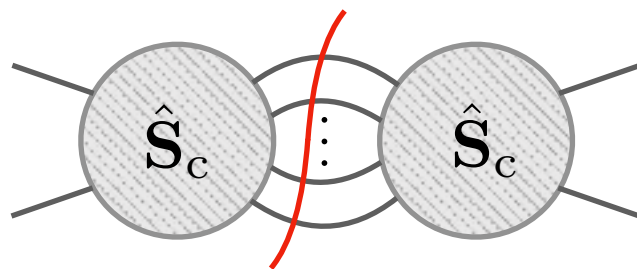
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Notice the multiple Cutkosky cuts, as opposed to the usual definition of the connected S-matrix

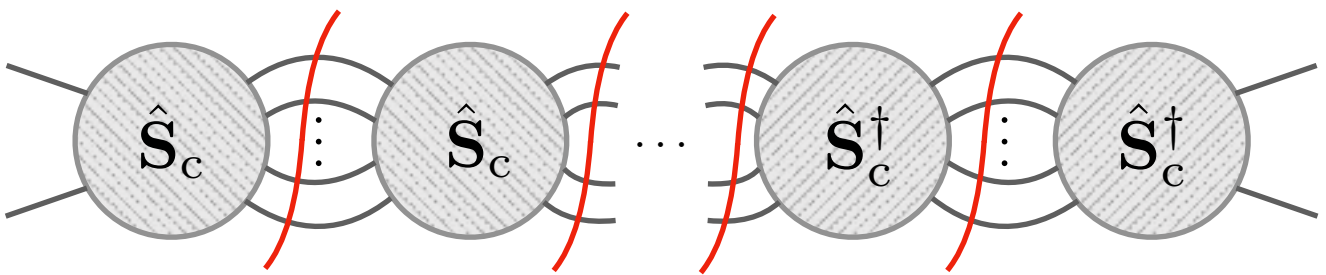
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+

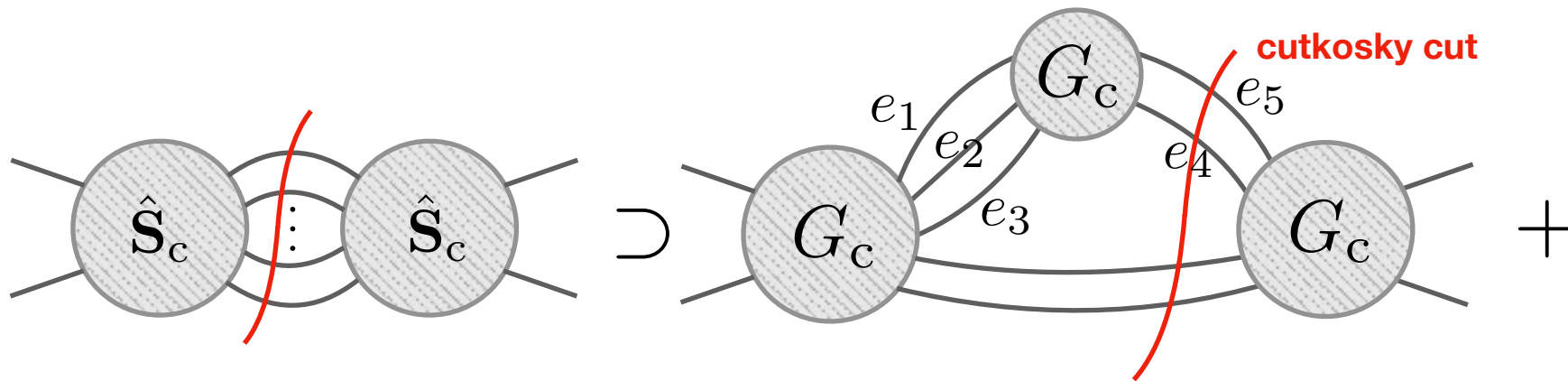
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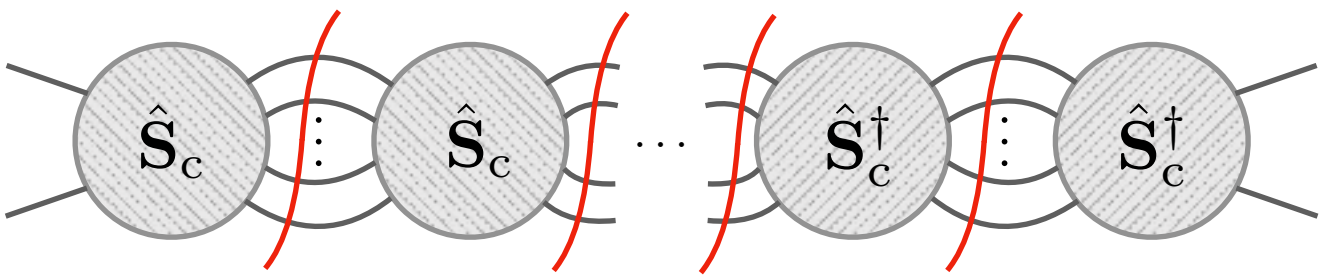
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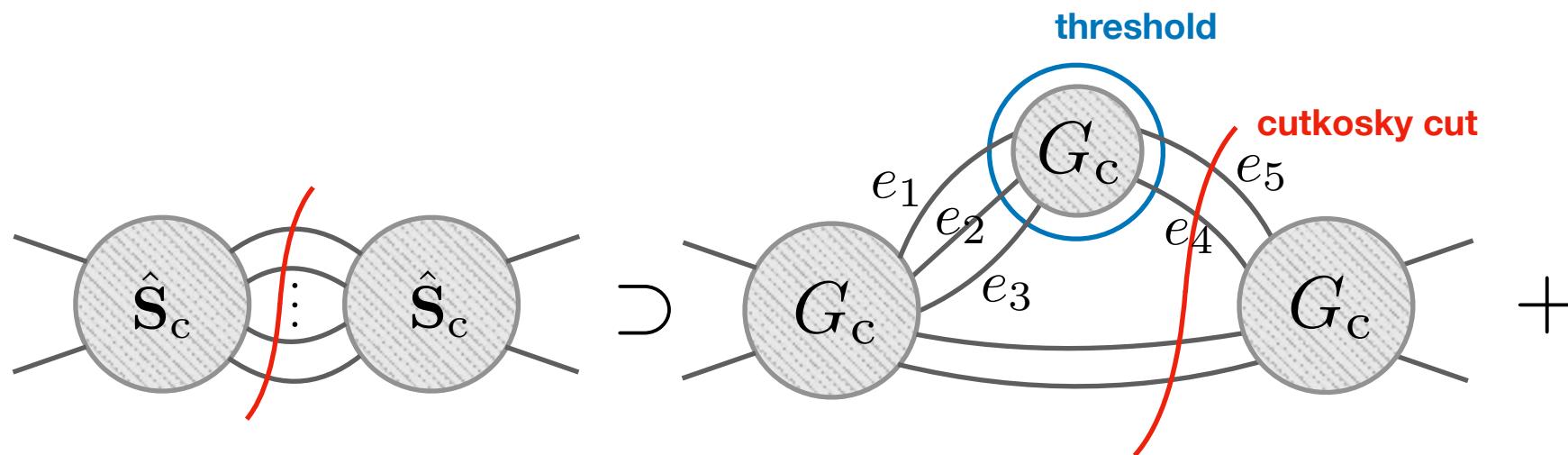
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The diagram shows a sequence of shaded circular nodes representing scattering matrices. The first two nodes are labeled $\hat{S}_c$ and are connected by a vertical line with three dots. This is followed by an ellipsis, then two more nodes labeled $\hat{S}_c^\dagger$, also connected by a vertical line with three dots. Red curved lines, representing Cutkosky cuts, are drawn across the connections between the nodes.

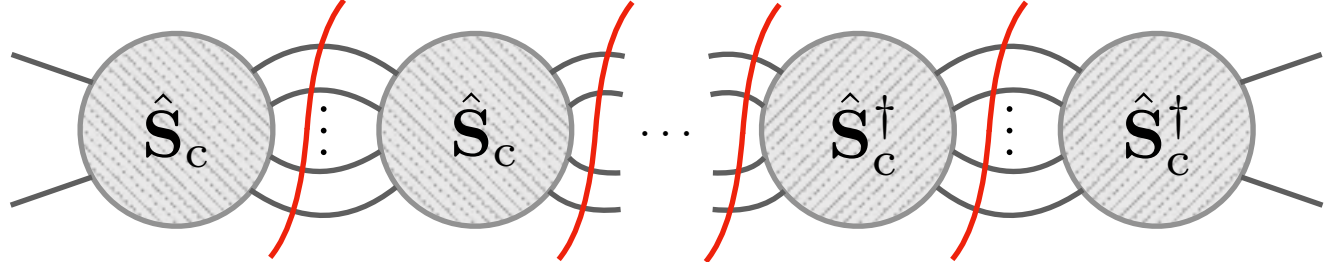
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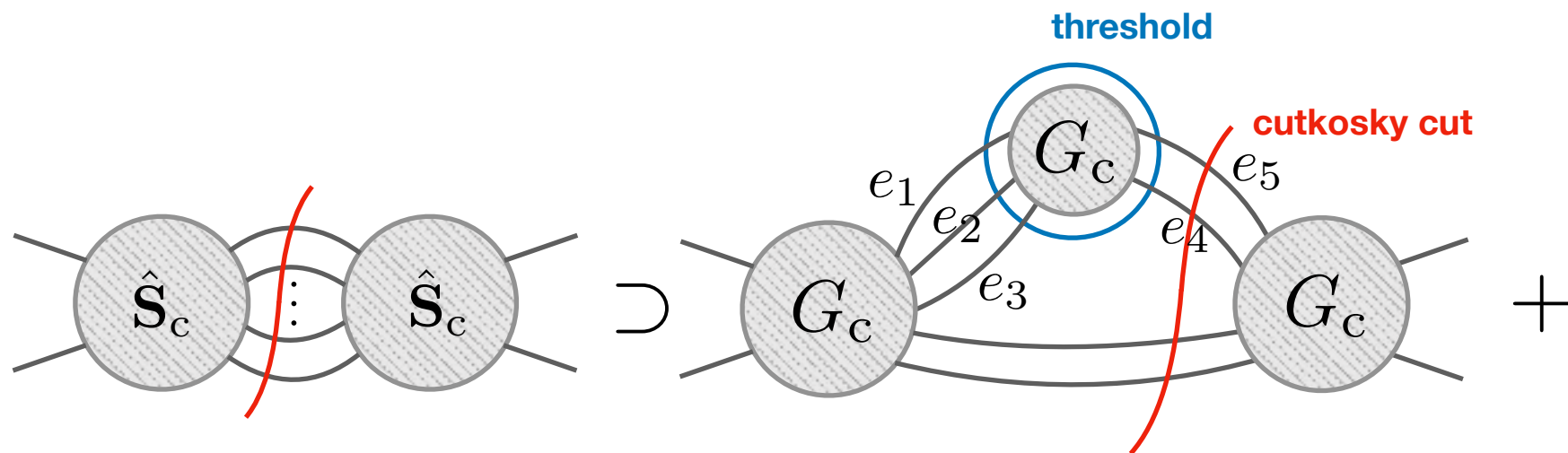
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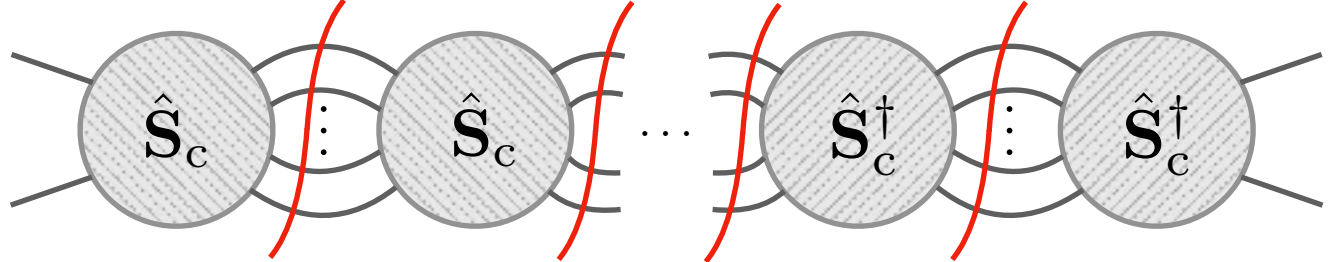
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$$\supset \frac{\delta(E_4 + E_5 - p_{12}^0)}{E_1 + E_2 + E_3 - E_4 - E_5}$$



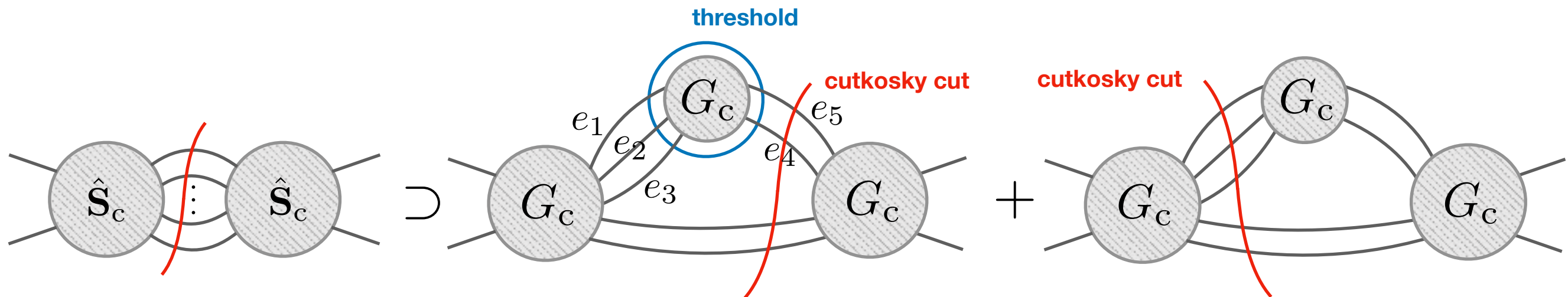
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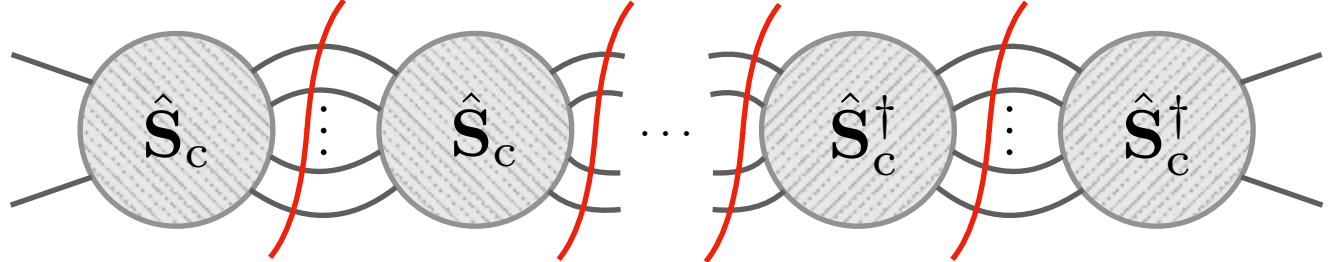
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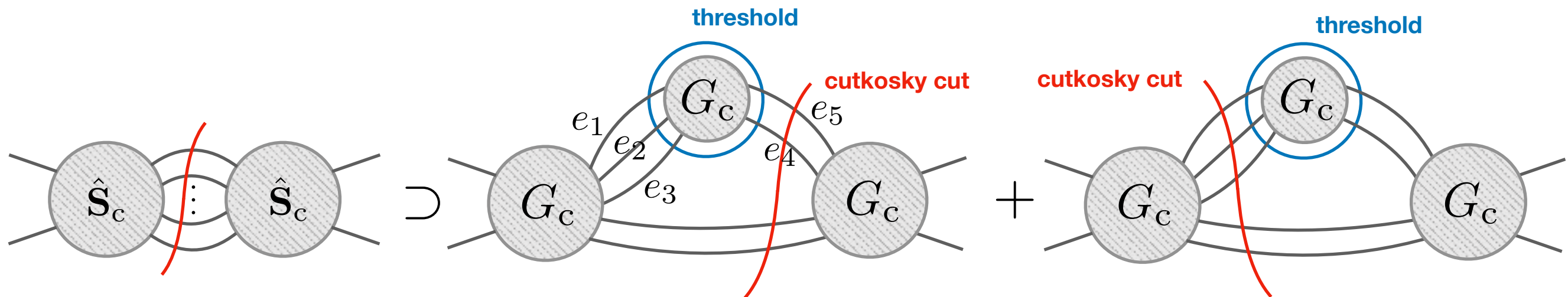
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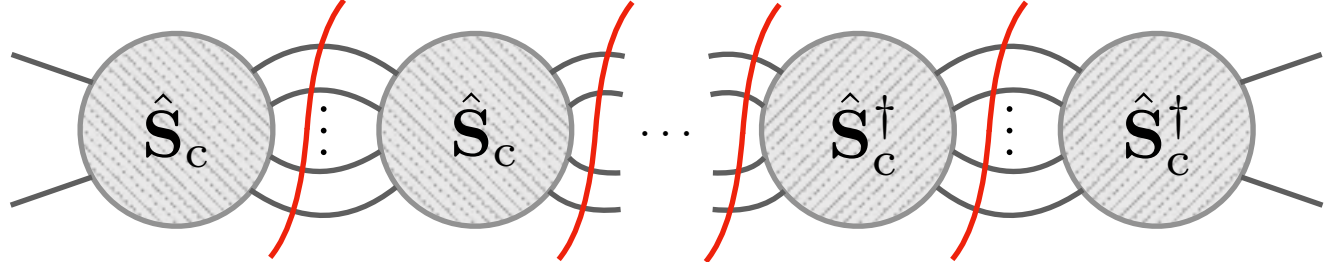
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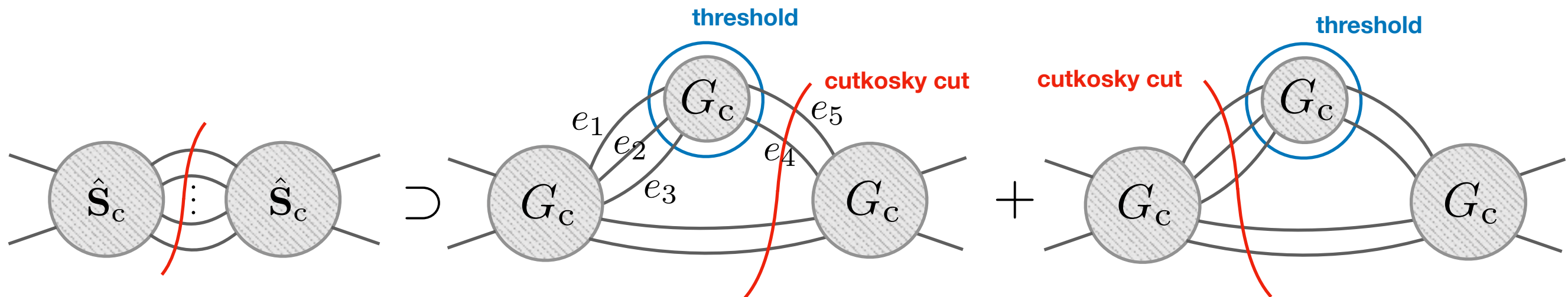
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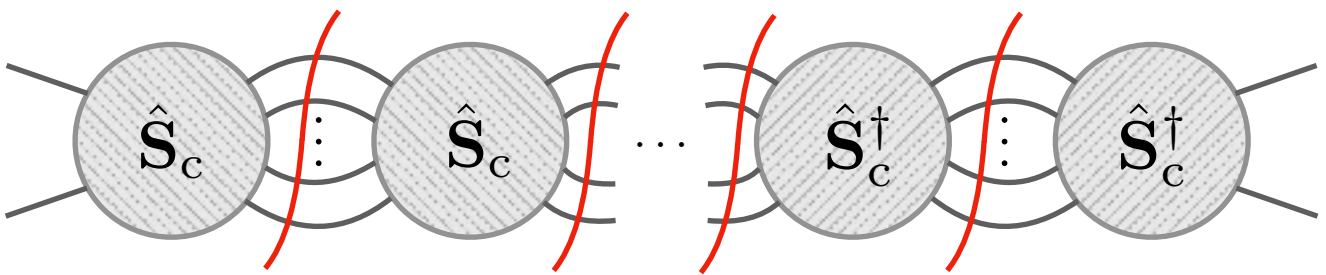
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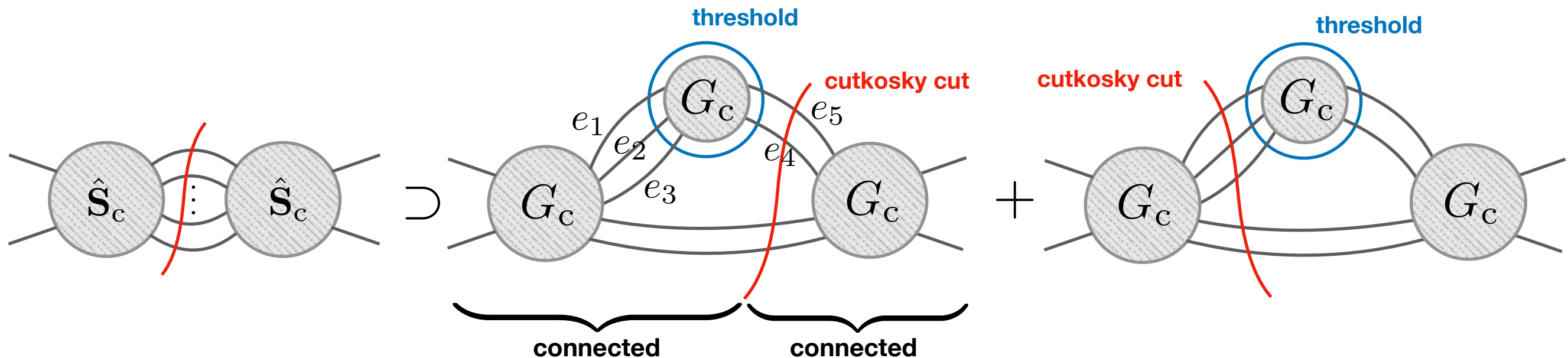
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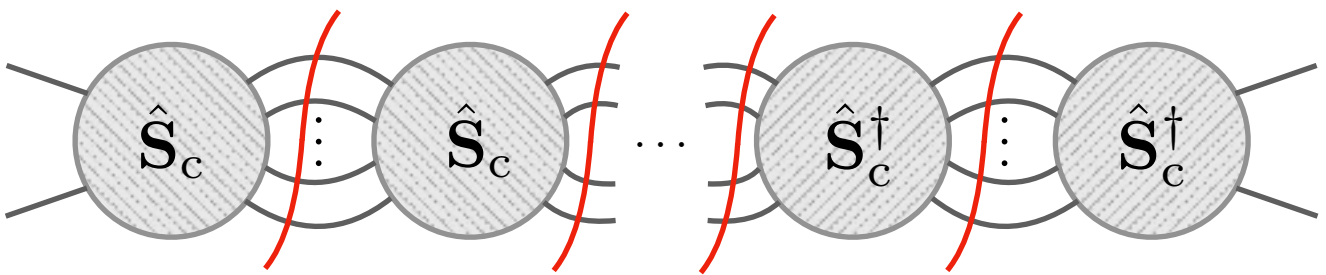
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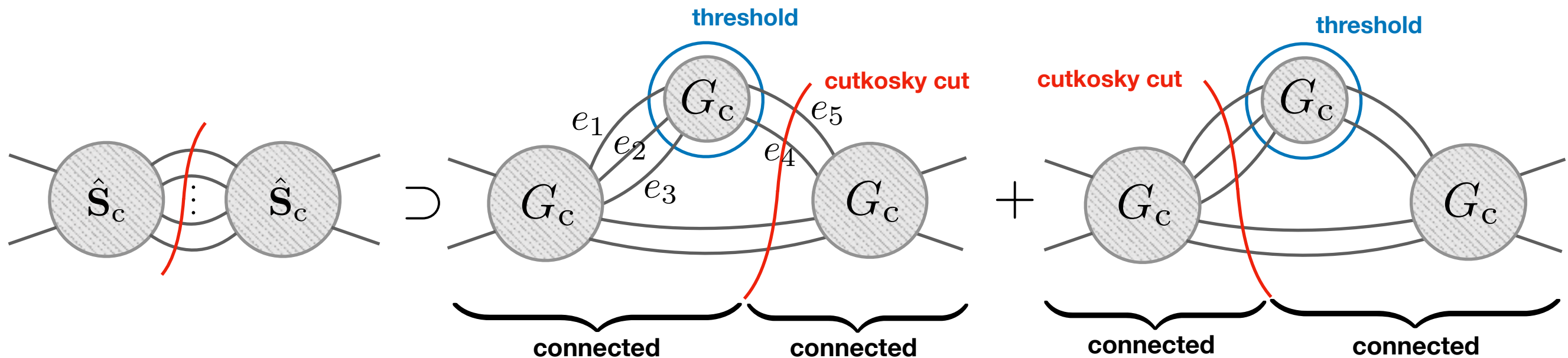
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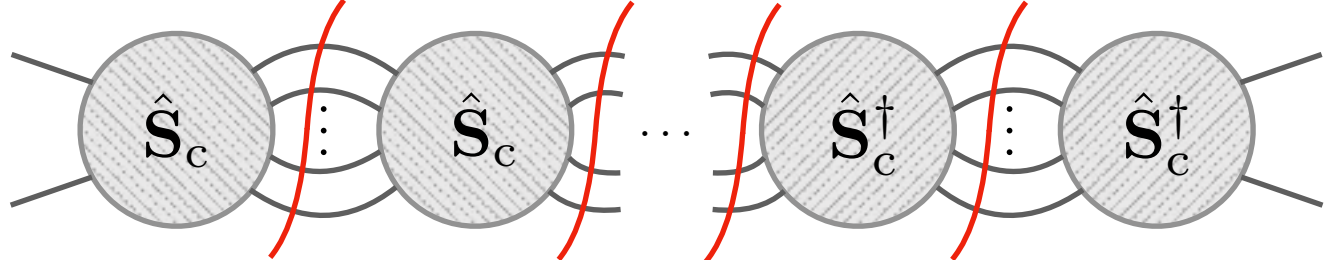
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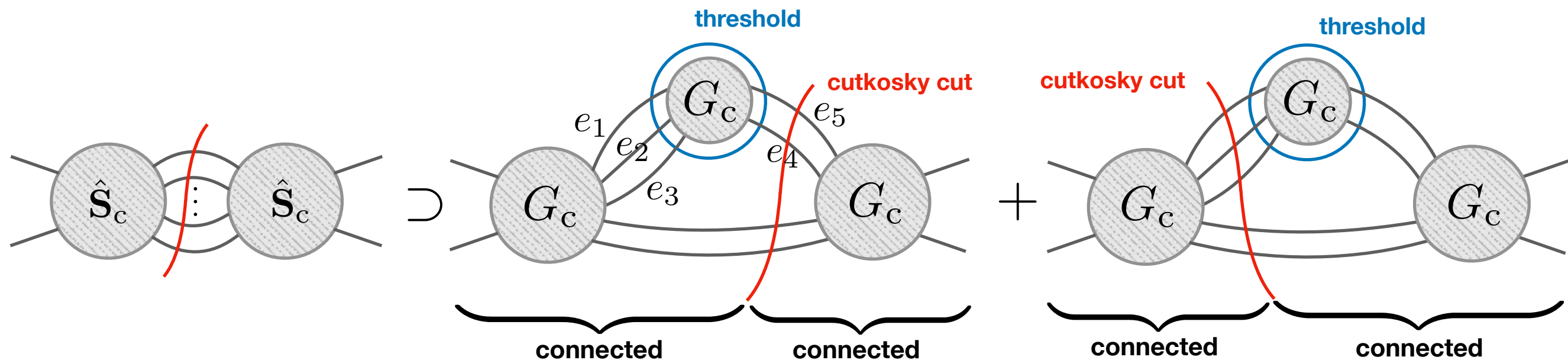
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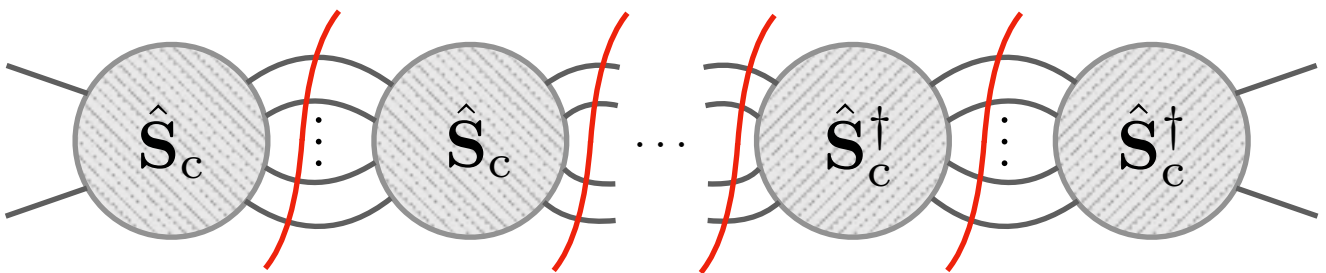
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$$\supset \frac{\delta(E_4 + E_5 - p_{12}^0)}{E_1 + E_2 + E_3 - E_4 - E_5} + \frac{\delta(E_1 + E_2 + E_3 - p_{12}^0)}{E_4 + E_5 - E_1 - E_2 - E_3}$$



It is finite when $E_4 + E_5 - E_1 - E_2 - E_3 \rightarrow 0$ (delta argument becomes same). In general:

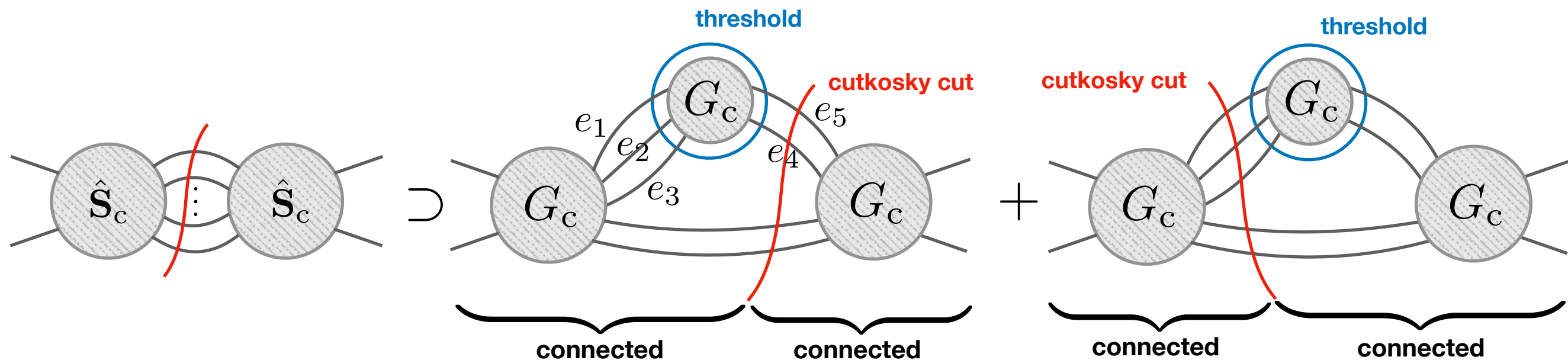
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Is IR finite for fixed n and m, that is *for fixed number of connected components!*

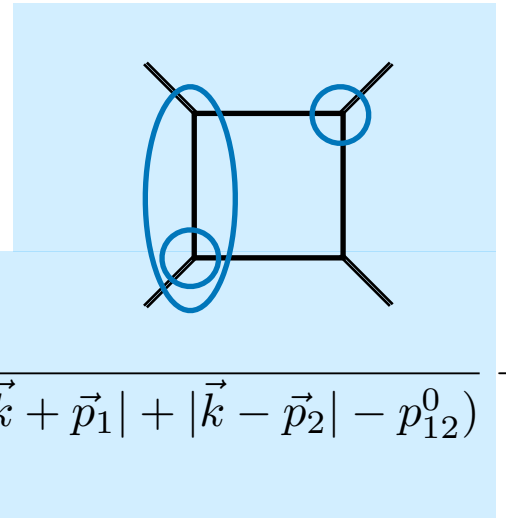
Backup slides

How to compute discontinuities

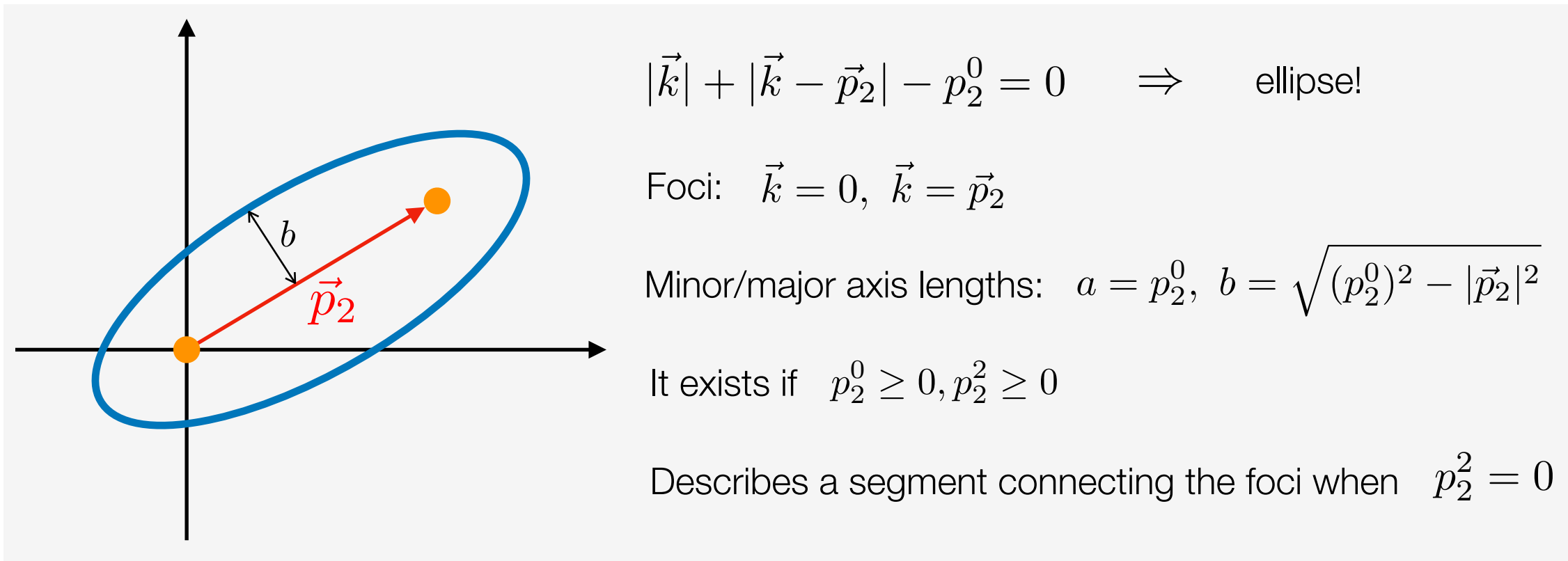
Thresholds in three-dimensional space

We kept on-shell energy dependence implicit. Let's re-instate it

$$\text{Box Diagram} = \dots + \int d^3 \vec{k} \frac{1}{\prod_{i=1}^4 2|\vec{k} + \vec{p}_i|} \frac{i^3}{(|\vec{k}| + |\vec{k} - \vec{p}_2| - p_2^0)(|\vec{k} + \vec{p}_1| + |\vec{k} + \vec{p}_1 - \vec{p}_3| - p_3^0)(|\vec{k} + \vec{p}_1| + |\vec{k} - \vec{p}_2| - p_{12}^0)} + \dots$$

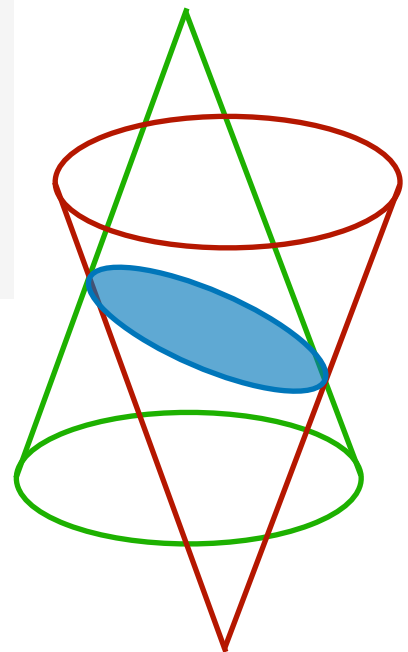


We can now draw the threshold locations in the space of spatial loop momenta

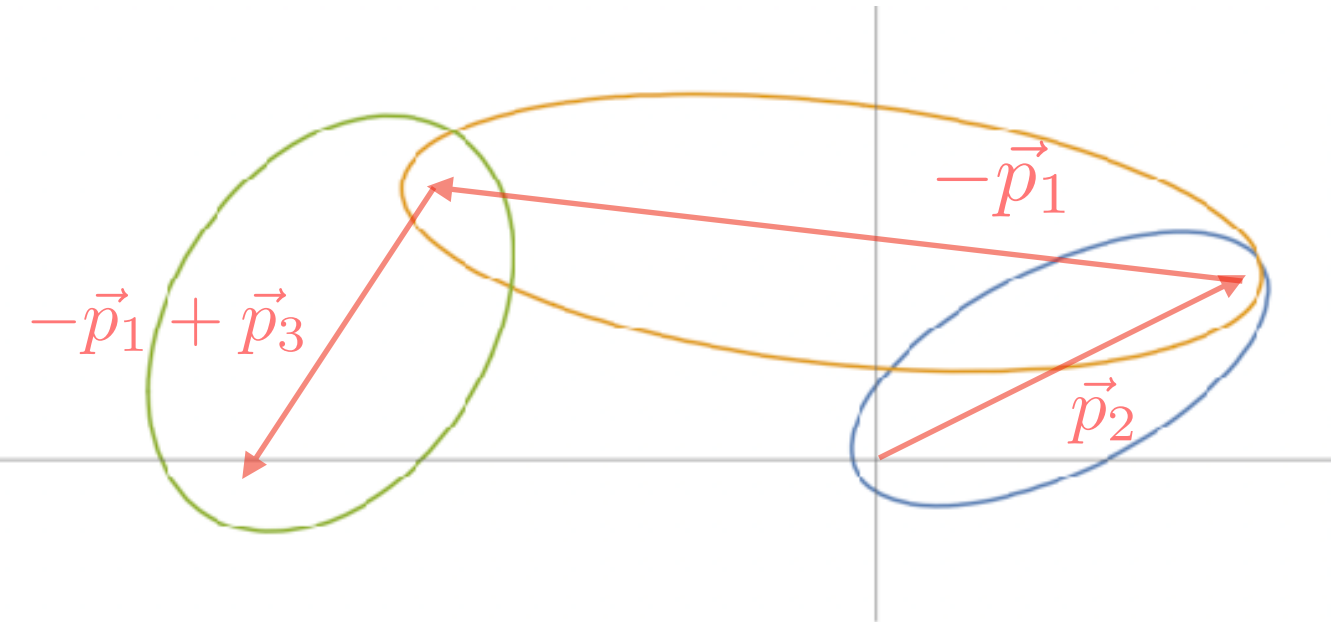


Conic obtained by intersecting a forward light cone with a backward one

$$\{k^2 = 0, k^0 > 0\} \cap \{(k - p_2)^2 = 0, k^0 - p_2^0 < 0\}$$



Now we can draw all the remaining thresholds



$$\eta_1 = |\vec{k}| + |\vec{k} - \vec{p}_2| - p_2^0 = 0$$

$$\eta_2 = |\vec{k} + \vec{p}_1| + |\vec{k} - \vec{p}_2| - p_{12}^0 = 0$$

$$\eta_3 = |\vec{k} + \vec{p}_1| + |\vec{k} + \vec{p}_1 - \vec{p}_3| - p_3^0 = 0$$

We now want to regulate the thresholds through a contour deformation $\vec{k} \rightarrow \vec{k} - i\kappa$

κ has to satisfy constraints

{

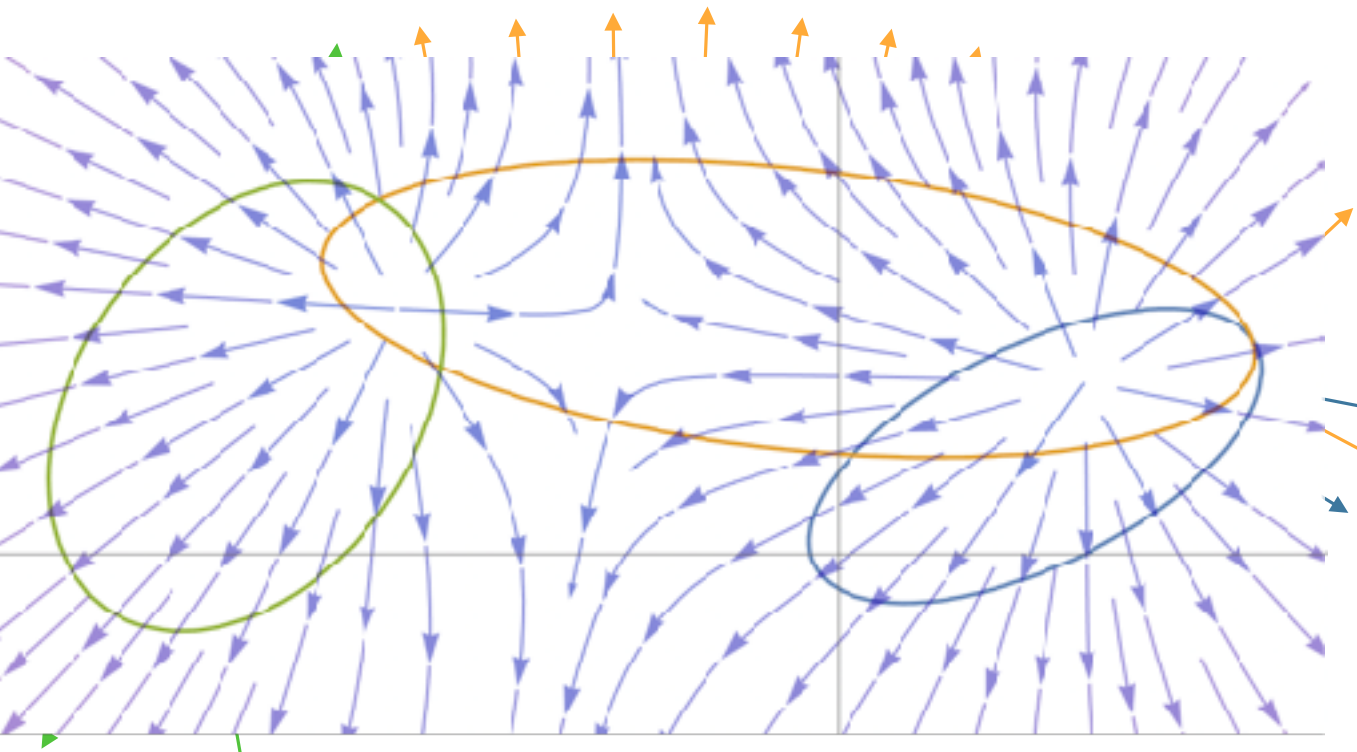
No crossing of branch-cuts

Stay on well-defined Riemann sheet

Satisfy causal prescription $\kappa \cdot \nabla \eta_i > 0$, when $\eta_i = 0$

(In Feynman parameter space, $\kappa \propto \nabla \mathcal{F}$)

Focus on the last one



The vector field

$$\kappa = \lambda_1(\vec{k})(\vec{k} - \vec{s}_1) + \lambda_2(\vec{k})(\vec{k} - \vec{s}_2)$$

satisfies the causal constraint for all thresholds!

1. Find out whether thresholds overlap
combinatorial problem
2. Find a point inside each overlap
convex problem

ZC, Hirschi, Kermanshah, Pelloni, Ruijl, [arXiv: 2009.05509](#) (2020)
Kermanshah, [arXiv: 2110.06869](#) (2020)

In general, the problem of finding the correct contour deformation decomposes

- 1. Combinatorial problem:** enumerate overlaps
- 2. Convex problem:** find point inside each overlap

When all internal propagators are massless, these problems can be solved in closed form, thanks to the factorisation formula!

This principle can be translated into a constraint on overlaps, establishing that there is one overlap for each spanning tree of the graph

For each of these overlaps, the point inside it, when internal propagators are massless, corresponds to the origin in the loop momentum basis identified by the spanning tree

$$\kappa = \sum_{\substack{\text{spanning} \\ \text{tree } T}} \lambda_T(\vec{k})(\vec{k} - \vec{s}_T)$$

This in turn shows that a deformation for massless internals always exists, and thus the Feynman diagram is well-defined!

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- 2. Convex problem:** find point inside each overlap

When all internal propagators are massless, these problems can be solved in closed form, thanks to the factorisation formula!

A set of n thresholds leads to a Δ^{-n} scaling only if the corresponding cuts divide the graph in exactly $n + 1$ connected components and not more

This principle can be translated into a constraint on overlaps, establishing that there is one overlap for each spanning tree of the graph

For each of these overlaps, the point inside it, when internal propagators are massless, corresponds to the origin in the loop momentum basis identified by the spanning tree

$$\kappa = \sum_{\substack{\text{spanning} \\ \text{tree } T}} \lambda_T(\vec{k})(\vec{k} - \vec{s}_T)$$

This in turn shows that a deformation for massless internals always exists, and thus the Feynman diagram is well-defined!

Local Unitarity

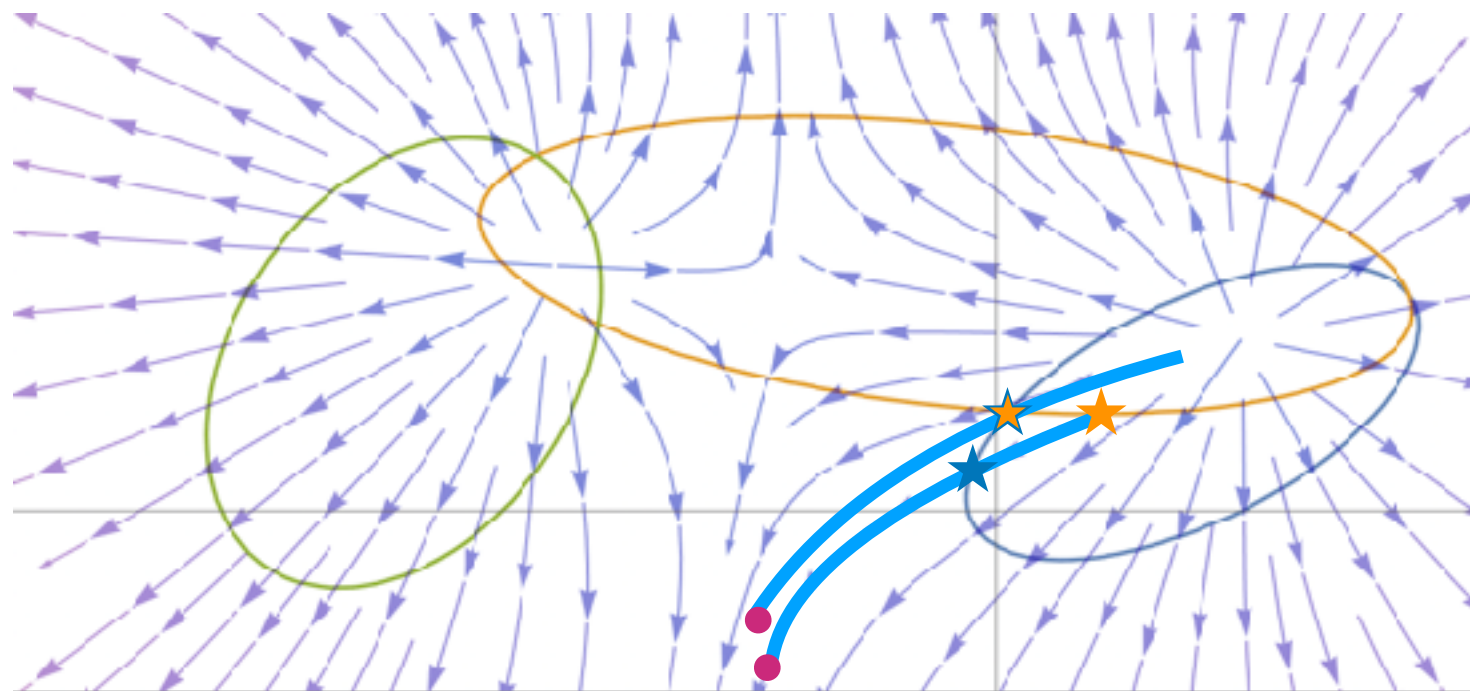
We can then use the deformation field to construct a parametrisation of the surfaces

Causal flow
$$\begin{cases} \partial_t \phi(t, \vec{k}) = \kappa(\phi(t, \vec{k})) \\ \phi(0, \vec{k}) = \vec{k} \end{cases}$$

Capatti, Hirschi, Pelloni, Ruijl,
arXiv:2010.01068 (2020)

Capatti, Hirschi, Ruijl,
arXiv:2203.11038 (2022)

We use the flow to transport us on the thresholds



$$\eta_i(\phi(t, \vec{k})) = 0, \quad \Rightarrow \quad t = t_i^*(\vec{k})$$

The function

$$\phi(t_i^*(\vec{k}), \vec{k})$$

assigns to each point a threshold point

$$\vec{k} \rightarrow \phi(t_i^*(\vec{k}), \vec{k}) \in \{\vec{k} \mid \eta_i = 0\}$$

The parametrisation are correlated so that they coincide at intersections!

$$\text{disc} \left[\text{diagram} \right] = \sum_{i=1}^3 \int d^3 \vec{k} \frac{\delta(\eta_i(\vec{k}))}{2E_1 2E_2 2E_3} \frac{\mathcal{O}_i}{\prod_{j \neq i}^3 \eta_j} = \int d^3 \vec{k} \sum_{i=1}^3 \frac{(\kappa \cdot \nabla \eta_i)^{-1}}{2E_1 2E_2 2E_3} \frac{\mathcal{O}_i}{\prod_{j \neq i}^3 \eta_j} \Big|_{\phi(t_i^*(\vec{k}), \vec{k})}$$

The right-hand side is integrable, and corresponds to a **differential cross-section!**

The causal flow aligns the phase-space integration measure of interference diagrams