

# *QCD splitting functions at four loops*

**Sven-Olaf Moch**

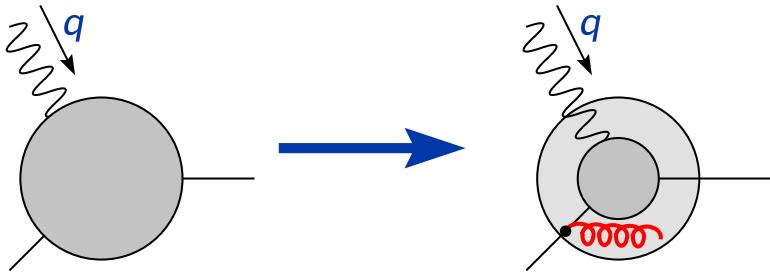
*Universität Hamburg*

## Based on work done in collaboration with:

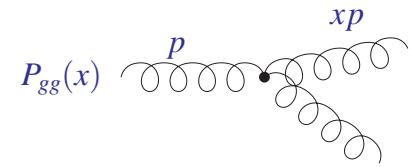
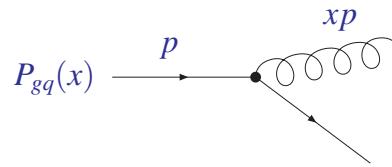
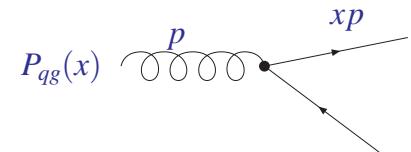
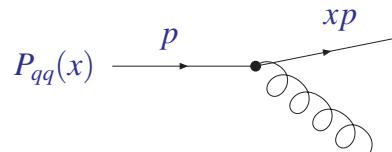
- *Four-loop splitting functions in QCD – The quark-quark case –*  
F. Herzog, G. Falcioni, S. M., and A. Vogt [arXiv:2302.07593](#)
- *Low moments of the four-loop splitting functions in QCD*  
S. M., B. Ruijl, T. Ueda, J. Vermaseren and A. Vogt [arXiv:2111.15561](#)
- *Five-loop contributions to low- $N$  non-singlet anomalous dimensions in QCD*  
F. Herzog, S. M., B. Ruijl, T. Ueda, J. Vermaseren and A. Vogt  
[arXiv:1812.11818](#)
- *On quartic colour factors in splitting functions and the gluon cusp anomalous dimension*  
S. M., B. Ruijl, T. Ueda, J. Vermaseren and A. Vogt [arXiv:1805.09638](#)
- *Four-Loop Non-Singlet Splitting Functions in the Planar Limit and Beyond*  
S. M., B. Ruijl, T. Ueda, J. Vermaseren and A. Vogt [arXiv:1707.08315](#)
- Many more papers of MVV and friends ... 2001 – ...

*QCD evolution at 1% precision*

# Parton evolution



- Feynman diagrams in leading order



- Proton in resolution  $1/Q$  → sensitive to lower momentum partons
- Evolution equations for parton distributions  $f_i$ 
  - predictions from fits to reference processes (universality)

$$\frac{d}{d \ln \mu^2} f_i(x, \mu^2) = \sum_j [P_{ij}(\alpha_s(\mu^2)) \otimes f_j(\mu^2)](x)$$

- Splitting functions  $P$  up to  $\text{N}^3\text{LO}$  (work in progress)

$$P_{ij} = \underbrace{\alpha_s P_{ij}^{(0)} + \alpha_s^2 P_{ij}^{(1)} + \alpha_s^3 P_{ij}^{(2)} + \alpha_s^4 P_{ij}^{(3)} + \dots}_{\text{NNLO: standard approximation}}$$

*Non-singlet*

# Operator matrix elements

- Quark operator of spin- $N$  and twist two

$$O_q^{\{\mu_1, \dots, \mu_N\}} = \bar{\psi} \gamma^{\{\mu_1} D^{\mu_2} \dots D^{\mu_N\}} \psi$$

- $N$  covariant derivatives

$$D_{\mu,ij} = \partial_\mu \delta_{ij} + ig_s (t_a)_{ij} A_\mu^a$$

sandwiched between quark fields  $\psi, \bar{\psi}$

- Evaluation of operators in matrix elements  $A_{qq}$  with external quark states

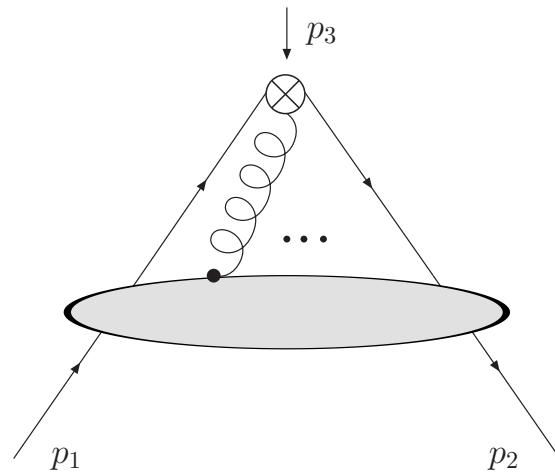
$$A_{qq}^{\{\mu_1, \dots, \mu_N\}} = \langle \psi(p_1) | O_q^{\{\mu_1, \dots, \mu_N\}}(p_3) | \bar{\psi}(p_2) \rangle$$

- Anomalous dimensions  $\gamma(\alpha_s, N)$  govern scale dependence of renormalized operators

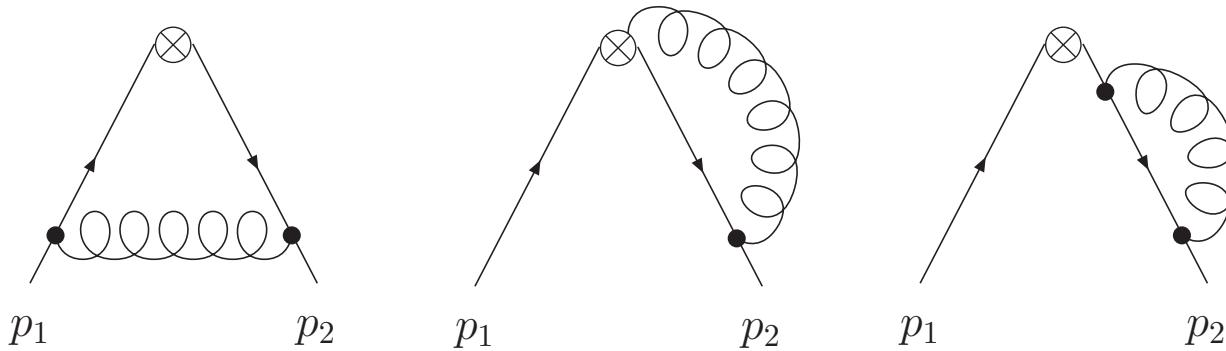
$$\frac{d}{d \ln \mu^2} O^{\text{ren}} = -\gamma O^{\text{ren}}$$

$$\gamma(N) = - \int_0^1 dx x^{N-1} P(x)$$

- Zero-momentum transfer through operator reduces problem to computation of propagator-type diagrams



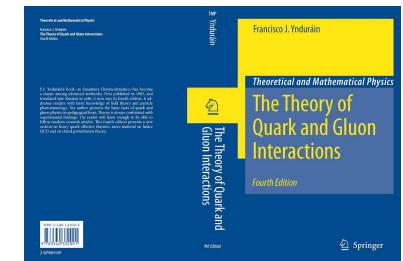
# One-loop computation



- Computation of loop integral in  $D = 4 - 2\epsilon$  dimensions and expansion in  $\epsilon$ 
  - anomalous dimension  $\gamma(N)$  from ultraviolet divergence

$$\begin{aligned} \Delta_{\mu_1} \dots \Delta_{\mu_N} \langle \psi(p_1) | O_q^{\{\mu_1, \dots, \mu_N\}}(0) | \bar{\psi}(-p_1) \rangle &= \\ &= 1 + \frac{\alpha_s}{4\pi} \frac{1}{\epsilon} \gamma^{(0)}(N) + \mathcal{O}(\alpha_s \epsilon^0) + \mathcal{O}(\alpha_s^2) \\ &= 1 + \frac{\alpha_s}{4\pi} \frac{1}{\epsilon} \left\{ C_F \left( 4S_1(N) + \frac{2}{N+1} - \frac{2}{N} - 3 \right) \right\} + \mathcal{O}(\alpha_s \epsilon^0) + \mathcal{O}(\alpha_s^2) \end{aligned}$$

- One-loop result with harmonic sum  $S_1(N) = \sum_{i=1}^N \frac{1}{i}$
- Details in *The Theory of Quark and Gluon Interactions*  
F.J. Yndurain

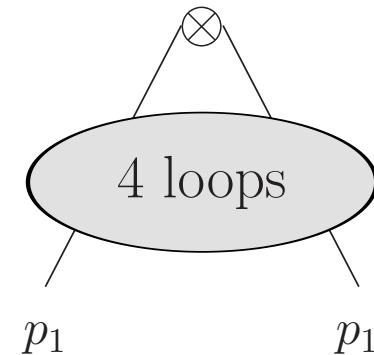


# Four-loop computation

- Feynman diagrams for operator matrix elements generated up to four loops with **Qgraf** Nogueira ‘91
- Parametric reduction of four-loop massless propagator diagrams with integration-by-parts identities encoded in **Forcer** Ruijl, Ueda, Vermaseren ‘17
- Symbolic manipulations with **Form** Vermaseren ‘00; Kuipers, Ueda, Vermaseren, Vollinga ‘12 and multi-threaded version **TForm** Tentyukov, Vermaseren ‘07
- Diagrams of same topology and color factor combined to meta diagrams
- Non-singlet anomalous dimension
  - 1 one-, 7 two-, 53 three- and 650 four-loop meta diagrams for  $\gamma_{\text{ns}}^{\pm}$
  - 1 three- and 29 four-loop meta diagrams for  $\gamma_{\text{ns}}^{\text{s}}$

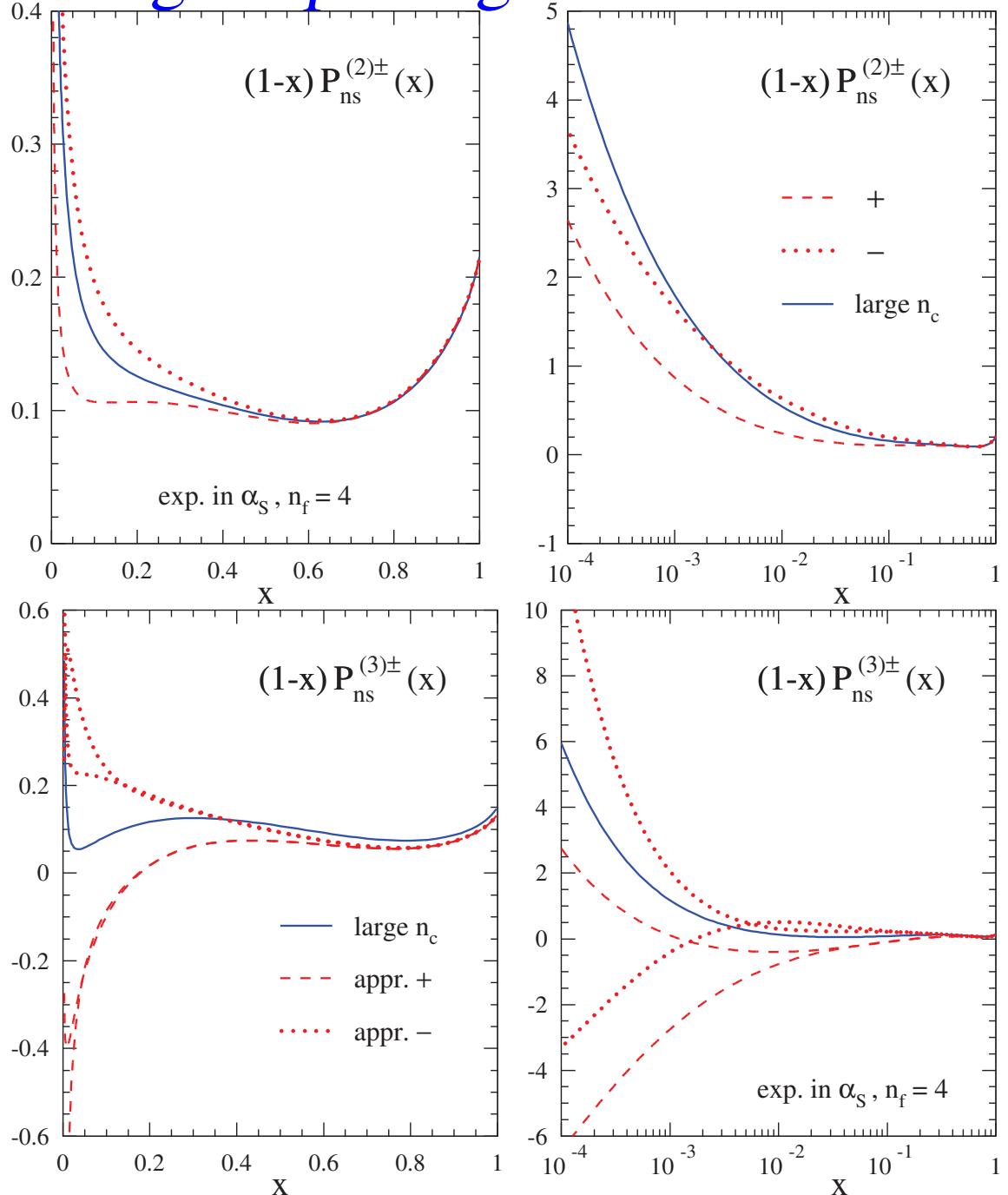
## Fixed Mellin moments

- Computation of anomalous dimensions  $\gamma(N)$  for Mellin moments mostly up to  $N = 18$ 
  - sometimes higher for complicated topologies ( $N = 19, N = 20, \dots$ )
  - much higher for “easy” topologies, e.g.,  $n_f$ -dependent ( $N \simeq 80, \dots$ )



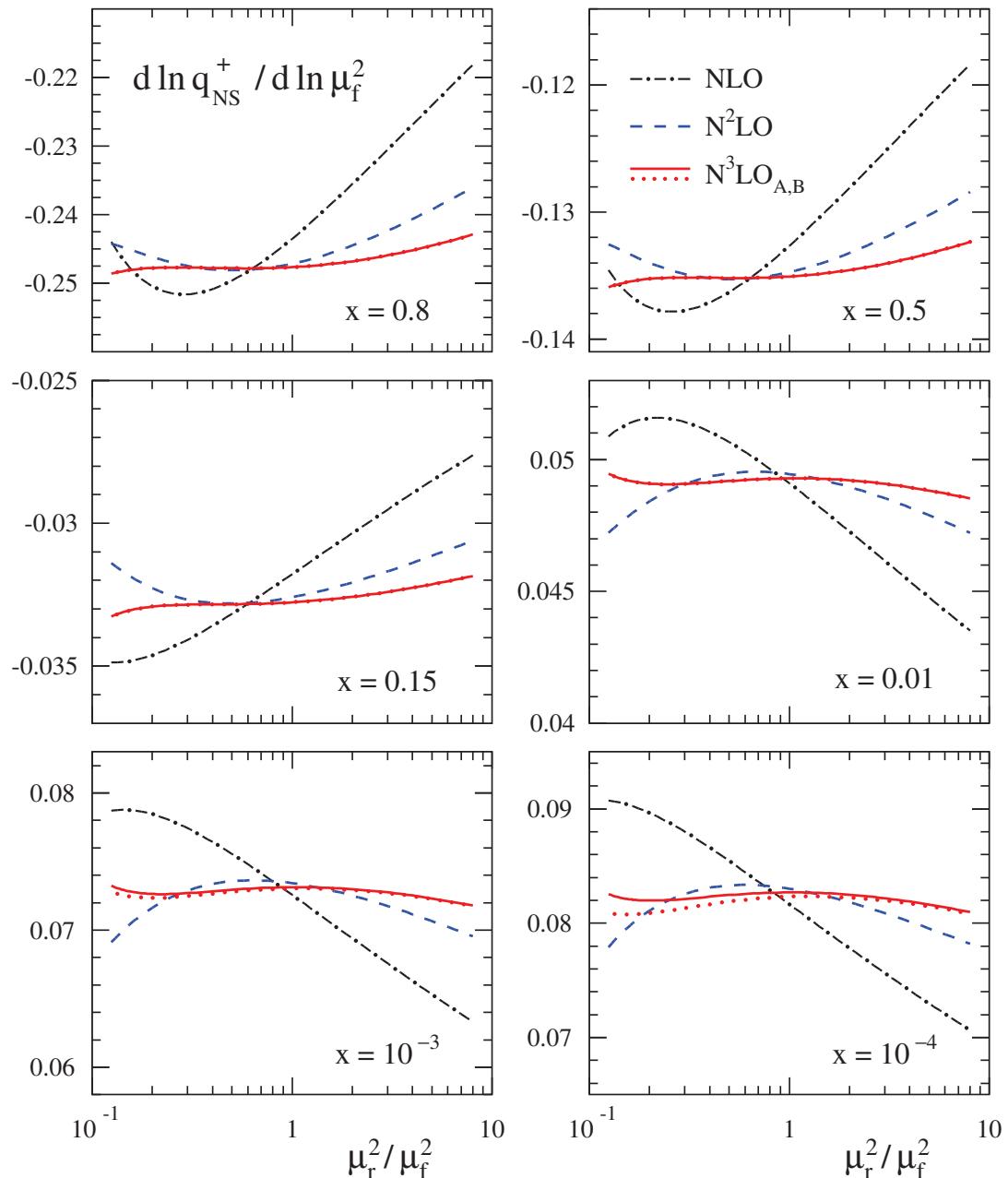
# Four-loop non-singlet splitting functions

- Top:  
three-loop  $P_{\text{ns}}^{(2)\pm}(x)$   
and large- $n_c$  limit  
with  $n_f = 4$
- Bottom:  
four-loop  $P_{\text{ns}}^{(3)\pm}(x)$   
and uncertainty bands  
beyond large- $n_c$  limit  
with  $n_f = 4$



# Scale stability of evolution

- Renormalization scale dependence of evolution kernel  $d \ln q_{\text{ns}}^+ / d \ln \mu_f^2$ 
  - non-singlet shape  
 $xq_{\text{ns}}^+(x, \mu_0^2) = x^{0.5}(1-x)^3$
- NLO, NNLO and N<sup>3</sup>LO predictions
  - remaining uncertainty of four-loop splitting function  $P_{\text{ns}}^{(3)+}$  almost invisible

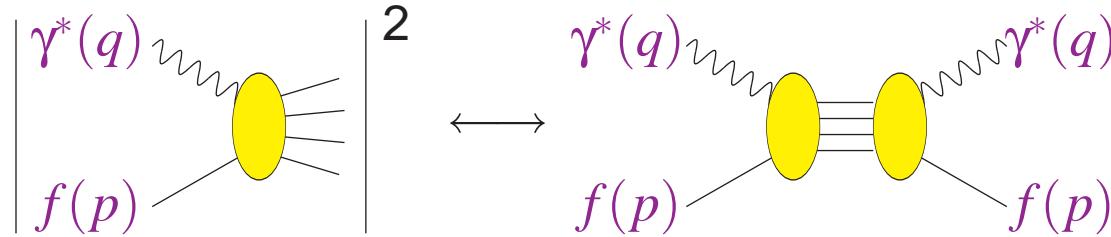


*Singlet*

# Operator product expansion (I)

## Optical theorem

- Total cross section related to imaginary part of Compton amplitude
  - momentum transfer  $Q^2 = -q^2$
  - Bjorken variable  $x = Q^2/(2p \cdot q)$



- Optical theorem relates hadronic tensor  $W_{\mu\nu}$  to imaginary part of Compton amplitude  $T_{\mu\nu}$

$$W_{\mu\nu} = e_{\mu\nu} \frac{1}{2x} F_L(x, Q^2) + d_{\mu\nu} \frac{1}{2x} F_2(x, Q^2) + i\epsilon_{\mu\nu\alpha\beta} \frac{p^\alpha q^\beta}{p \cdot q} F_3(x, Q^2)$$

- OPE of  $T_{\mu\nu}$  for short distances  $z^2 \simeq 0$  in Bjorken limit  $Q^2 \rightarrow \infty$ ,  $x$  fixed  
Wilson '72; Christ, Hasslacher, Mueller '72

$$T_{\mu\nu} = i \int d^4 z e^{iq \cdot z} \langle P | T \left( j_\mu^\dagger(z) j_\nu(0) \right) | P \rangle$$

# Operator product expansion (II)

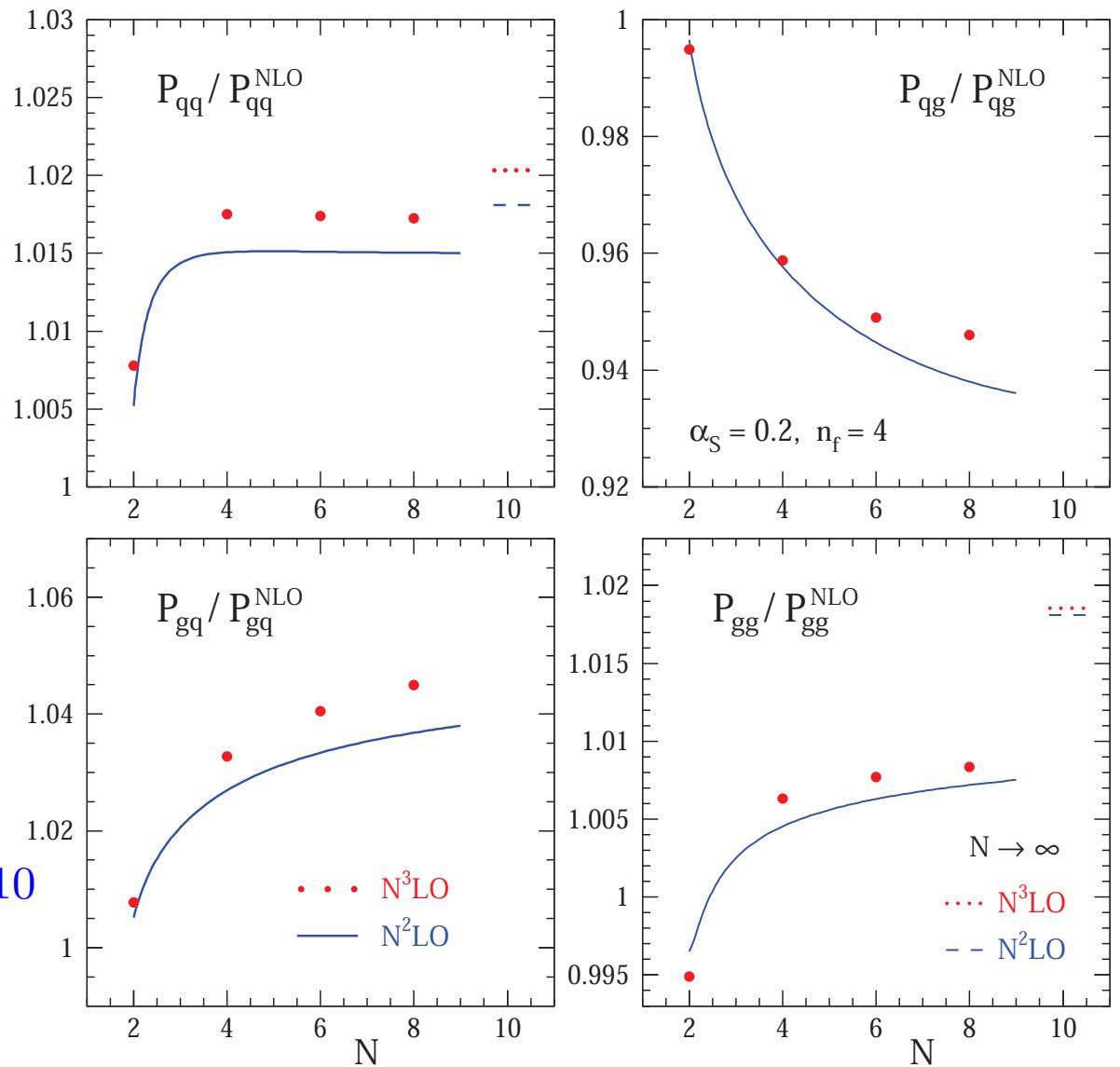
- Operator product expansion with coefficient functions in Mellin space  $C_{a,i}^N$

$$T_{\mu\nu} = \sum_{N,j} \left( \frac{1}{2x} \right)^N \left[ e_{\mu\nu} C_{L,j}^N \left( \frac{Q^2}{\mu^2}, \alpha_s \right) + d_{\mu\nu} C_{2,j}^N \left( \frac{Q^2}{\mu^2}, \alpha_s \right) + i \epsilon_{\mu\nu\alpha\beta} \frac{p^\alpha q^\beta}{p \cdot q} C_{3,j}^N \left( \frac{Q^2}{\mu^2}, \alpha_s \right) \right] A_{P,N}^j(\mu^2) + \text{higher twists}$$

- Operator matrix elements  $A_{P,N}^i = \langle P | O_i^N | P \rangle$  in nucleon state
  - leading twist physical partonic operators, e.g., quark operator  
 $O_q^{\{\mu_1, \dots, \mu_N\}} = \bar{\psi} \gamma^{\{\mu_1} D^{\mu_2} \dots D^{\mu_N\}} \psi$
- Compton amplitude  $T_{\mu\nu}$  for parton states yields coefficient functions and anomalous dimensions  $\gamma(\alpha_s, N)$  after mass factorization
  - established computational approach through four loops  
 one loop Buras '80; two loops Kazakov, Kotikov '90; S.M., Vermaseren '99 three loops S.M., Vermaseren, Vogt '04; four loops Davies, Vogt, Ruijl, Ueda, Vermaseren '17; S.M., Ruijl, Ueda, Vermaseren, Vogt to appear
  - photon-DIS  $\rightarrow \gamma_{qq}, \gamma_{qg}$ , Higgs (scalar)-DIS  $\rightarrow \gamma_{gq}, \gamma_{gg}$
  - graviton-DIS  $\rightarrow \Delta \gamma_{ij}$  (polarized quantities) S.M., Vermaseren, Vogt '14

# Four-loop singlet Mellin moments

- Singlet moments at NNLO (lines) and  $N^3\text{LO}$  (even- $N$  points) normalized to NLO results  
S. M., Ruijl, Ueda, Vermaseren, Vogt '21
  - $\alpha_s(\mu_f) = 0.2$  and  $n_f = 4$
- Large- $N$  limits in  $qq$ - and  $gg$ -channel
- Now moments up to  $N = 10$  to appear



# Operator matrix elements

- Scalar singlet operators of spin- $N$  and twist two from contraction with light-like vector  $\Delta_\mu$

$$O_q = \bar{\psi} \Delta D^{N-2} \psi$$

$$O_g = F_\nu^a D_{ab}^{N-2} F^{\nu;b}$$

- notation

$$F^{\mu;a} = \Delta_\nu F^{\mu\nu;a}, \quad A^a = \Delta_\mu A^{\mu;a},$$

$$D = \Delta_\mu D^\mu, \quad \partial = \Delta_\mu \partial^\mu$$

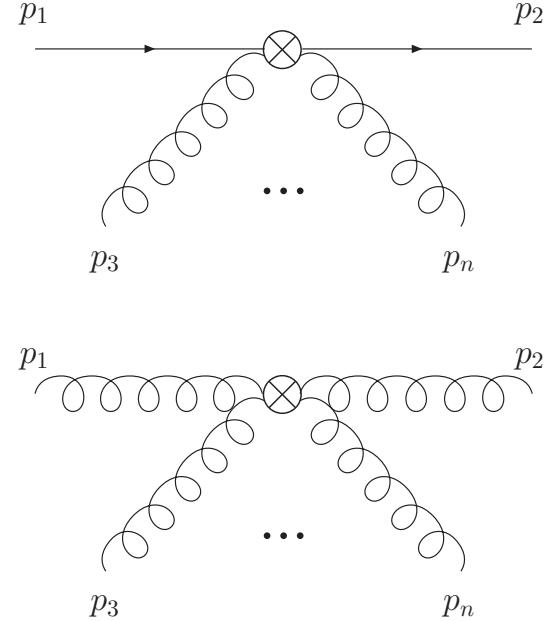
- Physical operators mix under renormalization with other operators (equation-of-motion operators and ghost operators)

- classic work on general theory of renormalization

Dixon, Taylor '74; Kluberg-Stern, Zuber '75; Joglekar, Lee '76

- Applications in QCD

- at two loops Floratos, Ross, Sachrajda '79 and with correct renormalization of gluon operator in covariant gauge Hamberg, van Neerven '92; Matiounine, Smith, van Neerven '98; Blümlein, Marquard, Schneider, Schönwald '22
- at three loops Gehrmann, von Manteuffel, Yang '23
- general procedure (based on BRST invariance) Falcioni, Herzog '22



# Alien operators

- Sets of alien operators  $O_A^i = O_q^i + O_g^i + O_c^i$  with  $i = I, II, \dots$

$$O_q^I = \eta g \bar{\psi} \Delta t^a \psi \left( \partial^{N-2} A_a \right) ,$$

$$O_g^I = \eta (D.F)^a \left( \partial^{N-2} A_a \right) ,$$

$$O_c^I = -\eta (\partial \bar{c}^a) \left( \partial^{N-1} c_a \right) ,$$

$$O_q^{II} = g^2 \bar{\psi} \Delta t_a \psi \sum_{i+j=N-3} \kappa_{ij} f^{abc} \left( \partial^i A_b \right) \left( \partial^j A_c \right) ,$$

$$O_g^{II} = g (D.F)_a \sum_{i+j=N-3} \kappa_{ij} f^{abc} \left( \partial^i A_b \right) \left( \partial^j A_c \right) ,$$

$$O_c^{II} = -g \sum_{i+j=N-3} \eta_{ij} f^{abc} (\partial \bar{c}_a) \left( \partial^i A_b \right) \left( \partial^{j+1} c \right)$$

- Class  $O_A^I$  alien operators with coupling  $\eta$  (function of  $N$  and  $\alpha_s$ )
- Class  $O_A^{II}$  alien operators with couplings  $\eta_{ij}$  and  $\kappa_{ij}$ 
  - constraints from (anti-)BRST symmetry [Falcioni, Herzog '22](#)
- Class  $O_A^{III}, O_A^{IV},$  etc alien operators with additional polynomials  $(\partial^i A_a)$

# Renormalization

- Physical operators  $O_q$  and  $O_g$
- Alien operators  $O_A^i = O_q^i + O_g^i + O_c^i$ 
  - gluon and quark equation-of-motion operators and ghost operators

$$\begin{pmatrix} O_q \\ O_g \\ O_A \end{pmatrix} = \begin{pmatrix} Z_{qq} & Z_{qg} & Z_{qA} \\ Z_{gq} & Z_{gg} & Z_{gA} \\ Z_{Aq} & Z_{Ag} & Z_{AA} \end{pmatrix} \begin{pmatrix} O_q^{\text{ren}} \\ O_g^{\text{ren}} \\ O_A^{\text{ren}} \end{pmatrix}$$

- Multiplicative renormalization of operators ( $Z$ -factors)

$$\mu^2 \frac{d}{d\mu^2} Z_{ij} = \left( \beta(\alpha_s) \frac{\partial}{\partial \alpha_s} + \gamma_3 \xi \frac{\partial}{\partial \xi} \right) Z_{ij} = -\gamma_{ik} Z_{kj}$$

- gauge parameter  $\xi$  with  $\xi = 1$  for Feynman gauge
- QCD  $\beta$ -function and gluon anomalous dimension  $\gamma_3$  (known to five loops) Baikov, Chetyrkin, Kühn '17; Herzog, Ruijl, Ueda, Vermaseren, Vogt '17; Luthe, Maier, Marquard, Schröder '17; Chetyrkin, Falcioni, Herzog, Vermaseren '17
- $Z_{ij}$  involving alien operators can be gauge dependent
- $Z_{Aq}$  and  $Z_{Ag}$  have to vanish  
(alien operators cannot mix into set of physical operators)

# Four-loop results

- Moments  $N = 2, \dots, 20$  for pure-singlet anomalous dimension  $\gamma_{\text{ps}}^{(3)}(N)$

$$\gamma_{\text{ps}}^{(3)}(N=2) = -691.5937093 n_f + 84.77398149 n_f^2 + 4.466956849 n_f^3,$$

$$\gamma_{\text{ps}}^{(3)}(N=4) = -109.3302335 n_f + 8.776885259 n_f^2 + 0.306077137 n_f^3,$$

$$\gamma_{\text{ps}}^{(3)}(N=6) = -46.03061374 n_f + 4.744075766 n_f^2 + 0.042548957 n_f^3,$$

$$\gamma_{\text{ps}}^{(3)}(N=8) = -24.01455020 n_f + 3.235193483 n_f^2 - 0.007889256 n_f^3,$$

...

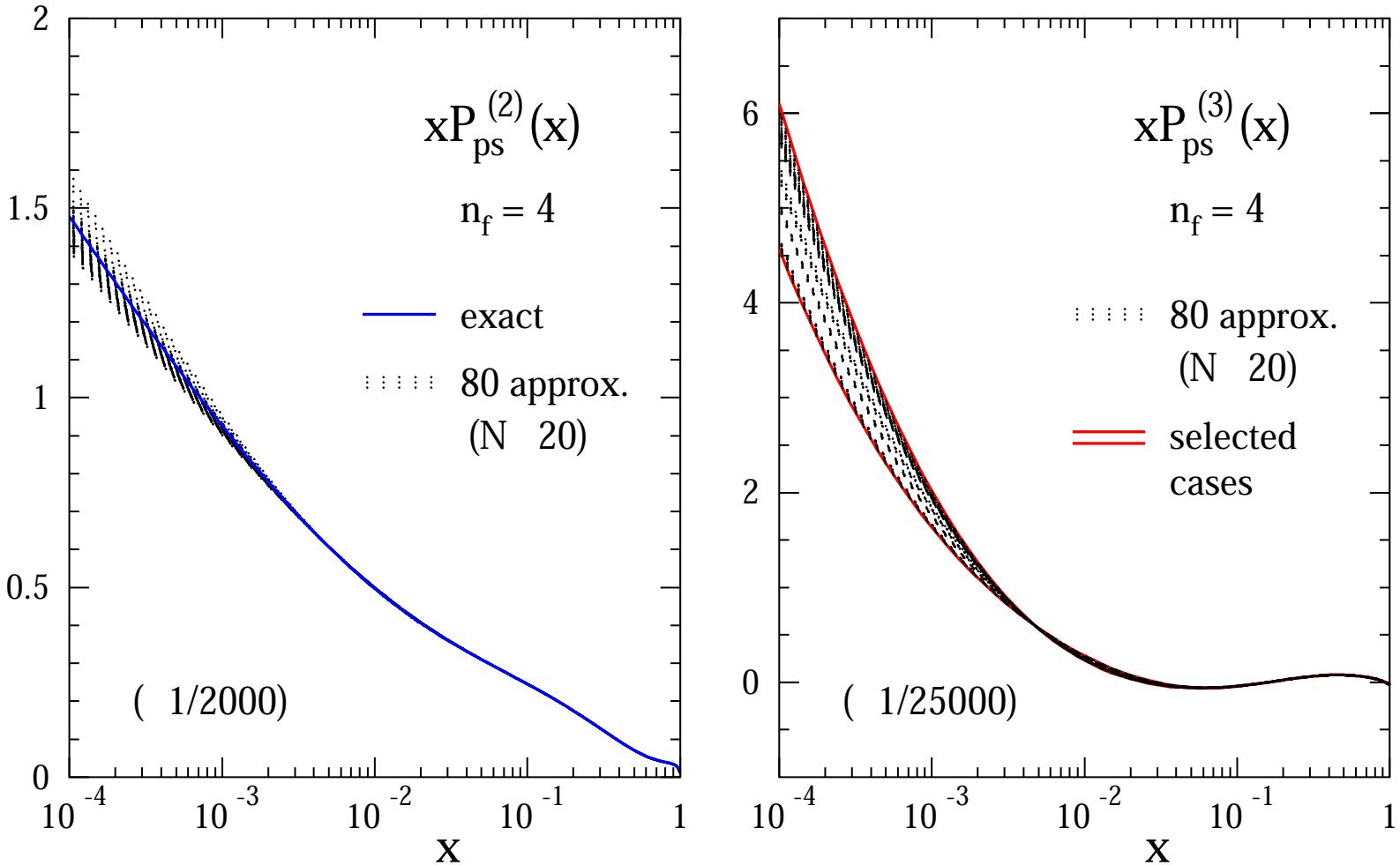
$$\gamma_{\text{ps}}^{(3)}(N=20) = -0.442681568 n_f + 0.805745333 n_f^2 - 0.020918264 n_f^3.$$

- Results  $N \leq 8$  agree with inclusive DIS S.M., Ruijl, Ueda, Vermaseren, Vogt '21 (also for  $N = 10$  and  $N = 12$ )
- Quartic color terms  $d_R^{abcd} d_R^{abcd}$  agree with S.M., Ruijl, Ueda, Vermaseren, Vogt '18
- Large- $n_f$  parts agree with all- $N$  results Davies, Vogt, Ruijl, Ueda, Vermaseren '17;
- $\zeta_4$  terms in  $\gamma_{\text{ps}}^{(3)}(N)$  agree with Davies, Vogt '17 based on no- $\pi^2$  theorem Jamin, Miravilllas '18; Baikov, Chetyrkin '18
- Renormalization constants involving alien operators (required to three loops) agree with Gehrmann, von Manteuffel, Yang '23

# Approximations in $x$ -space

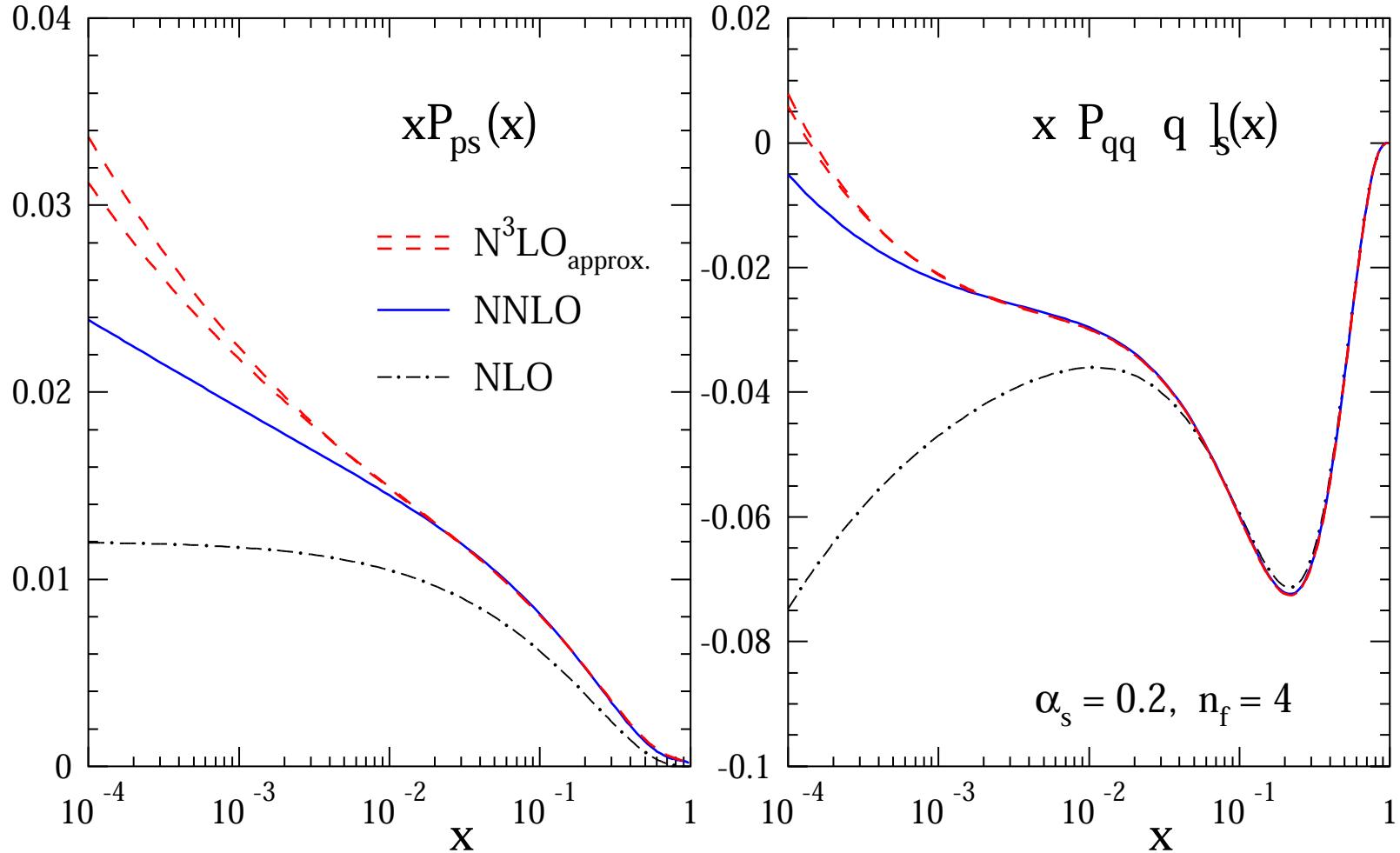
- Large- and small- $x$  information about four-loop splitting function  $P_{\text{ps}}^{(3)}(x)$ 
  - leading logarithm  $(\ln^2 x)/x$  Catani, Hautmann '94
  - sub-dominant logarithms  $\ln^k x$  with  $k = 6, 5, 4$  Davies, Kom, S.M., Vogt '22
  - leading large- $x$  terms  $(1 - x)^j \ln^k(1 - x)$  with  $j \geq 1$  and  $k \leq 4$  with  $k = 4, 3$  known Soar, S.M., Vermaseren, Vogt '09
- Approximation of four-loop splitting function  $P_{\text{ps}}^{(3)}(x)$  with suitable ansatz
  - unknown leading small- $x$  terms:  $(\ln x)/x, 1/x$
  - unknown sub-dominant logarithms:  $\ln^k x$  with  $k = 3, 2, 1$
  - two remaining large- $x$  terms  $(1 - x) \ln^k(1 - x)$  with  $k = 2, 1$
  - different two-parameter polynomials together one function (dilogarithm  $\text{Li}_2(x)$  or  $\ln^k(1 - x)$  with  $k = 2, 1$ , suppressed as  $x \rightarrow 1$ )

# Pure-singlet splitting function



- Approximations to pure-singlet splitting function  $P_{ps}^{(n)}(x)$  at  $n_f = 4$  with 80 trial functions
  - left: three-loops ( $n = 2$ ) with comparison to known result
  - right: three-loops ( $n = 3$ ) with remaining uncertainty

# Pure-singlet splitting function



- Left: NLO, NNLO and  $N^3LO$  approximations for  $P_{ps}(x)$  with  $\alpha_s = 0.2$  fixed and  $n_f = 4$
- Right: Contribution to evolution kernel  $d \ln q_s / d \ln \mu_f^2$  at NLO, NNLO and  $N^3LO$  for typical quark-singlet shape

$$xq_s(x, \mu_0^2) = 0.6 x^{-0.3} (1-x)^{3.5} (1 + 5.0 x^{0.8})$$

# Analytic reconstruction (I)

- Mellin moments suffice for reconstruction of analytic all- $N$  expression for parts of  $\gamma_{\text{ps}}^{(3)}(N)$ 
  - harmonic sums and Riemann  $\zeta_n$  terms up to total weight  $w = 7$
- Terms proportional to  $\zeta_5$  are particularly simple
  - $N$ -dependent terms respect reciprocity relation (invariance under mapping  $N \rightarrow -N - 1$ )
  - functions  $\eta = \frac{1}{N} - \frac{1}{N+1}$  and  $\nu = \frac{1}{N-1} - \frac{1}{N+2}$

$$\begin{aligned}\left. \gamma_{\text{ps}}^{(3)}(N) \right|_{\zeta_5} &= 160 n_f C_F^3 \left( 9\eta + 6\eta^2 - 4\nu \right) + 80/3 n_f C_A C_F^2 \left( -9\eta - 6\eta^2 + 4\nu \right) \\ &\quad + 40/9 n_f C_A^2 C_F \left( -1 - 214\eta - 144\eta^2 + 104\nu \right) \\ &\quad + 320/3 n_f \frac{d_R^{abcd} d_R^{abcd}}{n_c} \left( -1 + 56\eta + 36\eta^2 - 16\nu \right)\end{aligned}$$

- Inverse Mellin transformation generates additional terms with  $\zeta_n$ 
  - $\zeta_n$  in  $N$ -space  $\neq \zeta_n$  in  $x$ -space

# Analytic reconstruction (II)

- Moments  $N = 2, \dots, 16$  with quartic Casimir terms at four loops known for all singlet anomalous dimensions  $\gamma_{\text{qq}}, \gamma_{\text{qg}}, \gamma_{\text{gq}}$  and  $\gamma_{\text{gg}}$   
S.M., Ruijl, Ueda, Vermaseren, Vogt '18
  - now also  $N = 18, \dots, 22$  to appear
- Quartic Casimir terms at four loops are effectively ‘leading-order’
  - $d_{xy}^{(4)} \equiv d_x^{abcd} d_y^{abcd}$  for representations labels  $x, y$  with generators  $T_r^a$   
$$d_r^{abcd} = \frac{1}{6} \text{Tr} ( T_r^a T_r^b T_r^c T_r^d + \text{five } bcd \text{ permutations} )$$
  - anomalous dimensions fulfil relation for  $\mathcal{N} = 1$  supersymmetry  
$$\gamma_{\text{qq}}^{(3)}(N) + \gamma_{\text{gq}}^{(3)}(N) - \gamma_{\text{qg}}^{(3)}(N) - \gamma_{\text{gg}}^{(3)}(N) \stackrel{Q}{=} 0$$
- Eigenvalues of singlet splitting functions (conjectured to be) composed of reciprocity-respecting sums
  - quartic Casimir terms fulfil stronger condition Belitsky, Müller, Schäfer '99  
$$\gamma_{\text{qg}}^{(0)}(N) \gamma_{\text{gq}}^{(3)}(N) \stackrel{Q}{=} \gamma_{\text{gq}}^{(0)}(N) \gamma_{\text{qg}}^{(3)}(N)$$

# Analytic reconstruction (III)

- Reconstruction of analytic all- $N$  expressions for  $\zeta_5$  terms from solution of Diophantine equations

- example for  $\gamma_{gg}^{(3)}$  with  $\eta = \frac{1}{N} - \frac{1}{N+1}$  and  $\nu = \frac{1}{N-1} - \frac{1}{N+2}$

$$\left. \gamma_{gg}^{(3)}(N) \right|_{\zeta_5 d_{AA}^{(4)}/n_A} = \frac{64}{3} \left( 30 (12\eta^2 - 4\nu^2 - S_1(4S_1 + 8\eta - 8\nu - 11) - 7\nu) + 188\eta - \frac{751}{3} - \frac{1}{6} N(N+1) \right)$$

- Recall large- $N$  limit of anomalous dimensions

$$\left. \gamma_{ii}^{(k)}(N) \right|_{N \rightarrow \infty} = A_{n,i} \ln(N) + \mathcal{O}(\text{const}_N)$$

- Terms  $S_1(N)^2 \sim \ln(N)^2$  and  $N(N+1)$  proportional to  $\zeta_5$  must be compensated in large- $N$  limit

# Universal anomalous dimension

- Universal anomalous dimension  $\gamma_{\text{uni}}$  in  $N = 4$  SYM to three loops  
Kotikov, Lipatov, Onishchenko, Velizhanin '04
  - One-loop example:  $\gamma_{\text{uni}}^{(0)}(N) = 4n_c S_1$  emerges from
$$\gamma_{\text{qq}}^{(0)}(N) = C_F \left( -3 + 2 \frac{1}{N+1} - 2 \frac{1}{N} + 4S_1 \right) \quad \text{or}$$
$$\gamma_{\text{gg}}^{(0)}(N) = C_A \left( -\frac{11}{3} - \frac{4}{N-1} - \frac{4}{N+1} + \frac{4}{N+2} + \frac{4}{N} + 4S_1 \right) + \frac{2}{3} n_f$$
- Starting at four loops wrapping corrections to complement asymptotic Bethe ansatz
  - four-loop Bajnok, Janik, Lukowski '08, five-loop Lukowski, Rej, Velizhanin '09, six-loop [...], ...
  - $\gamma(N)^{\text{wrap},(4)} \simeq S_1(N)^2 f^{\text{wrap}}(N)$ 
$$f^{\text{wrap}}(N) = 5\zeta_5 - 2S_{-5} + 4S_{-2}\zeta_3 - 4S_{-2,-3} + 8S_{-2,-2,1} + 4S_{3,-2} - 4S_{4,1} + 2S_5$$
- Three-loop Wilson coefficient  $c_{\text{ns}}^{(3)}(N)$  S.M., Vermaseren, Vogt '05
  - $c_{\text{ns}}^{(3)}(N) \simeq C_F \left( C_F - \frac{C_A}{2} \right)^2 \{ N(N+1) f^{\text{wrap}}(N) \}$
- Non-planar part of  $\gamma_{\text{uni}}$  in  $N = 4$  SYM at four loops Kniehl, Velizhanin '21

# *Outlook on new four-loop results*

- Moments  $N = 2, \dots, 20$  for quark-gluon anomalous dimension  $\gamma_{\text{qg}}^{(3)}(N)$  to appear

$$\gamma_{\text{qg}}^{(3)}(N=2) = -654.4627782 n_f + 245.6106197 n_f^2 - 0.924990969 n_f^3,$$

$$\gamma_{\text{qg}}^{(3)}(N=4) = 290.3110686 n_f - 76.51672403 n_f^2 - 4.911625629 n_f^3,$$

$$\gamma_{\text{qg}}^{(3)}(N=6) = 335.8008046 n_f - 124.5710225 n_f^2 - 4.193871425 n_f^3,$$

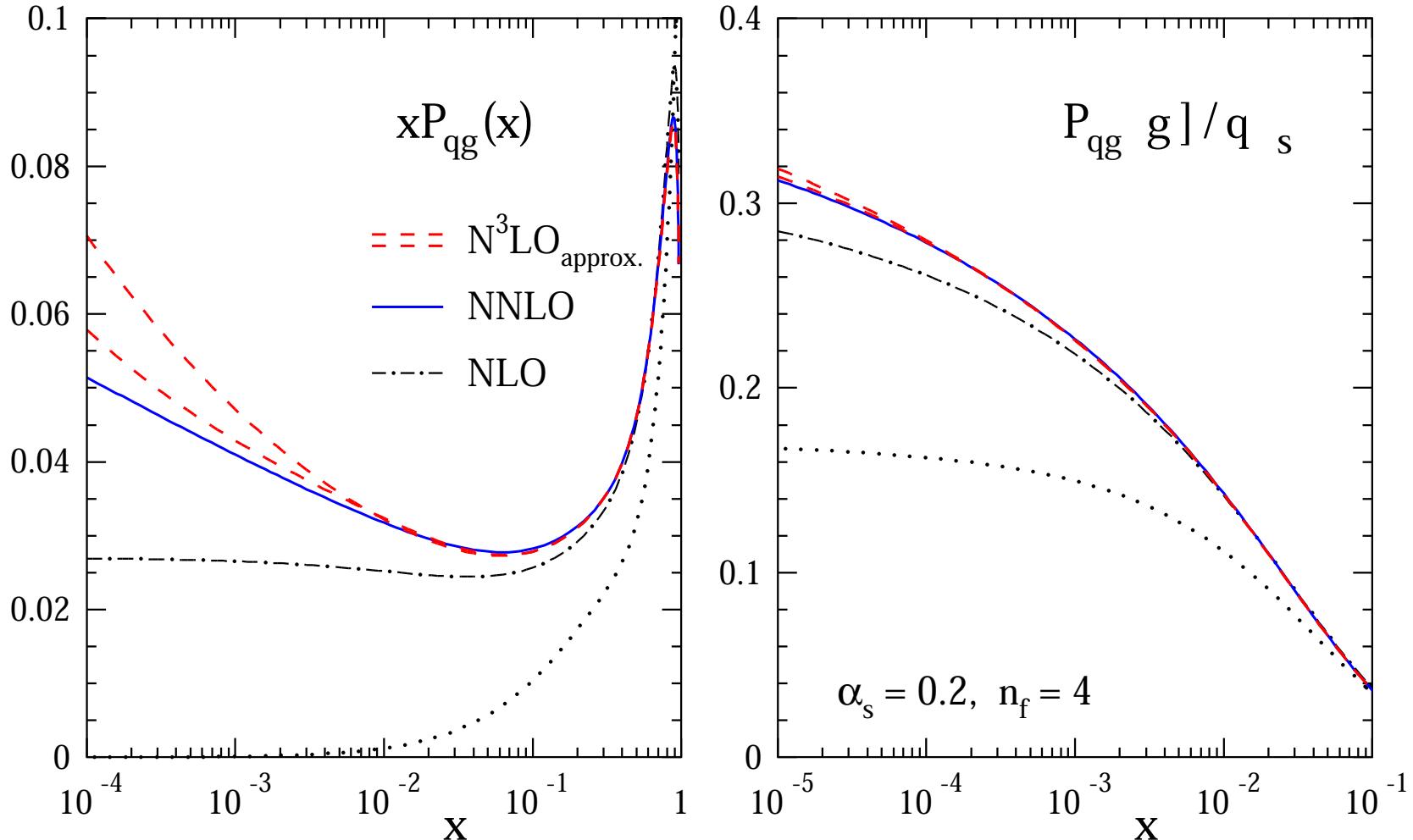
$$\gamma_{\text{qg}}^{(3)}(N=8) = 294.5876830 n_f - 135.3767647 n_f^2 - 3.609775642 n_f^3,$$

...

$$\gamma_{\text{qg}}^{(3)}(N=20) = 52.24329555 n_f - 109.3424891 n_f^2 - 2.153153725 n_f^3.$$

- Approximation of four-loop splitting function  $P_{\text{qg}}^{(3)}(x)$  again with known large- and small- $x$  information and suitable ansatz

# Quark-gluon splitting function



- Left: NLO, NNLO and  $N^3LO$  approximations for  $P_{qg}(x)$  with  $\alpha_s = 0.2$  fixed and  $n_f = 4$
- Right: Contribution to evolution kernel  $d \ln q_s / d \ln \mu_f^2$  at NLO, NNLO and  $N^3LO$  for typical gluon shape

$$xg(x, \mu_0^2) = 1.6 x^{-0.3} (1-x)^{4.5} (1 - 0.6 x^{0.3})$$

# Summary

- Experimental precision of  $\lesssim 1\%$  motivates computations at higher order in perturbative QCD
  - theoretical predictions at NNLO in QCD nowadays standard
- Push for theory results at  $N^3\text{LO}$  and  $N^4\text{LO}$ 
  - evolution equations expected to achieve percent-level
  - massive use of computer algebra
- Four-loop splitting functions approximated from moments  $N = 2, \dots, 20$ 
  - residual uncertainties negligible in wide kinematic range of  $x$  probed at current and future colliders
  - $P_{\text{qq}} = P_{\text{ns}}^+ + P_{\text{ps}}$  and  $P_{\text{qg}}$  done
  - $P_{\text{gq}}$  and  $P_{\text{gg}}$  to come
- Novel structural insights into QCD from integrability and conformal symmetry
  - Key parts of QCD inherited from  $N = 4$  Super Yang-Mills theory
  - Conformal symmetry in parts of QCD evolution equations