A first look at the two-mass QCD contributions to the four-loop ρ parameter

Arnd Behring

CERN
Theory Department

based on work in progress in collaboration with:
S. Abreu, A. McLeod, B. Page

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Two-mass corrections to the ρ parameter

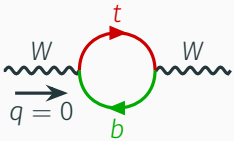
- The ρ parameter is an important precision observable in the Standard Model
  [Ross, Veltmann '75]

$$\rho = \frac{m_W^2}{m_Z^2 \cos^2 \theta_W} = 1 + \delta\rho$$

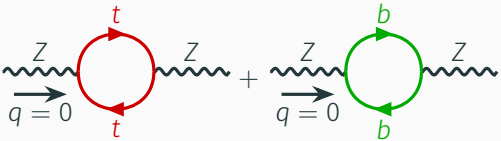
- Tests the structure of the electro-weak sector (SM at tree-level: $\delta\rho = 0$)
- Can be expressed in terms of W and Z boson self energies

$$\delta\rho = \frac{\Sigma_Z(0)}{m_Z^2} - \frac{\Sigma_W(0)}{m_W^2}$$

W self energy:



Z self energy:



State of the art

Long history of higher-order corrections to the ρ parameter in the Standard Model

$O(\alpha)$

[Veltman '77]

$O(\alpha\alpha_s)$

[Djouadi, Vezegnassi '87]
[Djouadi '87]
[Kniehl et al. '88]
[Kniehl '90]

$O(\alpha\alpha_s^2)$

[Anselm et al. '93]
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$O(\alpha\alpha_s^3)$

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$m_b=0$
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$m_b\neq 0$
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Our goal: $O(\alpha\alpha_s^3)$ with $m_b \neq 0$

Evaluation of the ρ parameter

- Self-energies with vanishing external momentum $q^2 = 0$

$$\delta\rho = \frac{\Sigma_Z(0)}{m_Z^2} - \frac{\Sigma_W(0)}{m_W^2}$$

- Map to tadpole integrals (with different "mass colourings": m_b, m_t and $m = 0$)

- Use standard tools:
  - integration-by-parts reduction to master integrals
  - differential equations in mass ratio

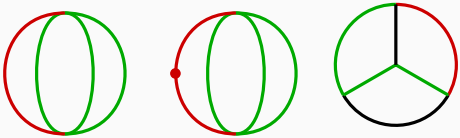
$$x = \frac{m_b}{m_t}$$

  - numerical evaluation of master integrals via AMFlow
    [Liu, Ma, Wang '17] [Liu, Ma '21] [Liu, Ma '22a] [Liu, Ma '22b]

- Numerical impact of four-loop two-mass corrections is expected to be very small

Analytic structure of the two-mass QCD contributions to the ρ parameter

- Up to two-loop order: only harmonic polylogarithms
- Three-loop order: elliptic integrals appeared (two curves, do not mix)
  [Grigo et al. '12] [Blümlein et al. '18] [Abreu et al. '19]



(+ $m_b \leftrightarrow m_t$)

- Four-loop: what comes next?
  - More elliptic curves?
  - Higher-genus curves, higher-dimensional geometries?

Identifying interesting sectors

Problem

- At four-loop order: 230 master integrals, organised into 152 sectors
- Many integrals are still just polylogarithmic
- How to find "interesting" integrals?

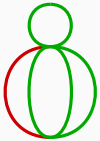
Solution

- Analyse differential equations
- New structures arise from homogeneous part ($\varepsilon \to 0$, no sub-sectors)
- Derive Picard-Fuchs operator L [Müller-Stach et al. '12] by uncoupling first-order systems into higher-order scalar differential equations [Zürcher '94] [Gerhold '02]
- Factor homogeneous differential operator L (e.g., using Maple's `DFactor` [van Hoeij '97])

$$L = [\partial_x^2 + f_1(x)\partial_x + f_2(x)]\,[\partial_x + f_3(x)]\ldots[\partial_x + f_k(x)]$$

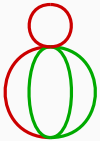
$\to$ higher-order factors can generate elliptic integrals and beyond

Potentially elliptic sectors



2×2 system

$L = \partial_x^2 + \#\partial_x + \#$



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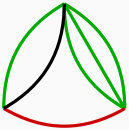

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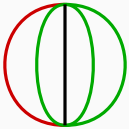

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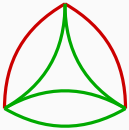

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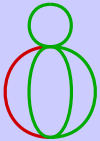

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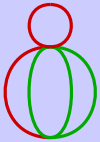

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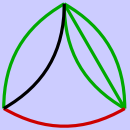

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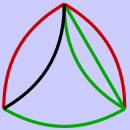

2×2 system

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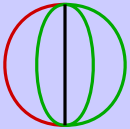
Have "twin" with $\mathsf{m_b} \leftrightarrow \mathsf{m_t}$



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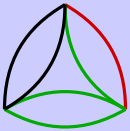

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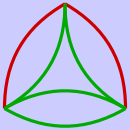

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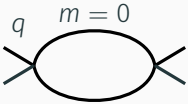
Symmetric under $\mathsf{m_b} \leftrightarrow \mathsf{m_t}$

Maximal cut analysis: idea

Idea

- Maximal cut fulfills the homogeneous differential equation (maximal cut eliminates subtopologies)
- Analyse maximal cuts (in $d = 2$) to find homogeneous solutions [Primo, Tancredi '16] [Primo, Tancredi '17]
- Work in loop-by-loop Baikov representation [Baikov '96] [Frellesvig, Papadopoulos '17]
  $\rightarrow$ reuse building blocks to construct maximal cuts
- Identify geometric structure underlying the integrals that go beyond polylogarithms. E.g., elliptic curves show up as $\int \frac{dz}{y}$ with $y^2 = P(z)$ and $P(z)$ third- or forth-order polynomial.

Maximal cut analysis: bubble and sunrise examples



$$\sim \int \mathrm{d}z_1 \mathrm{d}z_2 \frac{(q^2)^{(2-d)/2}}{z_1 z_2} \left[(q^2)^2 - 2q^2(z_1+z_2) + (z_1-z_2)^2\right]^{(d-3)/2}$$

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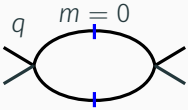

$$\sim \frac{1}{q^2}$$

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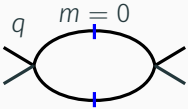

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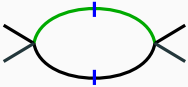
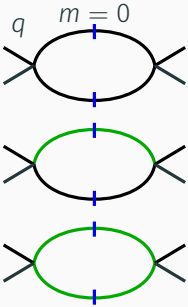


$$\sim \frac{1}{q^2 - m_b^2}$$

Maximal cut analysis: bubble and sunrise examples



$$\sim \frac{1}{q^2}$$

$$\sim \frac{1}{q^2 - m_b^2}$$

$$\sim \frac{1}{\sqrt{q^2[q^2 - 4m_b^2]}}$$

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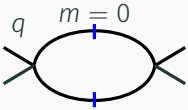

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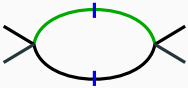

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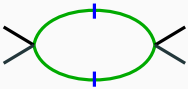

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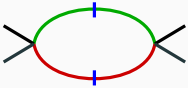

$$\sim \frac{1}{\sqrt{[q^2 - (m_b - m_t)^2]\,[q^2 - (m_b + m_t)^2]}}$$

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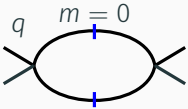

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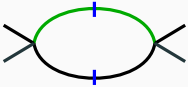

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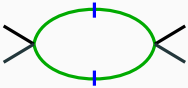

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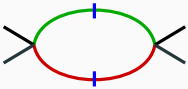

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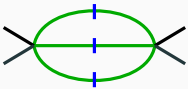
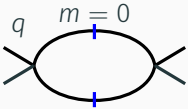


$$\sim \int \frac{\mathrm{d}z}{\sqrt{[z - (\sqrt{q^2} - m_b)^2]\,[z - (\sqrt{q^2} + m_b)^2]}\sqrt{z\,[z - 4m_b^2]}}$$

Maximal cut analysis: bubble and sunrise examples



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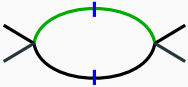

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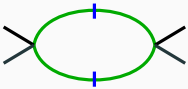

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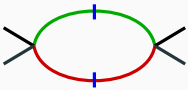

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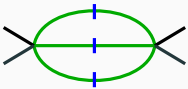


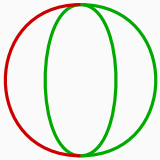
$$\sim \int \frac{dz}{\sqrt{[z - (\sqrt{q^2} - m_b)^2]\,[z - (\sqrt{q^2} + m_b)^2]}\sqrt{z\,[z - 4m_b^2]}}$$

→ Maximal cut of sunrise is an elliptic integral [Primo, Tancredi '16] [Primo, Tancredi '17]

→ Is annihilated by sunrise differential operator

Maximal cut analysis: three-loop example

Example: Analyse one of the elliptic tadpole integral from the three-loop ρ parameter



$$\sim \int \mathrm{d}^d k \, B_{bt}(k^2) \, B_{bb}(k^2)$$

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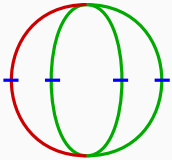
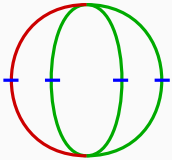


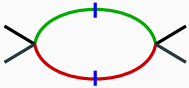
$$\sim \int \frac{\mathrm{d}k^2}{\sqrt{[k^2-(m_t-m_b)^2]\,[k^2-(m_t+m_b)^2]}\sqrt{k^2[k^2-4m_b^2]}}$$

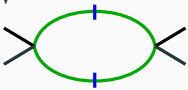
Maximal cut analysis: three-loop example

Example: Analyse one of the elliptic tadpole integral from the three-loop ρ parameter



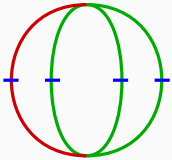
$$\sim \int \frac{\mathrm{d}k^2}{\sqrt{[k^2-(m_t-m_b)^2]\,[k^2-(m_t+m_b)^2]}\sqrt{k^2[k^2-4m_b^2]}}$$





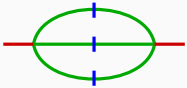
Maximal cut analysis: three-loop example

Example: Analyse one of the elliptic tadpole integral from the three-loop ρ parameter



$$\sim \int \frac{\mathrm{d}k^2}{\sqrt{[k^2-(m_t-m_b)^2]\,[k^2-(m_t+m_b)^2]}\sqrt{k^2[k^2-4m_b^2]}}$$

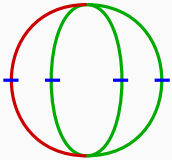
This is exactly the maximal cut of the on-shell ($q^2 = m_t^2$) two-loop sunrise diagram:



$$\sim \int \frac{\mathrm{d}z}{\sqrt{[z-(m_t-m_b)^2]\,[z-(m_t+m_b)^2]}\sqrt{z\,[z-4m_b^2]}}$$

Maximal cut analysis: three-loop example

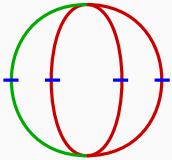
Example: Analyse one of the elliptic tadpole integral from the three-loop ρ parameter



$$\sim \int \frac{\mathrm{d}k^2}{\sqrt{[k^2-(m_t-m_b)^2]\,[k^2-(m_t+m_b)^2]}\sqrt{k^2[k^2-4m_b^2]}}$$

$\rightarrow$ "sunrise curve"

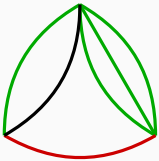
The $\mathsf{m_b} \leftrightarrow \mathsf{m_t}$ "twin" has a similar, but non-isomorphic curve:



$$\sim \int \frac{\mathrm{d}k^2}{\sqrt{[k^2-(m_t-m_b)^2]\,[k^2-(m_t+m_b)^2]}\sqrt{k^2[k^2-4m_t^2]}}$$

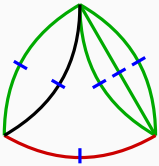
$\rightarrow$ "sunrise twin curve"

Maximal cut analysis: four-loop examples



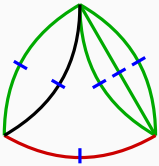
$$\sim \int \mathrm{d}^d k \, B_{b0}(k^2) \frac{1}{k^2 - m_t^2} S_{bbb}(k^2)$$

Maximal cut analysis: four-loop examples

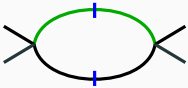


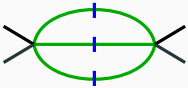
$$\sim \int \mathrm{d}k^2 \, \frac{\delta(k^2 - m_t^2)}{k^2 - m_b^2} \int \frac{\mathrm{d}z}{\sqrt{[z - (\sqrt{k^2} - m_b)^2]\,[z - (\sqrt{k^2} + m_b)^2]}\,\sqrt{z\,[z - 4m_b^2]}}$$

Maximal cut analysis: four-loop examples

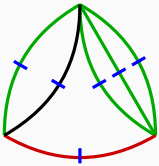


$$\sim \int \mathrm{d}k^2 \, \frac{\delta(k^2 - m_t^2)}{k^2 - m_b^2} \int \frac{\mathrm{d}z}{\sqrt{[z - (\sqrt{k^2} - m_b)^2]\,[z - (\sqrt{k^2} + m_b)^2]}\,\sqrt{z\,[z - 4m_b^2]}}$$





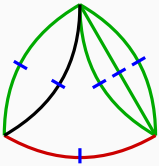
Maximal cut analysis: four-loop examples



$$\sim \frac{1}{m_t^2 - m_b^2} \int \frac{\mathrm{d}z}{\sqrt{[z-(m_t-m_b)^2]\,[z-(m_t+m_b)^2]}\sqrt{z\,[z-4m_b^2]}}$$

$\rightarrow$ “sunrise curve”

Maximal cut analysis: four-loop examples

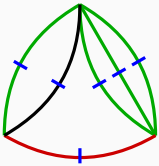


$$\sim \frac{1}{m_t^2 - m_b^2} \int \frac{\mathrm{d}z}{\sqrt{[z-(m_t-m_b)^2]\,[z-(m_t+m_b)^2]}\sqrt{z\,[z-4m_b^2]}}$$

$\rightarrow$ “sunrise curve”

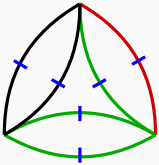
$$\sim \int \mathrm{d}^d k\, B_{00}(k^2)\, B_{bb}(k^2)\, B_{bt}(k^2)$$

Maximal cut analysis: four-loop examples



$$\sim \frac{1}{m_t^2 - m_b^2} \int \frac{\mathrm{d}z}{\sqrt{[z-(m_t-m_b)^2]\,[z-(m_t+m_b)^2]}\sqrt{z\,[z-4m_b^2]}}$$

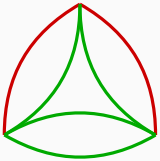
$\rightarrow$ “sunrise curve”



$$\sim \int \frac{\mathrm{d}z}{z\sqrt{[z-(m_t-m_b)^2]\,[z-(m_t+m_b)^2]}\sqrt{z\,[z-4m_b^2]}}$$

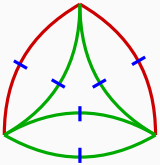
$\rightarrow$ “sunrise curve”

Maximal cut analysis: four-loop examples (cont.)



$$\sim \int \mathrm{d}^d k \, B_{bt}^2(k^2) \, B_{bb}(k^2)$$

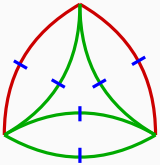
Maximal cut analysis: four-loop examples (cont.)



$$\sim \int \frac{\mathrm{d}z}{[z-(m_t-m_b)^2]\,[z-(m_t+m_b)^2]\sqrt{z\,[z-4m_b^2]}}$$

$\rightarrow$ algebraic solution, not elliptic!

Maximal cut analysis: four-loop examples (cont.)

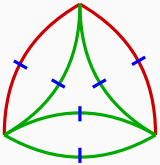


$$\sim \int \frac{\mathrm{d}z}{[z-(m_t-m_b)^2]\,[z-(m_t+m_b)^2]\sqrt{z\,[z-4m_b^2]}}$$

$\rightarrow$ algebraic solution, not elliptic!

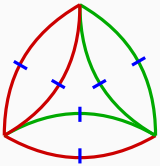
$$\sim \int \mathrm{d}^d k\, B_{tt}(k^2)\, B_{bb}(k^2)\, B_{bt}(k^2)$$

Maximal cut analysis: four-loop examples (cont.)



$$\sim \int \frac{\mathrm{d}z}{[z-(m_t-m_b)^2]\,[z-(m_t+m_b)^2]\sqrt{z\,[z-4m_b^2]}}$$

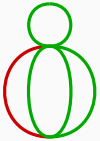
$\rightarrow$ algebraic solution, not elliptic!



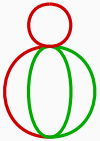
$$\sim \int \frac{\mathrm{d}z}{z\,\sqrt{[z-4m_t^2]}\sqrt{[z-4m_b^2]}\sqrt{[z-(m_t-m_b)^2]\,[z-(m_t+m_b)^2]}}$$

$\rightarrow$ “new curve”

Maximal cut analysis: results



→ Sunrise curve



→ Sunrise curve

→ Sunrise curve

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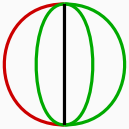
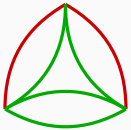


→ Sunrise curve

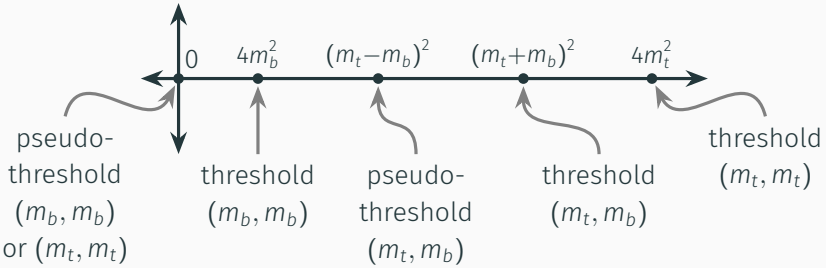
→ Sunrise curve



→ Algebraic, not elliptic

→ New curve

Interpretation of the branch points



- Maximal cut picture lends itself to physical interpretation of the branch points
- Each curve uses four of the five possible points

Sunrise curve: $y^2 = z\,[z - 4m_b^2]\,[z - (m_t - m_b)^2]\,[z - (m_t + m_b)^2]$

Sunrise twin curve: $y^2 = z\,[z - (m_t - m_b)^2]\,[z - (m_t + m_b)^2]\,[z - 4m_t^2]$

New curve: $y^2 = [z - 4m_b^2]\,[z - (m_t - m_b)^2]\,[z - (m_t + m_b)^2]\,[z - 4m_t^2]$

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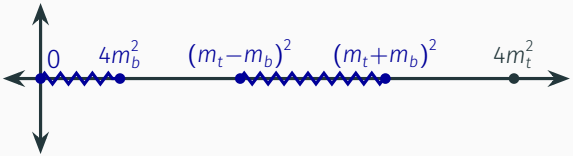

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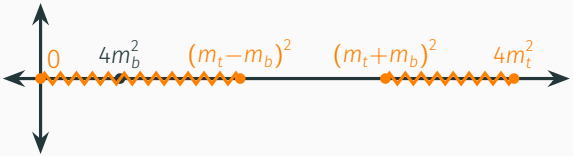

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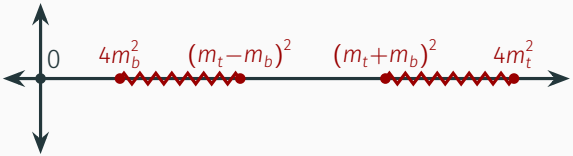
Interpretation of the branch points



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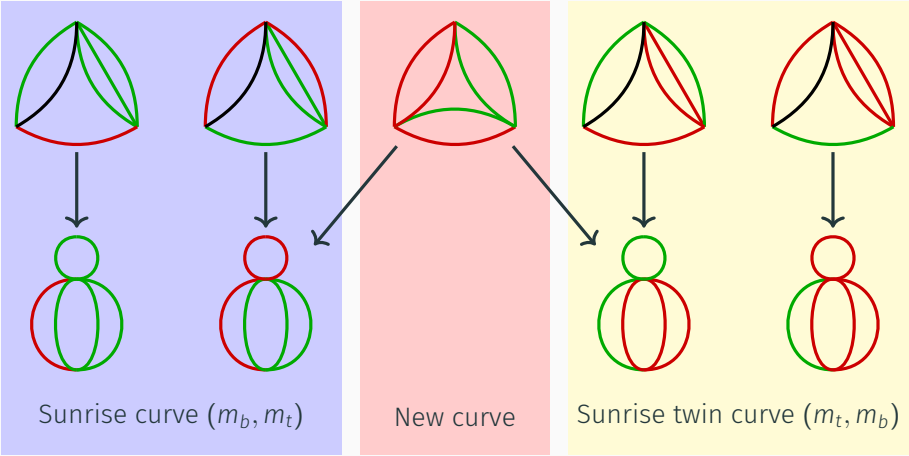
New curve:	$y^2 = [z - 4m_b^2]\,[z - (m_t - m_b)^2]\,[z - (m_t + m_b)^2]\,[z - 4m_t^2]$

Interaction of the elliptic curves

- Homogeneous differential equations only generate one elliptic curve per sector
- Inhomogeneous terms can mix different curves
  $\rightarrow$ also for sectors that are themselves elliptic



Conclusions and outlook

Conclusions

- We calculate the two-mass QCD contributions to the four-loop ρ parameter
- Interesting testing ground for special functions beyond GPLs
- We find three elliptic curves contributing to the integrals
- The three elliptic curves mix in the differential equation

Outlook

- Finish calculation of numerical value for four-loop corrections
- Find good description for expressing iterated integrals over these curves