Looping the loops: a tale of *elliptic* dual Feynman integrals

based on [hep-th:2210.09898] +

WIP with Pokraka, Porkert and Sohnle

Mathieu Giroux

McGill University

Radcor 2023

Crieff, Scotland

May 30, 2023

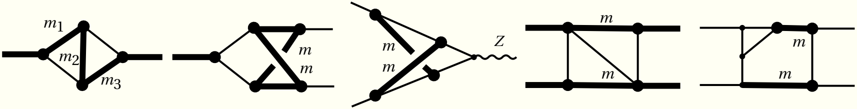
Motivating question

Why bother with elliptic[1] Feynman integrals?

[1]Family of integrals in which elliptic curve(s) is(are) lurking

Perturbative QFT

Elliptic integrals appear in the early stages of pQFT



Already rich literature at two-loop!

[Sabry ; 61, Broadhurst ; 90, Laporta, Remiddi; 05, Adams, Bogner, Schweitzer, Weinzierl; 16, Broedel, Duhr, Dulat, Penante, Tancredi; 19, Duhr, Dulat, Mistlberger; 20, Frellesvig; 21, Duhr, Smirnov, Tancredi; 21, Wilhelm, Zhang; 22 **and many more**]

$\Downarrow$

An *essential* step in opening a gateway to more precise perturbative calculations of cross-sections

The missing piece we are after: canonical bases

A *canonical basis* $\boldsymbol{I}$ satisfies a differential equation we "know" how to solve order-by-order in the dim-reg ε [Henn; 13]

$$\mathrm{d}\boldsymbol{I} = \varepsilon\, \boldsymbol{\Omega} \cdot \boldsymbol{I}$$

The missing piece we are after: canonical bases

A *canonical basis* $\boldsymbol{I}$ satisfies a differential equation we "know" how to solve order-by-order in the dim-reg ε [Henn; 13]

$$\mathrm{d}\boldsymbol{I} = \varepsilon\, \boldsymbol{\Omega} \cdot \boldsymbol{I}$$

For many processes (e.g., massless), such basis is *systematically* derived by normalizing a naive basis with leading singularities

$$(\text{Gram determinant})^{\#}$$

The missing piece we are after: canonical bases

A *canonical basis* $\boldsymbol{I}$ satisfies a differential equation we "know" how to solve order-by-order in the dim-reg ε [Henn; 13]

$$\mathrm{d}\boldsymbol{I} = \varepsilon\, \boldsymbol{\Omega} \cdot \boldsymbol{I}$$

For many processes (e.g., massless), such basis is *systematically* derived by normalizing a naive basis with leading singularities

$$(\text{Gram determinant})^{\#}$$

Introducing internal masses usually changes the story...

The missing piece we are after: canonical bases

A *canonical basis* $\boldsymbol{I}$ satisfies a differential equation we "know" how to solve order-by-order in the dim-reg ε [Henn; 13]

$$\mathrm{d}\boldsymbol{I} = \varepsilon\, \boldsymbol{\Omega} \cdot \boldsymbol{I}$$

For many processes (e.g., massless), such basis is *systematically* derived by normalizing a naive basis with leading singularities

$$(\text{Gram determinant})^{\#}$$

Introducing internal masses usually changes the story...

In fully massive examples, the definition of leading singularities is somewhat ambiguous, and so is the path to canonical form...
Active area of research [Brödel et al. | Bourjaily, Kalyanapuram | Wilhelm, Zhang | Frellesvig | Frellesvig, Weinzierl | Dlapa et al. | Görges et al.]

The missing piece we are after: canonical bases

A *canonical basis* $\boldsymbol{I}$ satisfies a differential equation we “know” how to solve order-by-order in the dim-reg ε [Henn; 13]

$$\mathrm{d}\boldsymbol{I} = \varepsilon\, \boldsymbol{\Omega} \cdot \boldsymbol{I}$$

For many processes (e.g., massless), such basis is *systematically* derived by normalizing a naive basis with leading singularities

$$(\text{Gram determinant})^{\#}$$

Introducing internal masses usually changes the story...

In fully massive examples, the definition of leading singularities is somewhat ambiguous, and so is the path to canonical form...

This talk's query:

Can we *systematically* derive canonical bases for *multi-loop* integrals with *generic* mass scales?

Honest answer…

Not *quite* systematically, but (I think) our community is slowly getting there!

[Talks by Wang and Weinzierl]

Our modest tool box

- **Unitarity** and **geometry**: innately enclosed into the framework of *dual forms* [Caron-Huot, Pokraka; 21]
- **Looping loops**: to first approximation, a multi-loop problem is a bunch of (coupled) one-loop problems
- **Modular ($\mathrm{SL}(2,\mathbb{Z})$) symmetry**: focus on *elliptic* classes of Feynman integrals

The dual paradigm

The inevitable “Feynman integrals” slide

A loop diagram + Feynman rules $\Longrightarrow$ A Feynman integral

The inevitable “Feynman integrals” slide

A loop diagram + Feynman rules $\implies$ A Feynman integral

In *dimensional regularisation*, they correspond to twisted *periods*

Multivalued *twist*

$$I = \int_X u(\varepsilon)\ \phi$$

Algebraic differential n-form

Space of loop momenta invariants

The inevitable “Feynman integrals” slide

A loop diagram + Feynman rules $\Longrightarrow$ A Feynman integral

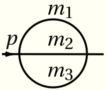
In *dimensional regularisation*, they correspond to *twisted periods*

Multivalued *twist*

$$I = \int_X u(\varepsilon)\ \phi$$

Algebraic differential n-form

Space of loop momenta invariants



$$\sim \int \frac{\mathrm{d}^D \ell_1}{\pi^{D/2}} \frac{\mathrm{d}^D \ell_2}{\pi^{D/2}} \frac{\mathrm{d}^D \ell_3}{\pi^{D/2}} \frac{\delta^D(\ell_3 - \ell_1 + \ell_2 - p)}{(\ell_1^2 + m_1^2)(\ell_2^2 + m_2^2)(\ell_3^2 + m_3^2)}$$

$$\sim \int \sqrt{\ell_{1,\perp}^2 + p^2(x_1+1)^2} \left(\ell_{1,\perp}^2\right)^{1/2-\varepsilon} \left(\ell_{2,\perp}^2\right)^{1/2-\varepsilon} \phi\left(x_1, x_2, \ell_{1,\perp}^2, \ell_{2,\perp}^2\right)$$

The inevitable "Feynman integrals" slide

A loop diagram + Feynman rules $\Longrightarrow$ A Feynman integral

In *dimensional regularisation*, they correspond to *twisted periods*

Multivalued *twist*

$$I = \int_X u(\varepsilon)\ \phi$$

Algebraic differential
n-form

Space of loop
momenta invariants

For each topology, there exists a *finite* set of *spanning* integrals

[Smirnov, Petukhov; 10]

$\Downarrow$

This set forms a *vector space* closed under differentiation

[Frellesvig, Gasparotto, Mandal, Mastrolia, Mattiazzi, Mizera; 19]

The dual technology

The space of *dual forms* $\{\check{\phi}\}$ is defined s.t. the *intersection pairing*

$$\langle\check{\phi}|\phi\rangle \sim \int_{\mathbb{C}^n} (\check{u} \times u)\, \check{\phi} \wedge \phi$$

[Caron-Huot, Pokraka; 21]
[See Hjalte's talk]

"makes sense"

The dual technology

The space of *dual forms* $\{\check{\phi}\}$ is defined s.t. the *intersection pairing*

$$\langle\check{\phi}|\phi\rangle \sim \int_{\mathbb{C}^n} (\check{u} \times u)\, \check{\phi} \wedge \phi$$

[Caron-Huot, Pokraka; 21]
[See Hjalte's talk]

"makes sense"

1. Single-valuedness of intersection pairing

$\check{u} \times u$ is an algebraic function

The dual technology

The space of *dual forms* $\{\check{\phi}\}$ is defined s.t. the *intersection pairing*

$$\langle\check{\phi}|\phi\rangle \sim \int_{\mathbb{C}^n} (\check{u}\times u)\,\check{\phi}\wedge\phi$$

[Caron-Huot, Pokraka; 21]
[See Hjalte's talk]

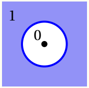
"makes sense"

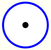
1. Single-valuedness of intersection pairing

$\check{u}\times u$ is an algebraic function

2. Finiteness of intersection pairing

$\check{\phi}$ supported away from ϕ's *unregulated* poles (propagators = 0)

$\theta_i = \theta\,(\mathsf{D}_i) \overset{\text{supp}}{\rightsquigarrow}$ 

$\mathrm{d}\theta_i = \mathrm{d}\theta\,(\mathsf{D}_i) \overset{\text{supp}}{\rightsquigarrow}$ 

The dual technology

The space of *dual forms* $\{\check{\phi}\}$ is defined s.t. the *intersection pairing*

$$\langle \check{\phi} | \phi \rangle \sim \int_{\mathbb{C}^n} (\check{u} \times u)\, \check{\phi} \wedge \phi$$

[Caron-Huot, Pokraka; 21]
[See Hjalte's talk]

"makes sense"

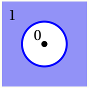
1. Single-valuedness of intersection pairing

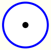
$\check{u} \times u$ is an algebraic function

2. Finiteness of intersection pairing

For a given Feynman graph, the rule of thumb is

$\check{\phi}$ come with a $\mathrm{d}\theta_i$ for each internal propagator D_i

$\theta_i = \theta(\mathsf{D}_i) \overset{\text{supp}}{\rightsquigarrow}$ 

$\mathrm{d}\theta_i = \mathrm{d}\theta(\mathsf{D}_i) \overset{\text{supp}}{\rightsquigarrow}$ 

Looping the loops

A *simple* idea

A multiloop problem is a ‘bunch’ of easier (but coupled) 1-loop problems

A *simple* idea

A multiloop problem is a 'bunch' of easier (but coupled) 1-loop problems

True for differential equations too!
Construct differential equations one loop at a time

Mathematical setup and differential equations

If our total space X locally looks like $F \times B$ then [Serre, 51]

(p + q)-form on X

p-form on B

$$\check{\phi}_j = \check{\phi}_{F,i} \wedge \check{\phi}_{B,ij}$$

q-form on F

Mathematical setup and differential equations

If our total space X locally looks like $F \times B$ then [Serre, 51]

$$\underbrace{\check{\phi}_j}_{(p+q)\text{-form on } X} = \underbrace{\check{\phi}_{F,i}}_{q\text{-form on } F} \wedge \underbrace{\check{\phi}_{B,ij}}_{p\text{-form on } B}$$

The goal is to obtain DEs on X from the ones on F and B

$$\check{\nabla}\check{\boldsymbol{\phi}} = \overset{(1)}{\boxed{\check{\nabla}\Big(\check{\boldsymbol{\phi}}_F}} \mathbin{\bar{\wedge}} \underline{\check{\boldsymbol{\phi}}}_B\Big)$$

This step computes the fibre DE: $\check{\nabla}\check{\boldsymbol{\phi}}_F \simeq \check{\boldsymbol{\phi}}_F \mathbin{\bar{\wedge}} \check{\boldsymbol{\Omega}}_F$

Mathematical setup and differential equations

If our total space X locally looks like $F \times B$ then [Serre, 51]

$$\underbrace{\check{\phi}_j}_{(p+q)\text{-form on } X} = \underbrace{\check{\phi}_{F,i}}_{q\text{-form on } F} \wedge \underbrace{\check{\phi}_{B,ij}}_{p\text{-form on } B}$$

The goal is to obtain DEs on X from the ones on F and B

$$\check{\nabla}\check{\boldsymbol{\phi}} = \overset{(1)}{\check{\nabla}\left(\check{\boldsymbol{\phi}}_F \mathbin{\dot{\wedge}} \underline{\check{\boldsymbol{\phi}}}_B\right)} \simeq \check{\boldsymbol{\phi}}_F \mathbin{\dot{\wedge}} \overset{(2)}{\check{\mathbb{W}}\,\underline{\check{\boldsymbol{\phi}}}_B}$$

With the induced covariant derivative: $\check{\mathbb{W}} = \boldsymbol{d} + \check{\boldsymbol{\omega}}(\check{\boldsymbol{\Omega}}_F) \mathbin{\dot{\wedge}} \ldots$

Mathematical setup and differential equations

If our total space X locally looks like $F \times B$ then [Serre, 51]

$$\overbrace{\check{\phi}_j}^{(p+q)\text{-form on } X} = \underbrace{\check{\phi}_{F,i}}_{q\text{-form on } F} \wedge \overbrace{\check{\phi}_{B,ij}}^{p\text{-form on } B}$$

The goal is to obtain DEs on X from the ones on F and B

$$\check{\nabla}\check{\boldsymbol{\phi}} = \overset{(1)}{\check{\nabla}\Big(\check{\boldsymbol{\phi}}_F} \mathbin{\dot\wedge} \underline{\check{\boldsymbol{\phi}}}_B\Big) \simeq \check{\boldsymbol{\phi}}_F \mathbin{\dot\wedge} \overset{(2)}{\check{\mathbb{W}}\;\underline{\check{\boldsymbol{\phi}}}_B} \simeq \check{\boldsymbol{\phi}} \mathbin{\dot\wedge} \check{\boldsymbol{\Omega}}_B$$

… we compute the base DE: $\check{\mathbb{W}}\,\underline{\check{\boldsymbol{\phi}}}_B \simeq \underline{\check{\boldsymbol{\phi}}}_B \mathbin{\dot\wedge} \check{\boldsymbol{\Omega}}_B$

Mathematical setup and differential equations

If our total space X locally looks like $F \times B$ then [Serre, 51]

$(p+q)$-form on X

p-form on B

$$\check{\phi}_j = \check{\phi}_{F,i} \wedge \check{\phi}_{B,ij}$$

q-form on F

Loop-by-loop constraints

The goal is to obtain DEs on X from the ones on F and B

$$\check{\nabla}\check{\boldsymbol{\phi}} = \overset{(1)}{\boxed{\check{\nabla}\Big(\check{\boldsymbol{\phi}}_F}} \mathbin{\dot{\wedge}} \underline{\check{\boldsymbol{\phi}}}_B\Big) \simeq \check{\boldsymbol{\phi}}_F \mathbin{\dot{\wedge}} \overset{(2)}{\boxed{\check{\mathbb{W}}\ \underline{\check{\boldsymbol{\phi}}}_B}} \simeq \check{\boldsymbol{\phi}} \mathbin{\dot{\wedge}} \check{\boldsymbol{\Omega}}_B$$

Loop-by-loop comes with *strong* constraints on bases choices!

Mathematical setup and differential equations

If our total space X locally looks like $F \times B$ then [Serre, 51]

(p + q)-form on X

p-form on B

$$\check{\phi}_j = \check{\phi}_{F,i} \wedge \check{\phi}_{B,ij}$$

q-form on F

Loop-by-loop constraints

1. The fibre basis chosen s.t. $\check{\boldsymbol{\omega}} \subset \check{\mathbb{W}}$ is in ε-form

The goal is to obtain DEs on X from the ones on F and B

$$\check{\nabla}\check{\boldsymbol{\phi}} = \overset{(1)}{\check{\nabla}\Big(\check{\boldsymbol{\phi}}_F} \,\dot{\wedge}\, \underline{\check{\boldsymbol{\phi}}}_B\Big) \simeq \check{\boldsymbol{\phi}}_F \,\dot{\wedge}\, \overset{(2)}{\check{\mathbb{W}}\;\underline{\check{\boldsymbol{\phi}}}_B} \simeq \check{\boldsymbol{\phi}} \,\dot{\wedge}\, \check{\boldsymbol{\Omega}}_B$$

Loop-by-loop comes with *strong* constraints on bases choices!

Mathematical setup and differential equations

If our total space X locally looks like $F \times B$ then [Serre, 51]

$(p+q)$-form on X

p-form on B

$$\check{\phi}_j = \check{\phi}_{F,i} \wedge \check{\phi}_{B,ij}$$

q-form on F

Loop-by-loop constraints

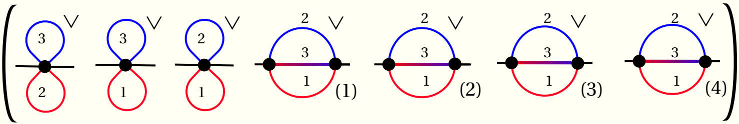
1. The fibre basis chosen s.t. $\check{\boldsymbol{\omega}} \subset \check{\mathbb{W}}$ is in ε-form
2. The base basis chosen s.t. $\check{\phi}_j = \check{\phi}_{F,i} \wedge \check{\phi}_{B,ij}$ is single-valued (algebraic)

The goal is to obtain DEs on X from the ones on F and B

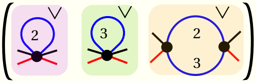
$$\check{\nabla}\check{\boldsymbol{\phi}} = \overset{(1)}{\check{\nabla}\Big(\check{\boldsymbol{\phi}}_F} \mathbin{\dot{\wedge}} \underline{\check{\boldsymbol{\phi}}}_B\Big) \simeq \check{\boldsymbol{\phi}}_F \mathbin{\dot{\wedge}} \overset{(2)}{\check{\mathbb{W}}\, \underline{\check{\boldsymbol{\phi}}}_B} \simeq \check{\boldsymbol{\phi}} \mathbin{\dot{\wedge}} \check{\boldsymbol{\Omega}}_B$$

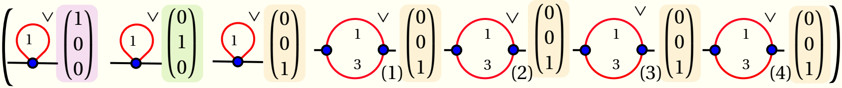
Loop-by-loop comes with *strong* constraints on bases choices!

Example: *The 3-scale sunrise*

$\{\check{\phi}_j\}_{j=1}^{7} =$ 

$\check{\phi}_j = \check{\phi}_{F,i} \wedge \check{\phi}_{B,ij}$ ↓ Loop-by-loop splitting

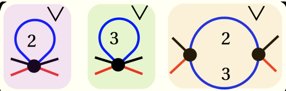
$\{\check{\phi}_{F,i}\}_{1}^{3} =$ 

$\{\check{\boldsymbol{\phi}}_{B,j}\}_{1}^{7} =$ 

Step 1: fibre basis and canonical differential equation

The normalized basis in [Caron-Huot, Pokraka; 21]

$$\{\check{\phi}_{F,i}\}_1^3 = \Big(\quad\Big)$$



$$\check{\phi}_{F,1} = \frac{2\varepsilon}{q\sqrt{\ell_{1\perp}^2}} \frac{\mathrm{d}\theta_2 \wedge \mathrm{d}\ell_{2\parallel}}{\ell_{2\perp}^2\big|_2} \qquad \check{\phi}_{F,2} = \frac{2\varepsilon}{q\sqrt{\ell_{1\perp}^2}} \frac{\mathrm{d}\theta_3 \wedge \mathrm{d}\ell_{2\parallel}}{\ell_{2\perp}^2\big|_3}$$

$$\check{\phi}_{F,3} = \frac{1}{q\sqrt{\ell_{1\perp}^2}} \frac{\mathrm{d}\theta_2 \wedge \mathrm{d}\theta_3}{\sqrt{\ell_{2\perp}^2\big|_{23}}}$$

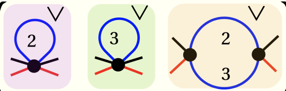
$\ell_{1\perp}^2$ = Gram determinant on the 2nd loop
$q = \sqrt{(p+\ell_1)^2}$ (fibre external momentum)

satisfies a dlog-form differential equation $\check{\boldsymbol{\Omega}}_F$ such that

$$\check{\mathbb{W}} \supset \check{\boldsymbol{\omega}}(\check{\boldsymbol{\Omega}}_F) = \mathcal{O}(\varepsilon)$$

The normalized basis in [Caron-Huot, Pokraka; 21]

$$\{\check{\phi}_{F,i}\}_1^3 = \Big($$



$$\Big)$$

$$\check{\phi}_{F,1} = \frac{2\varepsilon}{q\sqrt{\ell_{1\perp}^2}} \frac{\mathrm{d}\theta_2 \wedge \mathrm{d}\ell_{2\parallel}}{\ell_{2\perp}^2\big|_2} \qquad \check{\phi}_{F,2} = \frac{2\varepsilon}{q\sqrt{\ell_{1\perp}^2}} \frac{\mathrm{d}\theta_3 \wedge \mathrm{d}\ell_{2\parallel}}{\ell_{2\perp}^2\big|_3}$$

$$\check{\phi}_{F,3} = \frac{1}{q\sqrt{\ell_{1\perp}^2}} \frac{\mathrm{d}\theta_2 \wedge \mathrm{d}\theta_3}{\sqrt{\ell_{2\perp}^2\big|_{23}}}$$

$\ell_{1\perp}^2$ = Gram determinant on the 2^{nd} loop
$q = \sqrt{(p+\ell_1)^2}$ (fibre external momentum)

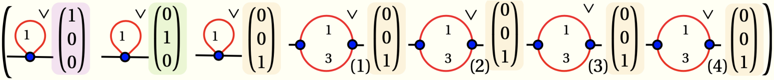
satisfies a dlog-form differential equation $\check{\boldsymbol{\Omega}}_F$ such that

$$\check{\mathbb{W}} \supset \check{\boldsymbol{\omega}}(\check{\boldsymbol{\Omega}}_F) = \mathcal{O}(\varepsilon)$$

Bubble denominator on last cut is the sunrise *elliptic curve Y*

$$Y^2 - \left[\ell_{1\perp}^2(\ell_1+p)^2\ell_{2\perp}^2\right]\Big|_{123} = 0$$

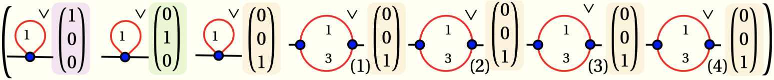
Step 2: base basis and *pre*-canonical differential equation

$\{\check{\phi}_{B,j}\}_1^7 =$ 

1. As close as possible to uniformly "transcendental"
2. *Second* loop-by-loop constraint : $\check{\phi}_j = \check{\phi}_{F,i} \wedge \check{\phi}_{B,ij}$ is algebraic
3. Linear differential equation: $\boldsymbol{\Theta} = \boldsymbol{\Theta}^{(0)} + \varepsilon\, \boldsymbol{\Theta}^{(1)}$, with $\boldsymbol{\Theta}^{(0)}$ lower triangular
4. $\boldsymbol{\Theta}$ is independent of a and b under $\mathrm{SL}(2,\mathbb{Z})$

Step 2: base basis and *pre*-canonical differential equation

$\{\check{\boldsymbol{\phi}}_{B,j}\}_1^7 =$



1. As close as possible to uniformly "transcendental"
2. *Second* loop-by-loop constraint : $\check{\phi}_j = \check{\phi}_{F,i} \wedge \check{\phi}_{B,ij}$ is algebraic
3. Linear differential equation: $\boldsymbol{\Theta} = \boldsymbol{\Theta}^{(0)} + \varepsilon\, \boldsymbol{\Theta}^{(1)}$, with $\boldsymbol{\Theta}^{(0)}$ lower triangular
4. $\boldsymbol{\Theta}$ is independent of a and b under $\mathrm{SL}(2,\mathbb{Z})$

Tadpoles:

$$\check{\boldsymbol{\phi}}_{B,1} = \mathrm{d}\log\left(\frac{1-\frac{i x}{\sqrt{r_1^2-x^2}}}{1+\frac{i x}{\sqrt{r_1^2-x^2}}}\right) \wedge \mathrm{d}\theta_1 \begin{pmatrix}1\\0\\0\end{pmatrix}$$

$$\check{\boldsymbol{\phi}}_{B,2} = \mathrm{d}\log\left(\frac{1-\frac{i x}{\sqrt{r_1^2-x^2}}}{1+\frac{i x}{\sqrt{r_1^2-x^2}}}\right) \wedge \mathrm{d}\theta_1 \begin{pmatrix}0\\1\\0\end{pmatrix}$$

$$\check{\boldsymbol{\phi}}_{B,3} = i\,\varepsilon\,\theta_1\,\mathrm{d}\log\left(\frac{p\,(x+1)+\sqrt{-\ell_{1\perp}^2}}{p\,(x+1)-\sqrt{-\ell_{1\perp}^2}}\right) \wedge \mathrm{d}\log\left(\frac{q_+-q_-}{q_++q_-}\right)\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$\ell_1^\mu = \ell_{1\parallel}^\mu + \ell_{1,\perp}^\mu = x\,p^\mu + \ell_{1\perp}^\mu, \quad \psi_1 \sim K$

$q_\pm = \sqrt{(p+\ell_1)^2 + m_\pm^2},\ m_\pm = m_2 \pm m_3$

c, ∞ = twisted singularities in $D=4$

Maximal-cut:

$$\check{\boldsymbol{\phi}}_{B,4} = \frac{\psi_1^2}{\pi\ \varepsilon\ W_0}\check{\nabla}_0 \check{\boldsymbol{\phi}}_{B,7}$$

$$\check{\boldsymbol{\phi}}_{B,5} = m_1^{-4\varepsilon}\ \mathrm{d}\theta_1 \wedge \frac{(x-r_1)\mathrm{d}x}{Y}\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$\check{\boldsymbol{\phi}}_{B,6} = m_1^{-4\varepsilon}\ \mathrm{d}\theta_1 \wedge \frac{Y(c)\mathrm{d}x}{(x-c)\,Y}\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$\check{\boldsymbol{\phi}}_{B,7} = m_1^{-4\varepsilon}\ \mathrm{d}\theta_1 \wedge \frac{\pi\,\mathrm{d}x}{\psi_1\,Y}\begin{pmatrix}0\\0\\1\end{pmatrix}$$

Step 3: modular symmetry and canonical form

Suppose $\boldsymbol{I}$ satisfies a *linear* differential equation[2]

$$\boldsymbol{\Gamma} = \boldsymbol{\Gamma}^{(0)} + \varepsilon \boldsymbol{\Gamma}^{(1)}$$

where $\boldsymbol{\Gamma}^{(0)}$ is lower-triangular and free of a and b under $\mathrm{SL}(2,\mathbb{Z})$

[2] $\boldsymbol{I}$ being a vector of Feynman integrals or dual forms is irrelevant

Step 3: modular symmetry and canonical form

Suppose $\boldsymbol{I}$ satisfies a *linear* differential equation[2]

$$\boldsymbol{\Gamma} = \boldsymbol{\Gamma}^{(0)} + \varepsilon\boldsymbol{\Gamma}^{(1)}$$

where $\boldsymbol{\Gamma}^{(0)}$ is lower-triangular and free of a and b under $\mathrm{SL}(2,\mathbb{Z})$

Proposal A gauge transformation $\boldsymbol{G} = \boldsymbol{U} \cdot \boldsymbol{I}$ such that

$$\boldsymbol{U} \cdot \boldsymbol{\Gamma} \cdot \boldsymbol{U}^{-1} + \mathrm{d}\boldsymbol{U} \cdot \boldsymbol{U}^{-1} = \varepsilon\tilde{\boldsymbol{\Gamma}}$$

Empirical observation: $\tilde{\boldsymbol{\Gamma}}$ only has simple poles

is *fixed* by modular symmetry

[2] $\boldsymbol{I}$ being a vector of Feynman integrals or dual forms is irrelevant

Step 3: modular symmetry and canonical form

Suppose $\boldsymbol{I}$ satisfies a *linear* differential equation[2]

$$\boldsymbol{\Gamma} = \boldsymbol{\Gamma}^{(0)} + \varepsilon \boldsymbol{\Gamma}^{(1)}$$

where $\boldsymbol{\Gamma}^{(0)}$ is lower-triangular and free of a and b under $\mathrm{SL}(2,\mathbb{Z})$

Proposal A gauge transformation $\boldsymbol{G} = \boldsymbol{U} \cdot \boldsymbol{I}$ such that

$$\boldsymbol{U} \cdot \boldsymbol{\Gamma} \cdot \boldsymbol{U}^{-1} + \mathrm{d}\boldsymbol{U} \cdot \boldsymbol{U}^{-1} = \varepsilon \tilde{\boldsymbol{\Gamma}}$$

Empirical observation: $\tilde{\boldsymbol{\Gamma}}$ only has simple poles

is *fixed* by modular symmetry

✓ **Non-trivial step toward systematization** By symmetry

linear $\iff$ canonical form

[2] $\boldsymbol{I}$ being a vector of Feynman integrals or dual forms is irrelevant

Step 3: modular symmetry and canonical form

Suppose $\boldsymbol{I}$ satisfies a *linear* differential equation[2]

$$\boldsymbol{\Gamma} = \boldsymbol{\Gamma}^{(0)} + \varepsilon \boldsymbol{\Gamma}^{(1)}$$

where $\boldsymbol{\Gamma}^{(0)}$ is lower-triangular and free of a and b under $\mathrm{SL}(2, \mathbb{Z})$

Proposal A gauge transformation $\boldsymbol{G} = \boldsymbol{U} \cdot \boldsymbol{I}$ such that

$$\boldsymbol{U} \cdot \boldsymbol{\Gamma} \cdot \boldsymbol{U}^{-1} + \mathrm{d}\boldsymbol{U} \cdot \boldsymbol{U}^{-1} = \varepsilon \tilde{\boldsymbol{\Gamma}}$$

Empirical observation: $\tilde{\boldsymbol{\Gamma}}$ only has simple poles

is *fixed* by modular symmetry

✓ **Non-trivial step toward systematization** By symmetry

linear $\Longleftrightarrow$ canonical form

Puzzle Systematic algorithm to pullback canonical form into a “ready to integrate” modular form in *generic* mass examples

Yet, the sunrise is simple enough to do so from educated ansätze!

[2] $\boldsymbol{I}$ being a vector of Feynman integrals or dual forms is irrelevant

Differential equation: result

$$
\breve{\mathbf{\Omega}}_B = \varepsilon \begin{pmatrix}
\mho_2^{(1,2)}\begin{bmatrix}-2\\0\\2\end{bmatrix} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \mho_2^{(1,2)}\begin{bmatrix}0\\-2\\2\end{bmatrix} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \mho_2^{(1,2)}\begin{bmatrix}-2\\-2\\4\end{bmatrix} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \varpi_2^{(1)}\begin{bmatrix}0\\0\\4\end{bmatrix} + \varpi_2^{(2)}\begin{bmatrix}2\\2\\-6\end{bmatrix} & \varpi_1^{(1)}\begin{bmatrix}\frac{1}{2i}\\\frac{1}{2i}\\0\end{bmatrix} & \varpi_1^{(1)}\begin{bmatrix}-\frac{1}{2i}\\\frac{1}{2i}\\0\end{bmatrix} & \frac{1}{4}\,\omega_0^{\mathrm{Kr}}(\tau) \\
\mho_2^{(1,2)}\begin{bmatrix}4i\\-4i\\-8i\end{bmatrix} & \mho_2^{(1,2)}\begin{bmatrix}-4i\\4i\\-8i\end{bmatrix} & \mho_2^{(1,2)}\begin{bmatrix}-2\\-2\\-4\end{bmatrix} & \varpi_3^{(1)}\begin{bmatrix}-6i\\-6i\\12i\end{bmatrix} & \varpi_2^{(1)}\begin{bmatrix}-1\\-1\\0\end{bmatrix} + \varpi_2^{(2)}\begin{bmatrix}2\\2\\-6\end{bmatrix} & \varpi_2^{(1)}\begin{bmatrix}3\\-3\\0\end{bmatrix} & \varpi_1^{(1)}\begin{bmatrix}\frac{3}{4i}\\\frac{3}{4i}\\0\end{bmatrix} \\
\mho_2^{(1,2)}\begin{bmatrix}-4i\\-4i\\0\end{bmatrix} & \mho_2^{(1,2)}\begin{bmatrix}4i\\4i\\0\end{bmatrix} & \mho_2^{(1,2)}\begin{bmatrix}2\\-2\\0\end{bmatrix} & \varpi_3^{(1)}\begin{bmatrix}-6i\\6i\\0\end{bmatrix} & \varpi_2^{(1)}\begin{bmatrix}1\\-1\\0\end{bmatrix} & \varpi_2^{(1)}\begin{bmatrix}-3\\-3\\4\end{bmatrix} + \varpi_2^{(2)}\begin{bmatrix}2\\2\\-6\end{bmatrix} & \varpi_1^{(1)}\begin{bmatrix}-\frac{1}{4i}\\\frac{1}{4i}\\0\end{bmatrix} \\
\mho_3^{(1,4)}\begin{bmatrix}16i\\-16i\\16i\end{bmatrix} & \mho_3^{(1,4)}\begin{bmatrix}-16i\\16i\\16i\end{bmatrix} & \mho_3^{(1,4)}\begin{bmatrix}-8i\\-8i\\8i\end{bmatrix} & \varpi_4^{(1)}\begin{bmatrix}48\\48\\48\end{bmatrix} & \varpi_3^{(1)}\begin{bmatrix}-4i\\-4i\\8i\end{bmatrix} & \varpi_3^{(1)}\begin{bmatrix}-12i\\12i\\0\end{bmatrix} & \varpi_2^{(1)}\begin{bmatrix}0\\0\\4\end{bmatrix} + \varpi_2^{(2)}\begin{bmatrix}2\\2\\-6\end{bmatrix}
\end{pmatrix}
$$

$$
+\,\varepsilon \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -6\eta_2(\tau) & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -6\eta_2(\tau) & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -6\eta_2(\tau) & 0 \\
0 & 0 & 0 & -288\eta_4(\tau) & 0 & 0 & -6\eta_2(\tau)
\end{pmatrix}
$$

Compact notation:

$$\varpi_n^{(K)}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} := \sum_{i=1}^{3} c_i\ \omega_n^{\mathrm{Kr}}(z_i|K\,\tau), \qquad K\in\mathbb{N}$$

$$\mho_n^{(K,m)}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} := \varpi_n^{(K)}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} + \varpi_n^{(2K)}\begin{bmatrix}-m\ c_1\\-m\ c_2\\-m\ c_3\end{bmatrix}$$

[See Yu's talk]

Relation to Feynman integrands

***Caveat*: to extract the boundary conditions, we still need to know a basis of Feynman integrals. This requires additional intersection calculations!**

Relation to Feynman integrands

***Caveat*: to extract the boundary conditions, we still need to know a basis of Feynman integrals. This requires additional intersection calculations!**

Up to the *constant rescaling* (R)

$$\mathrm{diag}(1, 1, i/2, 1/4, i/2, 1/2i, -1/16)$$

of the ε-form dual basis, our basis is *dual* to the basis of integrands presented in [Bogner et al.;19] , meaning that[3]

$$\left\langle \check{\phi}_i^{(\mathrm{R})} \middle| \phi_j \right\rangle \propto \delta_{ij}$$

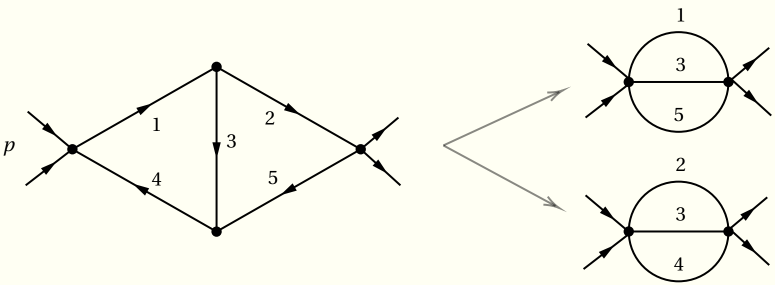
[3] Details of the calculation in [MG, A. Pokraka;22]

Soaring with the sunrise, the kite reached new heights

The 5-mass kite integral

Promising results based on the above ideas!

[WIP with Pokraka, Porkert and Sohnle]



- Most general two-point function
- Relevant to $\mathcal{O}(\alpha_s \alpha_W)$-corrections to $gg \to t\bar{t}$
- Mathematically interesting: 30 masters with *two* elliptic curves
- ✓ Canonical form in terms of energy and masses!
- ⚠ Missing the "ready to integrate" modular form: stay tuned!

Wrapping up

Closing Thoughts

- ✓ Extended dual forms to a multi-scale 2-loop problem
- ✓ Refined path to canonical forms in multi-scale examples:

  Proposed that having unitarity, geometry, and modular symmetry within a loop-by-loop model is adequate as a toolkit to build differential equations

- ✓ Full modular form for the 3-mass sunrise
- ✓ ε-form with simple poles for the 5-mass kite
- ⚠ Full modular form for the 5-mass kite

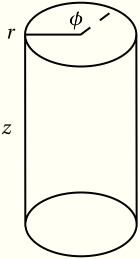
Thank you!

Backup slides

A one-loop example: *Bubble*

Using the *momentum space* parameterization in $D = 4 - 2\varepsilon$

$$\ell^\mu = \ell_\parallel^\mu + \ell_\perp^\mu, \quad \ell_\parallel \cdot \ell_\perp = 0 \rightsquigarrow \mathrm{d}^D\ell = \left(\mathrm{d}\Omega_{D-2} \wedge \left(\ell_\perp^2\right)^{\frac{D-3}{2}} \mathrm{d}\ell_\perp^2\right) \wedge \mathrm{d}\ell_\parallel$$



Volume form:

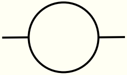
$$\mathrm{d}V = \left(\mathrm{d}\phi \wedge r\ \mathrm{d}r\right) \wedge \mathrm{d}z$$

A one-loop example: *Bubble*

Using the *momentum space* parameterization in $D = 4 - 2\varepsilon$

$$\ell^\mu = \ell_\parallel^\mu + \ell_\perp^\mu, \quad \ell_\parallel \cdot \ell_\perp = 0 \rightsquigarrow \mathrm{d}^D\ell = \left(\mathrm{d}\Omega_{D-2} \wedge \left(\ell_\perp^2\right)^{\frac{D-3}{2}} \mathrm{d}\ell_\perp^2\right) \wedge \mathrm{d}\ell_\parallel$$

the bubble integral is a twisted period over a 2-form



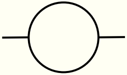
$$\sim \int \frac{\mathrm{d}^D\ell}{\mathsf{D}_1\ \mathsf{D}_2} = \underbrace{\left(\int \mathrm{d}\Omega_{D-2}\right)}_{=\ \text{hypersphere surface area}} \int \left(\ell_\perp^2\right)^{\frac{D-3}{2}} \frac{\mathrm{d}\ell_\parallel\ \mathrm{d}\ell_\perp^2}{\mathsf{D}_1\ \mathsf{D}_2}$$

A one-loop example: *Bubble*

Using the *momentum space* parameterization in $D = 4 - 2\varepsilon$

$$\ell^\mu = \ell^\mu_\parallel + \ell^\mu_\perp, \quad \ell_\parallel \cdot \ell_\perp = 0 \rightsquigarrow \mathrm{d}^D\ell = \left(\mathrm{d}\Omega_{D-2} \wedge \left(\ell_\perp^2\right)^{\frac{D-3}{2}} \mathrm{d}\ell_\perp^2 \right) \wedge \mathrm{d}\ell_\parallel$$

the bubble integral is a twisted period over a 2-form



$$\sim \int \frac{\mathrm{d}^D\ell}{\mathsf{D}_1\ \mathsf{D}_2} = \underbrace{\left(\int \mathrm{d}\Omega_{D-2}\right)}_{=\ \text{hypersphere surface area}} \int \left(\ell_\perp^2\right)^{\frac{D-3}{2}} \frac{\mathrm{d}\ell_\parallel\ \mathrm{d}\ell_\perp^2}{\mathsf{D}_1\ \mathsf{D}_2}$$

$$\left\{ \begin{array}{ll} \textit{Feynman form:} & \phi_{\text{bub}} = \dfrac{\mathrm{d}\ell_\parallel \wedge \mathrm{d}\ell_{1\perp}^2}{\mathsf{D}_1\ \mathsf{D}_2} \\ \textit{Gram determinant:} & G = \ell_\perp^2 \\ \textit{Twist:} & u = G^{\frac{D-3}{2}} \quad [\ (D-3)/2 \notin \mathbb{Z}\ ] \end{array} \right\}$$

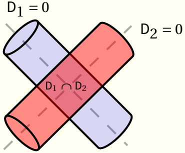
A one-loop example: *Dual bubble*

Localization of the intersection number (finiteness)

$$\langle\check{\phi}_{\text{bub}}|\phi_{\text{bub}}\rangle \sim \int_{\mathbb{C}^2} \check{\phi}_{\text{bub}} \frac{\mathrm{d}\ell_{1\parallel}\mathrm{d}\ell_{1\perp}^2}{\mathsf{D}_1\ \mathsf{D}_2}$$

$\Rightarrow \check{\phi}_{\text{bub}}$ supported on tubular n.b.h. of *unregulated* poles

$$\mathrm{d}\theta_1 \wedge \mathrm{d}\theta_2 = \mathrm{d}\theta(\mathsf{D}_1 = 0) \wedge \mathrm{d}\theta(\mathsf{D}_2 = 0) \rightsquigarrow$$



A one-loop example: *Dual bubble*

Localization of the intersection number (finiteness)

$$\langle \check{\phi}_{\text{bub}} | \phi_{\text{bub}} \rangle \sim \int_{\mathbb{C}^2} \check{\phi}_{\text{bub}} \frac{\mathrm{d}\ell_{1\parallel} \mathrm{d}\ell_{1\perp}^2}{\mathsf{D}_1 \ \mathsf{D}_2}$$

$\Rightarrow \check{\phi}_{\text{bub}}$ supported on tubular n.b.h. of *unregulated* poles

$$\therefore \quad \check{\phi}_{\text{bub}} \sim \mathrm{d}\theta_1 \wedge \mathrm{d}\theta_2$$

A one-loop example: *Dual bubble*

Localization of the intersection number (finiteness)

$$\langle \check{\phi}_{\text{bub}} | \phi_{\text{bub}} \rangle \sim \int_{\mathbb{C}^2} \check{\phi}_{\text{bub}} \frac{\mathrm{d}\ell_{1\parallel} \mathrm{d}\ell_{1\perp}^2}{\mathsf{D}_1\ \mathsf{D}_2}$$

$\Rightarrow \check{\phi}_{\text{bub}}$ supported on tubular n.b.h. of *unregulated* poles

$$\therefore \quad \check{\phi}_{\text{bub}} \sim \mathrm{d}\theta_1 \wedge \mathrm{d}\theta_2$$

$$\Downarrow$$

$$\langle \check{\phi}_{\text{bub}} | \phi_{\text{bub}} \rangle \sim \int_{\mathbb{C}^2} \frac{\mathrm{d}\theta_1 \mathrm{d}\ell_{1\parallel}}{\mathsf{D}_1} \frac{\mathrm{d}\theta_2 \mathrm{d}\ell_{1\perp}^2}{\mathsf{D}_2} = \oint_{\substack{\mathsf{D}_1=0 \\ \mathsf{D}_2=0}} \frac{\mathrm{d}\ell_{1\parallel}}{\mathsf{D}_1} \frac{\mathrm{d}\ell_{1\perp}^2}{\mathsf{D}_2} \sim 1$$

A one-loop example: *Dual bubble*

Localization of the intersection number (finiteness)

$$\langle \check{\phi}_{\text{bub}} | \phi_{\text{bub}} \rangle \sim \int_{\mathbb{C}^2} \check{\phi}_{\text{bub}} \frac{\mathrm{d}\ell_{1\parallel} \mathrm{d}\ell_{1\perp}^2}{\mathsf{D}_1 \; \mathsf{D}_2}$$

$\Rightarrow \check{\phi}_{\text{bub}}$ supported on tubular n.b.h. of *unregulated* poles

$$\therefore \quad \check{\phi}_{\text{bub}} \sim \mathrm{d}\theta_1 \wedge \mathrm{d}\theta_2$$

$$\Downarrow$$

$$\langle \check{\phi}_{\text{bub}} | \phi_{\text{bub}} \rangle \sim \int_{\mathbb{C}^2} \frac{\mathrm{d}\theta_1 \mathrm{d}\ell_{1\parallel}}{\mathsf{D}_1} \frac{\mathrm{d}\theta_2 \mathrm{d}\ell_{1\perp}^2}{\mathsf{D}_2} = \oint_{\substack{\mathsf{D}_1 = 0 \\ \mathsf{D}_2 = 0}} \frac{\mathrm{d}\ell_{1\parallel}}{\mathsf{D}_1} \frac{\mathrm{d}\ell_{1\perp}^2}{\mathsf{D}_2} \sim 1$$

***Rule of thumb for dual forms*:**

Dual forms come with a $\mathrm{d}\theta$ for each *cut* propagator

Summary: *Feynamn vs dual*

Feynman forms:	*Dual forms*:
$H^n_{\mathrm{dR}}(\mathbb{C}^n \backslash \{u=0,\infty\} \cup_a \{\mathsf{D}_a=0\};\nabla)$	$H^n_{\mathrm{alg}}(\mathbb{C}^n \backslash \{u=0,\infty\},\{\mathsf{D}=0\};\check{\nabla})$
Top dimensional holo forms	Top dimensional holo forms
Possible singularities on the locus $\{u=0,\infty\}$	
Possible singularities on the loci $\{\mathsf{D}_a=0\}$	Vanish on the loci $\{\mathsf{D}_a=0\}$

Differential equations

Both integrals and forms satisfy the same *differential equation* Ω

$$\mathrm{d}I$$

Differential equations

Both integrals and forms satisfy the same *differential equation* Ω

$$\Omega = \begin{array}{c}\text{matrix of}\\ \textit{kinematic}\text{ 1-forms}\end{array}$$

$$\mathrm{d}I \overset{\text{IBP}}{=} \Omega \cdot I$$

$$= \Omega \cdot \int u \, \phi$$

$$= \int u \, \Omega \mathbin{\dot\wedge} \phi$$

Differential equations

Both integrals and forms satisfy the same *differential equation* Ω

$$\mathrm{d}I$$

$\Omega =$ matrix of *kinematic* 1-forms

$$\overset{\text{IBP}}{=}\quad \Omega \cdot I = \Omega \cdot \int u\, \phi = \int u\, \Omega \,\dot{\wedge}\, \phi$$

$$= \quad \mathrm{d}\int u\, \phi = \int \mathrm{d}(u\, \phi) = \int u\, \big(\mathrm{d} + \mathrm{d}\log(u) \wedge\big)\, \phi = \int u\, \nabla\, \phi$$

Differential equations

Both integrals and forms satisfy the same *differential equation* Ω

$$\Omega = \text{matrix of } \textit{kinematic} \text{ 1-forms}$$

$$\mathrm{d}I$$

$$\Omega \cdot I \overset{\text{IBP}}{=} \mathrm{d}I = \mathrm{d}\int u\,\phi$$

$$= \Omega \cdot \int u\,\phi \qquad\qquad = \int \mathrm{d}(u\,\phi)$$

$$= \int u\,\Omega \mathbin{\overset{\cdot}{\wedge}} \phi \qquad\qquad = \int u\,\big(\mathrm{d} + \mathrm{d}\log(u)\wedge\big)\,\phi$$

$$= \int u\,\nabla\,\phi$$

Exercise Differentiating the intersection pairing yields

$$\check{\Omega} = -\Omega^{\top}$$

Differential equations

Both integrals and forms satisfy the same *differential equation* Ω

$$\mathrm{d}I$$

$$\Omega = \begin{matrix}\text{matrix of} \\ \textit{kinematic}\ \text{1-forms}\end{matrix}$$

$$\begin{aligned} \mathrm{d}I &\overset{\text{IBP}}{=} \Omega \cdot I = \Omega \cdot \int u\, \phi = \int u\, \Omega \mathbin{\dot{\wedge}} \phi \\ \mathrm{d}I &= \mathrm{d}\int u\, \phi = \int \mathrm{d}(u\, \phi) = \int u\, \left(\mathrm{d} + \mathrm{d}\log(u) \wedge\right) \phi = \int u\, \nabla\, \phi \end{aligned}$$

Exercise Differentiating the intersection pairing yields

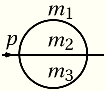
$$\check{\Omega} = -\Omega^{\top}$$

In most *analytic* calculations, solving

$$\nabla \phi \simeq \Omega \mathbin{\dot{\wedge}} \phi \quad \text{or} \quad \check{\nabla}\check{\phi} \simeq \check{\Omega} \mathbin{\dot{\wedge}} \check{\phi}$$

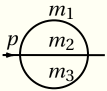
is *much* more systematic than brute force integration, provided Ω has *only simple poles* and is *linear in ε*

The *elliptic* sunrise integral



$$\sim \int \frac{\mathrm{d}^D \ell_1}{\pi^{D/2}} \frac{\mathrm{d}^D \ell_2}{\pi^{D/2}} \frac{\mathrm{d}^D \ell_3}{\pi^{D/2}} \frac{\delta^D(\ell_3 - \ell_1 + \ell_2 - p)}{\left(\ell_1^2 + m_1^2\right)\left(\ell_2^2 + m_2^2\right)\left(\ell_3^2 + m_3^2\right)}$$

The *elliptic* sunrise integral



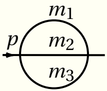
$$\sim \int \frac{\mathrm{d}^D \ell_1}{\pi^{D/2}} \frac{\mathrm{d}^D \ell_2}{\pi^{D/2}} \frac{\mathrm{d}^D \ell_3}{\pi^{D/2}} \frac{\delta^D(\ell_3 - \ell_1 + \ell_2 - p)}{\left(\ell_1^2 + m_1^2\right)\left(\ell_2^2 + m_2^2\right)\left(\ell_3^2 + m_3^2\right)}$$

in *momentum space* parameterization

$$\ell_i^\mu = \ell_{i\parallel}^\mu + \ell_{i\perp}^\mu, \quad \ell_{i\parallel} \cdot \ell_{i\perp} = 0, \quad \ell_{1\parallel}^\mu = x\, p^\mu$$

$$\begin{cases} \mathsf{D}_1 = \ell_{1\perp}^2 + \ell_{2\parallel}^2 + m_1^2 \\ \mathsf{D}_2 = \ell_{2\perp}^2 + x^2\, \ell_{1\perp}^2 + x^2 \left(\ell_{2\parallel}/p + 1\right)^2 p^2 + m_2^2 \\ \mathsf{D}_3 = \ell_{2\perp}^2 + (x+1)^2\, \ell_{1\perp}^2 + (x+1)^2 \left(\ell_{2\parallel}/p + 1\right)^2 p^2 + m_3^2 \end{cases}$$

The *elliptic* sunrise integral



$$\sim \int \frac{\mathrm{d}^D \ell_1}{\pi^{D/2}} \frac{\mathrm{d}^D \ell_2}{\pi^{D/2}} \frac{\mathrm{d}^D \ell_3}{\pi^{D/2}} \frac{\delta^D(\ell_3 - \ell_1 + \ell_2 - p)}{\left(\ell_1^2 + m_1^2\right)\left(\ell_2^2 + m_2^2\right)\left(\ell_3^2 + m_3^2\right)}$$

in *momentum space* parameterization

$$\ell_i^\mu = \ell_{i\parallel}^\mu + \ell_{i\perp}^\mu, \quad \ell_{i\parallel} \cdot \ell_{i\perp} = 0, \quad \ell_{1\parallel}^\mu = x\, p^\mu$$

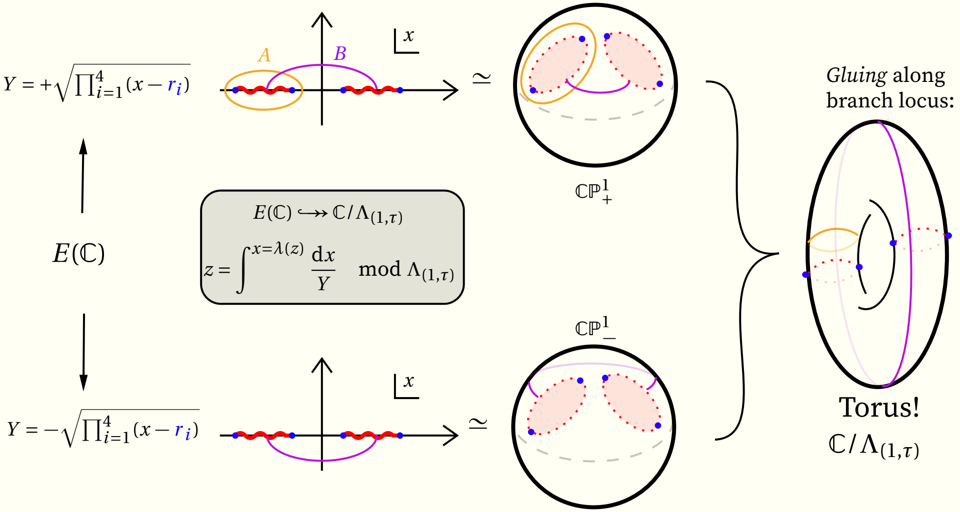
$$\begin{cases} \mathsf{D}_1 = \ell_{1\perp}^2 + \ell_{2\parallel}^2 + m_1^2 \\ \mathsf{D}_2 = \ell_{2\perp}^2 + x^2\, \ell_{1\perp}^2 + x^2 \left(\ell_{2\parallel}/p + 1\right)^2 p^2 + m_2^2 \\ \mathsf{D}_3 = \ell_{2\perp}^2 + (x+1)^2\, \ell_{1\perp}^2 + (x+1)^2 \left(\ell_{2\parallel}/p + 1\right)^2 p^2 + m_3^2 \end{cases}$$

Maximal-cut $\Longleftrightarrow$ **residue around** $\mathsf{D}_i = 0 \;\forall\; i$

In $D = 4$**, get an integral in** x **over** Y

$$E(\mathbb{C}) : Y^2 - (x - r_1)(x - r_2)(x - r_3)(x - r_4) = 0$$

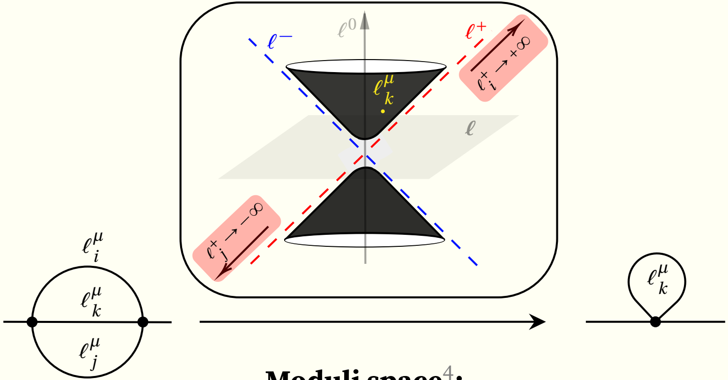
A *useful** isomorphism



*We will see soon that torus variables are the *natural* ones for our problem

DOFs on the torus

$\mathbb{C}/\Lambda_{(1,\tau)}$ comes with *marked points* inherited from $\binom{3}{2}$ = three special configurations of the sunrise graph



Moduli space[4]:

Torus with three marked points: $\{z_i = F(u_i)/K\}_{i=1}^3$

[4]One marked point is fixed by translational symmetry