FACTORIZATION AND RESUMMATION AT NEXT-TO-LEADING POWER

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OUTLINE

• Soft-collinear radiation at NLP
• Endpoint divergences at NLP
• NLP LLs in Thrust in the two-jet limit
• NLP NNLO in Drell-Yan near threshold

In collaboration with M. Beneke, A. Broggio, M. Garny, S. Jaskiewicz, R. Szafron, J. Strohm J. Wang,
Based on
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and to appear.
PARTICLE SCATTERING NEAR KINEMATIC LIMITS

- Consider Drell-Yan, DIS near partonic threshold and Thrust in the back-to-back jet limit:

\[(p_1 + p_2)^2 \equiv s, \quad z \equiv \frac{q^2}{s} \to 1\]
\[Q^2 = -q^2, \quad x \equiv \frac{Q^2}{2p_q} \to 1\]

- The partonic cross section has singular expansion

\[\Delta_{ab}(\xi) \sim \sum_{n=0}^{\infty} \left( \frac{\alpha_s}{\pi} \right)^n \left[ c_n \delta(1-\xi) + \sum_{m=0}^{2n-1} c_{nm} \left( \frac{\ln^m(1-\xi)}{1-\xi} \right) \right] + d_{nm} \ln^m(1-\xi) + \ldots,\]

with \(\xi = z\) for DY, \(\xi = x\) for DIS, and \(\xi = T\) for Thrust.

- Resummation of large logarithms relevant for precision phenomenology.
  - well understood at LP (up to N3LL),
  - progress toward resummation at NLP, yet no systematic approach so far.
PARTICLE SCATTERING NEAR KINEMATIC LIMITS

- Subject of intense work in the past few years!

- Within SCET:


- And “diagrammatic” methods:

  Del Duca, 1990; Laenen, Magnea, Stavenga, 2008, Laenen, Stavenga, White, 2008; Laenen, Magnea, Stavenga, White, 2010; Bonocore, Laenen, Magnea, LV, White, 2014, 2015, 2016; Bahjat-Abbas, Sinninghe Damsté, LV, White, 2018; Bahjat-Abbas, Bonocore, Sinninghe Damsté, Laenen, Magnea, LV, White, 2019; Liu, Penin, 2017/18; Anastasiou, Penin, 2020; Cieri, Oleari, Rocco, 2019; Oleari, Rocco 2020; van Beekveld, Beenakker, Laenen, White, 2019; van Beekveld, Laenen, Sinninghe Damsté, LV, 2021; + ...

- Several topics considered:

  LBKD theorem, operator bases, renormalization, N-jettiness subtraction, thrust distribution, Drell-Yan and Higgs production near threshold, DIS for x → 1, QED effects in B decays, New Physics decay, Higgs decay through bottom loops, TMD factorization, energy-energy correlation in N = 4 SYM, gravitation, ...
PARTICLE SCATTERING NEAR THRESHOLD

- Phenomenological analyses have shown LLs at NLP to be competitive with NNLLs at LP: relevant for precision physics.

van Beekveld, Laenen, Sinninghe Damsté, LV, 2021.

Beneke, Garny, Jaskiewicz, Szafron, LV, Wang, 2019
Consider Drell-Yan, DIS near partonic threshold and Thrust in the back-to-back jet limit:

- These limits involve a dynamical enhancement of soft and collinear radiation.
- Factorize soft and collinear radiation from the hard interaction:
  - by means of Soft-Collinear Effective Field Theory (SCET);
  - by means of a diagrammatic approach in QCD.

Here we discuss the first method.
SOFT-COLLINEAR EFFECTIVE FIELD THEORY

- Effective Lagrangian and operators made of collinear and soft fields.
  \[ \mathcal{L}_{\text{SCET}} = \sum_i \mathcal{L}_{c_i} + \mathcal{L}_s, \]

- Constructed to reproduce a scattering process as obtained with the method of regions.

- The cross section factorizes into a hard scattering kernel, and matrix elements of soft and collinear fields.
  \[ \sigma \sim \mathcal{H} \otimes \mathcal{I}_1 \otimes \ldots \otimes \mathcal{I}_n \otimes S. \]

- Renormalize UV divergences of EFT operators and obtain renormalization group equations.

- Each function depends on a single scale: solving the RGE resums large logarithms.
  \[ \text{See e.g. Becher, Neubert 2006} \]

\[ \mathcal{O}_n = \int dt_1 \ldots dt_n C(t_1, \ldots, t_n) \phi_1(t_1 n_{1+}) \ldots \phi_n(t_n n_{n+}). \]
**FACTORORIZATION IN SCET: LP**

- Factorization theorem at LP are “simple” due to soft-collinear decoupling:
- There is a single eikonal soft-collinear interaction at LP:

\[
\mathcal{L}_c^{(0)} = \bar{\xi} \left( i n_- D_c + g_s n_- A_s(x_-) + i \slashed{D}_c \frac{1}{i n_+ D_c} i \slashed{D}_c \right) \frac{\gamma_+}{2} \xi + \mathcal{L}_{c,\text{YM}},
\]

where

\[
i D_c = i \partial + g_s A_c, \quad x_\mu = n_+ \cdot x \frac{n_-^\mu}{2}.
\]

- This can be removed by means of a field redefinition:

\[
\xi(x) \rightarrow Y(x_-)\xi(x), \quad A_\mu^c(x) \rightarrow Y(x_-)A_\mu^c(x)Y^\dagger(x_-), \quad Y^\dagger i n_- D_s Y = i n_- \partial,
\]

with \( Y(x) = \mathcal{P} \exp \left( i g_s \int_{-\infty}^{0} ds \, n_- A_s(x + s n_-) \right) \).

- No soft-collinear interactions are left at LP:

\[
\mathcal{L}_c^{(0)} = \bar{\xi} \left( i n_- D_c + g_s n_- A_s(x_-) + i \slashed{D}_c \frac{1}{i n_+ D_c} i \slashed{D}_c \right) \frac{\gamma_+}{2} \xi + \mathcal{L}_{c,\text{YM}}.
\]


Bauer, Pirjol, Stewart, 2001
FACTORIZATION IN SCET: LP

- One obtains "classical" factorization theorem at LP:

$$\frac{d\sigma}{dQ^2} = |C^{A0}|^2 \times f_{a/A} \otimes f_{b/B} \otimes S_{DY}(Q(1-z)),$$

Becher, Neubert, Xu 2007;

$$F_2 = |C^{A0}|^2 \times Q^2 \times f_{a/A} \otimes J_{hc}^{(q)},$$

Becher, Neubert, Pecjak 2006;

$$\frac{d\sigma}{d\tau} = |C^{A0}|^2 \times J_{c}^{(q)} \otimes J_{\bar{c}}^{(\bar{q})} \otimes S_{LP}.$$

Becher, Schwartz, 2008
**FACTORIZATION IN SCET: NLP**

- **Soft-collinear interactions** are still present at subleading power, and one needs to take into account several effects: *Beneke, Chapovsky, Diehl, Feldmann, 2002; Beneke, Feldmann, 2002*

- **Subleading Lagrangian:**
  \[ \mathcal{L}_{c_i} = \mathcal{L}^{(0)}_{c_i} + \mathcal{L}^{(1)}_{c_i} + \mathcal{L}^{(2)}_{c_i} + \ldots \]
  where e.g.

  \[ \mathcal{L}^{(1)\text{gluon}}_{c_i}(x) = \bar{c} \left[ x_\bot^{\mu} n_\nu W_c g_s F^{gs}_{\mu \nu}(x) W_c^\dagger \right] \frac{\eta^\dagger + \xi}{2}, \]

  \[ \mathcal{L}^{(1)\text{quark}}_{c_i}(x) = \bar{q}(x) W_c^\dagger i \slashed{D} \bot_c \xi. \]

- **Power-suppressed operators**, e.g.

  \[ J^{A_0, A_1}_\rho(t, \bar{t}) = \bar{c} \bar{c}(\bar{t} n_\bot) n_\rho + \rho i \partial_\bot \chi_c(t n_\bot), \]

  \[ J^{A_0, B_1}_\rho(t_1, t_2, \bar{t}) = \bar{c} \bar{c}(\bar{t} n_\bot) n_\rho \mathcal{A}_\bot c(t_2 n_\bot) \chi_c(t_1 n_\bot). \]

*Beneke at Al, 2017-19; see also Stewart at Al., 2014-19*
FACTORIZATION IN SCET: NLP

- **Matrix elements** involve **convolutions** over momentum fractions/soft momenta components:

\[ k_s \sim \frac{n_+ k}{\lambda^2} + \frac{k_\perp}{\lambda^2} + \frac{n_- k}{\lambda^2} \]

\[ p_c \sim \frac{n_+ p}{\lambda^2} + \frac{p_\perp}{\lambda} + \frac{n_- p}{\lambda^2} \]

\[ \rightarrow \int d\omega J(\omega) S(\omega), \]

\[ \rightarrow \int_0^1 dz \left( \frac{\mu^2}{s_{qg} z \tilde{z}} \right)^\epsilon P_{qg}(s_{qg}, z) \bigg|_{s_{qg} = Q^2 \frac{1-x}{x}}. \]
FACTORIZATION IN SCET: NLP

- Convolutions are divergent in $d = 4$!


\[
\rightarrow \int_0^{\Omega} d\omega \left( n_p + p \omega \right)^{-\epsilon} \frac{1}{\omega^{1+\epsilon}} \frac{1}{\left( \Omega - \omega \right)^\epsilon};
\]

- Cannot apply the standard RGE methods directly to the collinear and soft functions.

\[
\rightarrow \int_0^1 dz \left( \frac{\mu^2}{s_{qg} z \bar{z}} \right)^\epsilon \frac{\alpha_s C_F}{2\pi} \frac{(1-z)^2}{z} \bigg|_{s_{qg}=Q^2 \frac{1-x}{x}}.
\]
• At LP only “diagonal” $q\bar{q}$ channel contributes, endpoint divergences relevant at NLL.

• The “off-diagonal” $g\bar{g}$, $qg$ channels start at NLP:

$$\Delta_{q\bar{q}}(z) = \Delta_{q\bar{q}}(z)|_{\text{LP}} + \underbrace{\Delta_{q\bar{q}}(z)|_{\text{NLP}}}_{\text{starts at NLL}} + \underbrace{\Delta_{q\bar{q}}^{\text{dyn}}(z)|_{\text{NLP}}}_{\text{starts at NLL}}$$

$$\sum_i \int \{d\omega\} J_i(\{\omega\}) S_i(\{\omega\})$$

$$\sim \frac{C_F}{4\pi} \sum_i \int \{d\omega\} \left[ S_i^{(n)}(\{\omega\}) + \sum_{m=1}^{n-1} J_i^{(m)}(\{\omega\}) S_i^{(n-m)}(\{\omega\}) \right]$$

$$\Delta_{g\bar{g}}(z) = \Delta_{g\bar{g}}(z)|_{\text{NLP}} + \underbrace{\Delta_{g\bar{g}}^{\text{dyn}}(z)|_{\text{NLP}}}_{\text{starts at NLL}}$$

$$\int \{d\omega\} J(\{\omega\}) S(\{\omega\})$$

• Off-diagonal channels have a simpler structure, but endpoint divergences relevant at LL.
Let’s investigate the endpoint divergence in off-diagonal gluon DIS in some more detail:

\[
\mathcal{P}_{gg}(s_{gg}, z)|_{1-\text{loop}} = \mathcal{P}_{gg}(s_{gg}, z)|_{\text{tree}} \alpha_s \frac{1}{\pi} \frac{1}{\epsilon^2} \times \left( T_1 \cdot T_0 \left( \frac{\mu^2}{z Q^2} \right)^\epsilon + T_2 \cdot T_0 \left( \frac{\mu^2}{z Q^2} \right)^\epsilon \right.
\]

\[
+ T_1 \cdot T_2 \left[ \left( \frac{\mu^2}{Q^2} \right)^\epsilon - \left( \frac{\mu^2}{z Q^2} \right)^\epsilon + \left( \frac{\mu^2}{z s_{gg}} \right)^\epsilon \right] \bigg) + \mathcal{O}(\epsilon^{-1}).
\]

The \textbf{T1.T2} term contains a single pole, but: promoted to leading pole after integration!

Compare exact integration:

\[
\frac{1}{\epsilon^2} \int_0^1 dz \frac{1}{z^{1+\epsilon}} (1 - z^{-\epsilon}) = -\frac{1}{2\epsilon^3},
\]

vs integration after expansion:

\[
\frac{1}{\epsilon^2} \int_0^1 dz \frac{1}{z^{1+\epsilon}} \left( \epsilon \ln z - \frac{\epsilon^2}{2!} \ln^2 z + \frac{\epsilon^2}{3!} \ln^3 z + \cdots \right) = -\frac{1}{\epsilon^3} + \frac{1}{\epsilon^3} - \frac{1}{\epsilon^3} + \cdots.
\]

Expansion in \(\epsilon\) not possible before integration! The pole associated to \textbf{T1.T2} does not originate from the standard cups anomalous dimension.

\textit{Beneke, Garny, Jaskiewicz, Szafron, LV, Wang 2020}
**BREAKDOWN OF FACTORIZATION NEAR THE ENDPOINT**

- What happens for $z \to 0$?

  - **Dynamic scale**: $zQ^2$.
  - In the **endpoint region** new counting parameter, $\lambda^2 \ll z \ll 1$.
  - **New modes** contribute: $z$-softcollinear.
  - Need **re-factorization**:

    $$ C^{B1}(Q, z) J^{B1}(z) \xrightarrow{z \to 0} C^{A0}(Q^2) \int d^4x \ T \left[ J^{A0}, \mathcal{L}_{qz-\overline{s}c}(x) \right] = C^{A0}(Q^2) D^{B1}(zQ^2, \mu^2) J^{B1}_{z-\overline{s}c}. $$

Consider the power-suppressed contribution to Thrust in the two-jet region:

\[ e^+ e^- \rightarrow \gamma^* \rightarrow [g]_c + [q\bar{q}]_c. \]

Within SCET one has two contributions:

“Direct” term (B-type) and time-ordered product soft-quark term (A-type):

\[
\bar{\psi} \gamma_\perp \psi(0) = \int dt \bar{c} \tau^{A_0}(t, \bar{t}) \chi_c(tn_+) \gamma_\perp \chi_c(\bar{t}n_-) + (c \leftrightarrow \bar{c}) \\
+ \sum_{i=1,2} \int dt \bar{t}_1 dt \bar{t}_2 \bar{\tilde{C}}_i^{B_1}(t, \bar{t}_1, \bar{t}_2) \bar{\chi}_c(\bar{t}_1 n_-) \Gamma_i^{\mu\nu} A_{c \perp \nu}(tn_+) \chi_c(\bar{t}_2 n_-) + \ldots
\]

\[ \mathcal{L}_{\xi q}(x) = \bar{q}_s(x_-) \mathcal{A}_{c \perp}(x) \chi_c(x) + \text{h.c.} \]

Beneke, Garny, Jaskiewicz, Strohm, Szafron, LV, Wang 2022
"Direct" B-type term expressed in hard, (anti-)collinear and soft function:

\[
\frac{1}{\sigma_0} \frac{d\sigma}{dM_R^2 dM_L^2} \bigg|_B \sim \int_0^1 dr \, dr' C^{B1}(r) C^{B1}(r') \times \mathcal{J}_c^{(q\bar{q})}(r, r') \otimes \mathcal{J}_c^{(g)} \otimes S^{(g)}.
\]

It develops endpoint divergences when the quark \((r \rightarrow 0)\) or anti-quark \((r \rightarrow 1)\) become soft:

\[
\frac{1}{\sigma_0} \frac{d\sigma}{dM_R^2 dM_L^2} \bigg|_B \propto \int_0^1 dr \left[ \frac{1}{r^{1+\epsilon}} + \frac{1}{(1-r)^{1+\epsilon}} \right].
\]

Time-ordered product A-type term expressed in hard, (anti-)collinear and soft function:

\[
\frac{1}{\sigma_0} \frac{d\sigma}{dM_R^2 dM_L^2} \bigg|_A \sim \int_0^\infty d\omega \, d\omega' \left| C^{A0} \right|^2 \times \mathcal{J}_c^{(\bar{q})} \otimes \mathcal{J}_c(\omega, \omega') \otimes S_{\text{NLP}}(\omega, \omega').
\]

It develops endpoint divergences when the soft quark or anti-quark become energetic \((\omega \rightarrow \infty)\):

\[
\frac{1}{\sigma_0} \frac{d\sigma}{dM_R^2 dM_L^2} \bigg|_A \propto 2 \int_{M_R^2/Q}^\infty d\omega \, \frac{1}{\omega^{1+\epsilon}}.
\]
OFF-DIAGONAL “GLUON” THRUST

- As for DIS, in the $r \to 0$ (or $r \to 1$) limit, the $B_1$ coefficient is a two-scale object, which refactorizes, since the intermediate state develops an on-shell pole:

\[
C_1^{B_1}(Q^2, r) = C_1^{A_0}(Q^2) \times \frac{D_{B_1}(rQ^2)}{r} + O(r^0).
\]

- In $d$ dimensions the $1/\epsilon$ poles from the divergent convolution integrals cancels. The integrands of $A$ and $B$ match in the asymptotic limits $\omega, \omega' \to \infty$ (A-type) and $r, r' \to 0(1)$ (B-type).

- This allows a rearrangement between the terms that makes them separately finite, provided two refactorization conditions hold for the soft and jet functions:
OFF-DIAGONAL “GLUON” THRUST

\[ \omega, \omega' \rightarrow \infty \]

\[ [J_c (p^2, \omega, \omega')] \rightarrow J_c^{(g)} (p^2) \frac{D_{B1}(\omega Q)}{\omega} \frac{D_{B1}^*(\omega' Q)}{\omega'} \]

\[ Q \tilde{J}_c^{(q)} (s_R) [\tilde{S}_{NLP}(s_R, s_L, \omega, \omega')] \]

\[ \rightarrow [[\tilde{J}_c^{qq}(8) (s_R, r, r')] [\tilde{S}^{(g)} (s_R, s_L)] \]

\[ [[C_{1}^{B1}(Q^2, r)]] \rightarrow C^{A0}(Q^2) \times \frac{D_{B1}(r Q^2)}{r} \]

\[ r, r' \rightarrow 0 \]

Beneke, Garny, Jaskiewicz, Strohm, Szafron, LV, Wang 2022
Refactorization can be achieved as follows: define the asymptotic (scaleless) integral

\[
0 = \frac{2C_F}{Q} f(\epsilon) |C^{A0}(Q^2)|^2 \tilde{\mathcal{J}}(q)(s_R) \tilde{\mathcal{J}}(g)(s_L) \\
\times \int_0^\infty d\omega d\omega' \frac{D^{B1}(\omega Q)}{\omega} \frac{D^{B1*}(\omega' Q)}{\omega'} \left[ \tilde{S}_{NLP}(s_R, s_L, \omega, \omega') \right].
\]

Then split the integral over the two regions

\[
0 = I_1 + I_2.
\]

by introducing a factorization parameter \( \Lambda \), and subtract \( I_1 \) from the B-type term and \( I_2 \) from the A-type term, which makes both endpoint-finite as \( d \to 4 \).

One can then proceed with standard resummation methods.
OFF-DIAGONAL "GLUON" THRUST

- Refactorized factorization formula:

\[
\frac{1}{\sigma_0 dS_R dS_L} |_{\text{A-type}} = \frac{2C_F}{Q} f(\epsilon) |C^{A0}(Q^2)|^2 \left( \tilde{J}_c(q) (s_R) \right) \int_0^\infty d\omega d\omega' \times \left\{ \tilde{J}_c(s_L, \omega, \omega') \tilde{S}_{\text{NLP}}(s_R, s_L, \omega, \omega') \right. \\
\left. - \theta(\omega - \Lambda) \theta(\omega' - \Lambda) \left[ [\tilde{J}_c(s_L, \omega, \omega')] \right] [\tilde{S}_{\text{NLP}}(s_R, s_L, \omega, \omega')] \right\}
\]

\[
\frac{1}{\sigma_0 dS_R dS_L} |_{\text{B-type}} = \frac{2C_F}{Q^2} f(\epsilon) \tilde{J}_c(g)(s_L) \tilde{S}(g)(s_R, s_L) \int_0^\infty dr dr' \times \left[ \theta(1 - r) \theta(1 - r') C_1^{B1*}(Q^2, r') C_1^{B1}(Q^2, r) \tilde{J}_c^{gq}(8)(s_R, r, r') \right. \\
\left. - [1 - \theta(r - \Lambda/Q) \theta(r' - \Lambda/Q)] \times \left[ [C_1^{B1*}(Q^2, r')]_0 \left[ [C_1^{B1}(Q^2, r)]_0 \left[ \tilde{J}_c^{gq}(8)(s_R, r, r') \right]_0 \right] \right] \right).
\]

- \( \Lambda \) dependence cancels between the two terms. Each separately independent of \( \text{dim reg } \mu \).
- In principle valid to any log accuracy. At \( \text{LL} \) only the subtraction terms contribute do to the extra log from large \( \omega \)/small \( r \).

Beneke, Garny, Jaskiewicz, Strohm, Szafron, LV, Wang 2022


**OFF-DIAGONAL “GLUON” THRUST**

- $D^{B_1}$ appears as a universal coefficient that renormalizes soft quark emission. Its double logarithms are proportional to the change of colour charge of the collinear particles:

$$\langle g_c^a(p_c)q_{sc}(p_{sc})| \int d^4x T\{\bar{\chi}_c(0), \mathcal{L}_{\xi q}(x)\}|0\rangle = g_s\bar{u}(p_{sc})t^a_c\frac{\ln p_c}{p^2} \frac{\Phi^-}{2} D^{B_1}(p^2).$$

- It appears in _Thrust, DIS, DY_, as well as _H → gg_. Up to one loop:

$$D^{B_1}(p^2) = 1 + \frac{\alpha_s}{4\pi} (C_F - C_A) \left( \frac{2}{\epsilon^2} - 1 - \frac{\pi^2}{6} \right) \left( \frac{\mu^2}{-p^2 - i\varepsilon} \right)^\epsilon + \mathcal{O}(\alpha_s^2).$$

- $D^{B_1}$ has (non-local) anomalous dimension

$$\frac{d}{d\ln \mu} D^{B_1}(p^2) = \int_0^\infty d\hat{p}^2 \gamma_D(\hat{p}^2, p^2) D^{B_1}(\hat{p}^2),$$

with

$$\gamma_D(\hat{p}^2, p^2) = \frac{\alpha_s}{\pi} (C_F - C_A) \delta(\hat{p}^2 - p^2) \ln \left( \frac{\mu^2}{-p^2 - i\varepsilon} \right)$$

$$+ \frac{\alpha_s}{\pi} \left( \frac{C_A}{2} - C_F \right) p^2 \left[ \frac{\theta(\hat{p}^2 - p^2)}{\hat{p}^2(\hat{p}^2 - p^2)} + \frac{\theta(p^2 - \hat{p}^2)}{p^2(p^2 - \hat{p}^2)} \right].$$

DRELL YAN AT NLP

- $D^{B1}$ is an example of new universal functions appearing at NLP.
- In general these functions are more involved to compute compared to their LP counterparts (not counting endpoint divergences), as they depend on more variables.
- On the other hands, refactorization conditions imposes additional constraints, as we have seen in case of thrust.
- Important to collect data on these functions. In this respect, in the past few years we have completed the calculation of all terms contributing to Drell-Yan at NLP, up to NNLO: this includes jet functions at NLO, and soft functions at NNLO.

\[ \Delta^{\text{dyn}}_{q\bar{q},gg}(z)|_{\text{NLP}} \sim \sum_i \int \{d\omega\} J_i(\{\omega\}) S_i(\{\omega\}). \]

$qq$: $J$ at one loop in Beneke, Broggio, Jaskiewicz, LV, 2019
$qg$: $J$ at one loop in Broggio, Jaskiewicz, LV, to appear
$qqb$: $S$ at two loops in Broggio, Jaskiewicz, LV, 2021
$qg$: $S$ at two loops in Broggio, Jaskiewicz, LV, to appear
DRELL YAN AT NLP: QG CHANNEL

- Focus on the $gqb$ channel: only a single term contribute, involving the emission of a soft quark:

\[
\Delta_{gq}|_{\text{NLP}}(z) = 8H(Q^2) \int d\omega \, d\omega' \, G_{\xi q}^*(x_a n+p_A; \omega') \, G_{\xi q}(x_a n+p_A; \omega) \, S(\Omega, \omega, \omega').
\]

- We have calculated the collinear function at one loop: it corresponds to the coefficient $D^{B1}$, and it is thus known to two loops.

_Broggio, Jaskiewicz, LV, to appear_
The soft function at two loops is more involved: there is a real-virtual contribution and a real-real term, that we calculate by means of standard Feynman parametrization and by means of differential equations:

\[
S_{gq}^{(2)1r1v}(\Omega, \omega, \omega') = \frac{\alpha_s^2 T_F}{(4\pi)^2} \left( 2C_F - C_A \right) \frac{e^{2e\gamma_E}}{\epsilon \Gamma[1+\epsilon]} \frac{\Gamma[1+\epsilon]}{\epsilon \Gamma[1-\epsilon]} \\
\times \text{Re} \left\{ \frac{1}{(-\omega)\omega'} \left[ \frac{\omega + \omega'}{\omega} 2F_1 \left( 1, 1+\epsilon, 1-\epsilon, \frac{\omega}{\omega'} \right) - 1 \right] \left( \frac{\mu^4}{(-\omega)\omega'(\Omega - \omega')^2} \right) \theta(-\omega) \\
+ \frac{2(\omega + \omega')}{\omega\omega'(\omega' - \omega)} \left( \frac{\mu^4}{(\omega' - \omega)^2(\Omega - \omega')^2} \right)^\epsilon \frac{\Gamma[1-\epsilon]^2}{\Gamma[1-2\epsilon]} \theta(\omega')\theta'(\Omega - \omega') \right\} \theta(\omega') \theta'(\Omega - \omega'),
\]

\[
S_{gq}^{(2)2r0v}(\Omega, \omega, \omega') = \frac{\alpha_s^2 T_F}{(4\pi)^2} \left\{ C_F \frac{e^{2e\gamma_E}}{\epsilon^2} \frac{\Gamma[1-\epsilon]}{\epsilon} \frac{1}{\omega} \left[ \frac{4}{\Gamma[1-3\epsilon]} \left( \frac{\mu^4}{\omega(\Omega - \omega)^3} \right)^\epsilon \right] \\
+ \frac{(4-\epsilon)\Gamma[2-\epsilon]}{(1-2\epsilon)\Gamma[1-2\epsilon]^2} \left( \frac{\mu^4}{\omega^2(\Omega - \omega)^2} \right)^\epsilon \delta(\omega - \omega') \theta(\Omega - \omega') \theta(\omega) \\
+ (C_A - 2C_F) \frac{2e^{2e\gamma_E}}{\epsilon \Gamma[1-2\epsilon]} \frac{\omega + \omega'}{\epsilon \omega'(\omega' - \omega)} \left( \frac{\mu^4}{\omega'(\omega' - \omega)(\Omega - \omega')^2} \right)^\epsilon \\
\times \left[ 2F_1 \left( 1, -\epsilon, 1-\epsilon, \frac{\omega}{\omega - \omega'} \right) - 1 \right] \theta(\omega) \theta(\omega') \theta'(\omega - \omega) \theta'(\Omega - \omega') \right\}.
\]

Broggio, Jaskiewicz, LV, to appear
DRELL YAN AT NLP: QG CHANNEL

• **Endpoint divergences:** at NLO these arise for $\omega \to 0$:

\[
\Delta_{gq}^{(1)}(z)|_{\text{NLO}} = 2 \int d\omega \, d\omega' \, S^{(1)}(\Omega, \omega, \omega'),
\]

with

\[
S^{(1)}_{gq}(\Omega, \omega, \omega') = \frac{\alpha_s T_F}{4\pi} \frac{e^{\gamma_E}}{\Gamma[1 - \epsilon]} \frac{1}{\omega} \left( \frac{\mu^2}{\omega (\Omega - \omega)} \right)^\epsilon \delta(\omega - \omega') \theta(\Omega - \omega)\theta(\omega).
\]

• For $\omega \to 0$ the (threshold-)soft momentum becomes (PDF)-soft-collinear:

\[
p \sim Q(\lambda^2, \lambda^2, \lambda^2) \xrightarrow{\omega, \omega' \to 0} Q(\lambda^2, \eta \lambda^2, \eta^2 \lambda^2), \quad \lambda^2 \sim 1 - z, \quad \eta \ll \lambda \ll 1.
\]

• The **endpoint divergence** needs to be removed by a corresponding divergent term in the PDF, which needs to be factorized near $x \to 1$.

• Even in absence of such term, the **subtraction procedure** is quite similar to the case of gluon thrust: we can formally write a **scaleless integral**

\[
[[\Delta_{gq}^{\text{NLO}}(z, \mu)]] = 8H(Q^2, \mu) \int d\omega \, d\omega' \, G^{*}_{\xi q}(Q; \omega'; \mu) \, G_{\xi q}(Q; \omega; \mu) \, [[S(\Omega, \omega, \omega', \mu)]],
\]

with

\[
[[S^{(1)}_{gq}(\omega, \omega')]] = \frac{\alpha_s T_F}{4\pi} \frac{e^{\gamma_E}}{\Gamma[1 - \epsilon]} \frac{1}{\omega} \left( \frac{\mu^2}{\omega \Omega} \right)^\epsilon \delta(\omega - \omega') \theta(\Omega)\theta(\omega).
\]
We can subtract the asymptotic contribution from both the “partonic cross section” and “PDF” contribution in the renormalized partonic cross section:

\[ \Delta_{gq}^{\text{ren}}(z) = \left[ (\Gamma^{-1})_{gg}^{\text{LP}} \otimes (\Gamma^{-1})_{qq}^{\text{LP}} \otimes \Delta_{gq}^{\text{bare}, \text{NLP}} \right](z) \]

\[ + \left[ (\Gamma^{-1})_{qq}^{\text{NLP}} \otimes (\Gamma^{-1})_{qq}^{\text{LP}} \otimes \Delta_{gq}^{\text{bare}, \text{LP}} \right](z), \]

to obtain

\[ \Delta_{gq}^{\text{ren}}(z) = H(Q^2) \left\{ 8(\Gamma^{-1})_{gg}^{\text{LP}} \otimes (\Gamma^{-1})_{qq}^{\text{LP}} \otimes \int_0^\infty d\omega \, d\omega' \, G_{\xi q}^*(\omega') \, G_{\xi q}(\omega) \right. \]

\[ \times \left. \left( S(\omega, \omega') - \theta(\delta - \omega)\theta(\delta - \omega')[[S(\omega, \omega')]] \right) \right. \]

\[ + \left. (\Gamma^{-1})_{qq}^{\text{NLP}} \otimes (\Gamma^{-1})_{qq}^{\text{LP}} \otimes Q \, S_{\text{DY}}^{\text{LP}} - 8(\Gamma^{-1})_{gg}^{\text{LP}} \otimes (\Gamma^{-1})_{qq}^{\text{LP}} \right. \]

\[ \otimes \int_0^\infty d\omega \, d\omega' \left[ 1 - \theta(\delta - \omega)\theta(\delta - \omega') \right] G_{\xi q}^*(\omega') \, G_{\xi q}(\omega) \left[[S(\omega, \omega')]] \right) \right\}. \]
It can be shown that the two terms are separately finite, the cutoff dependence cancels between the two terms, and the sum reproduces the PDF-renormalized cross section:

\[
2 \int_0^\infty d\omega \, d\omega' \left[ S^{(1)}(\omega, \omega') - \theta(\delta - \omega)\theta(\delta - \omega')[[S^{(1)}(\omega, \omega')]] \right] = 2 \int_\delta^\infty d\omega \, \hat{S}^{(1)}(\omega) \\
= \frac{\alpha_s T_F}{4\pi} \left\{ 4 \ln(1 - z) + 2 \ln \left( \frac{\mu^2}{\delta Q(1 - z)} \right) + \epsilon \left[ \frac{\pi^2}{3} - 4 \ln^2(1 - z) \right] + \ln^2 \left( \frac{\mu^2}{\delta Q(1 - z)} \right) \right\} + \mathcal{O}(\epsilon^2) \right\}.
\]

PDF contribution:

\[
\frac{\alpha_s}{4\pi} \frac{P^{(0),NLP}_{gg}(z)}{\epsilon} - 2 \int_0^\infty d\omega \, d\omega' \left[ 1 - \theta(\delta - \omega)\theta(\delta - \omega') \right] [[S^{(1)}(\omega, \omega')]] \\
= \frac{\alpha_s}{4\pi} \frac{P^{(0),NLP}_{gg}(z)}{\epsilon} - 2 \int_\delta^\infty d\omega [[\hat{S}^{(1)}(\omega)]] \\
= \frac{\alpha_s T_F}{4\pi} \left\{ - 2 \ln \left( \frac{\mu^2}{\delta Q(1 - z)} \right) + \epsilon \left[ \frac{\pi^2}{6} - \ln^2 \left( \frac{\mu^2}{\delta Q(1 - z)} \right) \right] + \mathcal{O}(\epsilon^2) \right\}.
\]
At NNLO the structure of endpoint divergences appear more involved:

\[ S^{(2)}(\omega, \omega') = \hat{S}^{(2)}(\omega) \delta(\omega - \omega') \theta(\Omega - \omega)\theta(\omega) + S^{(2A)}(\omega, \omega') \theta(-\omega)\theta(\omega')\theta(\Omega - \omega') + S^{(2B)}(\omega, \omega') \theta(\omega' - \omega)\theta(\omega')\theta(\Omega - \omega') + S^{(2C)}(\omega, \omega') \theta(\omega)\theta(\omega')\theta(\omega' - \omega)\theta(\Omega - \omega'). \]

A formally finite soft function can still be defined, but we need consistency conditions with PDF factorization for validation.
CONCLUSION

• In the past few years, a lot of work has been devoted to understand the structure of large logarithms at next-to-leading power.

• With some optimism, we have now frameworks (SCET and diagrammatic approach) which allows us to resum LLs at NLP, at least for color singlet processes.

• The next task is to understand whether this framework can be applied straightforwardly beyond NLP LLs.