Karlsruhe Institute of Technology

Semileptonic and nonleptonic *B* decays at NNLO

RADCOR 2023

Manuel Egner | Crieff, May 30, 2023

INSTITUTE FOR THEORETICAL PARTICLE PHYSICS

based on work with:
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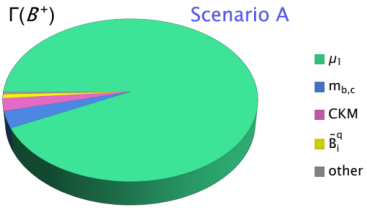
Outline

Motivation

Calculation of B-meson lifetimes in HQE:

$$\Gamma(B) = \Gamma_3 + \Gamma_5 \frac{\langle \mathcal{O}_5 \rangle}{m_b^2} + \Gamma_6 \frac{\langle \mathcal{O}_6 \rangle}{m_b^3} + \cdots + 16\pi^2 \left(\Gamma_6 \frac{\langle \tilde{\mathcal{O}}_6 \rangle}{m_b^3} + \Gamma_7 \frac{\langle \tilde{\mathcal{O}}_7 \rangle}{m_b^4} + \ldots \right).$$

- Heavy Quark Expansion (HQE): Decay width of B-meson as sum of decay width of a free b-quark (Γ_3) and corrections suppressed by the mass m_b.
- Decay channels:
  - Semileptonic: $b \to c l \overline{\nu}$
  - Nonleptonic: $b \to c\overline{u}d$ and $b \to c\overline{c}s$
- Error contributions on B-meson lifetimes dominated by uncertainty induced by renormalization scale μ
- Knowing the $\mathcal{O}\left(\alpha_s^2\right)$ contributions will reduce the uncertainty induced by the renormalization scale variation.



[Lenz, Piscopo, Rusov (2022)]

Motivation

- Goal: Hadronic decay channels $b \to c\bar{u}d$ and $b \to c\bar{c}s$ at $\mathcal{O}\left(\alpha_s^2\right)$.
- In this talk: Semileptonic decay channel $b \to cl\bar{\nu}$ at $\mathcal{O}\left(\alpha_s^2\right)$ as a proof of concept.
  - Use different methods to calculate master integrals
  - $\mathcal{O}\left(\alpha_s^2\right)$ corrections for semileptonic decay channel are already known
    $\to$ cross checks

$\Rightarrow$ $b \to cl\bar{\nu}$ calculation to set up a frame work for the calculation for the hadronic decay channels

What is known?

Previous calculations including finite charm quark mass for Γ_3:

Semileptonic decay channel:

- $\mathcal{O}\left(\alpha_s^1\right)$ [Nir (1989)]
- $\mathcal{O}\left(\alpha_s^2\right)$ [Czarnecki, Pak (2008)], [Dowling, Piclum, Czarnecki (2008)]
- $\mathcal{O}\left(\alpha_s^3\right)$ [Fael, Schönwald, Steinhauser (2021)] [Czakon, Czarnecki, Dowling (2021)]

$\mathcal{O}\left(\alpha_s^2\right)$ and $\mathcal{O}\left(\alpha_s^3\right)$ calculations as expansions in mass ratios, no exact analytic results.

Hadronic decay channel:

- $b \to c\bar{u}d$: $\mathcal{O}\left(\alpha_s^1\right)$ [Bagan, Ball, Braun, Gosdzinsky (1994)]
- $b \to c\bar{c}s$: $\mathcal{O}\left(\alpha_s^1\right)$ [Bagan, Ball, Fiol, Gosdzinsky (1995)]
- $\mathcal{O}\left(\alpha_s^2\right)$, including only massless quarks in the final state, $b \to u$, but only one operator [Czarnecki, Slusarczyk, Tkachov (2006)]

Calculation setup

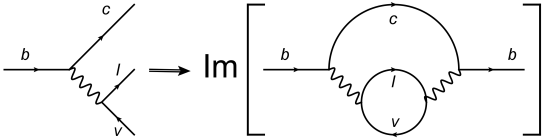
- Optical theorem:

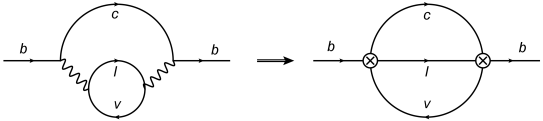
$$\Gamma = \frac{1}{m_b} \mathrm{Im}\left[\mathcal{M}(b \to b)\right]$$

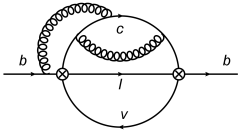
- Integrate out W-boson

$$\frac{1}{(m_W^2 - p^2)} \to \frac{1}{m_W^2}$$

- At $\mathcal{O}\left(\alpha_s^2\right)$ calculate imaginary part of 4-loop diagrams







Calculation setup

- Generate diagrams with `QGRAF` [Nogueira (1993)].
- Mapping on integral families with `TAPIR` [Gerlach, Herren, Lang (2022)] and `EXP` [Harlander, Seidensticker, Steinhauser (1998-1999)].
- Reduction to master integrals with `Kira` [Klappert, Lange, Maierhöfer, Usovitsch (2021)].
  - Choose good basis of master integrals, where ϵ and $x = m_c/m_b$ factorize, with `ImproveMasters.m` [Smirnov, Smirnov (2020)].
- Calculate only imaginary part of master integrals (optical theorem)

70 diagrams

$\Downarrow$

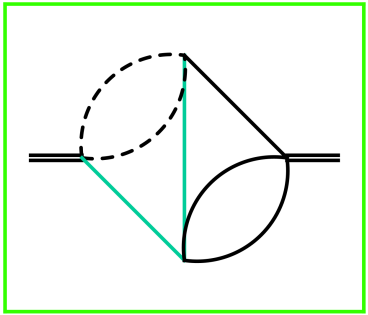
24 families

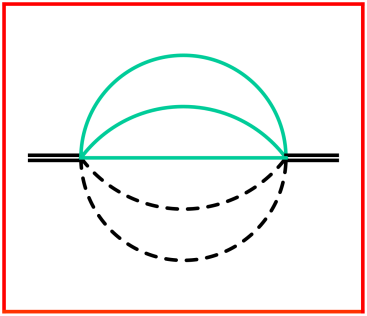
$\Downarrow$

129 master integrals in 9 families

Master integrals: analytic calculation

Two sets of master integrals:





Exact analytic calculation of imaginary part of 96 master integrals with cut through one charm quark.

Master integrals: analytic calculation

- Calculating master integrals via differential equations:
  - Determination of ϵ-form [Henn (2013)], [Lee (2014)] for 92 integrals with `Canonica` [Meyer (2018)] and `Libra` [Lee (2020)].
  - Remaining part of the differential equations (4 integrals) which we could not bring to ϵ-form: decouple system with `OreSys` [Gerhold (2002)], `Sigma.m` [Schneider (2007)].
  - Handling of iterated integrals with `HarmonicSums` [Ablinger (2010)].
- Fix constants $\vec{c}$ by matching to boundary conditions
- Boundary conditions: asymptotic expansion in $\delta = 1 - m_c^2/m_b^2$ the limit $m_c \approx m_b$

$$\frac{m_c}{m_b} = \frac{1-t^2}{1+t^2}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{I} = A(\epsilon, t)\cdot\vec{I}$$

$$\xrightarrow{\vec{I} = T\cdot\vec{J}} \quad \frac{\mathrm{d}}{\mathrm{d}t}\vec{J} = \epsilon\tilde{A}(t)\cdot\vec{J}$$

$$\rightarrow \quad \vec{J} = \int^t \epsilon\tilde{A}(t')\cdot\vec{J}\,\mathrm{d}t' + \vec{c}$$

Master integrals: analytic calculation

Similar to calculation of $\mathcal{O}\left(\alpha_s^2\right)$ [Dowling, Piclum, Czarnecki (2008)] and $\mathcal{O}\left(\alpha_s^3\right)$ [Fael, Schönwald, Steinhauser (2021)] corrections:

- Fix boundary condition in the limit $m_c \approx m_b$.
- Asymptotic expansion [Beneke, Smirnov (1997)] of master integrals in $\delta = 1 - m_c^2/m_b^2$.
- Two relevant momentum scalings:
  - hard: $|k^\mu| \sim m_b$
  - ultrasoft: $|k^\mu| \sim \delta \cdot m_b$

Master integrals: analytic calculation

- Obtain master integrals in terms of iterated integrals over the alphabet

$$\frac{1}{t}, \quad \frac{1}{1+t}, \quad \frac{1}{1-t}, \quad \frac{t}{1+t^2}, \quad \frac{t^3}{1+t^4}$$

- Exact analytic result for $\mathcal{O}\left(\alpha_s^2\right)$ contributions (new!)
- Exact analytic expression $\rightarrow$ allows for expansion around different kinematic limits:
  - Massless charm quark:

$$x = \frac{m_c}{m_b} = 0 \quad \leftrightarrow \quad t = 1$$

  - Heavy charm quark:

$$x = \frac{m_c}{m_b} = 1 \quad \leftrightarrow \quad \delta = 0 \quad \leftrightarrow \quad t = 0$$

Analytic results

$$\Gamma\left(b \to c l \bar{\nu}\right) = \frac{A_{ew} G_F^2 m_b^5 |V_{cb}|^2}{192\pi^3} \left(X_0 + \frac{\alpha_s}{\pi} C_F X_1 + \left(\frac{\alpha_s}{\pi}\right)^2 C_F X_2 + \mathcal{O}\left(\alpha_s^3\right) \right),$$

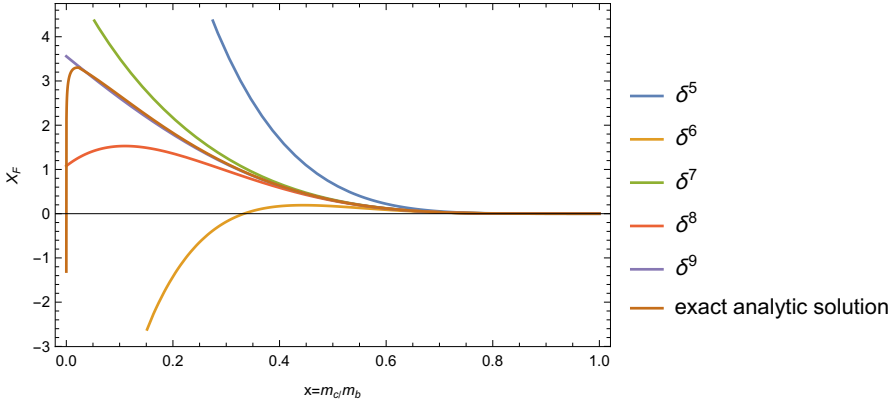
with

$$X_2 = C_F X_F + C_A X_A + T_F \left(n_l X_l + n_c X_c + n_b X_b\right).$$

- $n_l = 3$, $n_c = 1$, $n_b = 1$ number of light, charm and bottom quarks.
- All $\{X_F,\ X_A,\ X_l,\ X_c,\ X_b\}$ depending on the mass ratio m_c/m_b as functions of t, polynomials and iterated integrals up to weight five

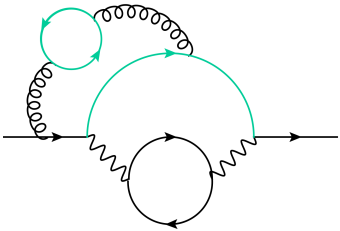
Comparing results

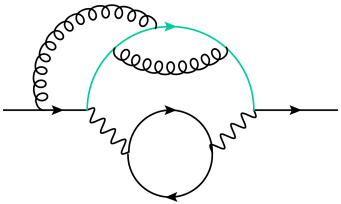
- Expand amplitude around $m_c \approx m_b$ after insertion of analytic master integrals
- Compare $\mathcal{O}\left(\alpha_s^2\right)$ contribution [Dowling, Piclum, Czarnecki (2008)]
  $\rightarrow$ Reproduce same results for all color factors!



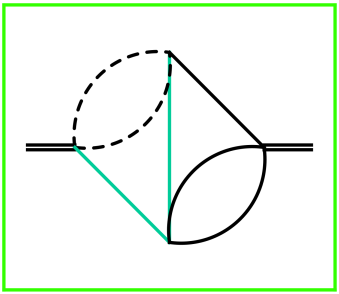
Comparing results

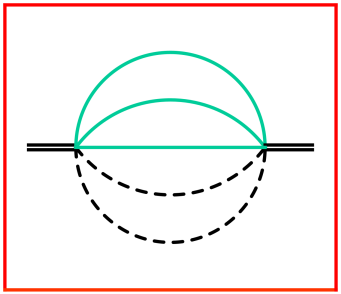
- Expand amplitude around $m_c \approx m_b$ after insertion of analytic master integrals
- Compare $\mathcal{O}\left(\alpha_s^2\right)$ contribution [Dowling, Piclum, Czarnecki (2008)]
  $\rightarrow$ Reproduce same results for all color factors!
- Results as an expansion in $x = m_c/m_b$ are known up to $\mathcal{O}\left(\alpha_s^2\right)$ [Czarnecki, Pak (2008)].
- Expand amplitude in x after insertion of analytic master integrals
- Compare $\mathcal{O}\left(\alpha_s^2\right)$ contribution
  $\rightarrow$ Reproduce same results for $T_F n_l$ and $T_F n_h$ but not for $T_F n_c$, C_F^2 and $C_F C_A$, because of 3 charm contributions





Cuts through 3 charm quarks





- Boundary conditions for differential equations at $m_c \approx m_b$
  - Cut through three charm quarks is trivially set to zero in this limit!
  - Expect this contribution to be small because of small phase space ($m_b \approx 4.5\mathrm{GeV}$, $3m_c \approx 3.9\mathrm{GeV}$)!

$\Rightarrow$ Alternative approach to calculate both, one and three charm cut contributions

Master integrals: Expand and Match

Calculation procedure: [Fael, Lange, Schönwald, Steinhauser (2021, 2022)]

- Strating point: DEQ for master integrals
- Make general expansion ansatz in $\delta = 1 - m_c^2/m_b^2$ around certain point δ_0 for integral

$$I_i(\delta, \delta_0) = \sum_{j=\epsilon_{\min}}^{\epsilon_{\max}} \sum_{m=0}^{j+4} \sum_{n=0}^{n_{\max}} c[i,j,m,n]\, \epsilon^j (\delta_0 - \delta)^n \log^m(\delta_0 - \delta)$$

$$\delta_0 = \frac{8}{9} \to \log\left(\frac{8}{9} - \delta\right) = \log\left(\left|\frac{8}{9} - \delta\right|\right) - i\pi,$$

- Insert ansatz in DEQ

$$\ldots c[i,j,m,n] \frac{\mathrm{d}}{\mathrm{d}\delta} (\delta_0 - \delta)^n + \cdots = \ldots c[i,j,m,n-1] (\delta_0 - \delta)^{n-1} + \ldots$$

$\to$ Linear equations for $c[i,j,m,n]$ for every power in δ

- Solve system of linear equations with `Kira` [Klappert, Lange, Maierhöfer, Usovitsch (2021)] and `FireFly` [Klappert, Lange(2020)]

Master integrals: Expand and Match

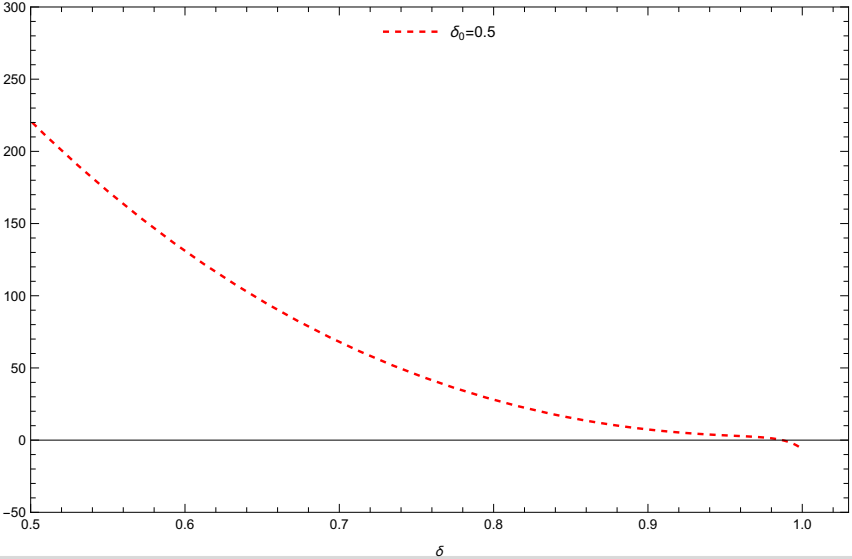
- Determine remaining coefficients by matching to numerical results (`AMFlow` [Liu, Ma (2022)])
- Expansions around several expansion points:
  - Expansion including Logarithms around $x = 1/3 \leftrightarrow \delta = 8/9$, $x = 0 \leftrightarrow \delta = 1$ and $x = 1 \leftrightarrow \delta = 0$.
  - Taylor expansions around other expansion points.

Example:

1. Taylor expansion around $\delta_0 = 0.5$
2. Obtain numerical values of integrals at $\delta = 0.5$ with `AMFlow`
3. Determine expansion coefficients by matching expansion of step 1 to numerical results of step 2
4. Taylor expansion around $\delta_0 = 0.7$
5. Evaluate expansions of step 1 at $\delta = 0.6$
6. Determine expansion coefficients of expansion of step 4 by matching to numerical results of step 5
7. repeat for next expansion point

Master integrals: Expand and Match

Real part of sample integral calculated by ”Expand and Match” method at $\mathcal{O}\left(\epsilon^2\right)$:



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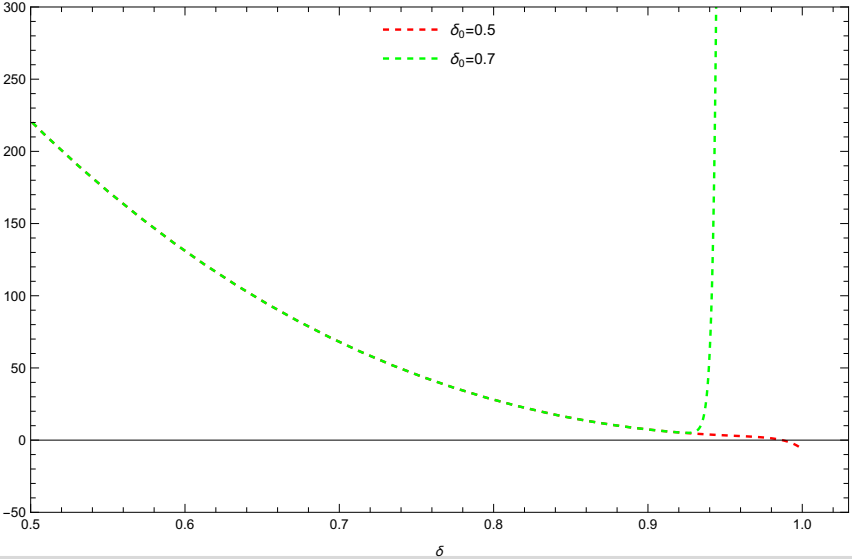

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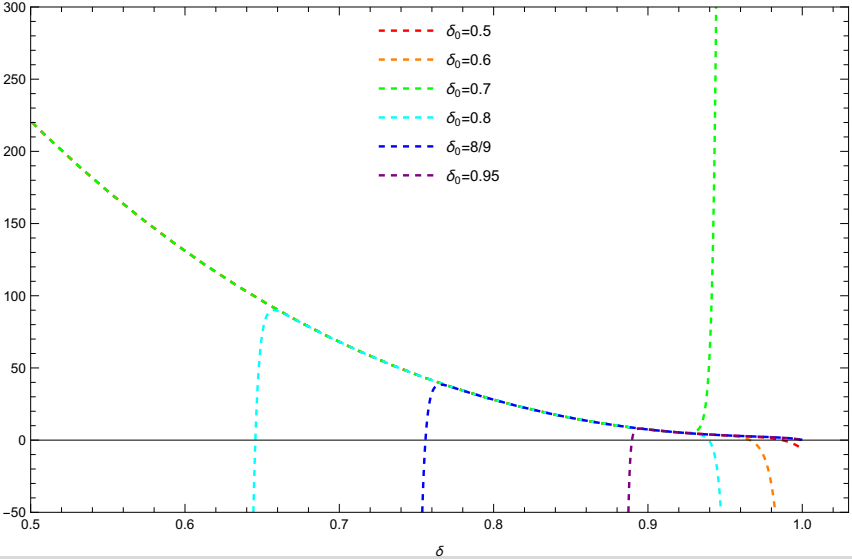

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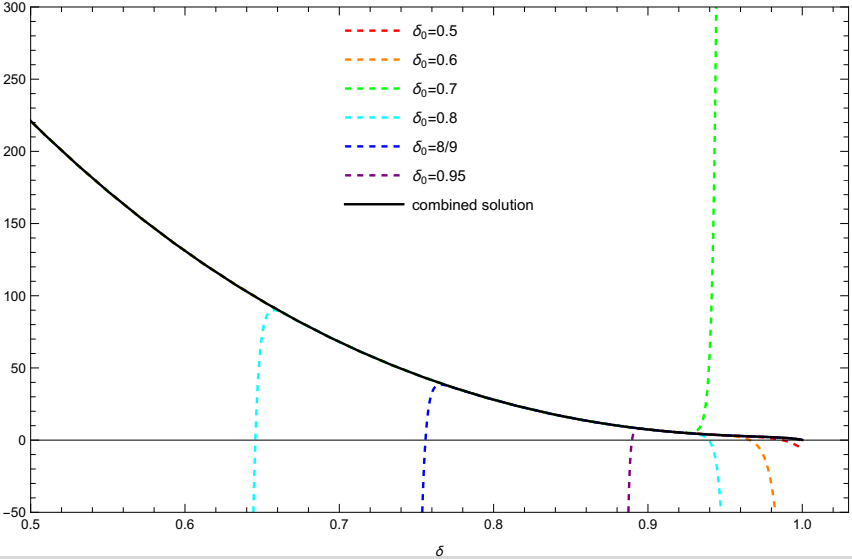
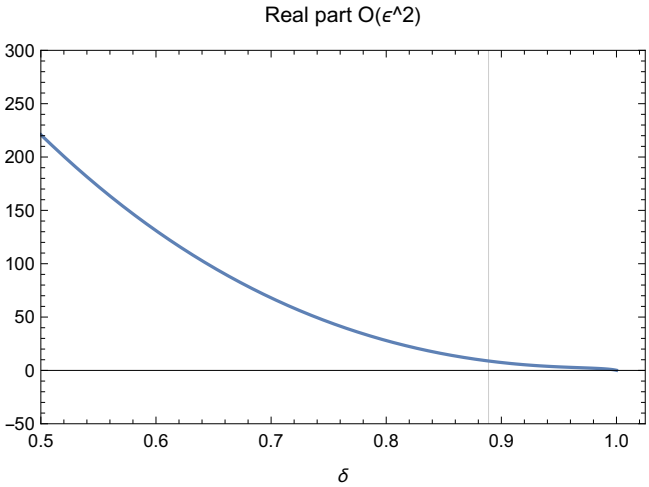
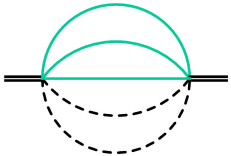


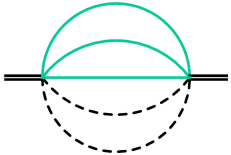
Master integrals: Expand and Match





$\delta = 1 - m_c^2/m_b^2$

Master integrals: Expand and Match



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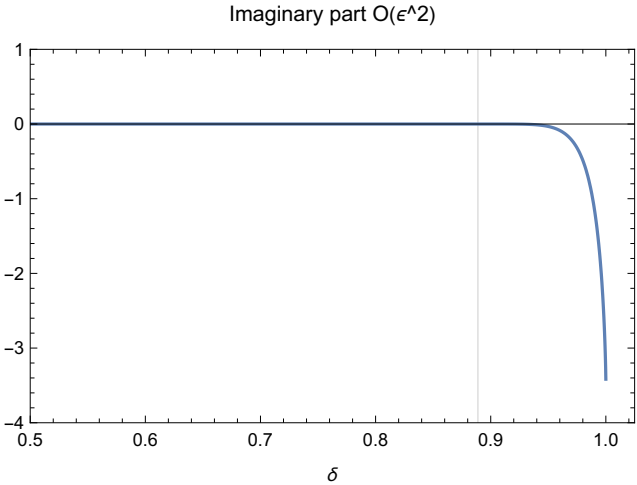
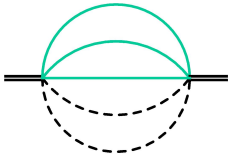


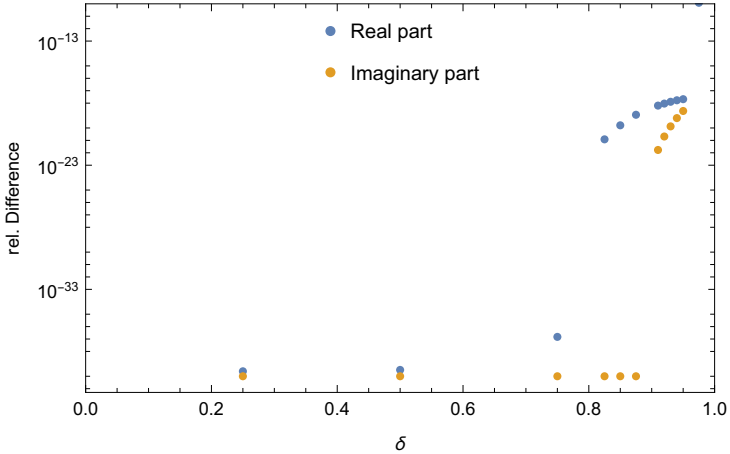
Master integrals: Expand and Match

Relative difference between numerical calculations with `AMFlow` and "Expand and Match" results at $\mathcal{O}\left(\epsilon^2\right)$:

$$\left|\frac{I_{\mathrm{E\&M}}(\delta) - I_{\mathrm{AMFlow}}(\delta)}{I_{\mathrm{AMFlow}}(\delta)}\right|$$



$\delta = 1 - m_c^2/m_b^2$



Results: Pole cancellation

Check results by combining bare NNLO amplitude ($\mathrm{A_{bare}}$) with Expand&Match master integrals and analytic counter terms CT

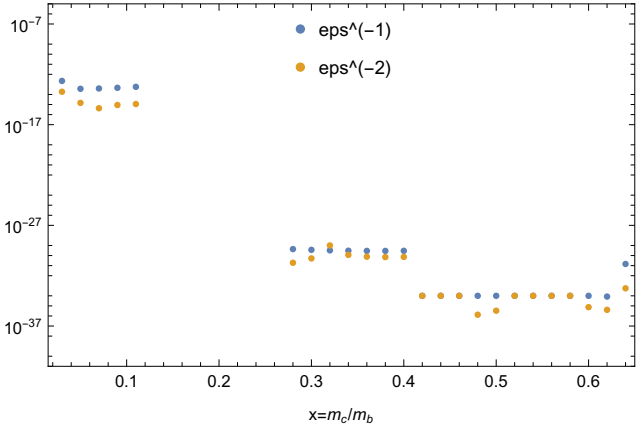
$$\frac{|\mathrm{A_{bare}} + \mathrm{CT}|}{\mathrm{CT}}$$

- Colour factors are set to numerical value
- Use different expansions:

$$\begin{aligned} x_0 &= 1/12 & x &\in [0.02, 0.12) \\ x_0 &= 1/6 & x &\in [0.12, 0.20) \\ x_0 &= 1/4 & x &\in [0.20, 0.28) \\ x_0 &= 1/3 & x &\in [0.28, 0.42) \\ x_0 &= 1/2 & x &\in [0.42, 1.00) \end{aligned}$$

- Starting point: Expansion around $x_0 = 1/2$

preliminary:



Comparing results

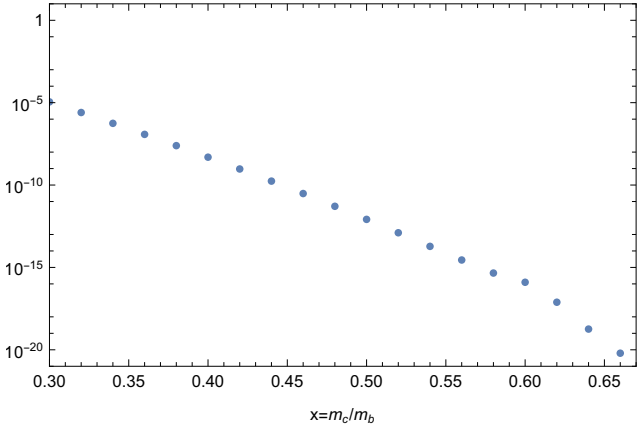
How good does the numerical and anlytic solution agree?

- Compare bare amplitude of numerical and analytic results:

$$100 \cdot \frac{|\mathrm{A_{ana}} - \mathrm{A_{num}}|}{\mathrm{A_{ana}}}$$

- Compare in region, where 3 charm cut is zero ($x > 1/3$)

preliminary:



3 charm cut

How big is the 3 charm contribution?

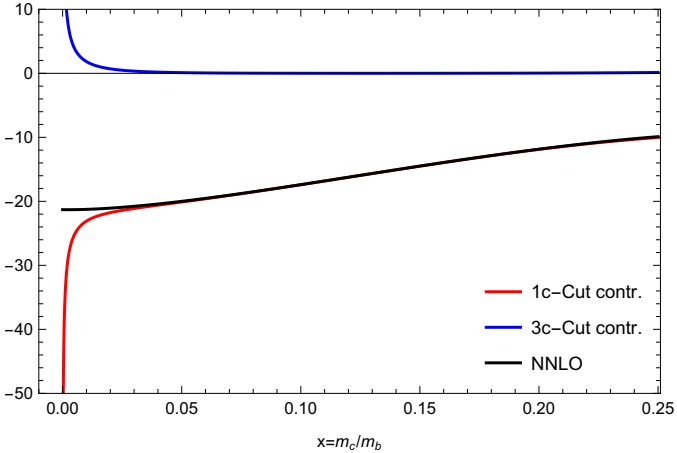
- In the physical region $x \in [0.2, 0.3]$

$$\frac{|\mathrm{A_{ana}} - \mathrm{A_{num}}|}{\mathrm{A_{ana}}}$$

is of the order of $10^{-1} \ldots 10^{-4}\%$

$\rightarrow$ 3 charm contribution is negligible in the physical region!

preliminary:



- Obtain expression for 3 charm contribution by subtracting expansion of exact analytic one charm result around $x = 0$ from [Czarnecki, Pak (2008)]

3 charm cut

$$\Gamma\left(b \to cccl\nu\right) = \frac{A_{ew} G_F^2 m_b^5 |V_{cb}|^2}{192\pi^3} \frac{\alpha_s^2}{\pi^2} \left(C_F^2 X_{F,3} + C_F C_A X_{A,3} + T_F n_c X_{c,3}\right)$$

with $L_x = \log(x)$

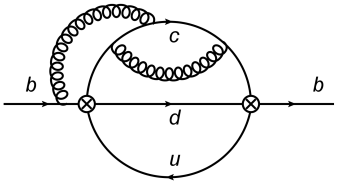
$$\begin{aligned}
X_{F,3} &= -\frac{409}{576} - \frac{349\pi^2}{288} - \frac{7\pi^4}{144} + \frac{19\pi^2 \log(2)}{6} + \frac{5\pi^2 x}{3} + \left(-\frac{13}{8} + \frac{\pi^2}{4} - \zeta_3\right) L_x - \frac{115\zeta_3}{24} \\
&\quad + x^2 \left(\frac{12083}{648} - \frac{103\pi^2}{36} - \frac{29\pi^4}{18} - \frac{4\pi^2 \log(2)}{3} + \left(-\frac{34}{9} - \frac{4\pi^2}{3}\right) L_x^2 + \frac{14}{3} L_x^3 + \frac{2}{3} L_x^4 \right. \\
&\quad \left. + \left(\frac{961}{54} - \frac{52\pi^2}{9} - 60\zeta_3\right) L_x - \frac{341\zeta_3}{3}\right) + x^3 \left(\frac{131\pi^2}{3} - \frac{56\pi^3}{3} - \frac{124\pi^2 L_x}{3}\right) + \mathcal{O}\left(x^4\right), \\
X_{A,3} &= -\frac{1}{2} X_{F,3}, \\
X_{c,3} &= -\frac{38225}{5184} + \frac{2\pi^2}{27} + \frac{4\zeta(3)}{3} + \frac{3\pi^2 x}{8} + \left(-\frac{415}{72} + \frac{\pi^2}{9}\right) L_x - \frac{5}{3} L_x^2 - \frac{2}{9} L_x^3 \\
&\quad + x^2 \left(\frac{9305}{162} + \frac{38\pi^2}{27} + \frac{340}{9} L_x + \frac{20}{3} L_x^2\right) + x^3 \left(\frac{209\pi^2}{72} + \frac{16\pi^2}{3} L_x\right) + \mathcal{O}\left(x^4\right).
\end{aligned}$$

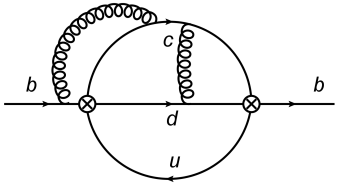
Conclusion

- We obtained an exact analytic expression for one charm cut contribution to the semileptonic decay width $b \to cl\bar{\nu}$ at $\mathcal{O}\left(\alpha_s^2\right)$.
- We compared our exact analytic results with previous known expansions and found agreement.
- We obtained a precise numerical result including one and three charm cut contributions.

What's next?

- Nonleptonic decay channels $b \to c\bar{u}d$ and $b \to c\bar{c}s$ with finite charm mass at $\mathcal{O}\left(\alpha_s^2\right)$.





Thank you for your attention!