Two-Loop Evanescent Integrals from Local Subtraction

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Radcor 2023

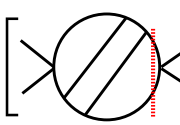
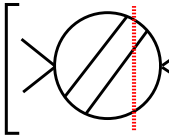
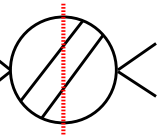
May 28th – June 2nd

In collaboration with Alessandro Georgoudis

based on [23xx.xxxxx] (in preperation)

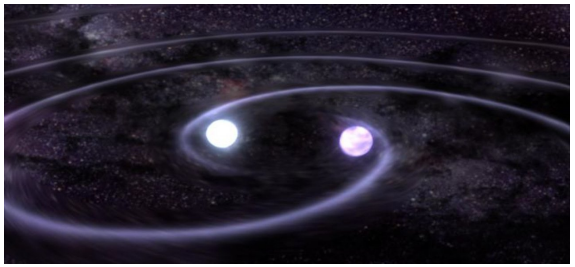
Multi-Loop Amplitudes are Ubiquitous

- Fundamental building block of collider predictions.

$\delta\sigma_n^{\mathrm{NNLO}} = \int d\Phi_n \Big[$  $+\cdots\Big] + \int d\Phi_{n+1}\Big[$  $+\cdots\Big] + \int d\Phi_{n+2}\Big[$  $+\cdots\Big].$

[many many talks]

- Of growing importance for gravitational wave observables.



[talks by Fernando and Michael]

Important to improve our understanding of scattering amplitudes.

The Canonical Structure of a Multi-Loop Computation

$$\mathcal{A}^{(l)}(p_1,\ldots,p_n) = \sum_j \underbrace{\mathcal{C}_j(p_1,\ldots,p_n,\epsilon)}_{\text{coefficients}} \underbrace{I_j^{(l)}(p_1,\ldots,p_n,\epsilon)}_{\text{integrals}}.$$

1. Compute integrals order by order in ϵ with differential eqs.
   [Kotikov '91; Gehrmann, Remiddi '01; Henn '13]
2. IBP reduction to master integrals: coefficients rational in ϵ.
   [Tkachov, Chetyrkin '81]
3. Expand everything in ϵ, find mysterious simplifications.

How do we understand and exploit simplifications in 4d limit?

see related [Lang, Pozzorini, Zhang, Zoller].

The Four-Dimensional Limit at One-Loop

- 4d limit at one loop has strong impact on analytic structure:

$$\mathcal{A}^{(1)} = \mathcal{A}^{\mathrm{cc}} + \mathcal{R} + O(\epsilon).$$

- $\mathcal{A}^{cc}$ is "cut-constructible": built using 4d unitarity cuts.

$$\mathcal{A}^{\mathrm{cc}} = \sum_i c_i^{(\bowtie)} I\left[\bowtie_i\right] + \sum_j c_j^{(\bowtie)} I\left[\bowtie_j\right] + \sum_k c_k^{(\bowtie)} I\left[\bowtie_k\right] + \sum_l c_l^{(\bowtie)} I\left[\bowtie_l\right].$$

- $\mathcal{R}$ is *rational piece*: built from evanescent integrals.

$$\mathcal{R} = \int \frac{d^D\ell}{i\pi^{D/2}}\left[\mu^2 R(\ell)\right], \quad \text{where} \quad \ell = \ell_4 + \underbrace{\mu}_{\epsilon\text{-dim}}, \quad \mu\cdot\ell_4 = 0.$$

This talk: What do evanescent integrals look like at two loops?

Outline

1. The Structure of Evanescent Integrals

2. Application: All-Plus Amplitudes from Local Subtraction

The Structure of Evanescent Integrals

Standard Technique at One Loop

- One-loop calculations use dimension-shift identities, e.g.

$$\int \mathrm{d}^{4-2\epsilon}\ell[\mu^2 f(\ell)] = -\epsilon \int \mathrm{d}^{6-2\epsilon}\ell[f(\ell)].$$

[Tarasov '96; Baikov '96]

- Only need pole part of $(6-2\epsilon)$-dimensional integral.

$$\int \mathrm{d}^{4-2\epsilon}\ell[\mu^2 f(\ell)] = -\epsilon \text{ poles} \left(\int \mathrm{d}^{6-2\epsilon}\ell[f(\ell)] \right) + \mathcal{O}(\epsilon).$$

- $(6-2\epsilon)$-dim integral only has UV pole, easy computation.

Technique does not apply to all evanescent integrals at two loop!

Local Subtraction for Evanescent Integrals

- [Anastasiou, Sterman '19] define a remainder for loop integrals.

$$\int [\mathrm{d}^D \vec{\ell}]\, f(\vec{\ell}) = \int [\mathrm{d}^D \vec{\ell}]\, f^{\mathrm{CT}}(\vec{\ell}) + \underbrace{\int [\mathrm{d}^4 \vec{\ell}_4] \left\{ f(\vec{\ell}_4) - f^{\mathrm{CT}}(\vec{\ell}_4) \right\}}_{I_{\mathrm{rem}}[f]} + O(\epsilon).$$

[Julia's talk]

- Counterterms locally cancel bad IR/UV power counting.

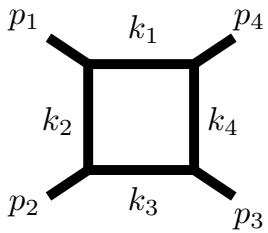
[Weinberg '60; Sterman '77] [Pavel's talk]

- Evanescent integrals trivially have zero finite remainder!

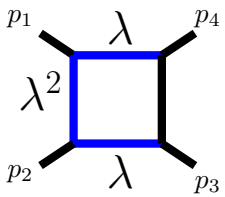
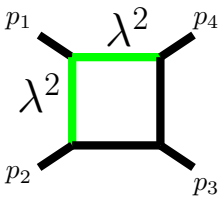
$$f_{\mathrm{ev}}(\vec{\ell}_4) = f_{\mathrm{ev}}^{\mathrm{CT}}(\vec{\ell}_4) = 0, \qquad \rightarrow \qquad I_{\mathrm{rem}}[f_{\mathrm{ev}}] = 0.$$

Up to $\mathcal{O}(\epsilon)$, evanescent integrals are the counterterms!

One-Loop Example: The Infrared



$[\mu^4]$, $\qquad p_i^2 = 0.$

Soft Region: $k_i \sim \lambda Q, \qquad d^D k_i \sim \lambda^D$	k_2 soft 
Collinear Region: $k_i = xp_j + \beta\eta + k_i^\perp,$ $\beta \sim \lambda^2, \; k_i^\perp \sim \lambda, \; d^D k_i \sim \lambda^D.$	k_1 collinear 

[Sterman '77]

Evanescent numerator suppresses infrared scaling: no counterterm.

$$\mu^4|_{k_i = k_i^{\mathrm{IR}}} = \mathcal{O}(\lambda^4).$$

One-Loop Example: The Ultraviolet

- Only relevant region at one-loop is the ultraviolet.

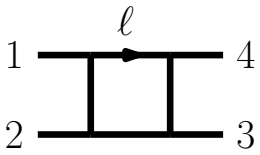
$$\ell \to \infty.$$

- Power Counting: Only divergent if $\omega \geq 0$

$$\int \mathrm{d}^D \ell f(\ell). \qquad f(\ell) \sim \ell^{\omega - D} + \mathcal{O}\left(\ell^{-1}\right).$$

[Weinberg '60]

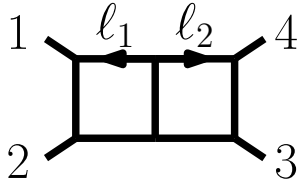
- Counterterm decomposition of evanescent box gives tadpole:



$$[\mu^4] = \int \frac{\mathrm{d}^D \ell}{i\pi^{D/2}} \frac{\mu^4}{D_1 D_2 D_3 D_4} = \int \frac{\mathrm{d}^D \ell}{i\pi^{D/2}} \frac{\mu^4}{(\ell^2 - M_{\mathrm{UV}}^2)^4} + \mathcal{O}(\epsilon)$$

$$= \ell[\mu^4] + \mathcal{O}(\epsilon) = -\frac{1}{6} + \mathcal{O}(\epsilon).$$

An Evanescent Double Box from Counterterms

$\mathcal{I}^{\text{ev}}_{\text{db}} =$ 

$[\mu_{22}^2]$, where $\mu_{22} = \mu_2 \cdot \mu_2$.

Relevant regions are one-loop like:

- ℓ_2 UV.
- $\ell_1 \to p_1$ (soft).
- p_1/p_2 collinear.

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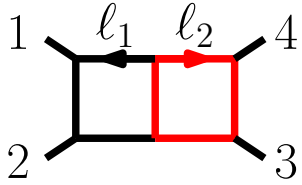
$\mathcal{I}^{\text{ev}}_{\text{db}} =$ 

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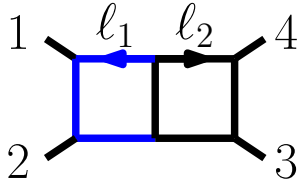
$\mathcal{I}^{\text{ev}}_{\text{db}} =$ 

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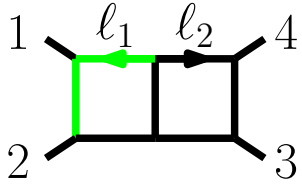
$\mathcal{I}^{\text{ev}}_{\text{db}} =$ 

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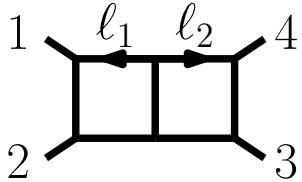
$\mathcal{I}^{\text{ev}}_{\text{db}} =$ 

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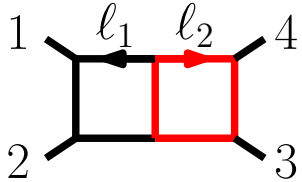
$\mathcal{I}^{\text{ev}}_{\text{db}} =$ 

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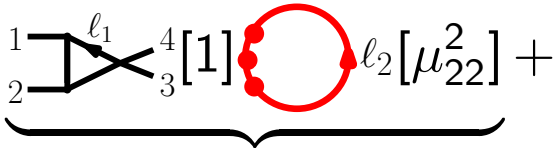
$\mathcal{I}^{\text{ev}}_{\text{db}} =$ 

$[\mu_{22}^2] +$

UV

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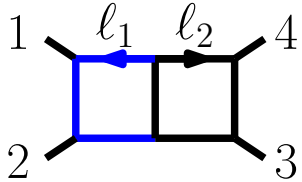
$\mathcal{I}^{\text{ev}}_{\text{db}} =$ 

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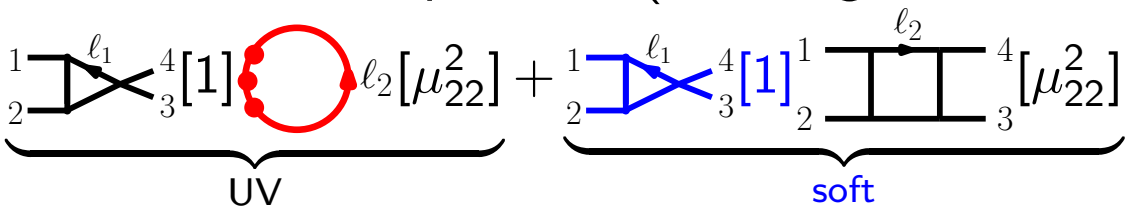
$\mathcal{I}^{\text{ev}}_{\text{db}} =$ 

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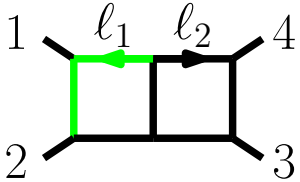
$\mathcal{I}^{\text{ev}}_{\text{db}} =$ 

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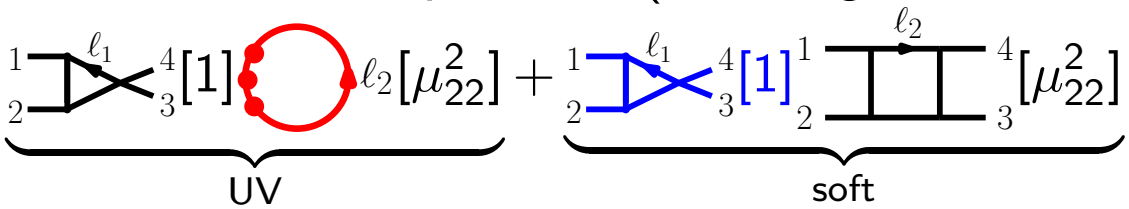
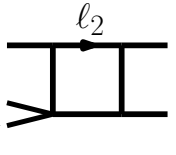
$\mathcal{I}^{\text{ev}}_{\text{db}} =$ 

$$\left[\underbrace{\frac{1}{\epsilon}\frac{1}{s_{12}}\int_0^1\left[\frac{\mathrm{d}x}{1-x}\right]_+}_{\text{collinear}} \right.$$


$$\left.[\mu_{22}^2] + \begin{pmatrix} 1 \leftrightarrow 2 \\ 3 \leftrightarrow 4 \end{pmatrix}\right] -$$

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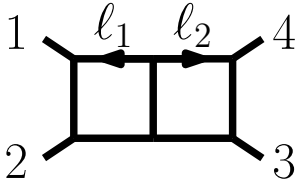
$\mathcal{I}^{\text{ev}}_{\text{db}} =$ 

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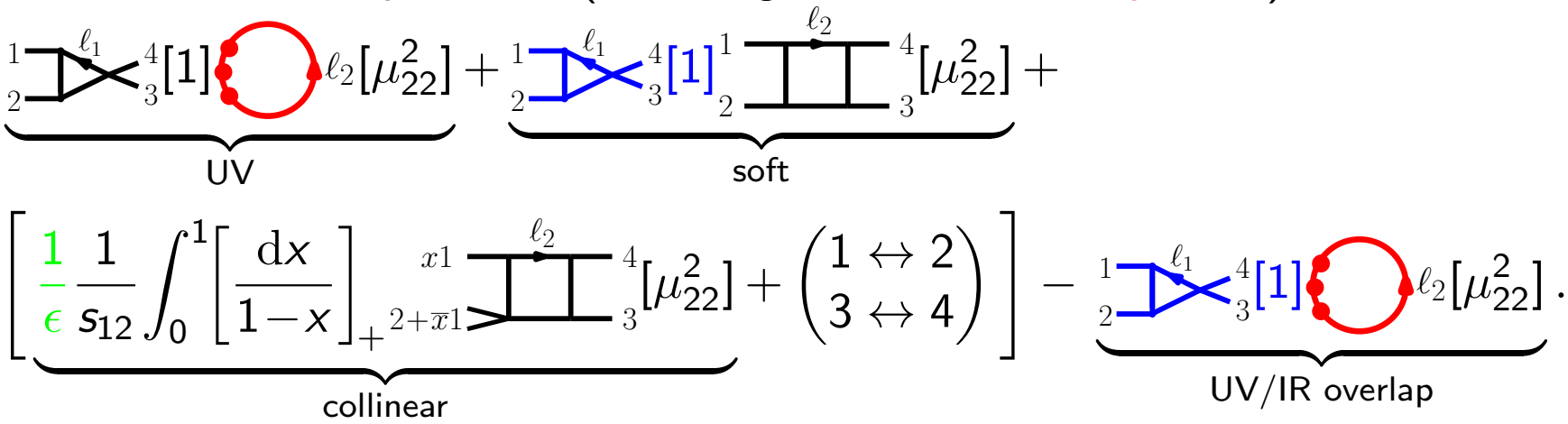
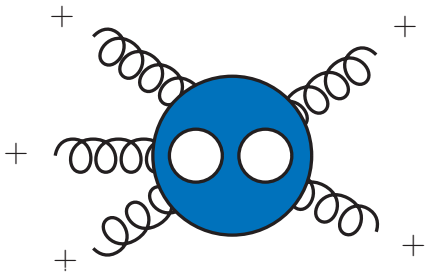
$\mathcal{I}^{\text{ev}}_{\text{db}} =$ 

Application: Two-Loop All-Plus Amplitudes

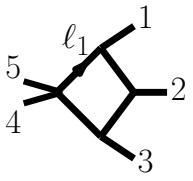


The Two-Loop Five-Point All-Plus Amplitude

- Surprisingly, 2-loop all-plus amplitudes are one-loop like!. E.g.:

$$\mathcal{A}_5^{(2)} = -\left(\frac{1}{\epsilon^2}\sum_{i=1}^{5}(-s_{i,i+1})^{-\epsilon}\right)\mathcal{A}_5^{(1)} + \mathcal{H}_5^{(2)} + \mathcal{O}(\epsilon),$$

$$\mathcal{H}_5^{(2)} = -\frac{(D_s-2)}{6}\sum_{\sigma\in Z_5}\sigma\circ\left[\frac{[45]^2}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}\right.$$



$$\left.\Big|_{D=6} + \text{rational}\right].$$

[Gehrmann, Henn, lo Presti '15]

- $\mathcal{A}_5^{(2)}$ is entirely evanescent!

$$\mathcal{A}_5^{(2)} = \int \mathrm{d}^D\ell_1\mathrm{d}^D\ell_2 A_5^{(2)}(\ell_1,\ell_2), \qquad A_5^{(2)}(\ell_1^4,\ell_1^4) = 0.$$

[Badger, Zhang, Frellesvig '13]

What can local subtraction tell us?

Counterterm Decomposition

- We organize $\mathcal{A}_5^{(2)}$ into classes of counterterm.

$$\mathcal{A}_5^{(2)} = \mathcal{A}_5^{(2),\text{soft}} + \mathcal{A}_5^{(2),\text{col}} + \mathcal{A}_5^{(2),\text{UV}} + \mathcal{O}(\epsilon).$$

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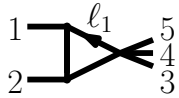
- Soft CT matches IR subtraction.

$$\mathcal{A}_5^{(2),\text{soft}} = \sum_{\sigma \in Z_5} \sigma \circ$$  $$[s_{12}] A_5^{(1)} = -\frac{1}{\epsilon^2} \sum_{i=1}^{5} (-s_{i,i+1})^{-\epsilon} A_5^{(1)}.$$

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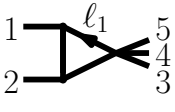
$$\mathcal{A}_5^{(2),\text{soft}} = \sum_{\sigma \in Z_5} \sigma \circ$$  $$[s_{12}] A_5^{(1)} = -\frac{1}{\epsilon^2} \sum_{i=1}^{5} (-s_{i,i+1})^{-\epsilon} A_5^{(1)}.$$

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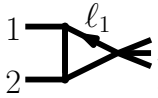
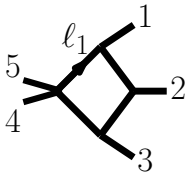
$\mathcal{A}_5^{(2),\text{soft}} = \sum_{\sigma \in Z_5} \sigma \circ$  $[s_{12}] A_5^{(1)} = -\frac{1}{\epsilon^2} \sum_{i=1}^{5} (-s_{i,i+1})^{-\epsilon} A_5^{(1)}.$

$\mathcal{A}_5^{(2),\text{col}} = 0.$ $\leftarrow$ non-standard cancellation mechanism

- Ultra-violet part matches finite remainder!:

$\mathcal{A}_5^{(2),\text{UV}} =$  $\ell [\mu^4](D_s - 2) \sum_{\sigma \in Z_5} \sigma \circ \left[\frac{[45]^2}{\langle 12 \rangle \langle 23 \rangle \langle 31 \rangle}$  $\Big|_{D=6} + \text{rational} \right].$

Summary and Conclusions

- Amplitudes simplify in limits. We are exploring simplifications that arise in the four-dimensional limit of evanescent integrals.
- Local subtraction shows that evanescent integrals localize onto singular configurations, giving a one-loop-like structure.
- Two-loop all-plus amplitudes are evanescent. Our technique shows that the finite remainder is of an ultra-violet nature.

Collinear Counterm

- Anastasiou/Sterman collinear counterterm is more involved:

$$\mathrm{CT}\left(\int \frac{\mathrm{d}^D\ell}{i\pi^{D/2}}\left[\frac{G(\ell)}{\ell^2(\ell-p)^2}\right]\right) = \int \mathrm{d}^D\ell\left[\frac{G(x[\ell]p)}{\ell^2(\ell-p)^2} - \frac{G(x[\ell]p)}{(\ell^2-M^2)([\ell-p]^2-M^2)}\right].$$

- First term is scaleless. Second term can be written as

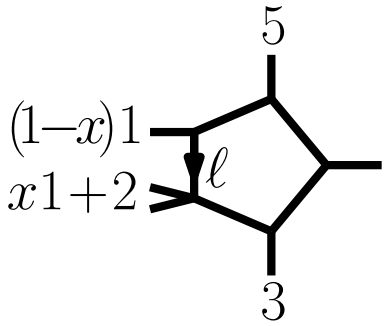
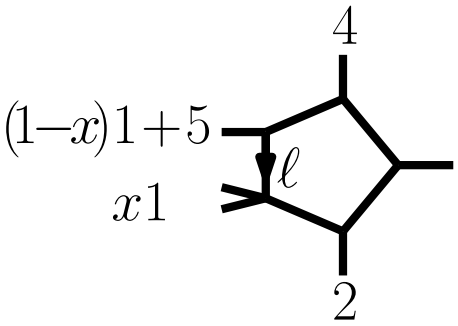
$$\int \mathrm{d}x G(xp_j)\underbrace{\left[\int \frac{\mathrm{d}^D\ell}{i\pi^{D/2}}\frac{\delta(x-\frac{\ell\cdot\eta}{p\cdot\eta})}{(\ell^2-M^2)([\ell-p]^2-M^2)}\right]}_{I_{\mathrm{col}}(x)}.$$

- The integral localizes to $0 \leq x \leq 1$.

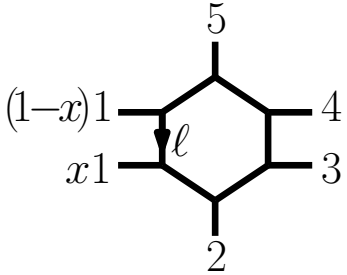
$$I_{\mathrm{col}}(x) = \Theta(0 \leq x \leq 1)\frac{\Gamma(1+\epsilon)}{\epsilon}M^{-2\epsilon}.$$

Special Collinear Identities

- We find "cross-topology" relations for collinear integrals.

$$\frac{1}{[1-(1-x)]_+}$$  $$+\frac{1}{[1-x]_+}$$  $$= 0.$$

- Can be understood as integrand relation from supertopology.



- Collinear hexagon legs $\Rightarrow$ linear relation of propagators.

$$\frac{1}{1-(1-x)}[(\ell - xp)^2 - \ell^2] + \frac{1}{1-x}[(\ell + [1-x]p)^2 - \ell^2] = 0.$$