Multi-point Energy Correlators in N=4 Super Yang-Mills Thoery

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Based on work with XY.Zhang, D.Chicherin,E.Sokatchev, YY.Zhu, et al

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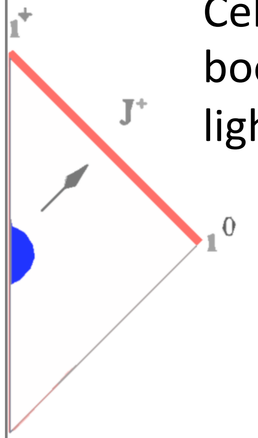
Celestial bootstrap; light-ray OPE



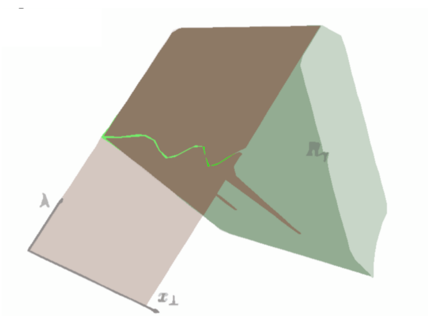
Energy correlators

Spatial correlation of flow operators

$$\langle \mathrm{E}(n_1)\mathrm{E}(n_2)\ldots\mathrm{E}(n_N)\rangle$$

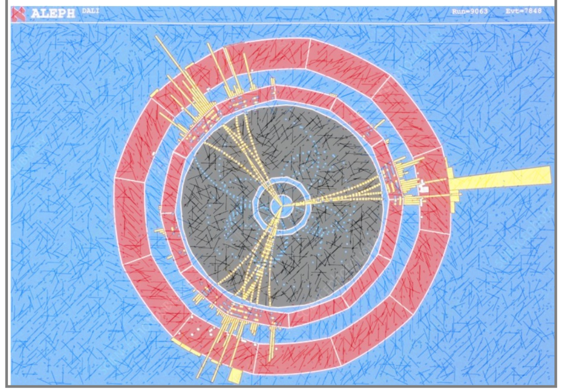
$$\mathcal{E}(n) = \int_{-\infty}^{+\infty} du \lim_{r\to\infty} r^2 T_{0i}(t = u + r, r\,\vec{n}) n^i$$

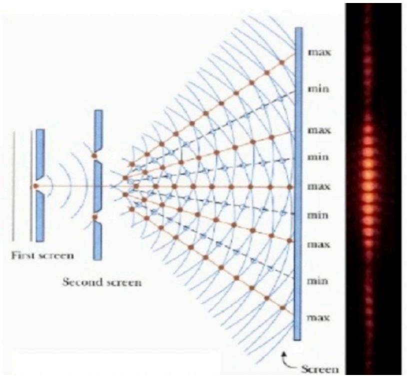
Hamiltonian flow in quantum information theory;



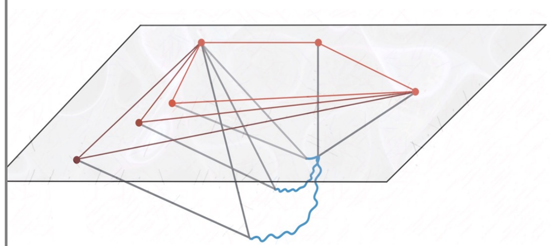
Precision collider physics

Jet substructure

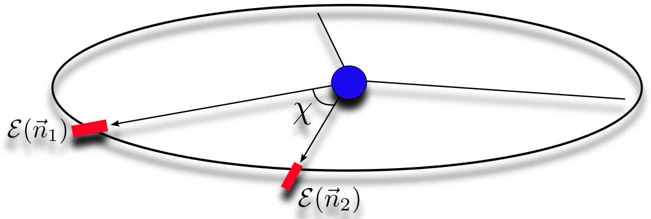


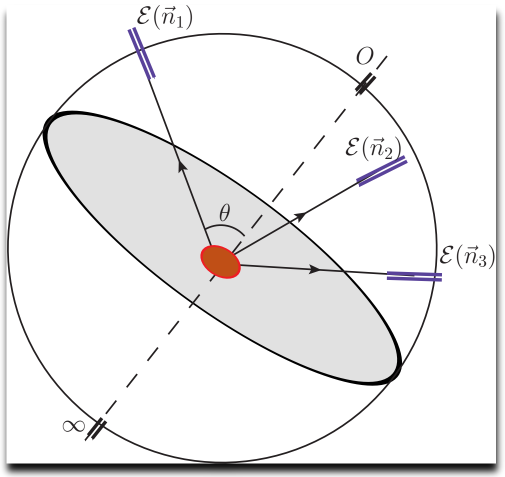


Cosmological correlator



Multi-point Energy correlators

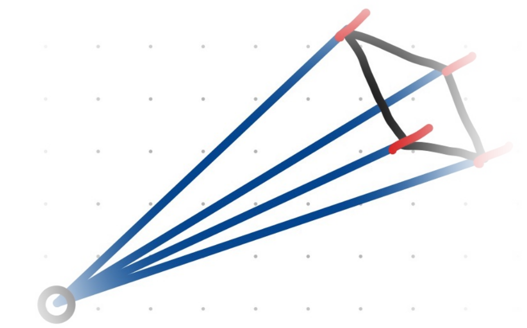


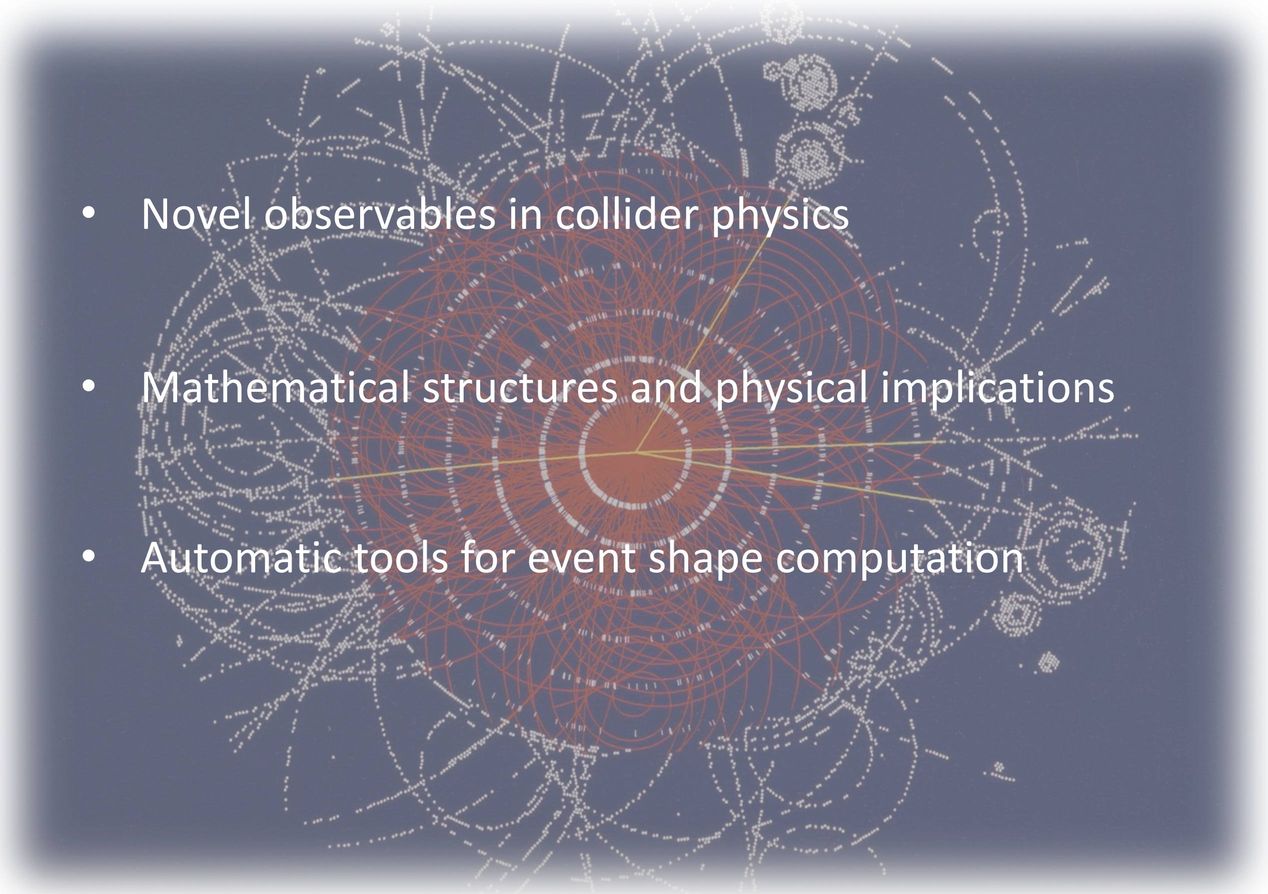


$\langle \mathrm{E}(n_1)\mathrm{E}(n_2)\ldots\mathrm{E}(n_n)\rangle$ analytic function of coordinates on the celestial sphere

Tree-level higher-point energy correlators: Energy integration over onshell higher-point Amplitudes/Form Factors

$$E^nC\Big|_{LO} \sim \int E_1 d\,E_1\ \ldots E_n d\,E_n \left|F^{(0)}_{n+1}\right|^2$$



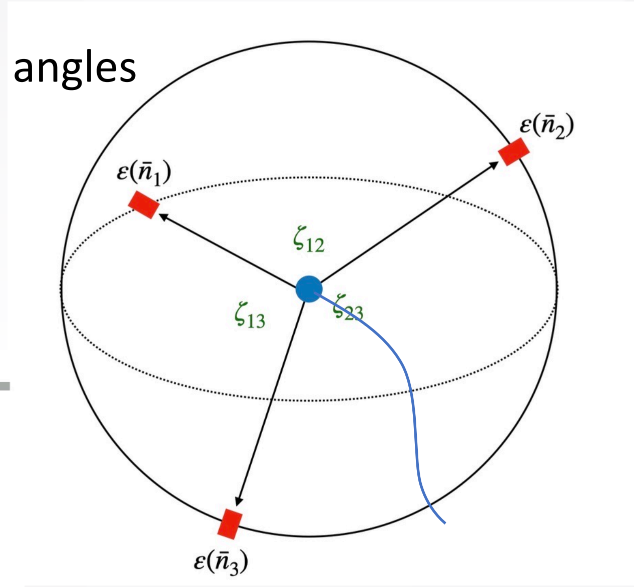


- Novel observables in collider physics
- Mathematical structures and physical implications
- Automatic tools for event shape computation

Three-point energy correlators EEEC $\langle \mathrm{E}(n_1)\mathrm{E}(n_2)\mathrm{E}(n_3)\rangle$ with arbitrary angle dependence

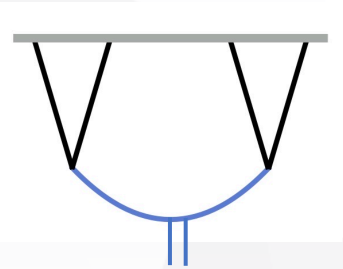
$$= \int d\,\sigma \sum_k \sum_i \sum_j E_i E_j E_k \,\delta(\,\zeta_{12} - \sin^2\frac{\theta_{ij}}{2})\delta(\,\zeta_{13} - \sin^2\frac{\theta_{jk}}{2})\delta(\,\zeta_{23} - \sin^2\frac{\theta_{ik}}{2})$$

geometric distribution of energy flow through 3 detectors as a function of three angles



On-shell computation at leading order only involves finite phase-space integration over

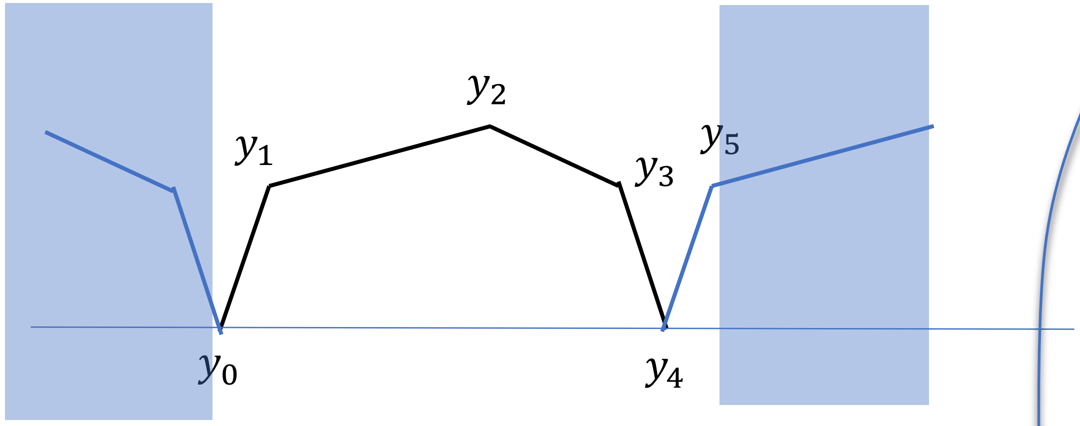
squared super Form factor $|\langle p_1 p_2 p_3 p_4 | tr\{\Phi^2\}(x)|0\rangle_{tree}|^2$



[Bork 16‘][Bianchia, Brandhubera, Paneraia,Travaglini 18‘]

A compact form of squared tree level 4-pt NMHV form factor given in terms of cross ratios of dual coordinates:

$$\left|F_{tr\{\Phi^2\}}\right|^2(1,2,3,4)+perm.=4|F_{MHV}|^2\left[\frac{1}{4}+\frac{y_{02}^2y_{35}^2}{y_{03}^2y_{25}^2}+\frac{y_{04}^2y_{13}^2}{y_{03}^2y_{14}^2}\right]+\text{perm.}$$



$$p_i=y_i-y_{i-1},\ p_{i+4}:=p_i,\ s_{i+1..k}=y_{ik}^2$$

Phase-space integrations generate logarithmic functions, with square-root letters in the first entry

$$q=p_1+p_2+p_3+p_4,\quad p_i=E_in_i,\ i=1,2,3$$

$$\int\frac{\prod_{i=1}^3[E_i^2d\,E_i]}{s_{12}s_{23}s_{34}s_{14}}\left[\frac{s_{1234}s_{23}}{s_{123}s_{234}}+\frac{s_{12}s_{14}}{s_{123}s_{134}}+perms.\right]\delta_+((q-E_1n_1-E_2n_2-E_3n_3)^2)$$

$$\frac{2-\zeta_{12}-\zeta_{23}-\zeta_{13}-\Delta_1}{2-\zeta_{12}-\zeta_{23}-\zeta_{13}+\Delta_1}\otimes\frac{\Delta_1}{(1-\zeta_{12})(1-\zeta_{23})(1-\zeta_{13})}$$

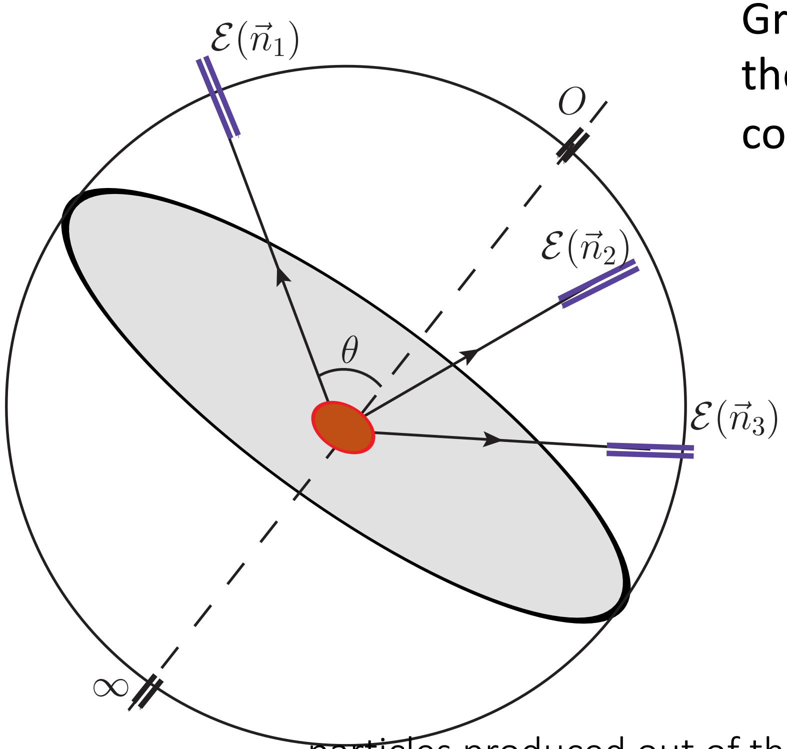
$$\frac{2-\zeta_{12}-\zeta_{23}-\zeta_{13}-\Delta_1}{2-\zeta_{12}-\zeta_{23}-\zeta_{13}+\Delta_1}\otimes\frac{\Delta_2-\Delta_1}{\Delta_2+\Delta_1}$$

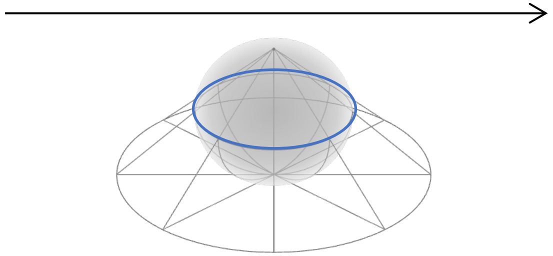
$$\Delta_1=\sqrt{\zeta_{12}^2+\zeta_{13}^2+\zeta_{23}^2-2\zeta_{12}\zeta_{13}-2\zeta_{12}\zeta_{23}-2\zeta_{13}\zeta_{23}+4\zeta_{12}\zeta_{13}\zeta_{23}}$$

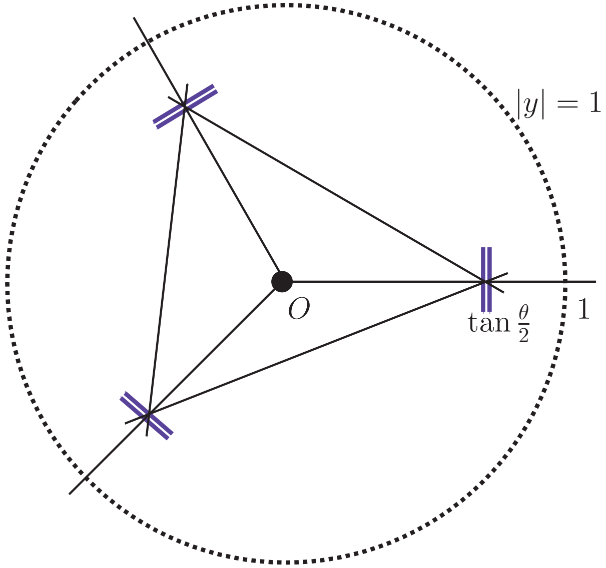
$$\Delta_2=\sqrt{\zeta_{12}^2+\zeta_{13}^2+\zeta_{23}^2-2\zeta_{12}\zeta_{13}-2\zeta_{12}\zeta_{23}-2\zeta_{13}\zeta_{23}}$$

Angle parametrization

Graphical representation of the three-point energy correlator





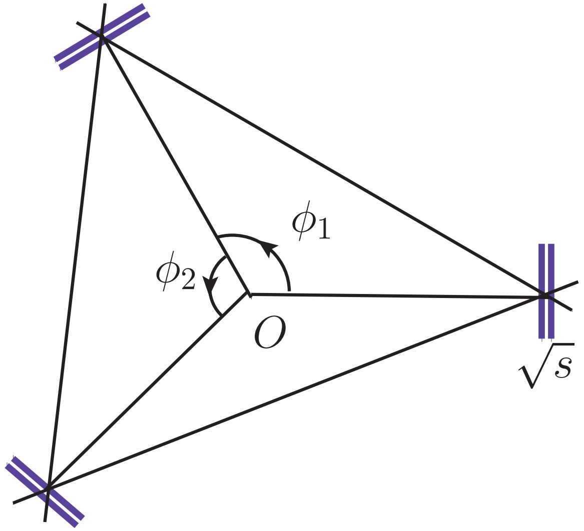


particles produced out of the vacuum by the source are captured by the three detectors located at spatial infinity in the directions of the unit vectors n1,n2 and n3.

triangle located on a circle with radius |y| = tan θ/2 on the celestial sphere. [sinθ (sin ϕ1 ,sin ϕ2 ,sin ϕ1+ϕ2)]

Single-valuedness constraints

three points on the celestial sphere which we put on a unit circle with radius $\sqrt{s}$ centered at the origin, Away from coplanar limit, we can set 0< s <1



$$s \coloneqq \tan^2\frac{\theta}{2}, \tau_1 \coloneqq e^{i\phi_{23}}, \tau_2 \coloneqq e^{i\phi_{13}}$$

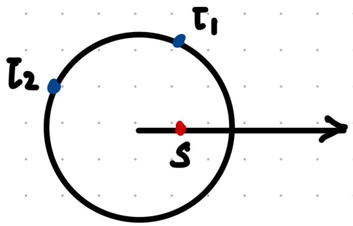
Generic angles correspond to $|\tau_1| = |\tau_2| = 1$, 0 < s < 1 (or s > 1), where EEEEC is real and single valued function, unambiguous under redefinition $(\tau_1, \tau_2) \to e^{\pm 2i\pi}(\tau_1, \tau_2)$, or $s \to e^{\pm 2i\pi}$s

$$\frac{2-\zeta_{12}-\zeta_{23}-\zeta_{13}-\Delta_1}{2-\zeta_{12}-\zeta_{23}-\zeta_{13}+\Delta_1} = \frac{(s+\tau_1)(s+\tau_2)(1+s\,\tau_1\tau_2)}{(1+s\,\tau_1)(1+s\tau_2)(s+\tau_1\tau_2)}$$

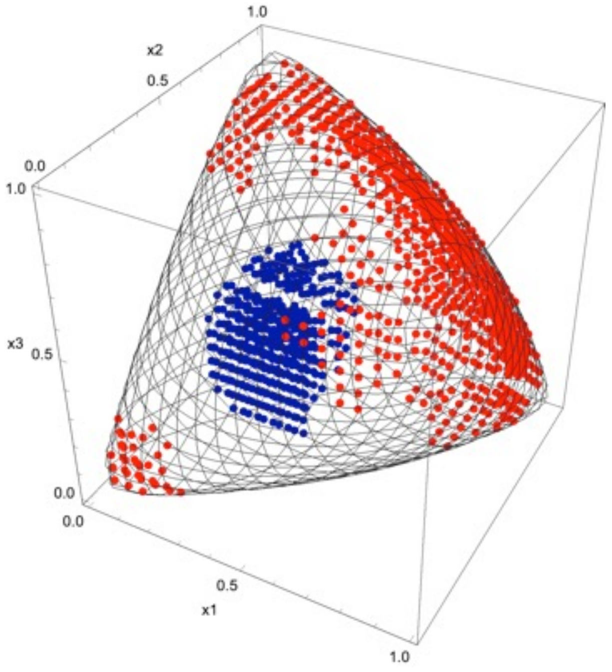
$$1-\zeta_{12} = \frac{(s+\tau_1)(1+s\tau_1)}{(1+s)^2\tau_1}$$ are allowed

$$\frac{\Delta_2-\Delta_1}{\Delta_2+\Delta_1} = s, \quad \zeta_{12} = \frac{-s(1-\tau_1)^2}{(1+s)^2\tau_1}$$

are forbidden in the first entry

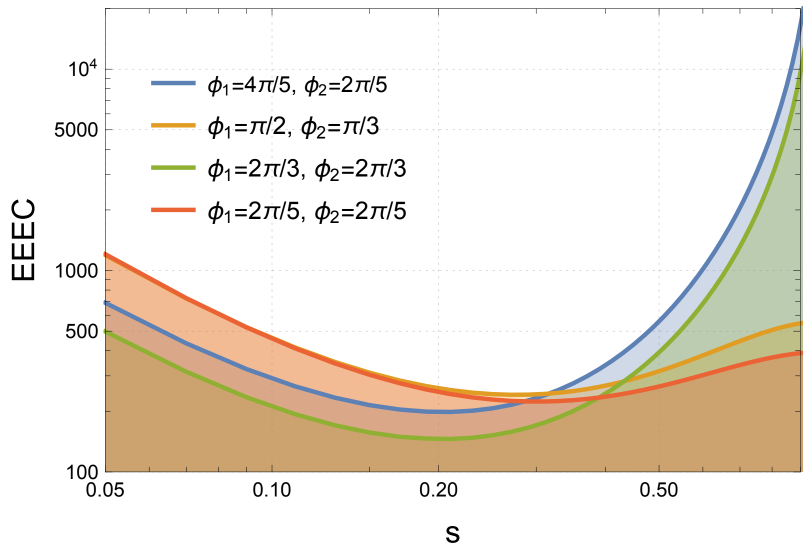


Kinematics and singularities

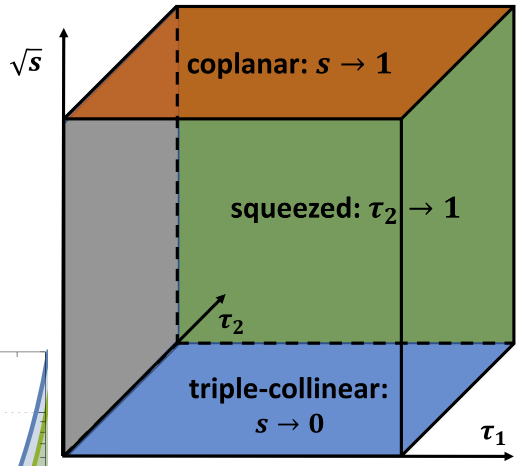


Phase-space parametrized by $\{\zeta_{12}, \zeta_{23}, \zeta_{13}\}$, *bounded by* $(\zeta_{12}-\zeta_{13}-\zeta_{23})^2 - 4(1-\zeta_{12})\zeta_{13}\zeta_{23} = 0$

physical singularities arise at the boundaries: triple-collinear; squeezed; coplanar limit corresponding to OPE limits of the light-ray operators



"Kinematic cube"

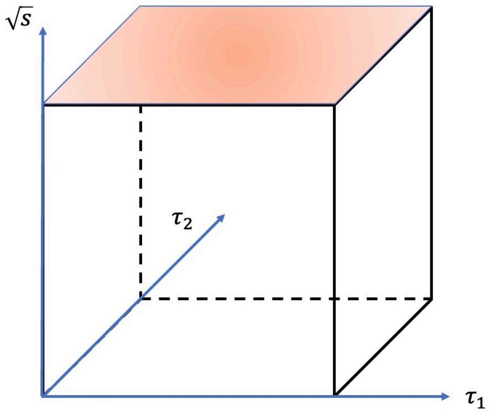


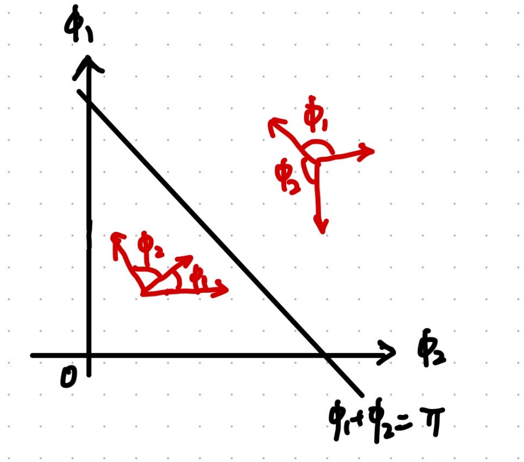
$s := \tan^2\frac{\theta}{2}, \tau_1 := e^{i\phi_{23}}, \tau_2 := e^{i\phi_{13}}$

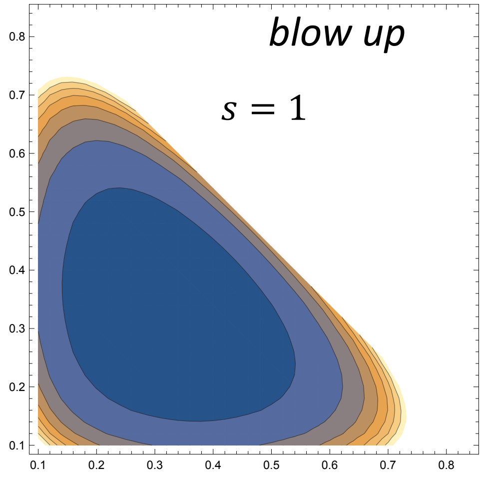
Singularity in the coplanar hypersurface

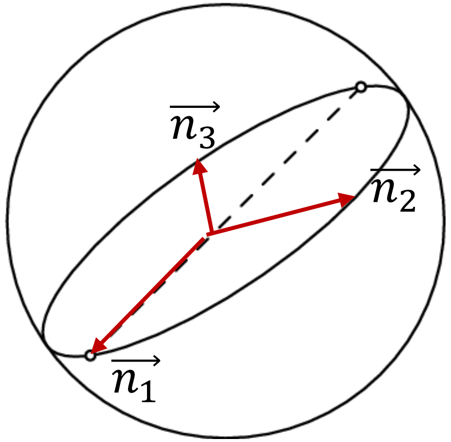
EEEC shape is peaked near coplanar limit (s=1) due to soft/collinear singularities

$$EEEC \sim \frac{f(\phi_1, \phi_2)}{1-s}\ln(1-s) + \cdots$$





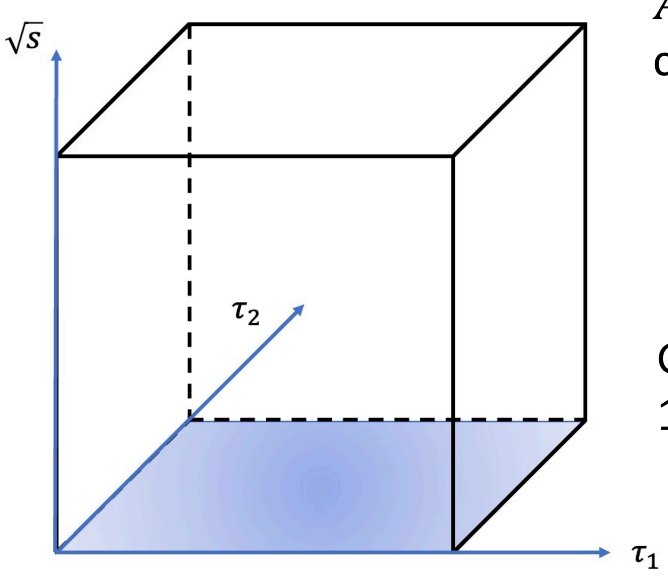




On the s=1 hypersurface, EEEC is no longer single-valued

Residue function at pole s=1 comes from its discontinuity , cancels with the IR singularity from virtual one-loop Form Factor.

Triple collinear limit

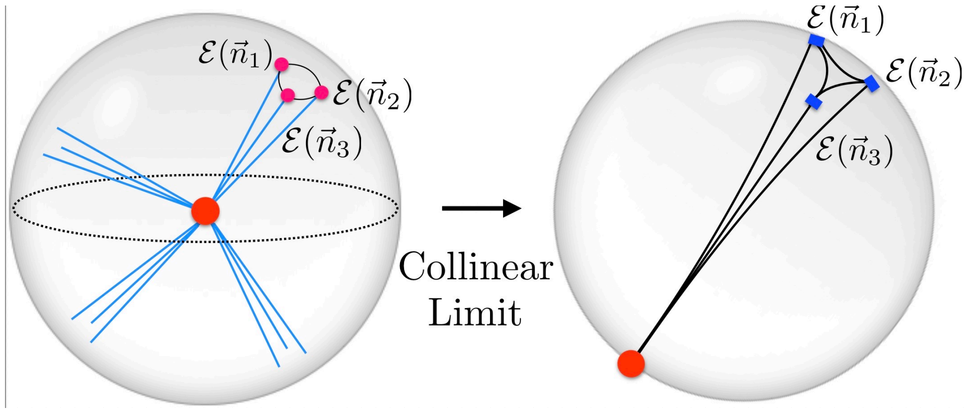


As $s \to 0$, we zoom into the triple collinear limit

$$EEEC \to \frac{1}{s^2} G(\tau_1, \tau_2)$$

G agrees with [H.Chen et al, 1912.11050], uppon setting

$$z_1 = \frac{1-\tau_1}{(1-1/\tau_2)}, \quad \bar{z_1} = \frac{1-1/\tau_1}{(1-\tau_2)}$$



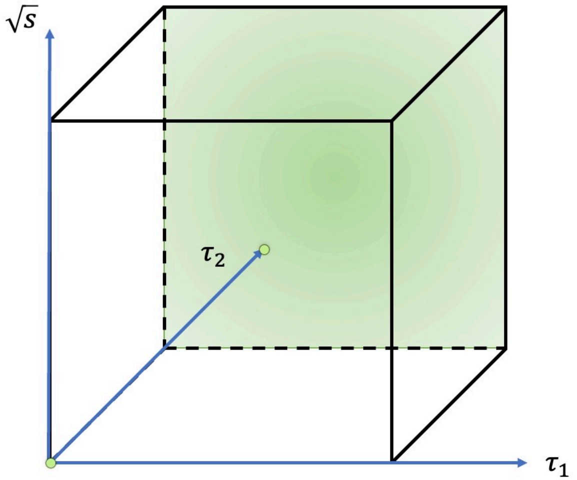
$$\mathrm{Li}_2(1 - |z_1|^2) + \frac{1}{2} \ln |z_1|^2 \ln |1 - z_1|^2$$

$$\mathrm{Li}_2(z_1) - \mathrm{Li}_2(\bar{z}_1) + \frac{1}{2} \ln |z_1|^2 \ln \frac{1 - z_1}{1 - \bar{z}_1}$$

$$\ln|z_1|^2 \qquad \ln|1 - z_1|^2$$

$s = 0$ is an isolated pole, analytic continuation around $s = 0$ is trivial

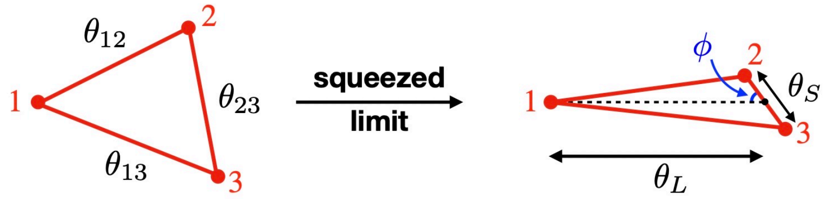
Squeezed limit



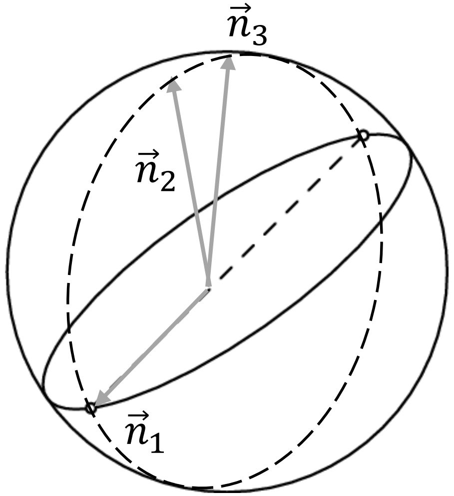
$$\langle E(n_1)E(n_2)E(n_3)\rangle \to \langle E(n_1)O^J(n_2)\rangle$$

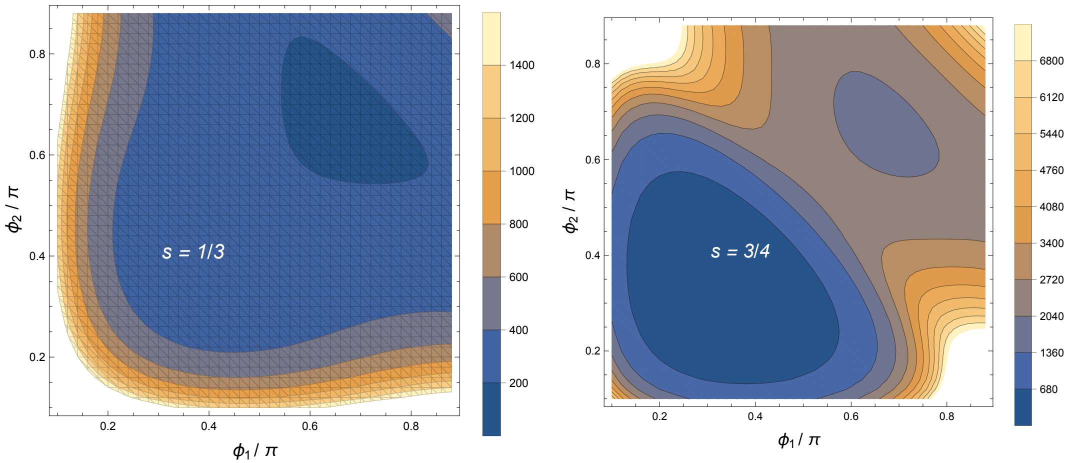
$$\zeta_{23} \sim 0,\ \zeta_{12} \sim \zeta_{23} \sim \zeta$$

$$EEEC \sim \frac{6}{\zeta_{23}}\left[\frac{\zeta + \ln(1-\zeta)}{(-1+\zeta)\zeta^3}\right]$$



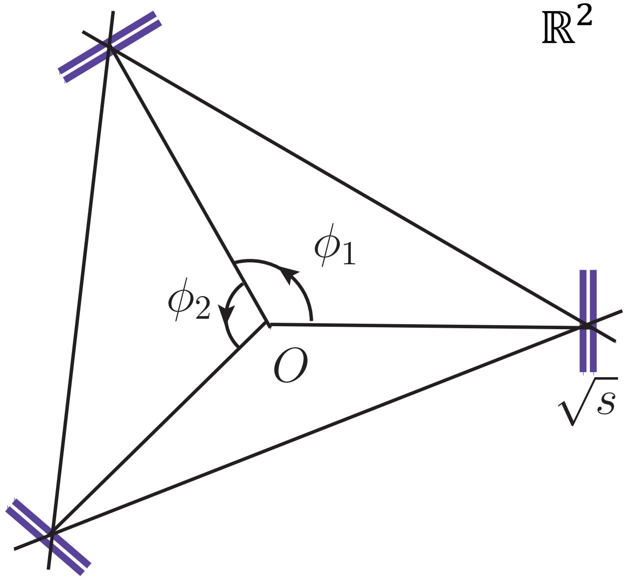
H.Chen et al,2202.04085



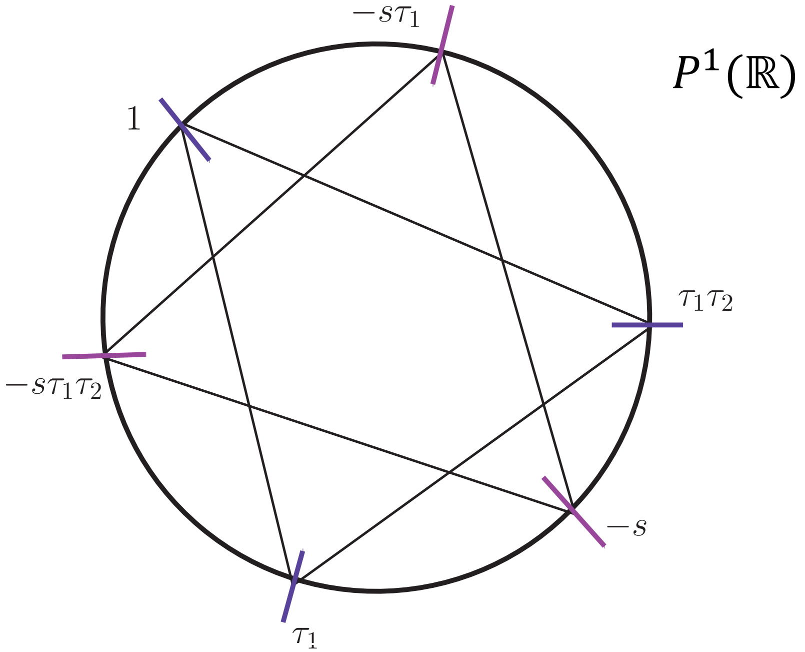


Embedding formalism

three points on the celestial sphere which we put on a unit circle with radius $\sqrt{s}$ centered at the origin



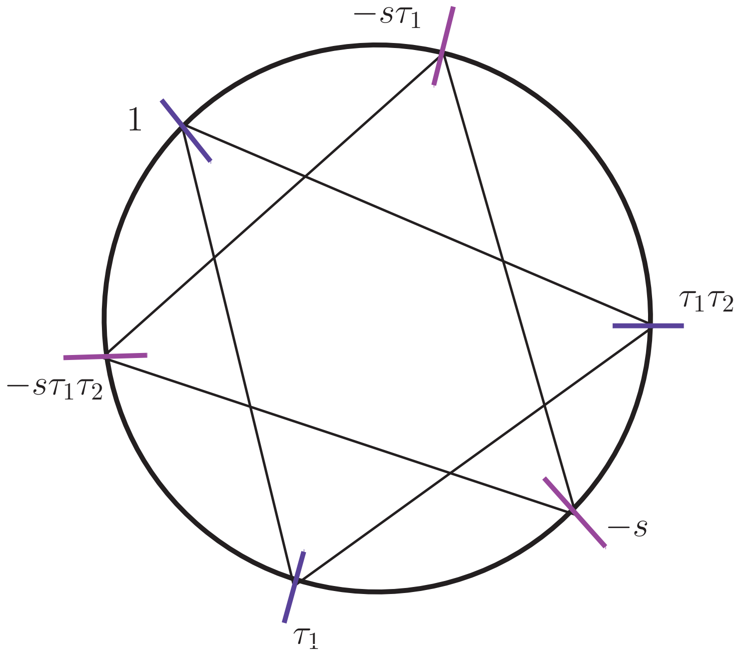
$$s \coloneqq \tan^2\frac{\theta}{2}, \tau_1 \coloneqq e^{i\phi_{23}}, \tau_2 \coloneqq e^{i\phi_{13}}$$



Kinematic data embedded in 6 points on unit circle; EEEC exhibits the dihedral symmetry of hexagon function

The geometries mapped onto partitions on unit circle centered at the origin

$$\{Z_1,..,Z_6\} := \{1, -sx_1x_2, x_1, -s, x_1x_2, -sx_1\} \qquad I := \infty$$



Plücker variables $\langle ab\rangle$ represents the distance between a and b

Dihedral group : D_6

Cyclic permutation $\sigma : \langle i| \to \langle i+2|$

Dihedral flip $\tau : \langle i| \to \langle 8-i|$ Parity $P : \langle i| \to \langle i+3|$

All 16 letters that appear can be written in terms of $\langle ij\rangle$

They form a closed set under the D6 group

$$\Big\{w\,,1+w\,,y\,,1+y\,,z\,,1+z\,,w+z\,,1+w+z\,,\\ y+z+yz\,,w+y+z+yz\,,1+w+z+yz\,,\\ 1+w+y+2z+yz\,,y+wy+y^2+z+2yz+y^2z\,,\\ 1+y+wy+y^2+z+2yz+y^2z\,,\\ 1+w+y+wy+y^2+z+2yz+y^2z\,,\\ 1+w+y+wy+y^2+2z+2yz+y^2z\Big\}$$

$$y=-\frac{\langle 31\rangle\langle 5I\rangle}{\langle 15\rangle\langle I3\rangle},\quad z=-\frac{\langle 13\rangle\langle 56\rangle}{\langle 35\rangle\langle 61\rangle},\quad w=\frac{\langle 51\rangle\langle 62\rangle\langle 43\rangle}{\langle 35\rangle\langle 16\rangle\langle 24\rangle}$$

EEEC as a 3-variable finite physical observable, fully analytic :

- Function space : single-valued function of three coordinates on the celestial sphere; encoding 16 alphabets

$$EEEC^{(1)}_{N=4\ SYM}(\zeta_{12},\zeta_{23},\zeta_{13}) = \sum_{i=1}^{14} b_i F_i + cyclic\ permutation$$

$$\{F_i\} \supset \{Li_2\left(\frac{\langle 14\rangle}{\langle 13\rangle}\right), Li_2\left(\frac{\langle 12\rangle\langle 45\rangle}{\langle 15\rangle\langle 24\rangle}\right), Li_2\left(\frac{\langle 12\rangle\langle 34\rangle}{\langle 14\rangle\langle 23\rangle}\right), Li_2\left(\frac{\langle 13\rangle\langle 45\rangle}{\langle 15\rangle\langle 43\rangle}\right),$$

$$Li_2\left(\frac{\langle 12\rangle\langle 34\rangle\langle 56\rangle}{\langle 23\rangle\langle 45\rangle\langle 61\rangle}+1\right), Li_2\left(\frac{\langle 14\rangle\langle 23\rangle\langle 56\rangle}{\langle 34\rangle\langle 25\rangle\langle 61\rangle}+1\right)\}$$ +parity conjugation + dihedral flips

- unexpected simplicity

- new type of single-valued polylogarithms

EEEC function space

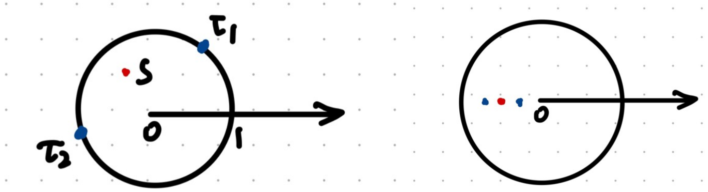
$Li_w(-S)$ *modulo* products of logarithms

The complete set S reads:

$$S := \{\frac{\langle 14\rangle\langle I3\rangle}{\langle 13\rangle\langle 4I\rangle}, \frac{\langle 12\rangle\langle 54\rangle}{\langle 15\rangle\langle 24\rangle}, \frac{\langle 12\rangle\langle 43\rangle}{\langle 14\rangle\langle 23\rangle}, \frac{\langle 13\rangle\langle 54\rangle}{\langle 15\rangle\langle 43\rangle}, \frac{\langle 12\rangle\langle 34\rangle\langle 56\rangle}{\langle 23\rangle\langle 45\rangle\langle 61\rangle}, \frac{\langle 14\rangle\langle 23\rangle\langle 56\rangle}{\langle 34\rangle\langle 25\rangle\langle 61\rangle}$$

$\}+ D_6$ images

$s := \tan^2\frac{\theta}{2}, \tau_1 := e^{i\phi_{23}}, \tau_2 := e^{i\phi_{13}}$



First entry conditions and physical implications:

$$\{\frac{2-\zeta_{12}-\zeta_{23}-\zeta_{13}-\Delta_1}{2-\zeta_{12}-\zeta_{23}-\zeta_{13}+\Delta_1}, \frac{\zeta_{12}(1-\zeta_{23})}{\zeta_{23}(1-\zeta_{12})}, 1-\zeta_{12}\}$$

- triple-collinear limit: s→ 0 or ∞ is a 'regular' phase-space point.
- Single-valued function away from coplanar plane: $|\tau_1|$ = $|\tau_2|$ = 1, $|s|$ < 1 (or |s| > 1)
- analytic in the region: $s = -\text{tahn}^2\left(\frac{\Theta}{2}\right), \tau_1 = -e^{-\varphi_{23}}, \tau_2 = -e^{-\varphi_{13}}$ $(\Theta = -i\,\theta, \varphi_{ij} = i(\pi - \phi_{ij})$.

Four-point Energy correlator in the quadruple collinear limit in N=4 super Yang-Mills Theory

We consider the 4-point energy correlator EEEEC in N=4 SYM, where four detectors are collinear.

$$EEEEC \sim \int \prod_{i=1}^{4} [x_i^2 dx_i] \delta\left(1 - \sum x_i\right) |\mathrm{Split}_{1\to 4}|(x_i, z_i, \bar{z}_i) + perm\{1,2,3,4\}$$

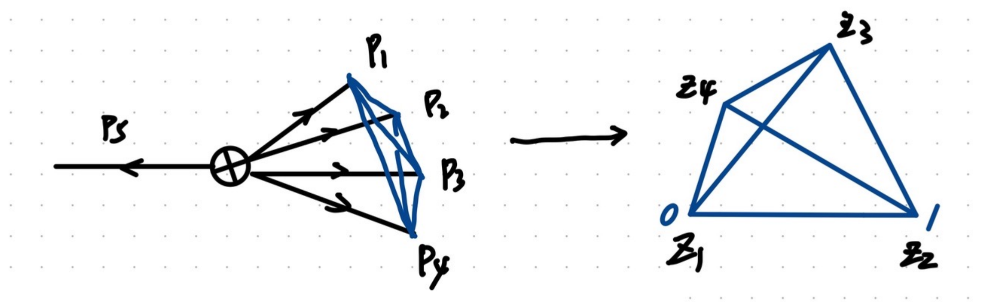
z_{ij}: 6 pairs of small angular separations

x_i: energy fractions

$$s_{ij} = Q^2 x_i x_j |z_i - z_j|^2$$

$$p_i^\mu \sim x_i \frac{Q}{2}(1, z_i + \bar{z}_i, -i(z_i - \bar{z}_i), 1)$$

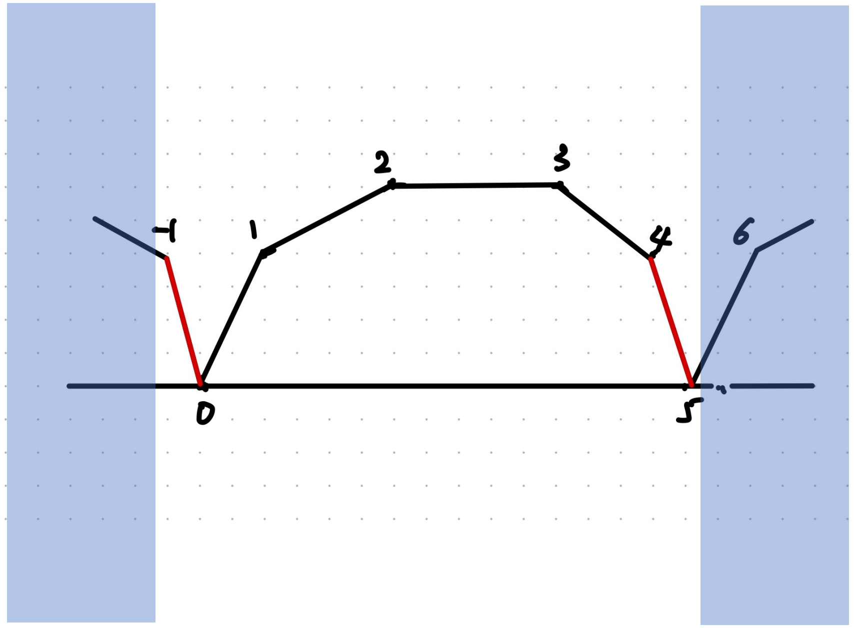
The tree-level splitting function can be obtained from the squared five-point form factor where p1,..p4 are collinear and p5 is anti-collinear.



$$\lim_{1\|2\|3\|4} |F_5(z_1 = 0, z_2 = 1, z_3, z_4)|^2$$

Chicherin, Moult, Sokatchev, Yan, TBA

Four-point splitting function in dual-coordinate space



$|F_5|^2$ can be expressed in terms of coordinates on a section in the periodic dual coordinate space

$$y_{ij}^2,\ i,j \in \{-1,1,\cdots,5\}, |i-j| \leq 5 \qquad y_{ij}^2 = s_{i+1\cdots j} = y_{i+5,j+5}^2$$

In the quadruple collinear limit, y_{-1}, y_5 are sent to infinity

$$y_{i5}^2/y_{05}^2 \to \frac{x_{i+2}+\cdots+x_4}{x_1+x_2+x_3+x_4}, \quad y_{-1i}^2/y_{-14}^2 \to \frac{x_1+\cdots+x_i}{x_1+x_2+x_3+x_4}$$

Compact form of the splitting function $\quad (a,b,c,d) \equiv \dfrac{y_{ab}^2 y_{cd}^2}{y_{ac}^2 y_{bd}^2}$

$$G := \frac{|F_5|_{NMHV}{}^2}{|F_5|_{MHV}{}^2} = \begin{aligned} &-1+(-1,1,2,5)+(-1,2,3,5)+(-1,4,3,0)+(4,1,0,5)+(-1,3,2,0)+(4,2,1,5) \\ &+(0,4,3,1)+(0,4,3,1)(-1,1,3,5)+(-1,4,3,1)(3,1,0,5) \\ &+(-1,4,2,0)(0,2,3,5)+(-1,1,2,4)(4,2,0,5)+(-1,3,2,0)(4,2,0,5)+(-1,4,2,0)(4,2,1,5) \\ &+(-1,4,3,0)(-1,1,2,4)+(4,1,0,5)(0,2,3,5)+(-1,4,3,1)(-1,4,2,0)+(3,1,0,5)(4,2,0,5) \\ &+(-1,4,3,1)(-1,1,2,5)+(3,1,0,5)(-1,2,3,5)+(-1,1,2,4)(-1,1,3,5)+(0,2,3,5)(-1,1,3,5) \end{aligned}$$

Energy integration over splitting function

$$\text{EEEC}(z_i) \sim (|z_{12}z_{23}z_{34}|^2)^{-1}\int_0^\infty \prod_{i=1}^{4} dx_i\, \delta(1-\sum_{i\in S} x_i)\,(x_1+x_2+x_3+x_4)^{-4}\mathcal{G}(\{(a,b,c,d)\})|_{\text{coll.}}$$

$$\mathcal{G}_4 = (-1,4,3,0) = \frac{s_{123}\,x_{1234}}{s_{1234}\,x_{123}}\,;$$

$$\mathcal{G}_6 = (-1,3,2,0) = \frac{s_{12}\,x_{123}}{s_{123}\,x_{12}},\quad \mathcal{G}_{12} = (-1,4,2,0)(0,2,3,5) = \frac{s_{12}^2\,x_{1234}x_4}{s_{1234}s_{123}\,x_{12}x_{34}}\,;$$

$$\mathcal{G}_{15} = (-1,4,3,0)(-1,1,2,4) = \frac{s_{123}s_{34}\,x_{1234}x_1}{s_{1234}s_{234}\,x_{12}x_{123}}\,,$$

$$\mathcal{G}_{21} = (-1,1,2,4)(-1,1,3,5) = \frac{s_{34}\,x_1^2x_4}{s_{234}\,x_{12}x_{123}x_{234}},\quad \mathcal{G}_{14} = (-1,4,2,0)(4,2,1,5) = \frac{s_{12}s_{34}\,x_{1234}x_{234}}{s_{1234}s_{234}\,x_{12}x_{34}}\,;$$

$$\mathcal{G}_{19} = (-1,4,3,1)(-1,1,2,5) = \frac{s_{23}\,x_1x_{34}x_{1234}}{s_{234}\,x_{12}x_{123}x_{234}},\quad \mathcal{G}_{18} = (-1,4,3,1)(-1,4,2,0) = \frac{s_{12}s_{23}\,x_{1234}^2}{s_{1234}s_{234}\,x_{12}x_{123}}\,.$$

$$\mathcal{G}_8 = (1,4,3,2) = \frac{s_{1234}s_{23}}{s_{123}s_{234}},\quad \mathcal{G}_9 = (0,4,3,1)(-1,1,3,5) = \frac{s_{1234}s_{23}}{s_{123}s_{234}}\frac{x_1x_4}{x_{123}x_{234}}$$

$$s_{ijk} = Q^2\left(x_ix_j|z_{ij}|^2 + x_ix_k|z_{ik}|^2 + x_jx_k|z_{jk}|^2\right) \qquad x_{i,\ldots,j} := x_i + \cdots + x_j$$

Multi kinematic scales and high degree poles in the integrand poses great challenge to partial fractioning and multi-fold integration.

$$\frac{s_{23}s_{1234}x_1x_4}{s_{123}s_{1234}x_{123}x_{234}x_{1234}^4}$$

Goal：

-lower the degree of denominators in target integrals,

transform them to simpler, manageable integrals with simple or at most double pole

Integration-by-part method can be designed to achieve these goals.

We develop an IBP algorithm suited for computing multi-fold energy integrals in the collinear limit

$$A = \int [dx] \frac{x_1^{q_1} x_2^{q_2} x_3^{q_3} x_4^{q_4}}{s_{123}^{a_1} \ldots x_{12}^{a_2} x_{123}^{a_3} x_{1234}^{a_n}}$$

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Energy integral families

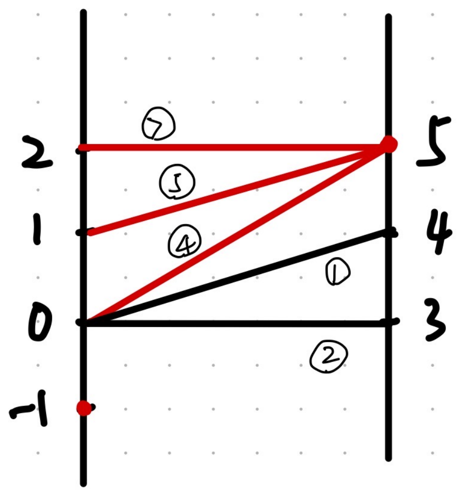
$$A_{a_1,a_2,\cdots,a_7} := \int d\,x_1 d\,x_2 d\,x_3 d\,x_4 \frac{\delta_{i\in S}(1-x_i)}{\prod D_i^{a_i}}$$

$$2a_1 + 2a_2 + 2a_3 + a_4 + a_5 + a_6 + a_7 - 4 = 0$$

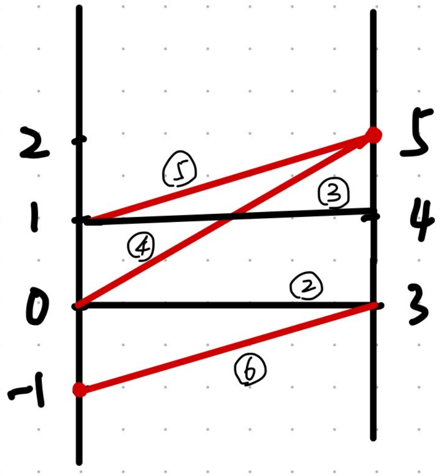
$$D_1 = s_{1234}, D_2 = s_{123}, D_3 = s_{234}, D_4 = x_{1234}, D_5 = x_{234}, D_6 = x_{123}, D_7 = x_{34}$$

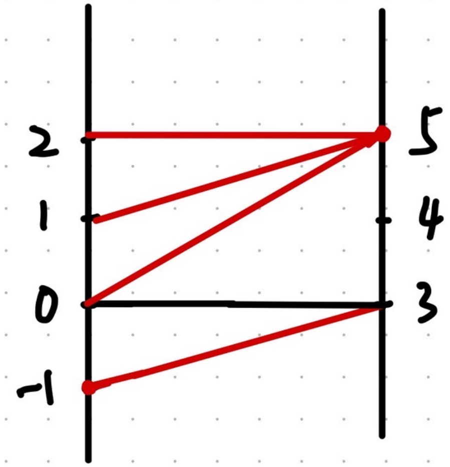
$$x_{i,..,j} := x_i + \cdots + x_j$$

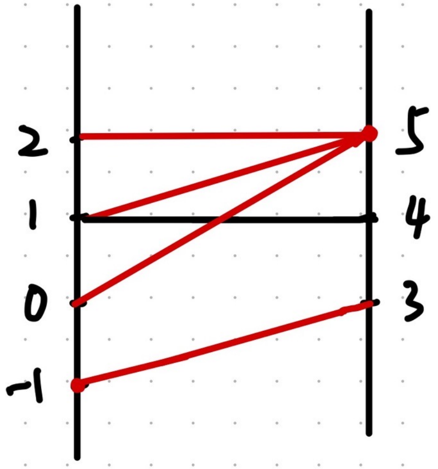
$A_{1,1,0,1,1,0,1}$



$A_{0,1,1,1,1,1,0}$







Features of the energy integral

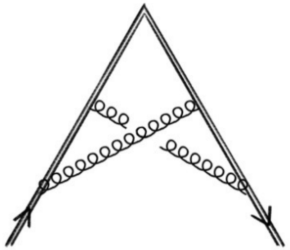
$$A_{a_1,a_2,\cdots,a_7} := \int d\,x_1 d\,x_2 d\,x_3 d\,x_4 \frac{\delta_{i\in S}(1-x_i)}{\prod D_i^{a_i}} \qquad 2a_1 + 2a_2 + 2a_3 + a_4 + a_5 + a_6 + a_7 - 4 = 0$$

Homogeneity:

integrand homogeneous w.r.t energy fractions, the last integration variable is localized

Finiteness:
convergent as any subset of the integration variables go to zero/infinity

Analog: Wilson-line web diagrams, with the absence of sub-divergences
No regulators are needed for the "leading" divergences
Four-dimensional IBP and DE method apply: *Henn, Ma, Yan, Zhang [2211.13967]*



$$\int \left(\prod_{i=1}^{L} \frac{d^{4-2\epsilon}k_i}{i\pi^{2-\epsilon}}\right) O\left(\frac{1}{D_1^{a_1}\ldots D_n^{a_n}}\right) = 0 \qquad O = \sum_{i=1}^{L} \frac{\partial}{\partial k_i^{\mu}} u_i^{\mu}$$

Graded IBP operators $\{k_i\} \to \lambda\{k_i\}, \quad O \to \lambda^{\beta_i} O$

The finite IBP method applies to the phase-space energy integrals

$$\int_0^\infty d\,x_1 d\,x_2 d\,x_3 d\,x_4\; O\left(\frac{\delta_{j\in S}(1-x_j)}{\prod D_i^{a_i}}\right), \qquad O := \frac{\partial}{\partial\,x_k} x_1^{q_1} x_2^{q_2} x_3^{q_3} x_4^{q_4}$$

$$\{x_i\} \to \lambda\{x_i\}, \qquad O \to \lambda^{\beta_i} O$$

Subtleties to be taken care of:

1. The propagators are not independent scalar products

E.g. for $\left\{\frac{x_1 x_{12}}{s_{123}^2 x_{123}}, \frac{x_1^2}{s_{123}^2 x_{123}}, \frac{x_{12}^2}{s_{123}^2 x_{123}}, \frac{x_1}{s_{123}^2}, \frac{x_{12}}{s_{123}^2}, \frac{1}{s_{123} x_{123}}\right\}$, there is relation $|z_{12}|^2 \frac{x_1(x_{12}-x_1)}{s_{123}^2 x_{123}} +$

$$|z_{23}|^2 \frac{(x_{12}-x_1)(x_{123}-x_{12})}{s_{123}^2 x_{123}} + |z_{13}|^2 \frac{x_1(x_{123}-x_{12})}{s_{123}^2 x_{123}} - \frac{1}{s_{123} x_{123}} = 0$$

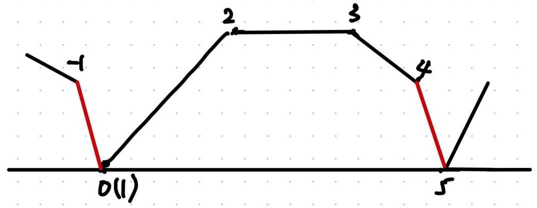
2. The finite IBPs generate boundary terms

$$\int \frac{\partial}{\partial x_i} \frac{x_1^{q_1} x_2^{q_2} x_3^{q_3} x_4^{q_4}}{s_{1234}^{a_1} \ldots x_{12}^{a_2} x_{123}^{a_3} x_{1234}^{a_n}} dx_1 dx_2 dx_3 dx_4 = \left(\int \frac{x_1^{q_1} x_2^{q_2} x_3^{q_3} x_4^{q_4}}{s_{1234}^{a_1} \ldots x_{12}^{a_2} x_{123}^{a_3} x_{1234}^{a_n}} dx_j dx_k dx_l\right)\Bigg|_{x_i=0}^{\infty} = \text{boundary term}$$

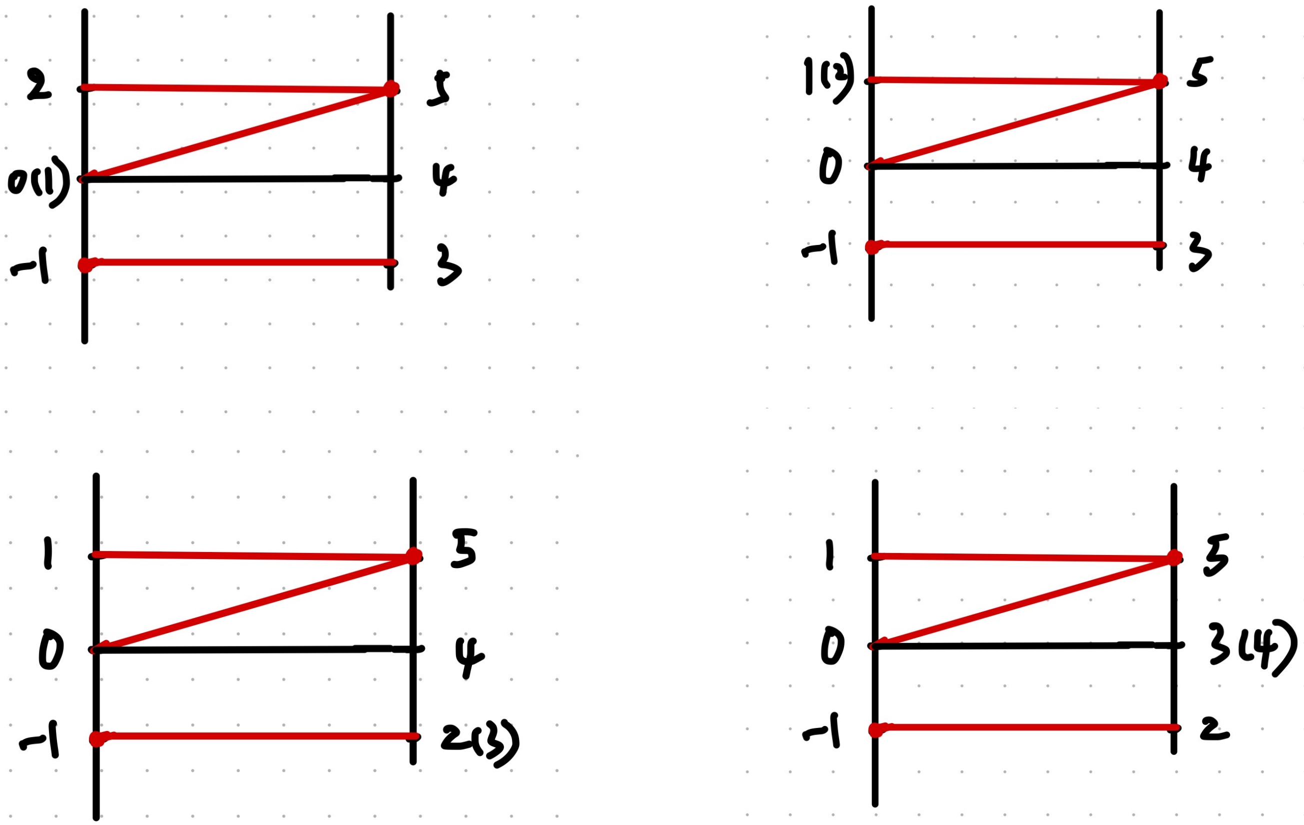
$= $ “lower loop integrals”

Boundary integral topology for $A_{1,1,0,1,1,0,1}$

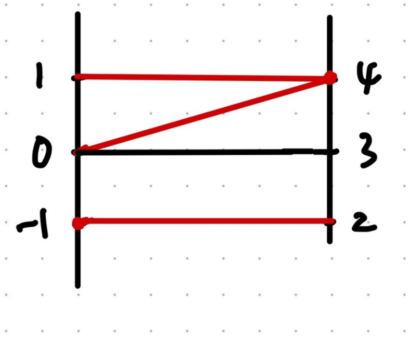
Boundary terms are generated by setting one particle to be soft.



landing onto the three-particle phase-space configuration.
The boundary integrals are those relavant for the three-point correlators.



Exercise: three-point correlators EEEC (triple collinear)



$$B_{a_1,a_2,a_3,a_4} = \int \frac{x_{12}^{-a_2}}{s_{123}^{a_1} x_{23}^{a_3} x_{123}^{a_4}} dx_1 dx_2 dx_3$$

$$s_{123} = x_1 x_2 |z_{12}|^2 + x_1 x_3 |z_{13}|^2 + x_1 x_2 |z_{23}|^2$$

$$z_2 = 0, z_3 = 1$$

Define linear mapping $I[B_{a_1,a_2,a_3,a_4}] = \sum B_i \ , \ B_i \in \{\frac{x_1^{q_1} x_2^{q_2} x_3^{q_3}}{s_{123}^{a_1} x_{23}^{a_3} x_{123}^{a_4}}\}$

On quotient space $V_B = \{B\}/\ker I[B]$ we generate finite IBPs and found 4 masters

B_1: $\frac{x_2}{s_{123} x_{23} x_{123}}$

B_2: $\frac{1}{s_{123} x_{123}}$ B_3: $\frac{x_2}{s_{123} x_{123}^2}$ B_4: $\frac{x_3}{s_{123} x_{123}^2}$

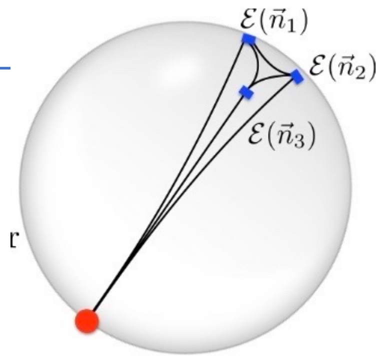
$$\mathrm{Li}_2(1-|z_1|^2) + \frac{1}{2}\ln|z_1|^2 \ln|1-z_1|^2$$

$$\mathrm{Li}_2(z_1) - \mathrm{Li}_2(\bar{z}_1) + \frac{1}{2}\ln|z_1|^2 \ln\frac{1-z_1}{1-\bar{z}_1}$$

$$\ln|z_1|^2 \qquad \ln|1-z_1|^2$$

Boundary terms are one-fold integrals which integrates to rational numbers.

EEEC is a linear sum of $B_{1..4}$ and 1 , agrees with[1912.11050]



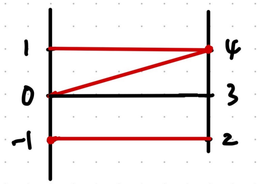
Algorithm for reducing multi-dimensional energy integrals

Define target integral family
Remove redundancy, define $V_A = \{A\}/\ker I[A]$

↓

Generate IBP relations on quotient space V_A satisfying:
homogeneity condition and finiteness conditions

$$\int \frac{\partial}{\partial x_i} \frac{x_1^{q_1} x_2^{q_2} x_3^{q_3} x_4^{q_4}}{s_{1234}^{a_1} \dots x_{12}^{a_2} x_{123}^{a_3} x_{1234}^{a_n}} dx_1 dx_2 dx_3 dx_4 = \left(\int \frac{x_1^{q_1} x_2^{q_2} x_3^{q_3} x_4^{q_4}}{s_{1234}^{a_1} \dots x_{12}^{a_2} x_{123}^{a_3} x_{1234}^{a_n}} dx_j dx_k dx_l \right) \Bigg|_{x_i=0}^{\infty} = \text{boundary term}$$



↓

Reduction for boundary terms (lower-dimensional integrals)

$$B = \int \frac{x_1^{q_1} x_2^{q_2} x_3^{q_3}}{s_{123}^{a_1} x_{12}^{a_2} x_{123}^{a_3}} dx_1 dx_2 dx_3$$

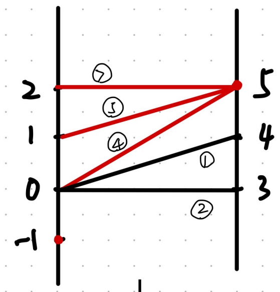
↓

Reduction for full family

↓

Functional reconstruction of reduction table

EEEEC Master Integrals



$$\frac{x_1x_2}{s_{1234}s_{123}x_{34}x_{1234}},\ \frac{x_2^2}{s_{1234}s_{123}x_{234}x_{1234}}$$

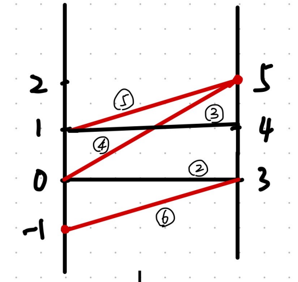
$$\frac{x_1}{s_{1234}s_{123}x_{1234}},\ \frac{x_2}{s_{1234}s_{123}x_{1234}},\ \frac{x_3}{s_{1234}s_{123}x_{1234}},$$

$$\frac{x_3}{s_{1234}s_{123}x_{34}},\ \frac{x_2}{s_{1234}s_{123}x_{234}},\ \frac{x_3}{s_{1234}s_{123}x_{234}},$$

$$\frac{x_2x_3}{s_{123}s_{234}x_{234}x_{1234}},\ \frac{x_2^2}{s_{123}s_{234}x_{234}x_{1234}}$$

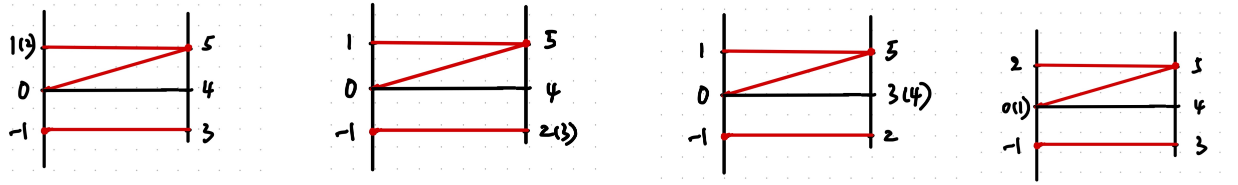
$$\frac{x_2}{s_{123}s_{234}x_{1234}},\ \frac{x_3}{s_{123}s_{234}x_{1234}},$$

$$\frac{x_1x_2}{s_{123}s_{234}x_{1234}^2},\ \frac{x_1x_3}{s_{123}s_{234}x_{1234}^2},\ \frac{x_1x_3}{s_{123}s_{234}x_{1234}^2}$$



$$\frac{1}{s_{1234}x_{34}x_{1234}},\ \frac{1}{s_{1234}x_{234}x_{1234}},\ \frac{1}{s_{123}x_{34}x_{1234}},\ \frac{x_3}{s_{123}x_{34}x_{1234}^2},\ \frac{1}{s_{123}x_{234}x_{1234}}$$

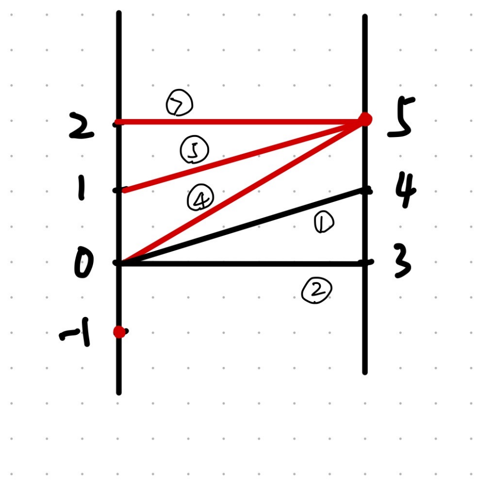
$$\frac{1}{s_{1234}^2x_{34}}$$



1

Symbol alphabets

$A_{1,1,0,1,1,0,1}$



$$\frac{|z_{12}|^2|z_{34}|^2}{|z_{13}|^2|z_{24}|^2} = Z\,\bar{Z}, \qquad \frac{|z_{14}|^2|z_{23}|^2}{|z_{13}|^2|z_{24}|^2} = (1-Z)(1-\bar{Z})$$

$$\frac{|z_{24}|^2}{|z_{23}|^2} = \frac{e\,\bar{e}}{(1+e)(1+\bar{e})}, \quad \frac{|z_{34}|^2}{|z_{23}|^2} = \frac{1}{(1+e)(1+\bar{e})}.$$

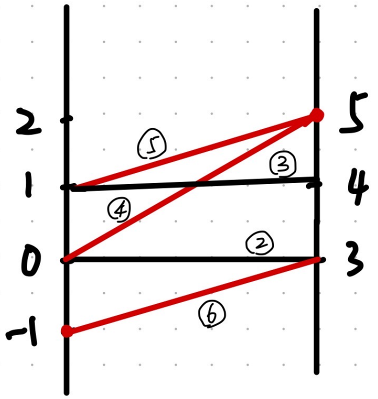
In this family, the symbol of integrals contain 23 letters

$$\{e, 1+e, \bar{e}, 1+\bar{e}, -e+\bar{e}, 1+e+\bar{e}, -1+Z, Z, 1+e\,Z,$$
$$1+\bar{e}\,Z, 1+e+\bar{e}+e\,\bar{e}\,Z, -1+\bar{Z}, \bar{Z}, -Z+\bar{Z}, 1+e\,\bar{Z},$$
$$1+\bar{e}\,\bar{Z}, -e\,Z+\bar{e}\,\bar{Z}, 1+e+\bar{e}+e\,\bar{e}\,\bar{Z},$$
$$e-\bar{e}-e\,Z-e\,\bar{e}\,Z+\bar{e}\,\bar{Z}+e\,\bar{e}\,\bar{Z},$$
$$Z+e\,Z-\bar{Z}-\bar{e}\,\bar{Z}-e\,Z\,\bar{Z}+\bar{e}\,Z\,\bar{Z}, -1+e\,\bar{e}\,Z\,\bar{Z},$$
$$-1-e-\bar{e}-e\,\bar{e}\,Z-e\,\bar{e}\,\bar{Z}+e\,\bar{e}\,Z\,\bar{Z},$$
$$-1+Z+\bar{Z}+e\,Z\,\bar{Z}+\bar{e}\,Z\,\bar{Z}+e\,\bar{e}\,Z\,\bar{Z}\}$$

Integrals involves cubic-root letters a, b, c

$$P[x\_] := -1 - x - \frac{x\, zz12}{zz13} - \frac{x^2\, zz12}{zz13} + \frac{x\, zz23}{zz13} + \frac{x\, zz23}{zz34} + \frac{x^2\, zz12\, zz23}{zz13\, zz34} - \frac{x\, zz24}{zz34} - \frac{x^2\, zz24}{zz34} - \frac{x^2\, zz12\, zz24}{zz13\, zz34} - \frac{x^3\, zz12\, zz24}{zz13\, zz34} + \frac{x^2\, zz23\, zz24}{zz13\, zz34} = -(-1+xa)(-1+xb)(-1+xc)$$

$A_{0,1,1,1,1,1,0}$



$$\frac{|z_{12}|^2}{|z_{23}|^2} = \frac{abc(abc+|e|^4)}{(ab-|e|^2)(ac-|e|^2)(bc-|e|^2)}, \frac{|z_{13}|^2}{|z_{23}|^2} = -\frac{|e|^2(abc+|e|^4)}{(ab-|e|^2)(ac-|e|^2)(bc-|e|^2)}$$

$$\frac{|z_{24}|^2}{|z_{23}|^2} = \frac{|e|^2(abc+|e|^4)}{(a+|e|^2)(b+|e|^2)(c+|e|^2)}, \frac{|z_{34}|^2}{|z_{23}|^2} = \frac{(abc+|e|^4)}{(a+|e|^2)(b+|e|^2)(c+|e|^2)}$$

In this family, 37 letters appear, 30 are independent, 16 involves the cubic roots

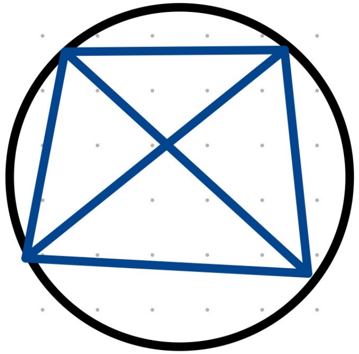
$$\{a, 1+a, b, 1+b, c, 1+c, e, 1+e, \text{-}a+e, \text{-}b+e, \text{-}c+e, \bar{e},$$
$$1+\bar{e}, \text{-}a+\bar{e}, \text{-}b+\bar{e}, \text{-}c+\bar{e}, \text{-}e+\bar{e}, a+e\,\bar{e}, b+e\,\bar{e},$$
$$c+e\,\bar{e}, Z, 1+e\,Z, \text{-}a+e\,Z, \text{-}b+e\,Z, \text{-}c+e\,Z, \bar{Z},$$
$$1+\bar{e}\,\bar{Z}, \text{-}a+\bar{e}\,\bar{Z}, \text{-}b+\bar{e}\,\bar{Z}, \text{-}c+\bar{e}\,\bar{Z}, \text{-}1+Z\bar{Z}, 1+e\,Z\bar{Z},$$
$$1+\bar{e}\,Z\bar{Z}, \text{-}1+e\,\bar{e}\,Z\bar{Z}, a+e\bar{e}\,Z\bar{Z}, b+e\bar{e}Z\bar{Z},$$
$$c+e\,\bar{e}\,Z\,\bar{Z}\}$$

On $Z = \bar{Z}$ hypersurface:

All algebraic-root letters rationalized by parameters (e, a, Z) , 20 letters .

$$\frac{|z_{12}|^2}{|z_{23}|^2} = \frac{a\, e\, Z^2\, (-1 + a - e - e\, Z)}{(1 + a\, Z)\,(a - e\, Z)\,(1 + e\, Z)},\quad \frac{|z_{13}|^2}{|z_{23}|^2} = \frac{a + a\, Z - e\, Z + a\, e\, Z}{(1 + a\, Z)\,(a - e\, Z)\,(1 + e\, Z)}$$

$$\frac{|z_{24}|^2}{|z_{23}|^2} = \frac{a\, e\, (-1 + a - e - e\, Z)}{(a - e)\,(1 + e)\,(a + Z)},\qquad \frac{|z_{34}|^2}{|z_{23}|^2} = -\frac{a + a\, Z - e\, Z + a\, e\, Z}{(a - e)\,(1 + e)\,(a + Z)}$$



$$\begin{gathered}\{a, e, 1 + e, -a + e, -1 + Z, Z, 1 + Z, a + Z, 1 + a\, Z,\\ 1 + e\, Z, -a + e\, Z, 1 - a + e + e\, Z, a + a\, Z - e\, Z + a\, e\, Z,\\ a^2 + a\, Z - e\, Z + a\, e\, Z - e^2\, Z + a\, e^2\, Z,\\ a - a^2 + 2\, a\, e + 2\, a\, e\, Z - e^2\, Z + a\, e^2\, Z,\\ 1 + e\, Z^2, -a - a\, Z + e\, Z - a\, e\, Z + a\, Z^2 - a^2 Z^2 + a\, e\, Z^2 +\\ a\, e\, Z^3, a - a\, Z - e\, Z + a\, e\, Z - a\, Z^2 - a^2 Z^2 + 2\, e\, Z^2 -\\ a\, e\, Z^2 - a^2\, e\, Z^2 + a\, e^2\, Z^2 + e^2\, Z^3,\\ a^2 + a\, Z - e\, Z - a\, e\, Z + 2\, a^2 e\, Z - e^2\, Z - a\, e^2\, Z + a\, e\, Z^2 -\\ a^2\, e\, Z^2 - a\, e^2 Z^2 + a\, e^2 Z^3,\\ a + a\, Z - e\, Z + a\, e\, Z + a\, e\, Z^2 - a^2\, e\, Z^2 + a\, e^2\, Z^2 + a\, e^2 Z^3\}\end{gathered}$$

EEEEC in N=4 SYM in the quadruple collinear limit :

$$EEEEC_{N=4\ SYM}\Big|_{coll.} = R_i\, A_i + r_j\, B_j + r_0 + \text{perms}(1,2,3,4)$$

R_i, r_j : Algebraic functions

A_i : 22 pure master integrals
17 weight-3 + 5 weight-2

B_j: 13 pure boundary master integrals
8 weight-2 + 5 weight-1

43 letters . First entry drawn from $\{ \frac{|z_{ij}|^2}{|z_{jk}|^2}, a, b, c\}$

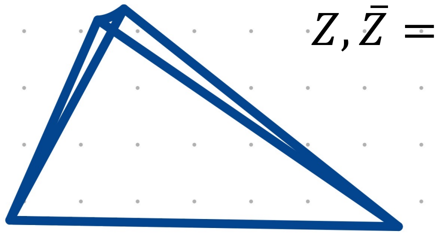
```
P[x_] := -1 - x - x zz12/zz13 - x^2 zz12/zz13 + x zz23/zz13 + x zz23/zz34 + x^2 zz12 zz23/(zz13 zz34) - x zz24/zz34 - x^2 zz24/zz34 - x^2 zz12 zz24/(zz13 zz34) - x^3 zz12 zz24/(zz13 zz34) + x^2 zz23 zz24/(zz13 zz34)
```

$$= -(-1 + xa)(-1 + xb)(-1 + xc)$$

Physical singularities can be read off from symbols, e.g.:

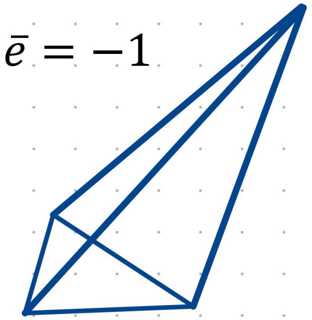
Two point coincide

$Z, \bar{Z} = 1$



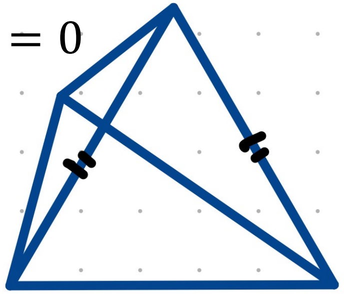
Three points coincide

$e, \bar{e} = -1$



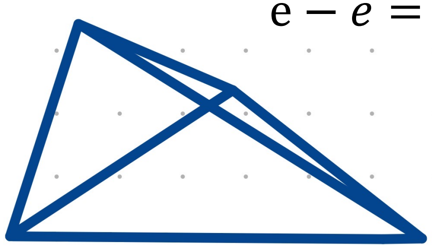
Equilateral triangle

$1 - |e|^2 = 0$



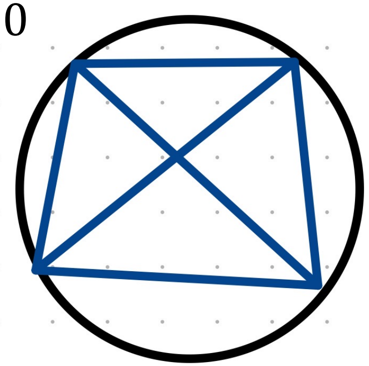
Triangle collapses

$e - \bar{e} = 0$



Four points on a circle

$Z - \bar{Z} = 0$



$$\frac{|z_{12}|^2 |z_{34}|^2}{|z_{13}|^2 |z_{24}|^2} = Z\,\bar{Z}, \quad \frac{|z_{14}|^2 |z_{23}|^2}{|z_{13}|^2 |z_{24}|^2} = (1 - Z)(1 - \bar{Z})$$

$$\frac{|z_{24}|^2}{|z_{23}|^2} = \frac{e\,\bar{e}}{(1+e)(1+\bar{e})}, \quad \frac{|z_{34}|^2}{|z_{23}|^2} = \frac{1}{(1+e)(1+\bar{e})}.$$

Discussions

What do we know about the EEEC function space?

- Interpretation of symbol alphabets, including the cubic roots
- Event shape bootstrap?

  Determine higher loop functions from physical constraints/geometries

Singularities of correlation function of light-ray correlators

- Leading and sub-leading power data in the asymptotics
  understand the corresponding OPE limits of light-ray operators

Generalization of phase-space integration algorithms

- Higher-order Phase-space integration
- Application to the study of Jet substructure observables

THANK YOU FOR YOUR ATTENTION !