Coupling versus Kinematics in a General Spontaneously Broken Gauge Theory

Based on ongoing work with Henrik Johansson, and 2204.13119 with Da Liu

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RADCOR, May 30, 2023

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Scattering amplitudes

The modern scattering amplitudes program

- Constructing amplitudes in purely on-shell ways
- Studying the properties of on-shell amplitudes

Why?

- Avoid redundancies of a local formulation, including EoM, field redefinitions, gauge invariance etc.
- Unveil properties of amplitudes obscured in a local formulation, e.g. the color-kinematics duality

The color-kinematics duality

Consider Yang-Mills

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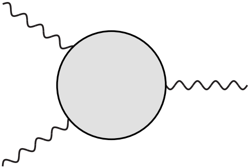
- Only controlled by a single coupling parameter (tensor): f^{abc}



$$\mathcal{A}(1^-2^-3^+) \quad = \quad f_{abc}\frac{\langle 12\rangle^3}{\langle 13\rangle\langle 23\rangle},$$

Spinor-helicity variables:

- Taking massless momentum p_μ:

$$p_{\alpha\dot\alpha} = p_\mu \sigma^\mu_{\alpha\dot\alpha}$$

- $\det p = 0$, thus $p_{\alpha\dot\alpha} = \lambda_\alpha \tilde\lambda_{\dot\alpha} = (|\lambda\rangle[\tilde\lambda|)_{\alpha\dot\alpha}$
- $\langle 12\rangle = \lambda_{1\alpha}\lambda_{2\beta}\varepsilon^{\alpha\beta}$, $[12] = \tilde\lambda_{1\dot\alpha}\tilde\lambda_{2\dot\beta}\varepsilon^{\dot\alpha\dot\beta}$

The color-kinematics duality

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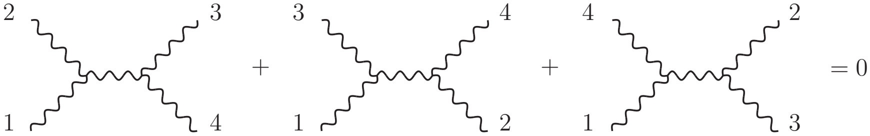
- At 4-pt, 3 factorization channels
- One can write

$$\mathcal{A}_4 = \sum_{I \in \{s,t,u\}} \frac{\mathsf{c}_I \; n_I}{d_I},$$

with

$$\mathsf{c}_s = f^{a_1 a_2 b} f^{b a_3 a_4}, \;\; \mathsf{c}_t = f^{a_1 a_4 b} f^{b a_2 a_3}, \;\; \mathsf{c}_u = f^{a_1 a_3 b} f^{b a_4 a_2},$$

and $c_s + c_t + c_u = 0$.



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- $\exists \; n_I$, s.t. $n_s + n_t + n_u = 0$.

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Bern, Carrasco, Johansson, 0805.3993

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This is nice because

- Simplifying computation for high-multiplicity/high-loop level
- Replacing c_g with n_g leads to gravity amplitudes:

$$\mathcal{M}_n = \sum_g \frac{n_g n_g}{d_g}.$$

Double copy: gravity $=$ (gauge theory)^2.

Review: Bern, Carrasco, Chiodaroli, Johansson, Roiban, 1909.01358

Questions:

The color-kinematics duality

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Question: Where does the Jacobi identity in YM come from?

Symmetry from unitarity

Massless spin-1: Tree unitarity $\rightarrow$ symmetry!

Benincasa, Cachazo, 0705.4305

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$$\begin{aligned} \mathcal{A}(1^-2^-3^+) &= f_{abc}\frac{\langle 12\rangle^3}{\langle 13\rangle\langle 23\rangle} \rightarrow \mathcal{O}(E), \\ \mathcal{A}(1^-2^-3^-) &= f'_{abc}\langle 12\rangle\langle 13\rangle\langle 23\rangle \rightarrow \mathcal{O}(E^3). \end{aligned}$$

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- Factorization

4-pt constraints lead to

$$f_{abe}f_{cde} + f_{bce}f_{ade} + f_{cae}f_{bde} = 0.$$

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Question: what is the most general renormalizable QFT with a finite spectrum of spin-0, $1/2$ and 1 states?

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PHYSICAL REVIEW D VOLUME 10, NUMBER 4 15 AUGUST 1974

Derivation of gauge invariance from high-energy unitarity bounds on the S matrix*

John M. Cornwall,[†] David N. Levin, and George Tiktopoulos
Department of Physics, University of California at Los Angeles, Los Angeles, California 90024
(Received 21 March 1974)

A systematic search is made for all renormalizable theories of heavy vector bosons. It is argued that in any renormalizable Lagrangian theory high-energy unitarity bounds should not be violated in perturbation theory (apart from logarithmic factors in the energy). This leads to the specific requirement of "tree unitarity": the N-particle S-matrix elements *in the tree approximation* must grow no more rapidly than E^{4-N} in the limit of high energy (E) at fixed, nonzero angles (i.e., at angles such that all invariants $p_i \cdot p_j$, $i \neq j$, grow like E^2). We have imposed this tree-unitarity criterion on the most general scalar, spinor, and vector Lagrangian with terms of mass dimension less than or equal to four; a certain class of nonpolynomial Lagrangians is also considered. It is proved that any such theory is tree-unitary if and only if it is equivalent under a point transformation to a spontaneously broken gauge theory, possibly modified by the addition of mass terms for vectors associated with invariant Abelian subgroups. Our result suggests that gauge theories are the only renormalizable theories of massive vector particles and that the existence of Lie groups of internal symmetries in particle physics can be traced to the requirement of renormalizability.

Gauge theory from unitarity

Question: what is the most general renormalizable QFT with a finite spectrum of spin-0, $1/2$ and 1 states?
Short answer: a (spontaneously broken) gauge theory

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Key observation: the Jacobi identity is just a special case of invariant tensor relations.

The Unbroken Phase

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- Couplings allowed by tree unitarity: f^{abc}, T^a_{ij}, L^a_{AB}, R^a_{AB}, P_{ijk}, K_{ijkl}, $(Y_i)_{AB}$.
- Relations dictated by the gauge group (emerging from the tree unitarity constraints):

$$
\begin{aligned}
& f^{abe}f^{cde} + f^{ace}f^{dbe} + f^{ade}f^{bce} = 0, \\
& [T^a, T^b] = i f^{abc} T^c, \\
& [L^a, L^b] = i f^{abc} L^c, \qquad [R^a, R^b] = i f^{abc} R^c, \\
& L^a Y_i - Y_i R^a - Y_j T^a_{ji} = 0, \\
& P_{ijl} T^a_{lk} + P_{jkl} T^a_{li} + P_{kil} T^a_{lj} = 0, \qquad K_{i_1 i_2 i_3 j} T^a_{j i_4} + \text{cycl} = 0.
\end{aligned}
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Example: VSF^2

The Yukawa coupling needs to be an invariant tensor:

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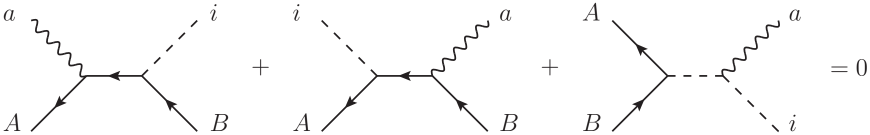

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- c_g satisfies the identity
- $\exists\ n_g$ satisfying the identity:

$$n_s = -2ip_3 \cdot \varepsilon_4 \bar{v}_{2L} u_{1R},\ \ n_t = i\bar{v}_{2L}\not{p}_3\not{\varepsilon}_4 u_{1R},\ \ n_u = i\bar{v}_{2L}\not{\varepsilon}_4\not{p}_3 u_{1R}.$$

The coupling-kinematics duality

Our claim: whenever there is an “invariant tensor” relation for the couplings, there are corresponding kinematic numerators that satisfy such a relation

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- The coupling relations have a kinematic origin
- The existence of kinematic numerators is related to UV constraints

The Broken Phase

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- The relations among couplings are more complicated because of the broken symmetry

Cornwall, Levin, Tiktopoulos, 1973; Llewellyn Smith, 1973

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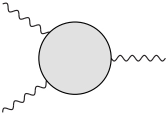
- An on-shell bootstrap can be done to study these relations, similar to the unbroken phase

Liu, **ZY**, 2204.13119

Tree unitary 3-pt amplitudes: V^3

V^3: $7 \to 1$



$$\frac{i\sqrt{2}C_{a_1a_2a_3}}{m_{a_1}m_{a_2}m_{a_3}}\left(m_{a_2}\langle\mathbf{12}\rangle\langle\mathbf{23}\rangle[\mathbf{31}]+\text{cycl}\right),$$

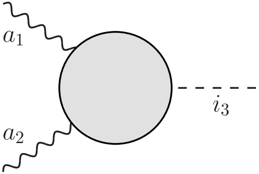
where C_{abc} has to be totally antisymmetric.

Little-group covariant massive spinor formalism, e.g. $\mathbf{1} = 1^I$.

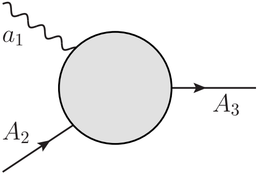
Arkani-Hamed, Huang, Huang, 1709.04891

Tree unitary 3-pt amplitudes: examples

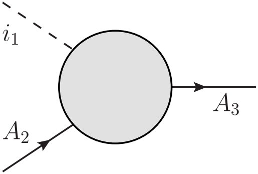
V^2S: $3 \to 1$



VF^2: $4 \to 2$



SF^2:



$$2F_{a_1 a_2 i_3}\frac{[\mathbf{12}]\langle\mathbf{21}\rangle}{m_{a_1}m_{a_2}},$$

where $F_{abi} = F_{bai}$.

$$\frac{\sqrt{2}}{m_{a_1}}\Big(R^{a_1}_{A_3A_2}[\mathbf{12}]\langle\mathbf{13}\rangle + L^{a_1}_{A_3A_2}\langle\mathbf{12}\rangle[\mathbf{13}]\Big).$$

$$(Y_{i_1})_{A_3A_2}[\mathbf{23}] + (Y^\dagger_{i_1})_{A_3A_2}\langle\mathbf{23}\rangle.$$

4-pt example: V^2F^2

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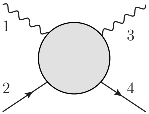

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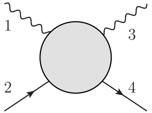

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- $\mathcal{O}(E^2)$ for $(0{-}0{+})$ and $(0{+}0{-})$, giving the relations:

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4-pt example: V^2F^2

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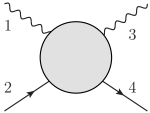

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$$\begin{aligned} & 2F_{a_1a_3i}(Y_i)_{A_4A_2} - m_{A_2}\left\{L^{a_1}, L^{a_3}\right\}_{A_4A_2} - m_{A_4}\left\{R^{a_1}, R^{a_3}\right\}_{A_4A_2} \\ & + \sum_B 2m_B\left(L^{a_1}_{A_4B}R^{a_3}_{BA_2} + L^{a_3}_{A_4B}R^{a_1}_{BA_2}\right) \\ = \ \ & \sum_b iC^{a_1a_3b}\frac{\left(m^2_{a_1} - m^2_{a_3}\right)}{m_b^2}\left(m_{A_2}L^b_{A_4A_2} - m_{A_4}R^b_{A_4A_2}\right). \end{aligned}$$

* $(0{-}0{-})$ gives the conjugate of the above.

4-pt example: V^2F^2

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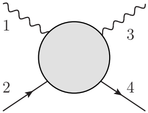

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$$L^aY_b - Y_bR^a - Y_{\tilde{i}}T^a_{\tilde{i}b} = 0,$$

if we recognise

$$\begin{aligned} T^a_{ib} &= -T^a_{bi} = \frac{i}{m_b}F_{abi}, \;\; T^a_{bc} = iC_{abc}\frac{m_a^2 - m_b^2 - m_c^2}{2m_bm_c}, \\ (Y_a)_{AB} &= \frac{i}{m_a}\left(m_BL^a - m_AR^a\right)_{AB}. \end{aligned}$$

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Goldstone boson equivalence

$$(Y_a)_{AB} = \frac{i}{m_a}(m_B L^a - m_A R^a)_{AB}\,.$$

Why such redefinitions? Consider the $(0{+}{+})$ components of the following amplitudes in the HE limit:

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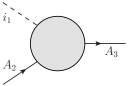

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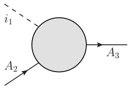

$$\Downarrow$$

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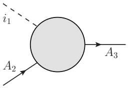

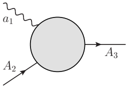

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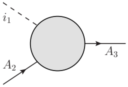
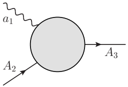


$\Downarrow$

$(Y_{i_1})_{A_2 A_3}[23]$



$\Downarrow$

$$\frac{i}{m_{a_1}}\,(m_{A_2} L^{a_1} - m_{A_3} R^{a_1})_{A_3 A_2}\,[23]$$

Goldstone boson equivalence

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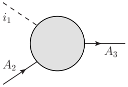
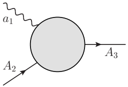




$$\Downarrow \qquad\qquad\qquad \Downarrow$$

$$(Y_{i_1})_{A_2A_3}[23] \qquad\qquad\qquad (Y_{a_1})_{A_2A_3}[23]$$

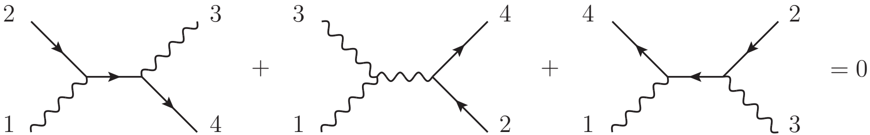
In the HE limit, the longitudinal component of the vectors are equivalent to (Goldstone) scalars, which together with (Higgs) scalars furnish some representation of G

Coupling-kinematics at the broken phase

$WFWF$:

- $\mathcal{O}(E^2)$ for $(0{-}0{+})$ and $(0{+}0{-})$, giving the relations:

  $iC_{a_1 a_3 b} L^b_{A_4 A_2} = [L^{a_1}, L^{a_3}]_{A_4 A_2}, \;\; iC_{a_1 a_3 b} R^b_{A_4 A_2} = [R^{a_1}, R^{a_3}]_{A_4 A_2}.$



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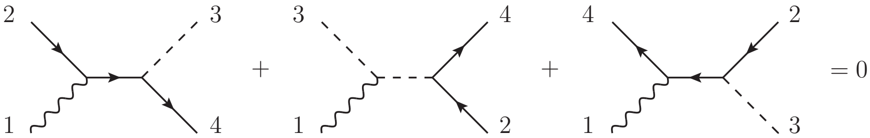
$$L^{a_1}_{A_4B}\left(Y_{a_3}\right)_{BA_2} - \left(Y_{a_3}\right)_{A_4B} R^{a_1}_{BA_2} - \left(Y_{\tilde{i}}\right)_{A_4A_2} T^{a_1}_{\tilde{i}a_3} = 0.$$



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4 sectors: e.g. in the s channel,

$$\sum_B \frac{c^B_{L,s}n_{L,s} + c^B_{R,s}n_{R,s} + \mathsf{f}^B_{L,s}n^{\mathsf{f}}_{L,s} + \mathsf{f}^B_{R,s}n^{\mathsf{f}}_{R,s}}{s - m_B^2},$$

with

$$c^B_{L,s} = L^{a_3}_{A_4B}L^{a_1}_{BA_2}, \qquad c^B_{R,s} = R^{a_3}_{A_4B}R^{a_1}_{BA_2},$$

$$\mathsf{f}^B_{L,s} = \left(Y_{a_3}\right)_{A_4B}L^{a_1}_{BA_2}, \qquad \mathsf{f}^B_{R,s} = \left(Y^\dagger_{a_3}\right)_{A_4B}R^{a_1}_{BA_2}.$$

Summary and outlook

- The coupling-kinematics duality: invariant tensor relations in renormalizable gauge theories correspond to kinematics numerators
- In spontaneously broken gauge theories, an amplitude of a massive gauge boson may involve multiple sets of numerators
- To explore: applications to SM, generalization to EFTs, double copy theories, higher spin