

Pentagon functions for one-mass scattering amplitudes

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based on work in collaboration with

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RADCOR 2023, Crieff
30 May 2023

Precision physics at hadron colliders

- Theoretical predictions (QCD corrections) at Next-to-Next-to-Leading-Order are required nowadays

$$\sigma = \sigma_{\text{LO}} + \alpha_s \sigma_{\text{NLO}} + \frac{\alpha_s^2 \sigma_{\text{NNLO}}}{\approx 1-10\%} + \mathcal{O}(\alpha_s^3)$$

- $2 \rightarrow 1$ and $2 \rightarrow 2$ extensively studied @NNLO
- Great interest in QCD corrections @NNLO for $2 \rightarrow 3$ production

$$pp \rightarrow V + 2j, \quad pp \rightarrow VV' + j, \quad pp \rightarrow H + 2j, \quad pp \rightarrow V + b\bar{b}, \quad pp \rightarrow t\bar{t} + j \\ pp \rightarrow t\bar{t} + \gamma, \quad pp \rightarrow t\bar{t} + W, \quad pp \rightarrow t\bar{t} + Z, \quad pp \rightarrow t\bar{t} + H, \quad pp \rightarrow \gamma\gamma\gamma$$

[from Les Houches 2021 wish list]

- Double-virtual corrections is an essential ingredient of NNLO calculations
⇒ Two-loop five-particle scattering amplitudes

Analytic, algebraic, numeric complexity of scattering amplitudes and Feynman integrals

Rapid growth of complexity with

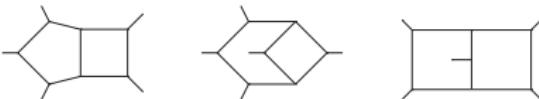
- # loops
- # legs
- # scales (also internal and external masses)

Efficient computational tools for complex multi-scale amplitudes are needed

Huge recent progress in calculation of five-particle two-loop amplitudes and four-particle three-loop amplitudes

Five-particle massless scattering @NNLO

- Feynman integrals



[Gehrman, Henn, Lo Presti '15] [DCh, Gehrman, Henn, Lo Presti, Mitev, Wasser '18]
[Abreu, Page, Zeng '18] [Abreu, Dixon, Herrmann, Page, Zeng '18]
[DCh, Gehrman, Henn, Wasser, Zhang, Zoia '18]

- Basis of special functions

[DCh, Sotnikov '20] [Gehrman, Henn, Lo Presti '18]
gitlab.com/pentagon-functions/pentagonfunctions.hepforge.org

- QCD amplitudes (two-loop, planar and non-planar)

$$q\bar{q} \rightarrow \gamma\gamma\gamma \text{ full color}$$

[Abreu, Page, Pascual, Sotnikov '20] [Chawdhry, Czakon, Mitov, Poncelet '20]
[Abreu, De Laurentis, Ita, Klinkert, Page, Sotnikov '23]

$$q\bar{q} \rightarrow g\gamma\gamma, qg \rightarrow q\gamma\gamma \text{ full color}$$

[Agarwal, Buccioni, von Manteuffel, Tancredi '21]

$$gg \rightarrow g\gamma\gamma \text{ full color}$$

[Badger, Brönnum-Hansen, DCh, Gehrman, Hartanto, Henn, Marcoli, Moodie, Peraro, Zoia '21]

$$gg \rightarrow ggg, qg \rightarrow qgg, \dots \text{ all planar five-parton}$$

[Abreu, Febres Cordero, Ita, Page, Sotnikov '21]

- QCD cross-sections @NNLO leading color

$$pp \rightarrow \gamma\gamma\gamma \text{ [Chawdhry, Czakon, Mitov, Poncelet '19][Kallweit, Sotnikov, Wiesemann '20]}$$

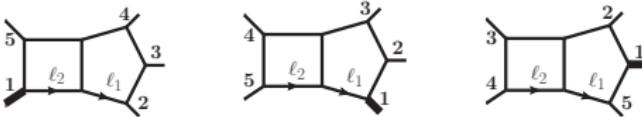
$$pp \rightarrow j\gamma\gamma \text{ [Chawdhry, Czakon, Mitov, Poncelet '21][Badger, Gehrman, Marcoli, Moodie '21]}$$

$$pp \rightarrow jjj \text{ [Czakon, Mitov, Poncelet '21][Chen, Gehrman, Glover, Huss, Marcoli '22]}$$

$$pp \rightarrow \gamma jj \text{ full color } \text{ [Badger, Czakon, Hartanto, Moodie, Tiziano Peraro, Poncelet, Zoia '23]}$$

Planar five-particle one-mass scattering @NNLO

- Feynman integrals



[Abreu, Ita, Moriello, Page, Tschernow, Zeng '20][Canko, Papadopoulos, Syrrakos '20]

[Papadopoulos, Tommasini, Wever '15]

- Basis of special functions

[DCh, Sotnikov, Zoia '21]

gitlab.com/pentagon-functions/

- QCD amplitudes for electroweak processes @leading color (two-loop, planar)

$$u\bar{d} \rightarrow W^+ b\bar{b}, pp \rightarrow b\bar{b}H$$

[Badger, Hartanto, Zoia '21][Badger, Hartanto, Kryś, Zoia '21]

$$pp \rightarrow W(\rightarrow \ell\nu)\gamma + j$$

[Badger, Hartanto, Kryś, Zoia '22]

$$4p + W \text{ planar: } gg \rightarrow q\bar{q} + W(\rightarrow \ell\bar{\ell}), Q\bar{Q} \rightarrow q\bar{q} + W(\rightarrow \ell\bar{\ell})$$

and also planar Z/γ^*

[Abreu, Febres Cordero, Ita, Klinkert, Page, Sotnikov '21]

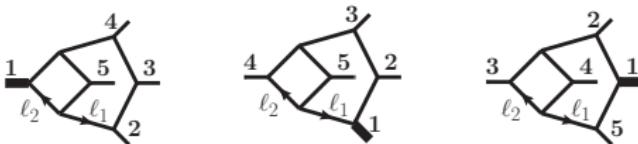
- QCD corrections @NNLO leading color

$$pp \rightarrow Wb\bar{b}$$

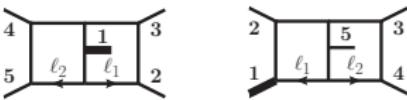
[Hartanto, Poncelet, Popescu, Zoia '22]

Non-planar five-particle one-mass scattering @NNLO

- Feynman integrals



[Abreu, Ita, Page, Tschernow '21][Kardos, Papadopoulos, Smirnov, Syrrakos, Wever '22]



- Basis of special functions

- ? QCD amplitudes for electroweak processes

$4p + Z/\gamma^*$ @leading color: two-loop, planar and **nonplanar**

- ? QCD corrections @NNLO for electroweak production

Finite remainders of amplitudes in the pentagon function basis

$$\mathcal{R} = \sum_a \underbrace{r_a(X)}_{\substack{\text{rational} \\ \text{coefficients}}} \underbrace{\text{mon}_a(\mathcal{F})}_{\substack{\text{special} \\ \text{functions} \\ [\text{universal}]}}$$

[depend on QFT,
type of scattered
particles and helicities]

Basis of special functions $\mathcal{F} := \{f_i^{(w)}(X)\}$ for five-particle scattering

Analytics

- compact expressions
- manifest analytic properties
- explicit cancellation of IR/UV poles
- avoid/control spurious cancellations
- embedded in the most efficient calculation strategies of QCD amplitudes

Numerics

- C++ implementation
- fast
- high precision
- stable across phase space

Pentagon function basis is a prerequisite for efficient rational reconstruction of amplitudes

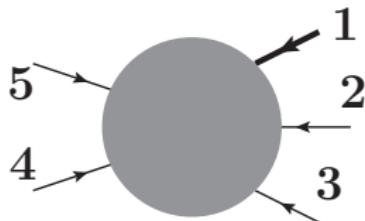
Calculation workflow

$$\begin{array}{ccc} \mathcal{A}^{(2)}(X, \epsilon) = \sum \text{Diagram} & \xrightarrow{\text{reduction}} & \sum_i \int d^D \ell \frac{\mathcal{N}_i(\ell)}{\prod_j \rho_j(\ell)} \\ & & \downarrow \text{IBP} \\ \sum_{w \geq -4} \epsilon^w \sum_a \underbrace{d_{a,w}(X)}_{\text{rational}} \text{mon}_a(\mathcal{F}) & \xleftarrow{\text{to funct basis}} & \sum_i \underbrace{c_i(X, \epsilon)}_{\text{rational}} \underbrace{\mathcal{I}_i(X, \epsilon)}_{\text{Master Integrals}} \\ & \downarrow \text{IR/UV subtraction} & \end{array}$$

$$\mathcal{R}^{(2)}(X) = \sum_a \underbrace{r_a(X)}_{\text{rational}} \text{mon}_a(\mathcal{F})$$

Modular arithmetics \mathbb{F}_p and rational reconstructions [von Manteuffel, Schabinger '14] [Peraro '16 '19] help to bypass complexity of intermediate steps

Kinematics of the five-particle one-mass scattering



Four light-like and one massive momenta

$$p_1^2 > 0, \quad p_2^2 = p_3^2 = p_4^2 = p_5^2 = 0, \quad \sum_i p_i = 0$$

Six independent Mandelstam variables $s_{ij} := (p_i + p_j)^2$,

$$X := (p_1^2, s_{12}, s_{23}, s_{34}, s_{45}, s_{15})$$

and a parity-odd invariant

$$4i\epsilon_{\alpha\beta\gamma\delta} p_1^\alpha p_2^\beta p_3^\gamma p_4^\delta = \pm \sqrt{\Delta_5}$$

Eight families of one-mass five-particle two-loop Feynman integrals

Feynman integrals : eight propagators and three ISP

$$I_{\vec{\nu}}(X, \epsilon) = \int \frac{d^D \ell_1}{i\pi^{\frac{D}{2}}} \int \frac{d^D \ell_2}{i\pi^{\frac{D}{2}}} \frac{\rho_9^{-\nu_9} \rho_{10}^{-\nu_{10}} \rho_{11}^{-\nu_{11}}}{\rho_1^{\nu_1} \rho_2^{\nu_2} \dots \rho_8^{\nu_8}}, \quad \nu_1, \nu_2, \dots, \nu_8 \in \mathbb{Z}, \\ \nu_9, \nu_{10}, \nu_{11} \in \mathbb{Z}_{\leq 0}$$

Laporta algorithm ['00] solves the Integration-by-Part identities (IBP) and finds a finite basis of **Master Integrals**

								
# MI	74	75	86	86	86	135	142	179
top sector	3	3	3	3	3	3	8	9

Efficient implementation in the finite-field framework **FiniteFlow**, **FIRE6**, **Kira 2.0**

[von Manteuffel,Schabinger '14; Peraro '16 '19; Smirnov,Chukharev '19; Klappert,Lange,Maierhofer,Usovitsch '20]

Polylogarithmic alphabet for all two-loop families of five-particle one-mass integrals

204-letter alphabet

$$\{W_i = W_i(X)\}_{i=1}^{204}$$

is closed under $4!$ permutations $\sigma \in S_4$ of the massless momenta

$$\sigma(d \log(W_i)) \in \langle d \log(W_1), \dots, d \log(W_{204}) \rangle_{\mathbb{Q}}$$

93 letters of the alphabet are linear or quadratic in Mandelstam variables, e.g.

$$W_1 = p_1^2, \quad W_2 = s_{12}, \dots, \quad W_{16} = s_{15} - s_{34}, \dots$$

$$W_{28} = s_{12}s_{15} - p_1^2 s_{34}, \dots, \quad W_{70} = s_{12}s_{15} - s_{12}s_{23} - p_1^2 s_{34}, \dots$$

and the remaining letters are algebraic and involve square roots

$1 + 3 + 6 = 10$ square roots in the alphabet

Analytic solution of the Master Integrals in terms of the iterated integrals

- DE is a powerful method to solve **Master Integrals** analytically
- DE takes the *canonical form* for a natural choice of **Master Integrals**

[Henn '13]

- **Easy:** formally solve DE in terms of the **iterated integrals**, i.e. iterated integrations of the $d \log$ forms

$$[W_{i_1}, \dots, W_{i_w}](X) := \int_{\gamma} [W_{i_1}, \dots, W_{i_{w-1}}] d \log(W_{i_w})$$

along a path γ linking point X and a reference point X_0

- **Automatized:** Fix boundary constants of the DE, e.g. evaluate numerically with AMFlow [Liu, Ma '22] the Master Integrals at the reference point X_0
- **Difficult:** find the **canonical basis** of **Master Integrals**

Algorithmic construction of the pentagon functions

- Series expand **canonical Master Integrals** in dim reg

$$g_\sigma(X, \epsilon) = \sum_{w \geq 0} \epsilon^w g_\sigma^{(w)}(X), \quad \sigma \in S_4$$

and solve them (for all eight families in S_4 permutations) as *pure* linear combinations of the iterated integrals (graded by the transcendental weight)

- Dependencies among $\{g_\sigma^{(w)}\}$? Iterated integrals satisfy the Shuffle algebra
- Pentagon functions : algebraically independent set of iterated integrals
 $\mathcal{F} := \{f_i^{(w)}\}$

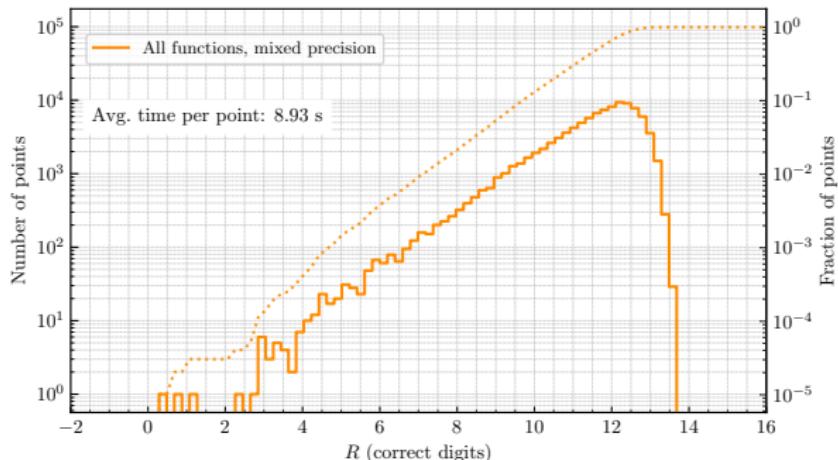
weight	P \cup PB	HB	DP	Total
1	11	0	0	11
2	25	10	0	35
3	145	72	0	217
4	675	305	48	1028

- **Canonical Master Integrals** in the pentagon function basis

$$g_\sigma^{(w)}(X) = \text{weight-}w \text{ polynomial in } \mathcal{F} := \{f_i^{(w)}\}, \zeta_2, \zeta_3$$

Efficient numerical evaluation of the one-mass pentagon functions

Implemented in the public C++ library `PentagonFunctions++` and ready for phenomenological applications



- ✓ Stable
- ✓ Fast
- ✓ Precise

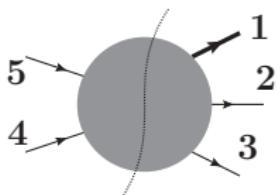
Physical simulation of 10^5 phase-space points evaluated in double precision

$$R(X) := \min_{f \in \mathcal{F}_{\text{planar}}} \left(-\log_{10} \left| \frac{f(X)_{\text{double}}}{f(X)_{\text{exact}}} - 1 \right| \right)$$

Evaluations of planar pentagon functions are much faster, $\approx 0.22s$ per point

Permutations of the Master Integrals vs Analytic continuation

Define pentagon functions $\mathcal{F} := \{f_i^{(w)}\}$ in the region $45 \rightarrow 123$



Master integrals/amplitudes/finite remainders in any scattering region with massive production are expressed in the same basis \mathcal{F} ,

$$g(\underbrace{X}_{\text{region } \sigma_4\sigma_5 \rightarrow 1\sigma_2\sigma_3}) = g_\sigma \left(X' := \underbrace{\sigma^{-1} \circ X}_{\text{region } 45 \rightarrow 123} \right)$$

↑
pure polynomial in \mathcal{F}

Permutations of the Master Integrals vs Analytic continuation

Difficult!

Analytically continue from
 $45 \rightarrow 123$ to all scattering regions \Leftrightarrow Solve DE in $4!$ permutations

Easy and Automatized!

Moreover, the pentagon functions are closed upon permutations

$$\sigma\left(f_i^{(w)}\right) = \text{weight-}w \text{ polynomial in } \mathcal{F} := \{f_i^{(w)}\}, \zeta_2, \zeta_3$$

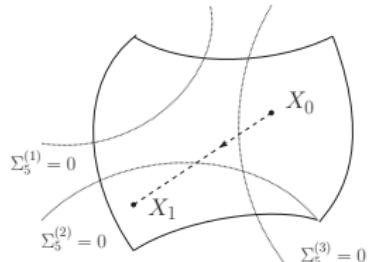
Finite remainders of amplitudes calculated in the region $45 \rightarrow 123$

$$\mathcal{R}(X) = \sum_a r_a(X) \operatorname{mon}_a(\mathcal{F})(X)$$

are automatically transferred to all regions

Singularities of the non-planar pentagon functions inside the scattering region

Six square-roots (related by S_4 permutations)

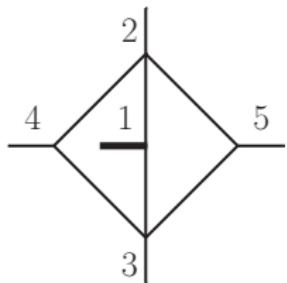


$$\Sigma_5^{(1)}, \Sigma_5^{(2)}, \dots, \Sigma_5^{(6)}$$

are singular surfaces

$$\begin{aligned} \Sigma_5^{(1)} := & (s_{12}s_{15} - s_{12}s_{23} - s_{15}s_{45} + s_{34}s_{45} + s_{23}s_{34})^2 \\ & - 4s_{23}s_{34}s_{45}(s_{34} - s_{12} - s_{15}) \end{aligned}$$

Example:



$$\sim \frac{1}{\sqrt{\Sigma_5^{(3)}}} \left[\frac{1}{\epsilon^2} f^{(2)} + \frac{1}{\epsilon} f^{(3)} + \mathcal{O}(\epsilon^0) \right]$$

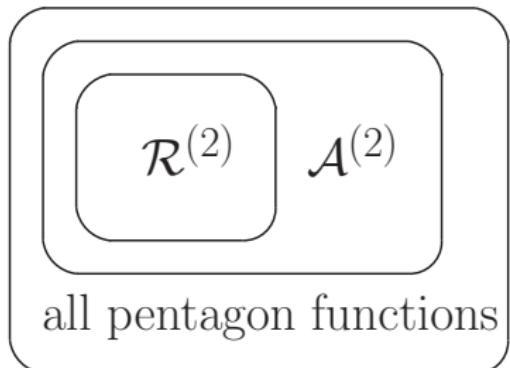
pentagon functions $f^{(w)}$ are singular at $\Sigma_5^{(3)} \sim 0$,

$$f^{(2)} \sim -4\pi^2, \quad f^{(3)} \sim 256\pi^2 \log(\Sigma_5^{(3)})$$

A subset of the pentagon functions is required for phenomenological applications

- Some alphabet letters drop out from the **amplitudes**

$$\sum_{w=0}^4 \epsilon^{-4+w} \mathcal{A}_{[w]}^{(2)}(X) + \mathcal{O}(\epsilon)$$



- and further on some more letters drop out from the **finite remainders**

$$\mathcal{R}^{(2)} = \mathcal{A}^{(2)} - \mathsf{I}^{(1)}\mathcal{A}^{(1)} - \mathsf{I}^{(2)}\mathcal{A}^{(0)} + \mathcal{O}(\epsilon)$$

- Conjecture: seven square-root letters are artifacts of dim-reg

$$\Delta_5, \Sigma_5^{(1)}, \dots, \Sigma_5^{(6)}$$

Conclusions

Pentagon Functions

- Basis of transcendental functions to describe five-particle scattering
- Transparent analytic properties and efficient numerical implementation
gitlab.com/pentagon-functions/
- A tool to facilitate QCD calculations @NNLO
- Successfully applied in numerous calculations

... and more generally, the approach is very efficient for multi-scale processes with many square roots