Two-loop five-particle scattering amplitudes

Simone Zoia

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Outline

Two-loop QCD amplitudes for $\mathrm{pp} \to \gamma + 2\mathrm{j}$ with full colour dependence (2304.06682)

with S. **Badger**, M. **Czakon**, H. **Bayu Hartanto**, R. **Moodie**, T. **Peraro**, R. **Poncelet**

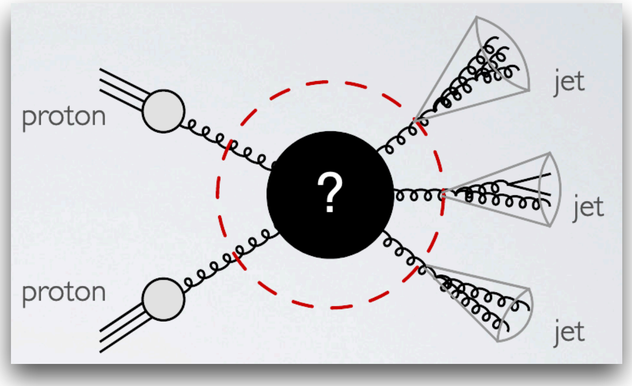
Immediate deployment in NNLO QCD phenomenology in **full colour**

➔ **Michał Czakon's talk**

General methodology for computing 2-loop multi-particle amplitudes

Urgent demand for NNLO QCD for LHC physics

Many observables probed at percent-level precision

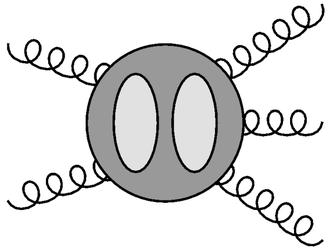


We **must** keep the theoretical uncertainties in line with the experimental ones

Current frontier: NNLO QCD corrections for $2 \to 3$ processes

Hot topic: talks by Becchetti, Buonocore, Chicherin, Czakon, Gambuti, Poncelet, Savoini…

Bottleneck: 2-loop 5-particle scattering amplitudes



Explosion of results in the last couple of years

➔ Giulio Gambuti's talk

- 3γ [Abreu, Page, Pascual, Sotnikov 2020; Chawdhry, Czakon, Mitov, Poncelet 2021; Abreu, De Laurentis, Ita, Klinkert, Page, Sotnikov, *yesterday*]
- $2\gamma + \mathrm{j}$ [Agarwal, Buccioni, von Manteuffel, Tancredi 2021; Chawdhry, Czakon, Mitov, Poncelet 2021; Badger, Brönnum-Hansen, Chicherin, Gehrmann, Hartanto, Henn, Marcoli, Moodie, Peraro, **SZ** 2021]
- $3\mathrm{j}$ (planar) [Abreu, Febres-Cordero, Ita, Page, Sotnikov 2021]
- $\mathrm{W} + \mathrm{b}\bar{\mathrm{b}}$ (planar) [Badger, Bayu Hartanto, **SZ** 2021; Bayu Hartanto, Poncelet, Popescu, **SZ** 2022]
- $\mathrm{W} + 2\mathrm{j}$ (planar) [Abreu, Febres Cordero, Ita, Klinkert, Page, Sotnikov 2022]
- $\mathrm{H} + \mathrm{b}\bar{\mathrm{b}}$ (planar) [Badger, Bayu Hartanto, Kryś, **SZ** 2021]
- $\mathrm{W} + \gamma + \mathrm{j}$ (planar) [Badger, Bayu Hartanto, Kryś, **SZ** 2022]
- $\gamma + 2\mathrm{j}$ [Badger, Czakon, Bayu Hartanto, Moodie, Peraro, Poncelet, **SZ** 2023]

Ready for deployment in NNLO QCD phenomenology

Leading colour @ 2 loops

$pp \to 3\gamma$ [Kallweit, Sotnikov, Wiesemann 2020; Chawdhry, Czakon, Mitov, Poncelet 2020]

$pp \to 2\gamma + j$ [Chawdhry, Czakon, Mitov, Poncelet 2021; Badger, Gehrmann, Marcoli, Moodie 2021]

$pp \to 3j$ [Czakon, Mitov, Poncelet 2021; Chen, Gehrmann, Glover, Huss, Marcoli 2022]

$pp \to W + b\bar{b}$ [Bayu Hartanto, Poncelet, Popescu, **SZ** 2022; Buonocore, Devoto, Kallweit, Mazzitelli, Rottoli, Savoini 2023]

$pp \to \gamma + 2j$ [Badger, Czakon, Bayu Hartanto, Moodie, Peraro, Poncelet, **SZ** 2023]

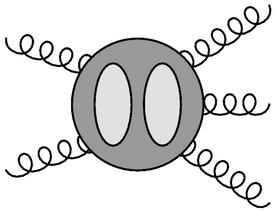
First calculation of a complete 2→3 hadron-collider process at NNLO QCD with <u>full colour</u> dependence

➔ **Michał Czakon's talk**

All massless 2→3 processes now analysed at NNLO QCD ✓

Algebraic and analytic complexity

$$\textbf{amplitude} = \sum \textbf{algebraic coeffs} \times \textbf{special funcs}$$



5 scalar invariants
1 square root

Many variables, complicated algebraic expressions

Intricate branch-cut structure
Functional identities
Difficult numerical evaluation

Algebraic and analytic complexity

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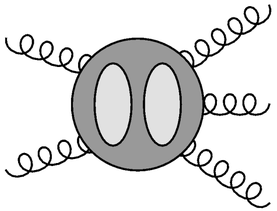

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Finite-field arithmetic & functional reconstruction

➔ **this talk**

Intricate branch-cut structure
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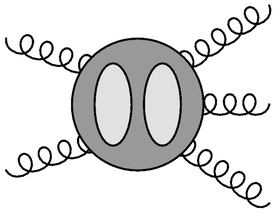

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↓

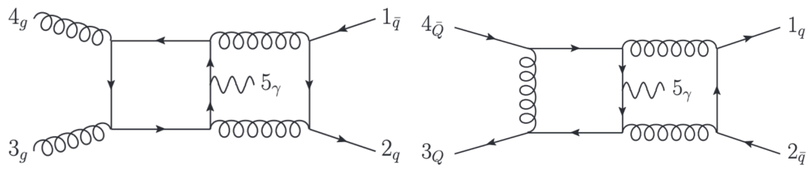
Canonical DEs &
Chen iterated integrals

➔ **Dmitry Chicherin's talk**

Amplitude workflow

Helicity partial amplitudes

$$A^{(2)}\left(\{p\},\epsilon\right) = \sum_i \text{Feynman diagram}_i$$



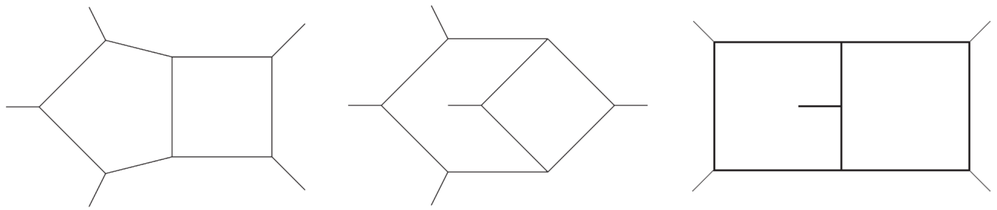
with QGRAF

[Nogueira]

FORM + Mathematica

$$A^{(2)}\left(\{p\},\epsilon\right) = \sum_i c_i\left(\{p\},\epsilon\right)\ \mathrm{I}_i\left(\{p\},\epsilon\right)$$

Scalar Feynman integrals



Integration-by-parts (IBPs)

$$A^{(2)}\left(\{p\},\epsilon\right) = \sum_i d_i\left(\{p\},\epsilon\right)\ \mathrm{MI}_i\left(\{p\},\epsilon\right)$$

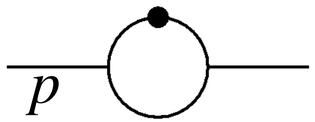
E.g. $\mathrm{I} = \int \frac{\mathrm{d}^D k_1\, \mathrm{d}^D k_2}{k_1^2 (k_1+p_1)^2 (k_1+p_1+p_2)^2 \ldots}$

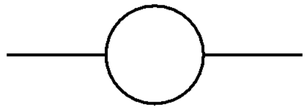
"Master integrals"

Integration-by-parts identities

[Chetyrkin, Tkachov '81]

A toy example:

$\mathrm{I}_{\mathrm{bubble}}(2,1) =$ 

$= \dfrac{3-D}{p^2} \times$  $= \dfrac{3-D}{p^2}\,\mathrm{I}_{\mathrm{bubble}}(1,1)$

Finite number of basis integrals, called *master integrals* (MIs)

$$\mathrm{I}_i\left(\{p\},\epsilon\right) = \sum_j W_{ij}\left(\{p\},\epsilon\right)\,\mathrm{MI}_j\left(\{p\},\epsilon\right)$$

Solve a very large linear system of equations (Laporta algorithm)

[Laporta 2000]

Complicated and bulky solution

Just an intermediate step 💡

Feynman diagrams

LARGE
intermediate expressions

Analytic expression of the amplitude

Finite-field arithmetic removes the bottleneck

[von Manteuffel, Schabinger 2015; Peraro 2016]

- Evaluate rational functions at numerical integer points $\left(\{p\}, \epsilon\right)$ modulo prime number
- Perform all intermediate rational operations numerically
- Reconstruct the analytic expression of the result from multiple numerical evaluations

Mathematica/C++ framework **FiniteFlow** [Peraro 2019]

Simplification from the ϵ-expansion

Amplitudes needed only up to a certain order in $\epsilon = (4 - D)/2$

- Significantly simpler expressions
- Check IR/UV poles (IR factorisation + UV renormalisation)

The Laurent expansion of the rational coefficients can be performed over finite fields

$$A^{(2)}\left(\{p\}, \epsilon\right) = \sum_i d_i\left(\{p\}, \epsilon\right) \, \mathrm{MI}_i\left(\{p\}, \epsilon\right)$$

We need the ϵ-expansion of the master integrals $\Rightarrow$ **Special functions!**

“Bases” of special functions

Special functions satisfy functional relations $\mathrm{Li}_2(z) + \frac{1}{2}\log^2(-z) + \mathrm{Li}_2\left(\frac{1}{z}\right) + \frac{\pi^2}{6} = 0$

$\Rightarrow$ Redundant representation, missed simplifications 👎

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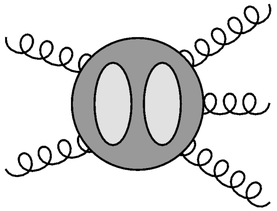
⇒ Redundant representation, missed simplifications 👎

Solution: map MIs onto a **basis** of special functions

Algebraically independent!

➔ **Dmitry Chicherin’s talk**

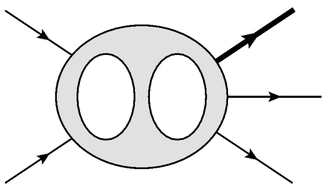


Massless 2-loop 5-point

[Gehrmann, Henn, Lo Presti 2018]
[Chicherin, Sotnikov 2020]

1-mass 2-loop 5-point

[Chicherin, Sotnikov, **SZ** 2022;
Abreu, Chicherin, Ita, Page,
Sotnikov, Tschernow, **SZ** *soon*]



⇒ Unique, simpler expression of the amplitudes 👍

Amplitude workflow

$$A^{(2)}\left(\{p\},\epsilon\right) = \sum_i \text{Feynman diagram}_i$$

IBP reduction

$$A^{(2)}\left(\{p\},\epsilon\right) = \sum_i d_i\left(\{p\},\epsilon\right)\ \text{MI}_i\left(\{p\},\epsilon\right)$$

ϵ expansion

$$A^{(2)}\left(\{p\},\epsilon\right) = \sum_{w=-4}^{\infty} \epsilon^w \sum_i c_i^{(w)}\left(\{p\}\right)\ \text{mon}_{w,i}\left[f\right]$$

IR/UV subtraction

$$F^{(2)}\left(\{p\}\right) = \sum_i e_i\left(\{p\}\right)\ \text{mon}_i\left[f\right]$$

Amplitude workflow

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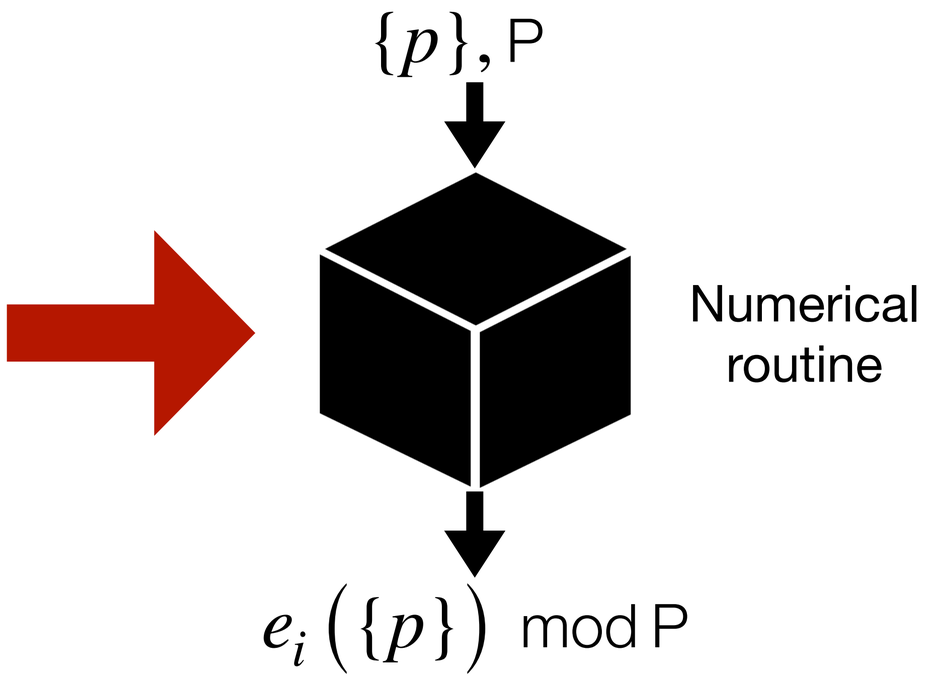

Finite-field arithmetic + rational reconstruction with FiniteFlow

[Peraro 2019]

Rational reconstruction

$$F^{(2)}\left(\{p\}\right) = \sum_i e_i\left(\{p\}\right)\ \mathrm{mon}_i\left[f\right]$$

Black box interpolation

$$\left(\{p\}, \mathrm{P}\right) \rightarrow \blacksquare \rightarrow e_i\left(\{p\}\right) \bmod \mathrm{P}$$

$e_i(\{p\})$ reconstructed in $\dfrac{\#\ \text{points} \times \text{eval. time}}{\#\ \text{CPUs}}$

Rational reconstruction

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$e_i(\{p\})$ reconstructed in $\dfrac{\#\ \text{points} \times \text{eval. time}}{\#\ \text{CPUs}}$ ⌛

Make a good ansatz

💰

Improve the IBP system

➔ Xiao Liu's talk

Improved IBPs through syzygy equations

[Gluza, Kajda, Kosower 2011; Ita 2016; Larsen, Zhang 2016]

Standard way to generate IBP relations: $$\int \mathrm{d}^D k_1 \, \mathrm{d}^D k_2 \, \frac{\partial}{\partial k_i^\mu} \frac{q_j^\mu}{D_1^{\nu_1} \ldots D_m^{\nu_m}} = 0$$

The resulting IBPs contain integrals which are not needed for the amplitudes

- Integrals with doubled propagators
- Dimensionally-shifted integrals

We generated optimised IBP systems by solving polynomial “syzygy” equations for all two-loop integral families

[Boehm, Georgoudis, Larsen, Schulze, Zhang 2017; von Manteuffel, Schabinger 2019]

$\Rightarrow$ speed-up in IBP solution + substantial reduction in RAM usage

Guessing the denominators

$$F^{(2)}\left(\{p\}\right) = \sum_i e_i\left(\{p\}\right)\ \mathrm{mon}_i\left[f\right] \qquad e_i(s) = \frac{N_i(s)}{D_i(s)} \qquad s := \{p\}$$

Ansatz for the denominators informed by singularities of Feynman integrals

$$\boxed{D_i = \prod_j w_j^{a_j}}$$

Determined by reconstructing the coefficients on univariate phase-space slices

[Abreu, Dormans, Febres Cordero, Ita, Page 2018]

$$\Rightarrow \quad e_i(s) = \frac{N_i(s)}{D_i(s)}$$

Degrees known

Entirely known ✓

Can we use this information to construct an ansatz for the numerators?

Univariate partial fraction decomposition

[Badger, Bayu Hartanto, **SZ** 2021; Badger, Brönnum-Hansen, Chicherin, Gehrmann, Hartanto, Henn, Marcoli, Moodie, Peraro, **SZ** 2021]

$$e(x,y) = \frac{-2x^4 - 4x^3y + 5x^2y^2 - xy^3 + 4y^4}{(x-y)y^2(x^2+y^2)} = -\frac{2x}{y^2} - \frac{6}{y} + \frac{1}{x-y} + \frac{3y}{x^2+y^2}$$

Construct ansatz based on the knowledge of degrees and denominators

$$e(x,y) = \frac{q_1(x)}{y^2} + \frac{q_2(x)}{y} + \frac{q_3(x)}{x-y} + \frac{q_4(x) + q_5(x)y}{x^2+y^2}$$

Linear fit to reconstruct the coefficients $q_i(x)$ →

- One fewer variables
- Lower degrees

Substantial drop in the polynomial degrees

Set $s_{12} = 1$

Most complicated 2-loop partial amplitudes for $0 \to \bar{q}qgg\gamma$

Most complicated @ leading colour

		4 variables			3 variables	
amplitude	helicity	original	stage 1	stage 2	stage 3	stage 4
$A^{(2),1}_{34;q}$	$-++-+$	94/91	74/71	74/0	22/18	22/0
$A^{(2),1}_{34;q}$	$-+-++$	93/89	90/86	90/0	24/14	18/0
$A^{(2),1/N_c^2}_{34;q}$	$-++-+$	90/88	73/71	73/0	23/18	22/0
$A^{(2),1/N_c^2}_{34;q}$	$-+-++$	90/86	86/82	86/0	24/14	19/0
$A^{(2),1/N_c}_{34;l}$	$-+-++$	89/82	74/67	73/0	27/14	20/0
$A^{(2),1/N_c}_{34;l}$	$-++-+$	85/81	61/58	60/0	27/18	20/0
$A^{(2),N_c^2}_{34;q}$	$-+-++$	58/55	54/51	53/0	20/16	20/0

$$e_i(s) = \frac{N_i(s)}{D_i(s)}$$

Linear relations

Partial fraction

Denominator guessing

Example:

$$e(x,y) = \frac{-2x^4 - 4x^3y + 5x^2y^2 - xy^3 + 4y^4}{(x-y)y^2(x^2+y^2)} \to 4/5$$

Ready for phenomenology!

We reconstructed the minimal set of independent partial helicity amplitudes

Other partial amplitudes for the colour/helicity-summed matrix elements obtained <u>numerically</u> by permutation of the momenta and parity conjugation

$$\mathscr{H}^{(2)} = 2\,\mathrm{Re} \sum_{\text{colour}} \sum_{\text{helicity}} \mathscr{A}^{(0)*}\mathscr{A}^{(2)} + \sum_{\text{colour}} \sum_{\text{helicity}} \mathscr{A}^{(1)*}\mathscr{A}^{(1)}$$

Efficient and stable numerical evaluation → C++ library `PentagonFunctions++` [Chicherin, Sotnikov 2020]

Amplitudes ready for immediate deployment in phenomenology!

Conclusions

Two-loop amplitudes for $\mathrm{pp} \to \gamma + 2\mathrm{j}$ in full colour

First calculation of a complete 2→3 process at NNLO QCD with <u>full colour</u> dependence

➔ **Michał Czakon's talk**

Efficient methodology for multi-particle computations based on finite-field arithmetic

All massless 2→3 processes now analysed at NNLO QCD

(at least at leading colour) 

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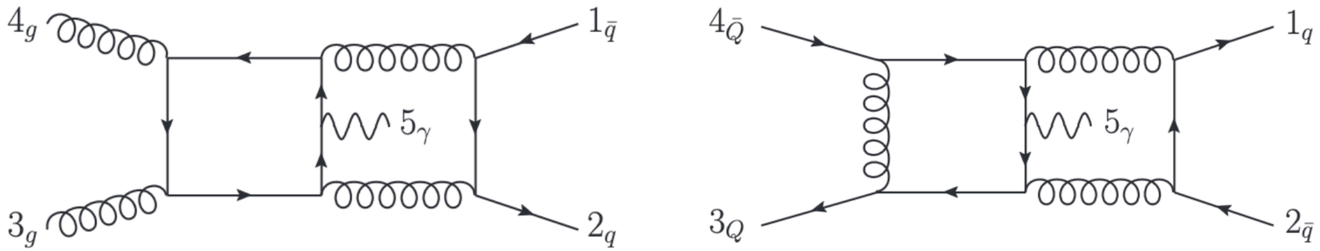
Thank you!

Amplitudes for $\mathrm{pp} \to \gamma + 2\mathrm{j}$

Two partonic channels:



Colour decomposition ➔ a lot of bookkeeping!

$$\mathcal{M}^{(L)}(1_{\bar{q}}, 2_q, 3_g, 4_g, 5_\gamma) = \sqrt{2}\, e\, g_s^2\, n^L \Big\{ (t^{a_3} t^{a_4})_{i_2}{}^{\bar{i}_1} \mathcal{A}_{34}^{(L)}(1_{\bar{q}}, 2_q, 3_g, 4_g, 5_\gamma)$$
$$+ (t^{a_4} t^{a_3})_{i_2}{}^{\bar{i}_1} \mathcal{A}_{43}^{(L)}(1_{\bar{q}}, 2_q, 3_g, 4_g, 5_\gamma) + \delta_{i_2}{}^{\bar{i}_1} \delta^{a_3 a_4} \mathcal{A}_{\delta}^{(L)}(1_{\bar{q}}, 2_q, 4_g, 3_g, 5_\gamma) \Big\},$$

$$\mathcal{A}_{34}^{(2)} = \mathcal{Q}_q N_c^2 A_{34;q}^{(2),N_c^2} + \mathcal{Q}_q A_{34;q}^{(2),1} + \mathcal{Q}_q \frac{1}{N_c^2} A_{34;q}^{(1),1/N_c^2} + \mathcal{Q}_q N_c n_f A_{34;q}^{(2),N_c n_f} + \mathcal{Q}_q \frac{n_f}{N_c} A_{34;q}^{(2),n_f/N_c}$$
$$+ \mathcal{Q}_q n_f^2 A_{34;q}^{(2),n_f^2} + \left(\sum_l \mathcal{Q}_l\right) N_c A_{34;l}^{(2),N_c} + \left(\sum_l \mathcal{Q}_l\right) \frac{1}{N_c} A_{34;l}^{(2),1/N_c} + \left(\sum_l \mathcal{Q}_l\right) n_f A_{34;l}^{(2),n_f},$$