Elliptic Moduli of Bhabha Scattering

arXiv:2306.???

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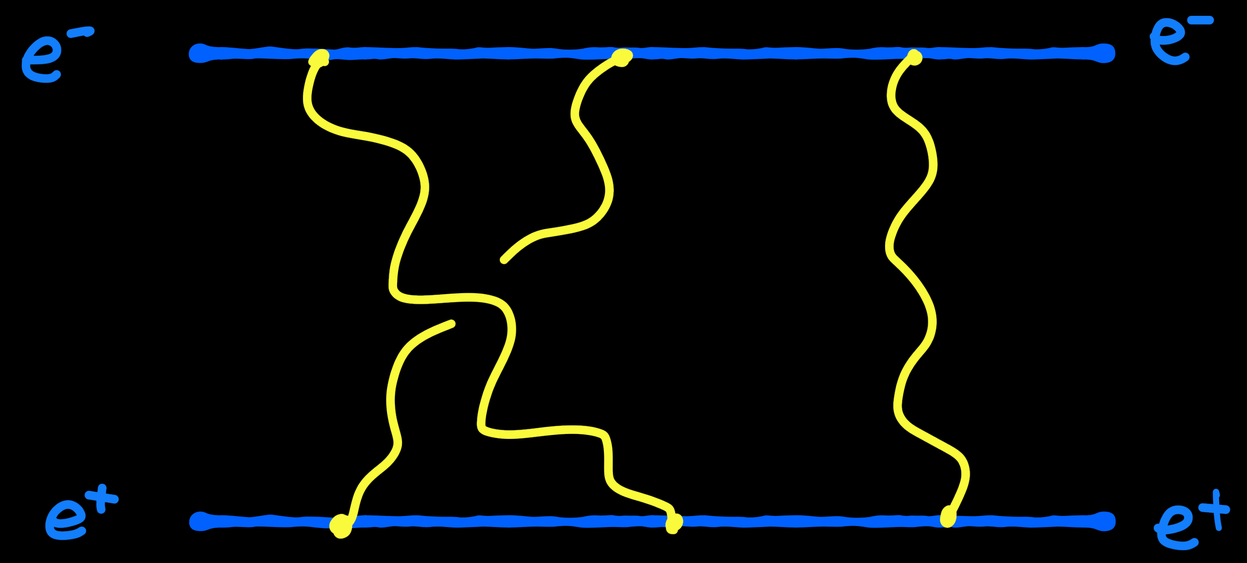

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Section 1: A discussion of the results

- **Brief histories**

  - [Bhabha,1936]
    $e^-(p_1) + e^+(p_2) \to e^-(p_3) + e^+(p_4)$

  - [Bern, Dixon and Ghinculov, 2001]
    Massless $p_i^2 = 0$

  - [R. Bonciani, J. Fleischer, Stefan's Actis, Michal Czakon, T. Riemann, Janus's Gluza, Thomas Becher … 2004 $\sim$ 2008 … Johannes M. Henn, Vladimir A. Smirnov, Lorenzo Tancredi, Claude Duhr 2013 $\sim$ 2021]
    Massive planar topologies, K3 surface

  - [Claude Duhr, Lorenzo Tancredi, Maximilian Delto, Yu Jiao Zhu, 2023]
    Nonplanar topologies, a family of elliptic curves

Is the finite remainder a physical quantity?

- **Abelian exponentiation and 1-loop exactness [Yennie, 1961]**

$$\mathscr{A}^{\mathrm{OS}}(\alpha, m, s, t, \epsilon) = e^{\frac{\frac{\alpha}{4\pi} Z_1^{\mathrm{IR}}}{\epsilon}} C(\alpha, m, s, t, \epsilon)$$

$$Z_1^{\mathrm{IR}} = \frac{4(-2m^2+s)}{\sqrt{-s}\sqrt{4m^2-s}} \ln\left(1 - \frac{1}{2}\frac{s}{m^2} - \frac{1}{2}\sqrt{\frac{-s}{m^2}}\sqrt{4-\frac{s}{m^2}}\right)$$
$$+ \frac{4(-2m^2+t)}{\sqrt{-t}\sqrt{4m^2-t}} \ln\left(1 - \frac{1}{2}\frac{t}{m^2} - \frac{1}{2}\sqrt{\frac{-t}{m^2}}\sqrt{4-\frac{t}{m^2}}\right)$$
$$- \frac{4(-2m^2+u)}{\sqrt{-u}\sqrt{4m^2-u}} \ln\left(1 - \frac{1}{2}\frac{u}{m^2} - \frac{1}{2}\sqrt{\frac{-u}{m^2}}\sqrt{4-\frac{u}{m^2}}\right) - 4$$

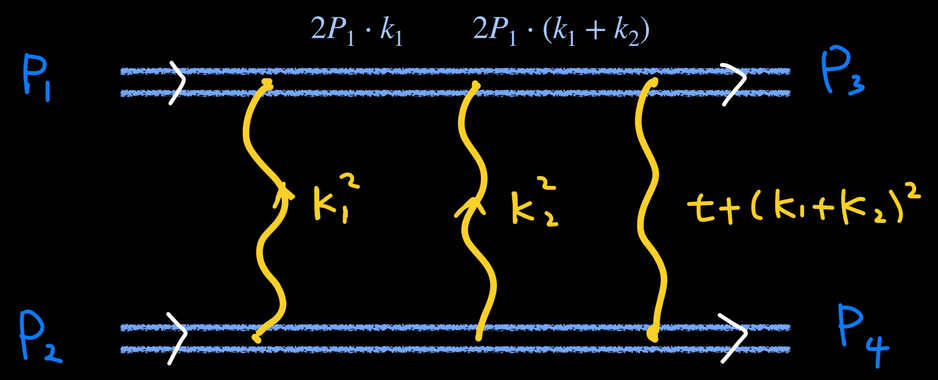
- **Scheme dependence [2011.13946]**

✓ $C_{\mathrm{CDR}} = C_{\mathrm{tHV}}$

✗ $C_{\mathrm{CDR}} = C_{\mathrm{tHV}} = C_{\mathrm{random}}$

Regge limit & mass expansions

- **Regge Limit $s, m^2 \gg t$ [Korchemsky 1996]**



$$\sim \left(\frac{t}{\mu}\right)^{-2\epsilon} \left(\frac{P_1^2 \, P_2^2}{(P_1 \cdot P_2)^2}\right)^{-2\epsilon}$$

$$t \equiv (P_3 - P_1)^2$$

- **Mass-expansion and region of convergence [Xiao Liu, Yan-Qing Ma, 2022]**
  - High energy: $E_{\text{CM}} \geq 1$ GeV and $\pm 2^\circ \leq \theta \leq \pm 178^\circ$,
    e.g., at $E_{\text{CM}} \approx 1$ GeV and $\theta \approx 4^\circ$, $|A^{[2]}| \simeq 10^{11}$, $|A^{[2]} - A^{[2]}_{\text{AMFlow}}| \simeq 10^{-3}$
  - Low energy: e.g., at $E_{\text{CM}} \approx 250$ MeV and $\theta \approx 36^\circ$, $|A^{[2]}| \simeq 10^{7}$, $|A^{[2]} - A^{[2]}_{\text{AMFlow}}| \simeq 10^{-9}$

Section 2: Symmetries of the amplitude

- SL(2,$\mathbb{C}$) Lorentz invariance (well-defined probability interpretations, regardless of any reference of frame)
- SU(3) or U(1) singlet (color-charge conservation)

$$e^{-i\theta\cdot T}|\mathscr{A}\rangle = |\mathscr{A}\rangle \Longleftrightarrow T^a|\mathscr{A}\rangle = 0\,, \quad T^a \equiv \sum_i T_i^a$$

- Linear operators in the Fock space
- SU(2) tensor (little group covariance)

$$P_{a\dot{a}} = {}_a|P^I\rangle[P_I|_{\dot{a}} = {}_a|P^I\rangle[P_I|_{\dot{a}} \Longrightarrow {}_a|P^I\rangle = {}_a|P^J\rangle R^I_J\,, \quad R \in SU(2)$$

- SL(2,$\mathbb{Z}$) modular symmetry for the symbol letters

SU(2) indices

$$\dot{a}\,|\,3^I\,]$$

$SL_2(\mathbb{C})$ indices

Branched symbol letters

- Amplitude through canonical bases

Residues and periods

$$\mathscr{A}(s,t) = \sum_i R_i(s,t) \times \mathrm{J}_i(s,t)$$

canonical bases

- Canonical form for the differential equations [Johannes M. Henn 2014]

$$\mathrm{d}\vec{\mathrm{J}} = \epsilon\, M(s,t) \cdot \vec{\mathrm{J}}$$

- Kernel $M(s,t)$ as linear array over the symbol letters

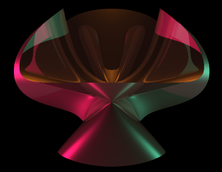
$$M(s,t) = \sum_i c_i \times \omega_i(s,t)\,, \quad c_i \in \mathbb{Q}\,, \quad \mathrm{d}\omega_i(s,t) = 0$$

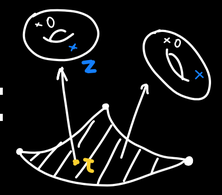
An typical symbol letter for the non-planar

$$
\begin{aligned}
\mathrm{ds}\times\Bigg\{&\frac{-4\sqrt{(t-4)t}(2s^2+3st-10s-2t+8)}{(s-4)s(4-t)(s+t-4)}T_2(s,t)\\
&+\frac{2s^3t^2-4s^3t+80s^3+s^2t^3+2s^2t^2+288s^2t-480s^2+4st^3+346st^2-1224st+640s+169t^3-776t^2+400t}{4(s-4)s(t-4)(s+t-4)(s+t)}T_1(s,t)\\
&+\frac{s^3t^2-2s^3t+8s^3+2s^2t^3-10s^2t^2+56s^2t-64s^2-2st^3+81st^2-260st+128s+49t^3-264t^2+272t}{(s-4)s(4-t)^2(s+t-4)}2\sqrt{(t-4)t}\Psi(s,t)\\
&+\left[\frac{6(t-4)\sqrt{(t-4)t}}{(s-4)s(4-t)^2t}T_1^2(s,t)+\frac{(s+1)(2s+t-4)}{(s-4)s(s+t-4)(s+t)}T_1(s,t)T_2(s,t)-\frac{\sqrt{(t-4)t}}{(s-4)s(4-t)t}T_2^2(s,t)\right]\frac{1}{\Psi(s,t)}\\
&+\left[\frac{2s+t-4}{4(s-4)st(s+t-4)(s+t)}T_1^3(s,t)+\frac{2s+t-4}{(s-4)st(s+t-4)(s+t)}T_1(s,t)T_2^2(s,t)\right]\frac{1}{\Psi^2(s,t)}\Bigg\}\\
+\mathrm{dt}\times\Bigg\{&2\frac{2s^2-st^2+11st-4s+7t^2-8t-16}{(4-t)^2(s+t-4)}\sqrt{(t-4)t}\Psi(s,t)+\frac{-s^2t^2+10s^2t+8s^2+12st^2+40st-32s+8t^3+39t^2-92t}{4(t-4)t(s+t-4)(s+t)}T_1(s,t)\\
&-\left[\frac{1}{4t^2(s+t-4)(s+t)}T_1^3(s,t)+\frac{1}{t^2(s+t-4)(s+t)}T_2^2(s,t)T_1(s,t)\right]\frac{1}{\Psi^2(s,t)}-\frac{s+1}{t(s+t-4)(s+t)}\frac{T_1(s,t)T_2(s,t)}{\Psi(s,t)}\\
&+\frac{4\sqrt{(t-4)t}}{(4-t)(s+t-4)}T_2(s,t)\Bigg\}
\end{aligned}
$$

Classification by moduli space

- **Review of planar topologies [1307.4083, 2108.03828]**

✗ Coordinates on framed curves: 

✓Coordinates on the moduli space of curves: 

- **A General Picture**
  - Classify and unify different scattering processes according to the underlying moduli space
  - Compute the uniformization groups *(deck transformations) for the symbol letters

Nonplanar topology—Preview

▸ Uniformization

- Given a punctured $\mathbb{CP}^2\backslash\Sigma$, what is its universal covering space and the corresponding covering map? What is the covering automorphism?

$$\mathrm{Deck}_\pi(\mathbb{C}\times\mathbb{H}) \simeq \overbrace{\mathbb{Z}^2 \rtimes \Gamma_1(4)}^{\pi} \simeq \pi_1(\mathbb{CP}^2\backslash\Sigma, \cdot\,) \simeq \overbrace{\mathrm{Deck}_f(\mathbb{C}\times\mathbb{H})}^{f}$$

$$\begin{tikzcd}
\mathbb{C}\times\mathbb{H} \arrow[d, "\pi"'] \arrow[dr, "f"] & \\
(\mathbb{Z}^2 \rtimes \Gamma_1(4))\backslash\mathbb{C}\times\mathbb{H} \arrow[r, shift left, "\tilde{f}"] & \mathbb{CP}^2\backslash\Sigma \arrow[l, shift left, "(z\,,\,\Psi_2/\Psi_1)"]
\end{tikzcd}$$

$\Sigma = \{\text{kinematic branch points}\}$

- Covering space quotient theorem
- Covering automorphism group structure theorem

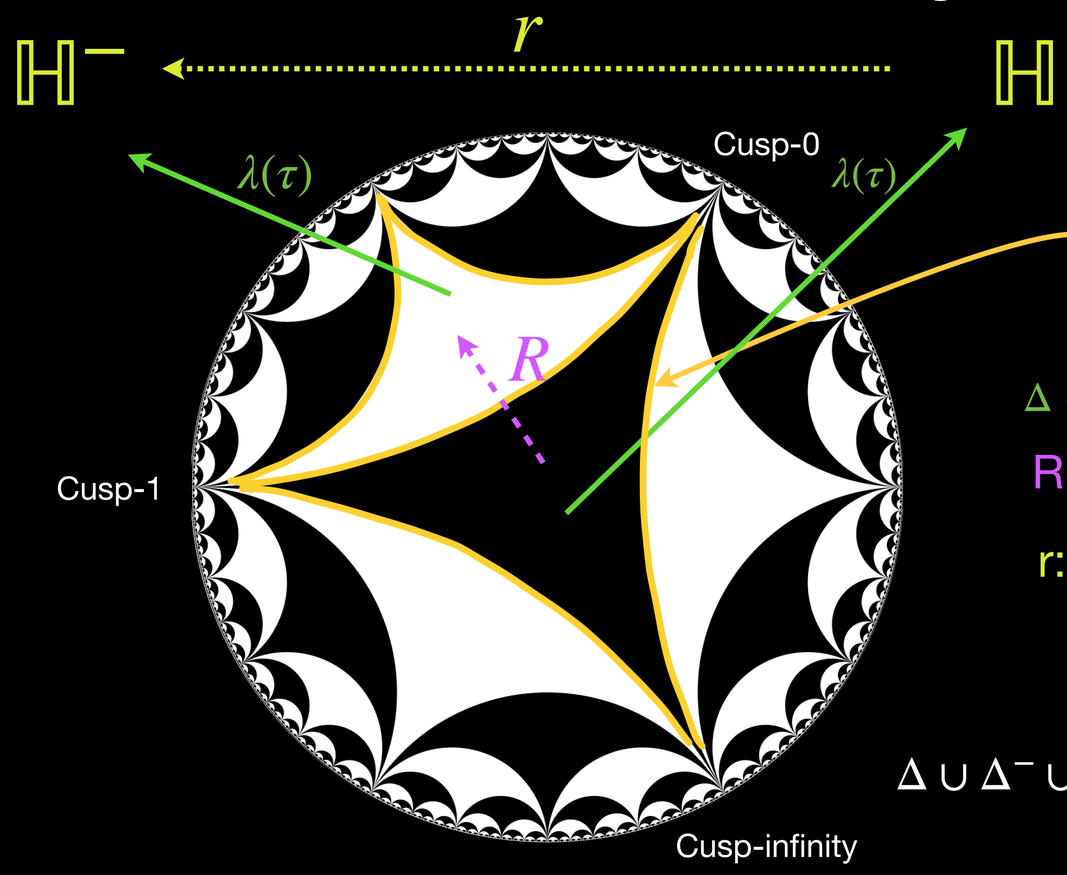
▸ Pullback of the closed 1-forms

- What are the closed 1-forms on moduli space of elliptic curves with level structures and extra marked points? Are they given by Kronecker differential forms? [D.Zagier 1991]

$$\omega_k^{\text{Kronecker}}(z,\tau) = (2\pi i)^{2-k}\left[g^{(k-1)}(z-c_j,\tau)dz + \left(k-1\right)g^{(k)}(z-c_j,\tau)\frac{d\tau}{2\pi i}\right]$$

Poincaré polygon theorem

Geometric construction by the Fuchsian Triangle Group $\Gamma_{\infty\infty\infty}$



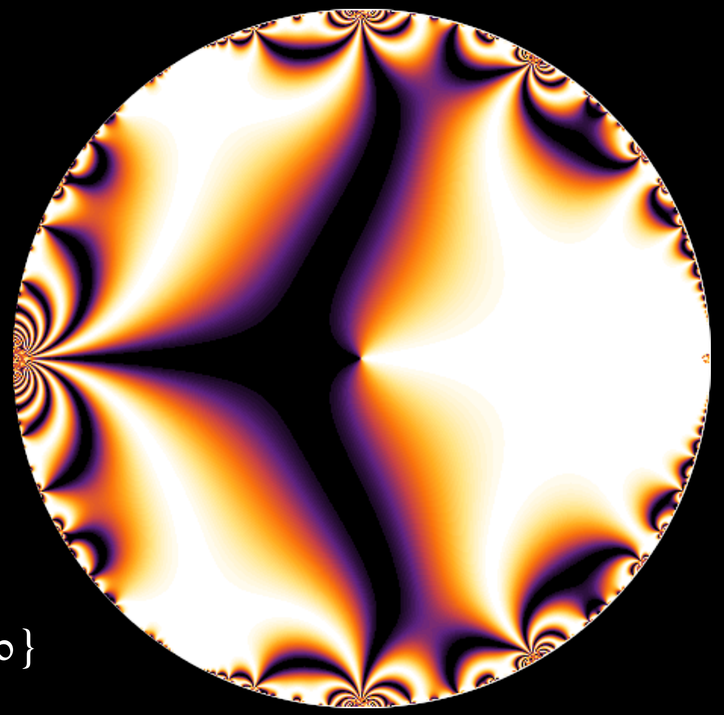
Δ = Fundamental triangle

$\Delta \overset{\lambda(\tau)}{\longleftrightarrow} \mathbb{H}$: Riemann Mapping Theorem

R: Schwarz Reflection

r: Schwarz Reflection Principle

$$\Delta \cup \Delta^- \cup \partial(\Delta \cup \Delta^-) \xrightarrow{\lambda(\tau)} \mathbb{H} \cup \mathbb{H}^- \cup \overline{\mathbb{R}} \backslash \{0, 1, \infty\}$$



Tiling by $\Gamma_{\infty\infty\infty} \simeq \Gamma(2) \simeq \mathbb{Z} * \mathbb{Z}$

$\lambda(\tau)$: Modular function for $\Gamma(2)$

$$\mathbb{CP}^1 \backslash \{0, 1, \infty\} \simeq \Gamma(2) \backslash \mathbb{D}$$

Uniformization of $\mathbb{CP}^1 \backslash \{0, 1, \infty\}$

What is the inverse function to $\tilde{\lambda}$?

$\lambda(\tau) = \lambda(\gamma \cdot \tau), \forall \gamma \in \Gamma(2) \Longrightarrow \tilde{\lambda}$ is a well defined holomorphic map on $Y(2) = \Gamma(2)\backslash\mathbb{H}$

$$\begin{tikzcd}
\mathbb{H} \simeq \mathbb{D} \arrow[d, "\pi"'] \arrow[dr, "\lambda"] & \\
Y(2) = \Gamma(2)\backslash\mathbb{H} \arrow[r, shift left, "\tilde{\lambda}"] & \mathbb{CP}^1 \backslash \{0, 1, \infty\} \arrow[l, shift left, "\tilde{\lambda}^{-1} = ?"]
\end{tikzcd}$$

Deck $= \Gamma(2)$

A family of curves over $\mathbb{CP}^1\backslash\{0,1,\infty\}$

- Period mappings for a family of elliptic curves

$$\begin{tikzcd} \mathbb{H} \arrow[d, "\pi"'] \arrow[dr, "\lambda"] & \\ \Gamma(2)\backslash\mathbb{H} \arrow[r, shift left, "\tilde{\lambda}"] & \mathbb{CP}^1\backslash\{0,1,\infty\} \arrow[l, shift left, "\Psi_2/\Psi_1"] \end{tikzcd}$$

$$E_\lambda : Y^2 = X(X-1)(X-\lambda)\,, \lambda \in \mathbb{CP}^1\backslash\{0,1,\infty\}$$

$$j(\lambda) = 256\frac{(1-\lambda(1-\lambda))^3}{\lambda^2(1-\lambda)^2}$$

$$\tau \equiv \frac{\Psi_2}{\Psi_1} = \frac{iK(1-\lambda)}{K(\lambda)}$$

which is ramified at $\lambda = 0\,,1$ and ∞, each with ramification index 2 so that $deg(j) = 6 = [\mathbb{P}\mathrm{SL}(2,\mathbb{Z}) : \Gamma(2)]$

$$\int \frac{dX}{Y} : H_1(E_\lambda) \to \mathbb{C}$$

$$\Psi_1(\lambda) \equiv \int_0^\lambda \frac{\mathrm{d}X}{Y} = 2K(\lambda)\,, \quad \Psi_2(\lambda) \equiv \int_1^\lambda \frac{\mathrm{d}X}{Y} = 2iK(1-\lambda)$$

Monodromy Group=Deck Transformation

- Picard-Fuchs differential equation

$$\left[4\lambda(1-\lambda)\frac{d^2}{d\lambda^2}+4(1-2\lambda)\frac{d}{d\lambda}-1\right]\Psi_i=0\,,\quad i=1\,,2$$

$$\begin{tikzcd}
\mathrm{Deck} \arrow[loop above] & \\
\mathbb{H} \arrow[d, "\pi"'] \arrow[dr, "\lambda"] & \\
\Gamma(2)\backslash\mathbb{H} \arrow[r, shift left, "\tilde{\lambda}"] & \mathbb{CP}^1\backslash\{0,1,\infty\} \arrow[l, shift left, "\Psi_2/\Psi_1"]
\end{tikzcd}$$

- Monodromy representations $\rho_{[\gamma_1][\gamma_2]}=\rho_{[\gamma_1]}\cdot\rho_{[\gamma_2]}$

$$\rho:\pi_1(X,\cdot)\to \mathrm{GL}_2(\mathbb{C})$$
$$[\gamma]\mapsto\rho_{[\gamma]}\,.$$

$$\tau\to\tau_{\circlearrowleft}\equiv\rho_{[\circlearrowleft]}\cdot\tau$$

- Images of the generators in $\mathbb{P}\mathrm{SL}(2,\mathbb{Z})$

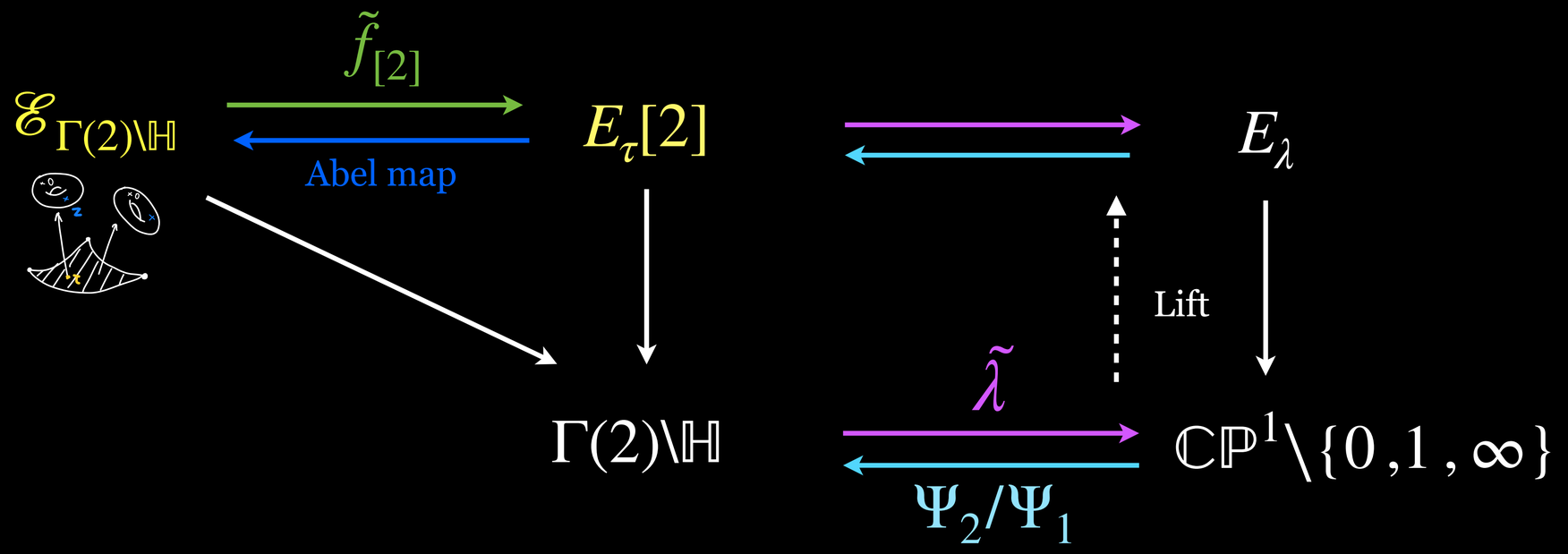
$$\rho_{[\circlearrowleft_0]}=\begin{pmatrix}1&2\\0&1\end{pmatrix},\quad \rho_{[\circlearrowleft_1]}=\begin{pmatrix}1&0\\-2&1\end{pmatrix}$$

$$\rho(\pi_1(X,\cdot))=\langle\rho_{[\circlearrowleft_0]},\rho_{[\circlearrowleft_1]}\rangle=\underbrace{\mathrm{Deck}_\pi(\mathbb{H})\simeq\Gamma(2)}_{\pi}\simeq\mathbb{Z}*\mathbb{Z}\simeq\mathrm{Deck}_\lambda(\mathbb{H})$$

- Covering automorphism group structure theorem
- Covering space quotient theorem

Uniformization & the universal family of curves

Are the two universal families $\mathscr{E}_{\Gamma(2)\backslash\mathbb{H}}$ and $E_\tau[2]$ isomorphic?



$$\mathscr{E}_{\Gamma(2)\backslash\mathbb{H}} \equiv (\mathbb{Z}^2 \rtimes \Gamma(2))\backslash\mathbb{C}\times\mathbb{H}$$

$$E_\tau[2] : Y^2 = X(X-1)(X-\lambda(\tau)), \tau \in \Gamma(2)\backslash\mathbb{H}$$

$$E_\lambda : Y^2 = X(X-1)(X-\lambda), \lambda \in \mathbb{C}\backslash\{0,1\}$$

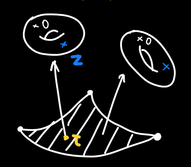
The isomorphism between $E_\tau[2] \simeq \mathscr{E}_{\Gamma(2)\backslash\mathbb{H}}$

- the isomorphism map

$$(z,\tau) \in \mathbb{C}\times\mathbb{H} \xmapsto{f_{[2]}} \left[X : \frac{1}{2\Psi_1(\tau)}\partial X/\partial z : 1\right] \in E_\tau[2]$$

$$\Psi_1(\tau) = \pi\theta_3^2(0,q)\,, \quad X(z) = \frac{\theta_2^2(0,q)\theta_4^2(0,q)}{\theta_3^2(0,q)\theta_1^2(\pi z,q)}$$

$$\begin{tikzcd}
\mathbb{C}\times\mathbb{H} \arrow[d, mapsto] \arrow[dr, dotted, green, "f_{[2]}"] & & \\
\mathscr{E}_{\Gamma(2)\backslash\mathbb{H}} \arrow[r, shift left, green, "\tilde{f}_{[2]}"] \arrow[dr] & E_\tau[2] \arrow[l, shift left, blue, "\text{Abel map}"] \arrow[d] \\
& \Gamma(2)\backslash\mathbb{H}
\end{tikzcd}$$



- invariance under $\mathbb{Z}^2 \rtimes \Gamma(2) \Longrightarrow \tilde{f}_{[2]}$ is well-defined

$$f_{[2]}[z\,,\tau] = f_{[2]}[((m,n),\gamma)\cdot(z\,,\tau)]\,, \forall (m\,,n) \in \mathbb{Z}^2\,, \gamma \in \Gamma(2)\,, \quad ((m,n),\gamma)\cdot(z\,,\tau) = \left(\frac{z+m\tau+n}{c\tau+d}\,,\gamma\cdot\tau\right)$$

The universal family of complex torus

$$\mathscr{E}_{\Gamma\backslash\mathbb{H}} \equiv (\mathbb{Z}^2 \rtimes \Gamma)\backslash\mathbb{C} \times \mathbb{H}$$

- $\mathbb{Z}^2 \rtimes \Gamma$ is isomorphic to the following action

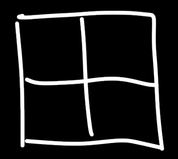
$$\begin{pmatrix} z \\ \tau \\ 1 \end{pmatrix} \mapsto \frac{1}{c\tau + d}\begin{pmatrix} 1 & m & n \\ 0 & a & b \\ 0 & c & d \end{pmatrix} = \begin{pmatrix} \frac{z + m\tau + n}{c\tau + d} \\ \gamma \cdot \tau \\ 1 \end{pmatrix}, \quad (m, n) \in \mathbb{Z}^2, \quad \gamma \in \Gamma$$

- Is each fiber a complex torus?

$-1 \notin \Gamma$ and that the action of Γ is free

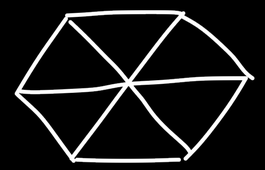
potential candidates: $\Gamma_1(N)$, with $N > 3$

$e^{i\frac{\pi}{2}}$



$\mathbb{Z}_4$

$e^{i\frac{\pi}{3}}$



$\mathbb{Z}_6$

A family of curves over punctured $\mathbb{CP}^2$

- The family of elliptic curves for Bhabha scattering, with coordinates $[s : t : m^2]$

$$E_4 : Y^2 = (X - e_1)(X - e_2)(X - e_3)(X - e_4)$$

$$e_1 = \frac{s}{m^2} - 4\,, \quad e_2 = -\frac{st + 2\sqrt{m^2 s\, t(s + t - 4m^2)}}{m^2(4m^2 - t)}\,, \quad e_3 = -\frac{st - 2\sqrt{m^2 s\, t(s + t - 4m^2)}}{m^2(4m^2 - t)}\,, \quad e_4 = \frac{s}{m^2}$$

- What is the base space ? Answer: equating the roots in all possible ways. But Why? Answer: cusps correspond to elliptic curves with nodes or monomial singularities

Union of the following linear varieties is deleted :

$$\Sigma = \langle s\,, s - 4\,, s + t\,, s + t - 4\,, t\,, t - 4 \rangle \cup \{[1 : 0 : 0]\}$$

$$\mathbb{CP}^2 \backslash \Sigma$$

The Mordell-Weil group for a family of curves

- Theorem of Mordell-Weil

For elliptic curves over $\mathbb{Q}$ (or its finite extensions), the group of rational points is finitely generated

- Sections of rational points $\{[n]p_0 \mid p_0 \in A(E_3) \simeq T \oplus r\mathbb{Z}, n \in \mathbb{Z}\} \simeq (\mathbb{Z}, +)$

$$E_3 : Y^2 = \prod_{i=1}^{3}(e_i - e_4)\left(X + \frac{e_i}{e_4(e_i - e_4)}\right)$$

$$p_0 = \left[\frac{s-4}{s(s+t)} : \frac{(s-4)(s+t-4)}{s(s+t)} : 1\right]$$

$$[2]p_0 = \left[\frac{16 + t(8 - 3t + s(s+t-4))}{4s(t-4)(s+t)} : \ldots : 1\right]$$

$$\vdots$$

Generators of the Mordell-Weil group as marked points

$$\begin{tikzcd} E_{\text{bhabha}} \arrow[d] & E_{\text{universal}} \arrow[d] \\ \mathbb{CP}^2 \backslash \Sigma \arrow[r] & \mathscr{M}_{1,2}[N] \end{tikzcd}$$

- The generator on Mordell-Weil group $A(E_{[s:t:m^2]}) \simeq T \oplus r\mathbb{Z}$

$$p_0 = [X : Y : 1] = \left[\frac{(s-4m^2)s}{-4m^2+2s+t} : \frac{(s-4m^2)s/m^2(s+t-4m^2)}{(2s+t-4m^2)^2/(s+t)} : m^2 \right]$$

- Mapping to a universal family of torus

Abel map: $\dfrac{(e_2-e_4)(e_1-X)}{(e_1-e_4)(e_2-X)} = \dfrac{\theta_2^2(0,q)}{\theta_3^2(0,q)} \dfrac{\theta_1^2(\pi z,q)}{\theta_4^2(\pi z,q)}$, Modular lambda: $\dfrac{4m^2}{2m^2+\sqrt{\dfrac{(-m^2-i0)s(s+t-4m^2)}{-t}}} = \dfrac{\theta_2^4(0,q)}{\theta_3^4(0,q)}$

Uniformization of punctured $\mathbb{CP}^2$

- Definition of the map $f_{[4]} : \mathbb{C} \times \mathbb{H} \mapsto \mathbb{CP}^2 \backslash \Sigma$

$$\begin{tikzcd} \mathbb{C} \times \mathbb{H} \arrow[d, "\pi"'] \arrow[dr, "f_{[4]}"] & \\ (\mathbb{Z}^2 \rtimes \Gamma_1(4)) \backslash \mathbb{C} \times \mathbb{H} \arrow[r, shift left, "\tilde{f}_{[4]}"] & \mathbb{CP}^2 \backslash \Sigma \arrow[l, shift left, "(z\,,\Psi_2/\Psi_1)"] \end{tikzcd}$$

$$\mathrm{s} = -\frac{4(-1+R)\times(-2+\lambda)}{-2+\lambda+R\times\lambda}, \quad \mathrm{t} = \frac{4(-1+R)\times R\times\lambda^2}{(-2+R\times\lambda)(-2+\lambda+R\times\lambda)}$$

$$R = \frac{\theta_2^2(0,q)}{\theta_3^2(0,q)}\frac{\theta_1^2(\pi z,q)}{\theta_4^2(\pi z,q)}, \quad \lambda = \frac{\theta_2^4(0,q)}{\theta_3^4(0,q)}$$

$\Sigma = \{$kinematic branch points$\}$

- $f_{[4]}$ is invariance under $\mathbb{Z}^2 \rtimes \Gamma_1(4) \Longrightarrow \tilde{f}_{[4]}$ is well-defined

$$f_{[4]}[z\,,\tau] = f_{[4]}[((m,n),\gamma)\cdot(z\,,\tau)]\,, \forall (m\,,n) \in \mathbb{Z}^2, \gamma \in \Gamma_1(4)\,, \quad ((m,n),\gamma)\cdot(z\,,\tau) = \left(\frac{z+m\tau+n}{c\tau+d}, \gamma\cdot\tau\right)$$

- **<u>The period is a modular form of weight 1 under the action of $\mathbb{Z}^2 \rtimes \Gamma_1(4)$</u>**

$$\Psi_1(z,\tau) \sim \theta_2^2(0,q)\frac{\theta_3(\pi z,q)\theta_4(\pi z,q)}{\theta_1(\pi z,q)\theta_2(\pi z,q)}, \quad \Psi_1|_{\bar{\gamma}}(z,\tau) = \frac{1}{c\tau+d}\Psi_1(z,\tau)\,, \forall \bar{\gamma} \in \mathbb{Z}^2 \rtimes \Gamma_1(4)$$

Pullback of the closed 1-forms on $\mathbb{CP}^2\backslash\Sigma$

$$\begin{tikzcd} f^*\omega & \mathbb{C}\times\mathbb{H} \arrow[d, "f_{[4]}"'] \\ \omega \arrow[u, "f^*"] & \mathbb{CP}^2\backslash\Sigma \end{tikzcd}$$

- Fundamental differentials

$$\omega_z = \mathrm{d}t\,\frac{-1}{4t^2(s+t-4m^2)(s+t)}\frac{\mathrm{T}_1(s,t)}{\Psi_1^2(s,t)} + \mathrm{d}s\left(\frac{2s+t-4m^2}{4s(s-4m^2)t(s+t)(s+t-4m^2)}\frac{\mathrm{T}_1(s,t)}{\Psi_1^2(s,t)} + \frac{2\sqrt{-t}\sqrt{4m^2-t}}{s(s-4m^2)t(t-4m^2)}\frac{1}{\Psi_1(s,t)}\right),$$

$$\omega_\tau = \frac{\mathrm{d}t\,(s-4m^2)s - \mathrm{d}s\,t(2s+t-4m^2)}{2st^2(s-4m^2)(s+t-4m^2)(s+t)\Psi_1^2(s,t)}$$

$$\omega_\tau \xmapsto{f^*} i\pi\mathrm{d}\tau \quad \text{and} \quad \omega_z \xmapsto{f^*} 2\pi\mathrm{d}z$$

$$\mathrm{T}_1(s,t) = \int \mathrm{d}s\left[\frac{-t}{s}\frac{4s^2+4s(t-4m^2)+t(t-4m^2)}{\sqrt{-t}\sqrt{4m^2-t}}\Psi_1 - 8t\frac{(s+t-4m^2)(s+t)}{\sqrt{-t}\sqrt{4m^2-t}\,(t+2s-4m^2)}\partial_s\Psi_1\right] + \mathrm{d}t\left[\frac{-t}{4m^2-t}\frac{-48m^4+4m^2s+2s^2+12m^2t+st}{\sqrt{-t}\sqrt{4m^2-t}\,(t+s-4m^2)}\Psi_1\right]$$

Pullback of the closed 1-forms on $\mathbb{CP}^2\backslash\Sigma$

- 4-dimensional cubic lattice $\mathbb{Z}^4$

$$\omega_{11} = \mathrm{d}t\frac{\sqrt{(s-4m^2)s}}{t\sqrt{(s+t-4m^2)(s+t)}} - \mathrm{d}s\frac{2s+t-4m^2}{\sqrt{(s-4m^2)s}\sqrt{(s+t-4m^2)(s+t)}}$$

$$\omega_{11} \overset{f^*}{\longmapsto} 2\,\Theta_{\mathbb{Z}^4}(e^{\pi i}q^2)\frac{\mathrm{d}q}{q} = 2\theta_3^4(e^{\pi i}q^2)\frac{\mathrm{d}q}{q} \in \mathscr{M}_2(\Gamma_1(4))$$

- Jacobi's four square theorem $\Omega = \mathbb{Z}^4$

$$\Theta_\Omega(\tau) = \sum_{x\in\Omega} e^{2i\pi\tau||x||^2} = \sum_{n=0}^{\infty} r(n\,,k)(e^{2\pi i\tau})^n\,, \quad r(n\,,k) = \#\left\{v\in\mathbb{Z}^k\,:\,n = v_1^2+\ldots+v_k^2\right\}\,, \quad \mathrm{Im}\tau > 0$$

$$\Theta_{\mathbb{Z}^4}(\tau) \equiv \theta_3^4(\tau) \Longrightarrow r(n,4) = 8\sum_{0<d|n\,,4\nmid d} d\,, \quad n\geq 1$$

Pullback of the closed 1-forms on $\mathbb{CP}^2\backslash\Sigma$

$$\omega_{41} = \mathrm{d}t\left[\frac{1}{2t^2(s+t-4m^2)(s+t)}\frac{\mathrm{T}_1^2(s,t)}{\Psi_1^2(s,t)} + \frac{2(s-4m^2)}{(t-4m^2)(s+t-4m^2)}\right]$$

$$+\,\mathrm{d}s\left[\frac{2s+t-4m^2}{2(s-4m^2)st(s+t-4m^2)(s+t)}\frac{\mathrm{T}_1^2(s,t)}{\Psi_1^2(s,t)} + \frac{\sqrt{t(t-4m^2)}}{(s-4m^2)s(4m^2-t)t}\frac{\mathrm{T}_1(s,t)}{\Psi_1(s,t)} - \frac{2t(2s^2+st+4m^2s+12m^2t-48m^4)}{(s-4m^2)s(t-4m^2)(s+t-4m^2)}\right]$$

- D_4 root lattice $\quad D_4 = \frac{1}{2}(1+\mathbf{i}+\mathbf{j}+\mathbf{k})\mathbb{Z} \oplus \mathbf{i}\,\mathbb{Z} \oplus \mathbf{j}\,\mathbb{Z} \oplus \mathbf{k}\,\mathbb{Z} = \frac{1}{2}\mathbb{Z} \oplus \frac{1}{2}\mathbf{i}\,\mathbb{Z} \oplus \frac{1}{2}\mathbf{j}\,\mathbb{Z} \oplus \frac{1}{2}\mathbf{k}\,\mathbb{Z}$

$$\omega_{41} \overset{f^*}{\longmapsto} 8\omega_2^{\mathsf{Kro}}(2z\,,q) - 8\omega_2^{\mathsf{Kro}}(2z\,,q^2) + \frac{4}{3}\frac{\mathrm{d}q}{q}\Theta_{D_4}(q^2)$$

$$\Theta_{D_4}(q^2) = \theta_3^4(q^2) + \theta_2^4(q^2) \in \mathscr{M}_2(\Gamma_0(2)) \subset \mathscr{M}_2(\Gamma_1(4))$$

Pullback of the closed 1-forms on $\mathbb{CP}^2\backslash\Sigma$

$$\omega_{63} = \mathrm{d}t\,\frac{1}{4t(-4+s+t)(s+t)}\times\left[-\frac{156t-55t^2+s^2(-24-2t+t^2)+s(96-72t+4t^2)}{-4+t}-4(1+s)\frac{\mathrm{T}_2(s,t)}{\Psi_1(s,t)}-\frac{1}{t}\frac{\mathrm{T}_1^2(s,t)+4\mathrm{T}_2^2(s,t)}{\Psi_1^2(s,t)}\right]$$
$$+\mathrm{d}s\,\frac{-4+2s+t}{4s(s-4)(-4+s+t)(s+t)}\times\left[\frac{-112t-136t^2+41t^3+s^3(16-4t+2t^2)+s^2(-96+64t+2t^2+t^3)+s(128-200t+58t^2+4t^3)}{(-4+t)(-4+2s+t)}\right.$$
$$\left.+\left(16\frac{(-4+s+t)(s+t)}{(-4+2s+t)}\frac{\mathrm{T}_1(s,t)}{\sqrt{(-4+t)t}}+4(1+s)\mathrm{T}_2(s,t)\right)\frac{1}{\Psi_1(s,t)}+\frac{1}{t}\frac{\mathrm{T}_1^2(s,t)+4\mathrm{T}_2^2(s,t)}{\Psi_1^2(s,t)}\right]$$

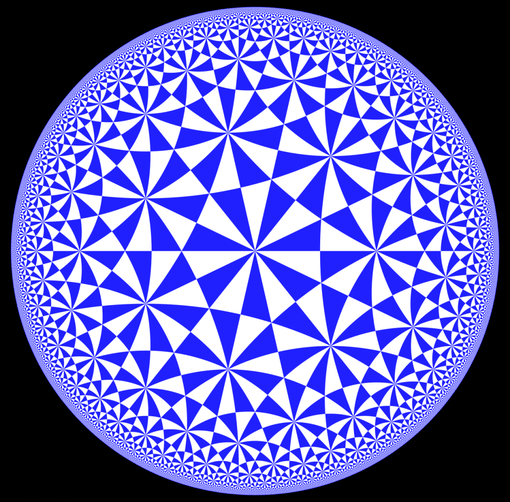
$$\omega_{63}\overset{f^*}{\longmapsto}-4\,\omega_2^{\mathsf{Kro}}(2z\,,q^2)+\Theta_?(q)\frac{\mathrm{d}q}{q}$$

- The (technical) problem of incompatibility of simultaneous uniformization

$$\frac{\mathrm{d}s}{\sqrt{-s}\sqrt{4-s}}\overset{\text{ramified covering}}{\underset{h}{\longmapsto}}\frac{\mathrm{d}x}{x},\quad s=h(x):=-\frac{(1-x)^2}{x}$$

$$x(z,\tau)=1-\frac{2\theta_3\theta_2(z)}{\sqrt{\lambda}}\frac{(2-\lambda)\theta_3\theta_2(z)+i\sqrt{2(2-\lambda)}\theta_4\theta_1(z)}{\theta_3^2\theta_3^2(z)+\theta_4^2\theta_4^2(z)}$$

Summary and Outlook



(2,3,7)

- **Phenomenology**
  - The first amplitude beyond genus 0 in QED
- **Connections between the amplitude and the arithmetic Groups**
  - Marked points as generators of Mordell-Weil group
  - A unified description of Bhabha scattering and sector 79 of top quark production through universal curves for $\Gamma_1(4)$
- **Future directions**
  - Dimension formula for the closed 1-forms on the universal curves
  - Amplitude beyond genus 1: the non-trivial Hurwitz automorphisms; the underlying connections to Hyperbolic Coxeter Groups