Two-loop Feynman integrals for top-quark pair plus jet production

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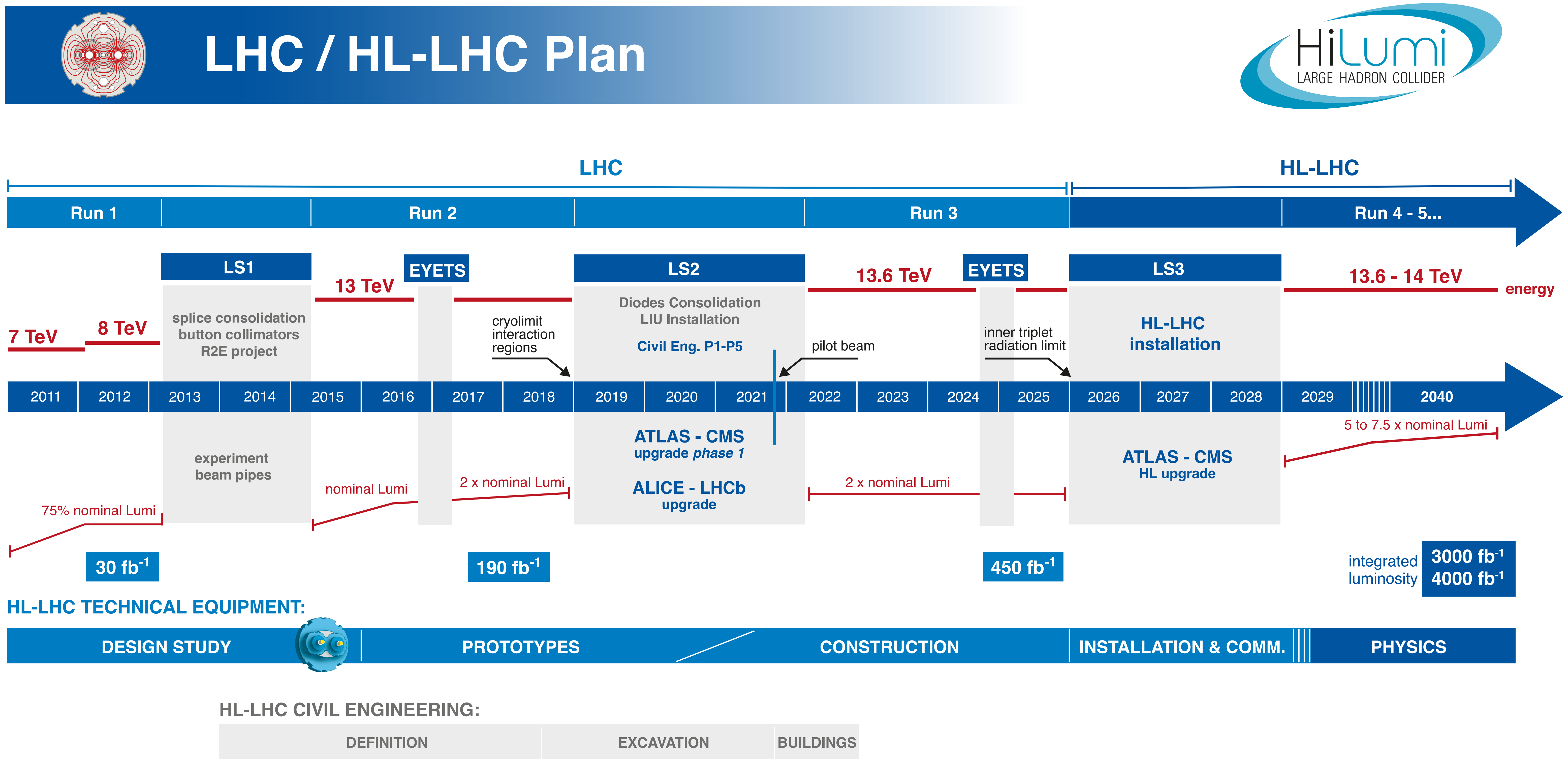
Outline

- *Motivation*
- *Status of multi-scale two-loop QCD corrections*
- *Feynman Integrals for Planar Topologies*
- *Outlook*

Motivation

- High-Luminosity LHC Plan



- Experimental precision $\sim \mathcal{O}(1\%)$ for many observables

- NNLO QCD Corrections required to reduce theoretical uncertainty

Motivation

- ttj production can be used to extract numerical value of top-quark mass m_t

[Alioli, Fernandez, Fuster, Irles, Moch, Uwer '13]

$$\mathcal{R}(m_t^R, \rho_s) = \frac{1}{\sigma_{t\bar{t}j}} \frac{d\sigma_{t\bar{t}j}}{d\rho_s}(m_t^R, \rho_s)$$

$$\rho_s = \frac{2m_0}{m_{t\bar{t}j}}$$

- Current theoretical predictions at NLO in QCD

[Dittmaier,Uwer,Weinzierl '07]
[Melnikov,Schulze '10]

- Theoretical uncertainty $\delta \sim O(10\%)$

[Alioli, Fuster, Garzelli, Gavardi, Irles, Melini '22]

- Higher-order corrections might be crucial to reduce uncertainties

- Analytic structure of two-loop five-point processes with internal massive propagators

NNLO corrections

- NNLO corrections require different ingredients

$$\sigma_{NNLO} = \sigma_{RR} + \sigma_{RV} + \sigma_{VV}$$

Solvable

Available numerically

Unknown

- Phase-space integration represents a hard-problem

Status of two-loop multi-scale computations

Gehrmannm, Henn, Lo Presti, Abreu, Dixon, Herrmann, Page, Zeng, Chicherin, Wasser, Zhang, Zoia, Sotnikov, Ita, Moriello, Tschernow, Canko, Papadopoulos, Syrrakos, Badger, Brønnum-Hansen, Hartanto, Peraro, Dormans, Febres Cordero, Heinrich, Pascual, Chawdhry, Mitov, Poncelet, Czakon, Agarwal, Buccioni, von Manteuffel, Tancredi, Kryś, Kallweit, Wiesemann, Marcoli, Moodie, Popescu Catani, Devoto, Grazzini, Mazzitelli, Savoini,…

		Complete Analytic Results	Public Numerical code	Cross Sections
$pp \to jjj$	Leading Color	✓	✓	✓
$pp \to \gamma\gamma j$	Leading Color (Planar)	✓	✓	✓
$pp \to \gamma\gamma\gamma$	Leading Color (Planar)	✓	✓	✓
$pp \to \gamma\gamma j$		✓		
$gg \to \gamma\gamma g$		✓	✓	✓
$pp \to Wb\bar{b}$	Leading Color	✓		✓
$pp \to Wjj$	Leading Color	✓		
$pp \to Zjj$	Leading Color (Planar)	✓		
$pp \to W\gamma j$	Leading Color (Planar)	✓		
$pp \to Hb\bar{b}$	Leading Color	✓		
$pp \to t\bar{t}H$	Two-Loop Approx.			✓

Adapted from Sotnikov

Five-point Two-loop Computations with internal and external massive particles

See Savoini's talk

- $pp \to t\bar{t}H$: NNLO computations done in the soft-Higgs boson approximation

[Catani,Devoto,Grazzini,Kallweit, Mazzitelli,Savoini '22]

$$\lim_{k\to 0} \mathcal{M}(\{p_i\},k) = F\left(\alpha_S(\mu_R); m/\mu_R\right) J^{(0)}(k)\mathcal{M}(\{p_i\})$$

Soft Limit of scalar heavy quark form factor

$$J^{(0)}(k) = \sum_i \frac{m_Q}{v}\frac{m_Q}{p_i \cdot k}$$

Two-loop Amplitude for $t\bar{t}$ production

See Buonocore's talk

- $pp \to Wb\bar{b}$: NNLO computations done with small mass approximation

[Buonocore,Devoto,Kallweit, Mazzitelli,Rottoli,Savoini '23]

$$\mathcal{M}(\{p_i\},m_b) = \prod_{i\in ext.\,legs}\left(Z_i^{(m_b|0)}\right)^{1/2}\mathcal{M}(p_i,m_b=0)+\mathcal{O}(m_b^k)$$

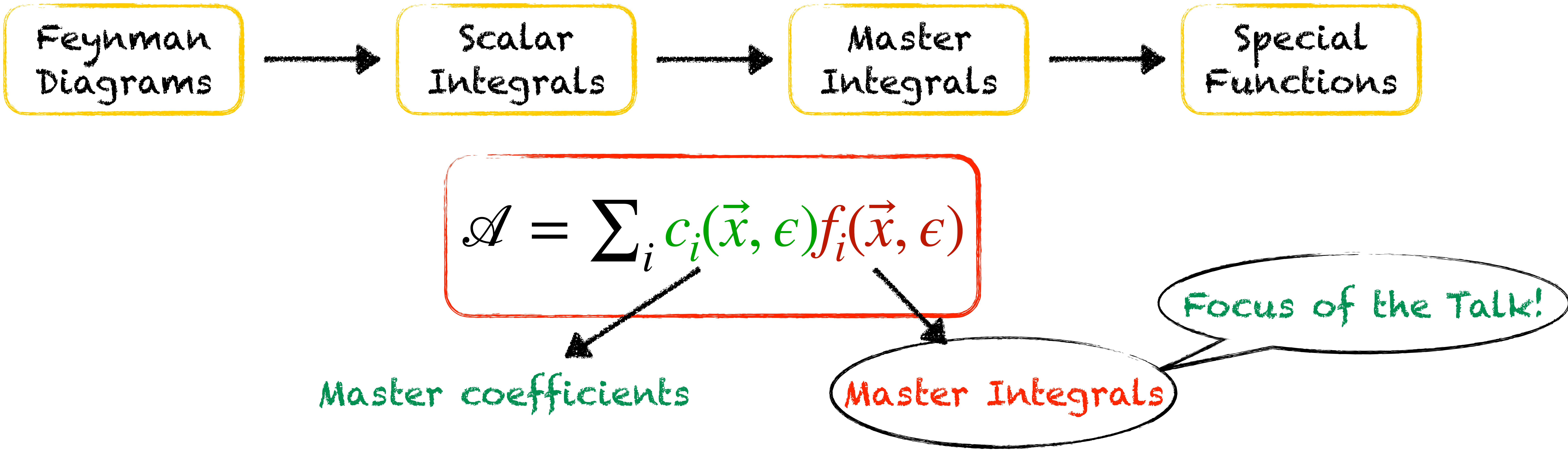
Ratio of massive/ massless quark form factors

Two-loop amplitude with $m_b = 0$

Corrections proportional to the quark mass

- No two-loop exact computations with internal and external massive particles available

Amplitude Computation Pipeline

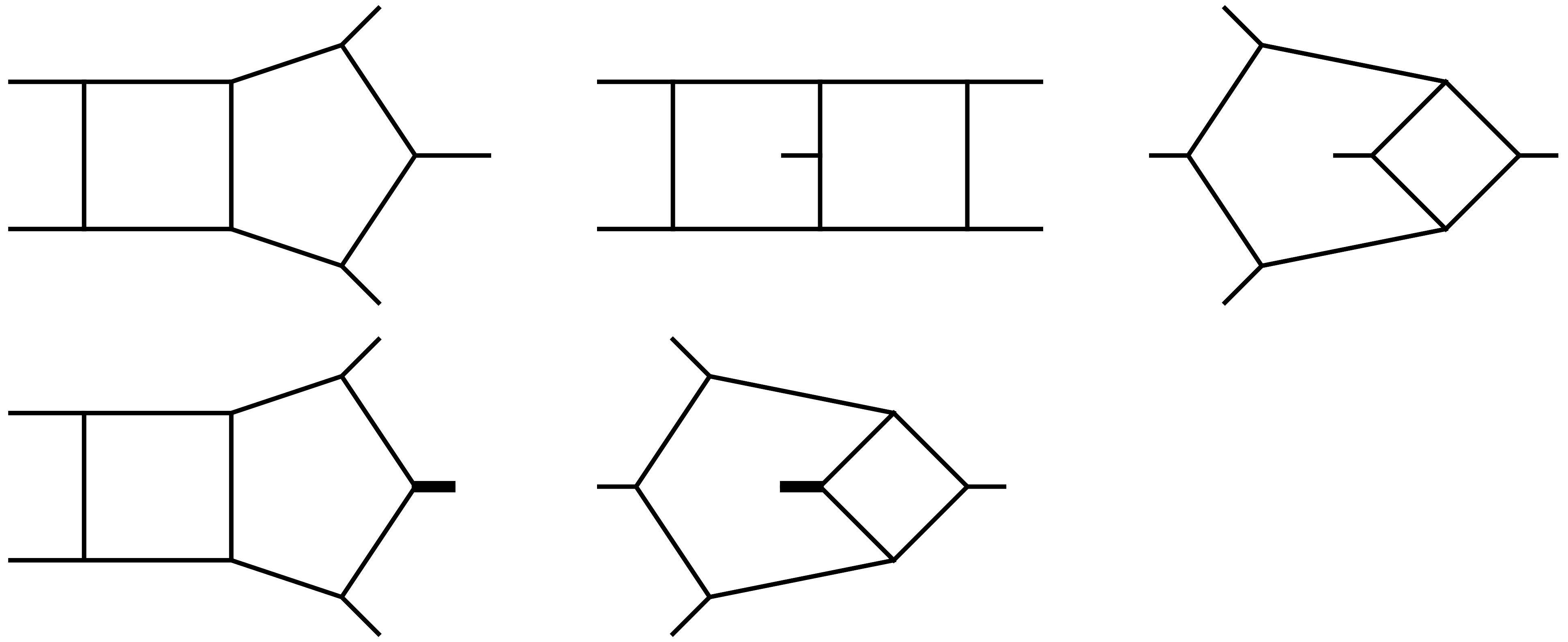


- Reduction to MIs challenging for high-multiplicity processes
- IBP reduction and amplitude reconstruction performed exploiting Finite Fields method

[von Manteuffel, Schabinger '14]
[Peraro '16]

Master Integrals Computation Status

- **Massless case:** [Gehrmann,Hennn,Lo Presti ’15] [Chicherin,Sotnikov ’20]
  - Canonical basis ✓
  - Alphabet ✓
  - Pentagon Function ✓



See Chicherin’s Talk

- **One-Mass case:**
  - Canonical basis ✓
  - Alphabet ✓
  - Pentagon Function ✓

[Abreu,Ita,Moriello,Page,Tschernow,Zeng ‘20]
[Canko,Papadopoulos,Syrrakos ’21]
[Chicherin,Sotnikov,Zoia ’22]

- **Canonical basis**

  Compact form of differential equations for MIs

- **Alphabet**

  Analytic structure of the Feynman Integrals

- **Pentagon Functions**

  Fast and reliable numerical evaluation

Two-Loop Planar Feynman Integrals for ttj: A First Look

[Badger,MB,Chaubey,Marzucca '22]

- **Scattering Kinematics**:

$$p_1^2 = p_2^2 = m_t^2 \quad p_3^2 = p_4^2 = p_5^2 = 0$$

- One Internal Massive propagator
- Topology described by 88 MIs
- Differential Equations:

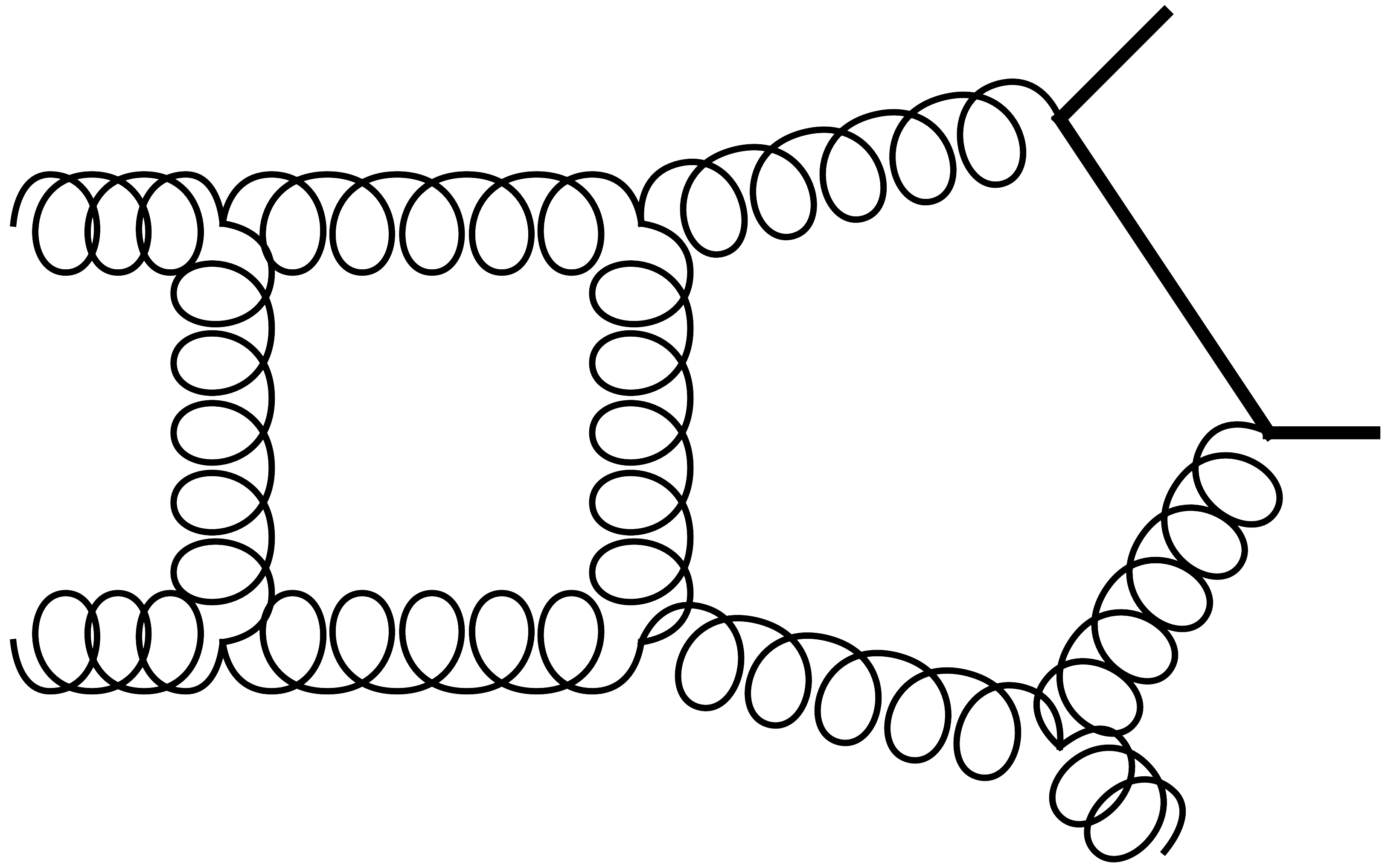
Canonical basis ✓

Alphabet ✓

See Liu's Talk

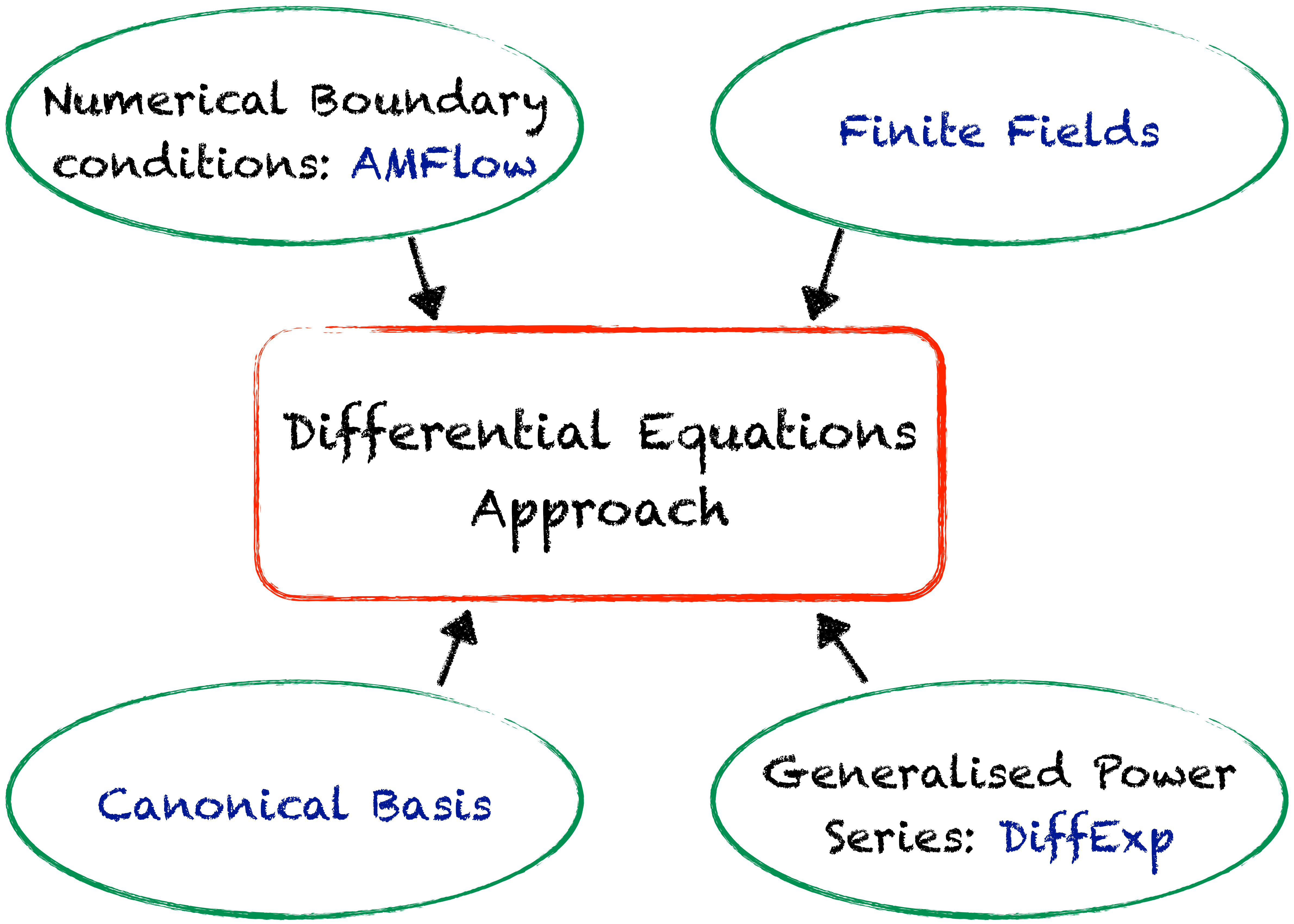
AMFlow: High-precision numerical boundary conditions [Liu,Ma,Wang '18] [Liu,Ma '23]

DiffExp: Semi-Analytic solution with generalised power series expansion [Moriello '19] [Hidding '20]



Contributes to the VV planar leading color

Strategy of the Computation



Numerical Evaluation of MIs

Analytic Information: Canonical Basis, Alphabet

A First Look at Planar sector of five-point two-loop scattering amplitude with massive propagators

Two-Loop Planar Feynman Integrals for ttj: A First Look

- Massive propagator increases complexity of calculation

	Planar Massless	Planar one-mass	Planar ttj
Number of MIs	61	74-86	88-123
Length of Alphabet	31	38-49	71
Number of roots	1	3	6

However, the structure of the Canonical Basis
and of the Alphabet are similar.....

The Alphabet

Canonical basis

[Henn '13]

$$d\vec{f}(\vec{x},\epsilon) = \epsilon\, dA(\vec{x})\,\vec{f}(\vec{x},\epsilon), \quad A(\vec{x}) = \sum_{i=1}^{71} c_i \log(w_i(\vec{x}))$$

Rational Letters

s_{ij} $\quad (\Delta_i)^2 \quad \mathrm{tr}(ij\cdots k) = \mathrm{tr}(\gamma_\mu p_i^\mu \gamma_\nu p_j^\nu \cdots \gamma_\rho p_k^\rho)$

Similar structure to five-point massless and one off-shell leg cases

Algebraic Letters

$\Omega_i(a,b) \quad \dfrac{\mathrm{tr}_+(i\cdots j)}{\mathrm{tr}_-(r\cdots s)} \quad \tilde{\Omega}_i(a,b,c)$

$$\Omega(a,b) := \frac{a+\sqrt{b}}{a-\sqrt{b}} \qquad \mathrm{tr}_\pm(i\cdots j) = \frac{1}{2}\mathrm{tr}((1\pm\gamma_5)\gamma_\mu p_i^\mu \cdots \gamma_\nu p_j^\nu)$$

$$\tilde{\Omega}(a,b,c) := \frac{(a+\sqrt{b}+\sqrt{c})(a-\sqrt{b}-\sqrt{c})}{(a+\sqrt{b}-\sqrt{c})(a-\sqrt{b}+\sqrt{c})}$$

Differential Equations with Finite Fields

[Peraro '19]

- We exploit Finite Fields method as implemented in FiniteFlow to obtain analytic DEQs

[Peraro '19]

- Given the complexity of the computation the reconstruction has to be done in a "good" basis of MIs

	Degrees	Reconstruction time
Non-UT	53/57	~3 Weeks★
Quasi-UT	15/15	~2 Hours

- Canonical basis depends on the set of square-roots:

$$\beta = \sqrt{1 - \frac{4m_t^2}{s_{12}}} \qquad \Delta_1 = \sqrt{\det G(p_{23}, p_1)} \qquad \Delta_2 = \sqrt{\det G(p_{15}, p_2)} \qquad \Delta_3 = \sqrt{1 - \frac{4 s_{45} m_t^2}{(s_{12} + s_{23} - m_t^2)^2}} \qquad \Delta_4 = \sqrt{1 + \frac{4 s_{34} s_{45} m_t^2}{s_{12}(s_{15} - s_{23})^2}}$$

$$\mathrm{tr}_5 = 4\sqrt{\det G(p_3, p_4, p_5, p_1)} \qquad G_{ij}(\vec{v}) = v_i \cdot v_j$$

- Square-roots have to be avoided within Finite Fields

★ 32 cores workstation

Differential Equations with Finite Fields

- We reconstruct DEQs with respect to a "Quasi"-UT basis

$$d\,\vec{J}(\vec{x},\epsilon) = d\left(\hat{A}^{(0)}(\vec{x}) + \epsilon\,\widehat{A}^{(1)}(\vec{x})\right)\vec{J}(\vec{x},\epsilon)$$

$\vec{J}$ MIs basis with no square-roots

$\widehat{A}^{(0)}$ diagonal rational matrix

- Canonical DEQs obtained by means of a diagonal transformation

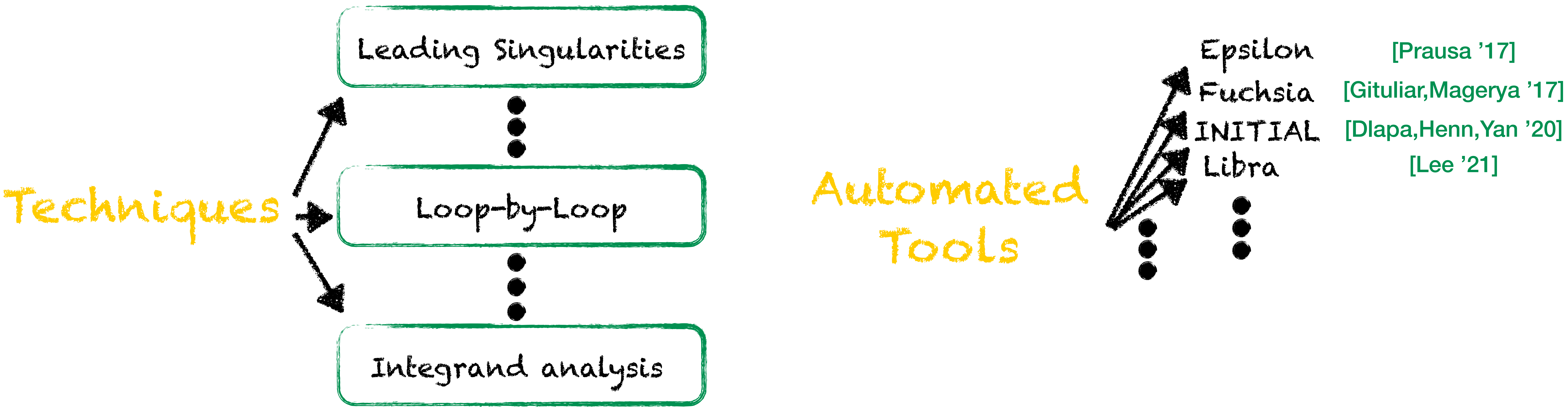
$$d\,\vec{I}(\vec{x},\epsilon) = \epsilon d\left(N(\vec{x})\,\widehat{A}^{(1)}(\vec{x})N^{-1}(\vec{x})\right)\vec{I}(\vec{x},\epsilon)$$

$I_i = N_{ij}(\vec{x})J_j$ Canonical basis of MIs

$N_{ij}(\vec{x})$ Diagonal matrix with square-roots

Building Canonical Basis of MIs

- Canonical Basis greatly improves effectiveness of DEQs for Feynman Integrals
- The construction of Canonical Bases is in general a hard problem



- Given the complexity of the kinematics automated approaches are difficult to apply
- We exploit emerging patterns for five-point processes in the construction of the UT basis

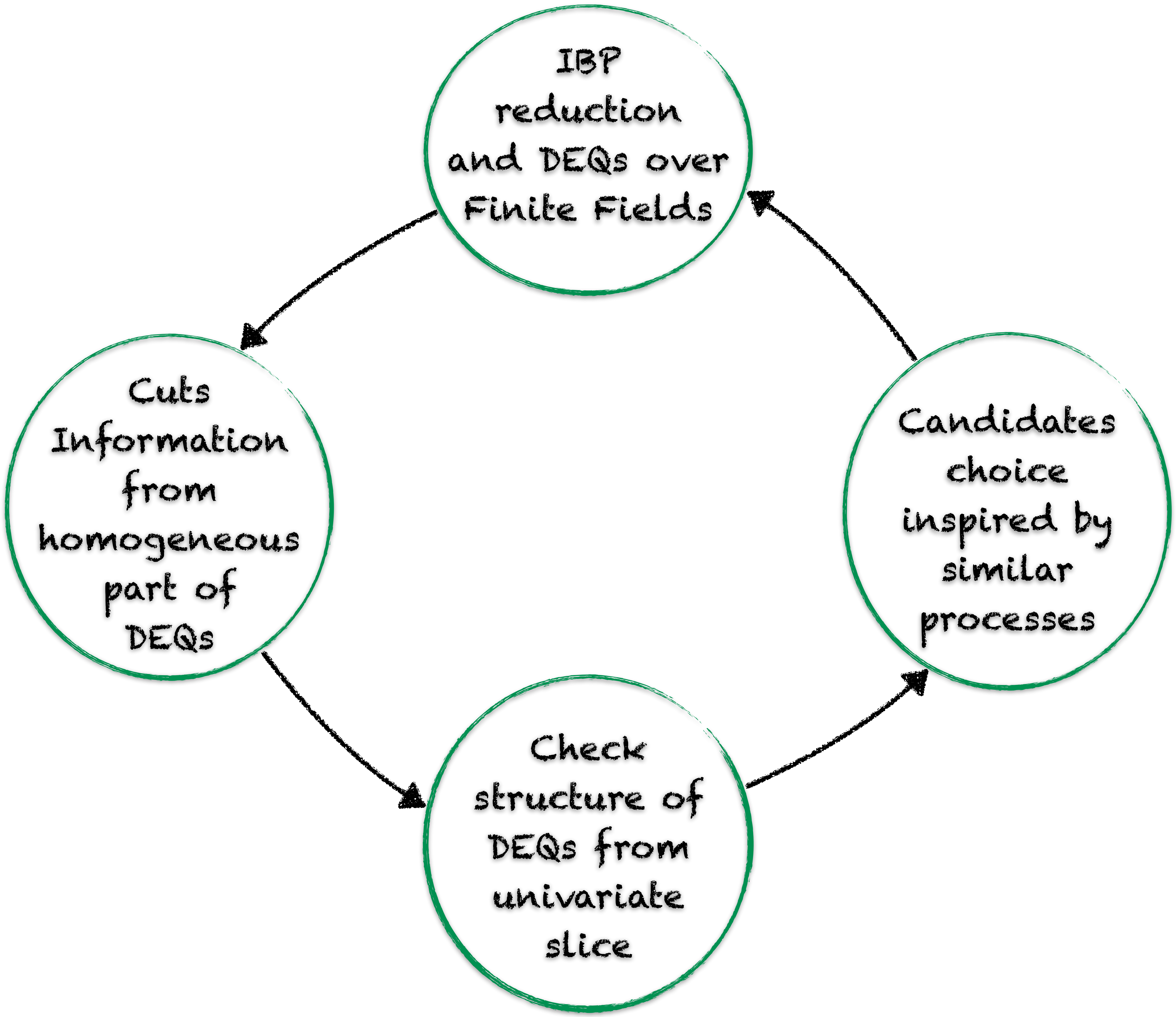
Emerging Patterns

five-point MIs	four-point MIs	3 and 2-point MIs
Scalar integrals with numerators and local integrand insertions	Scalar integrals with numerators and dotted denominators	Scalar integrals with dotted denominators

Local integrand: $\mu_{ij} = - k_i^{[-2\epsilon]} \cdot k_j^{[-2\epsilon]}$ Numerators: $\mathcal{N} = (k_i + q)^2$

- Good starting point to build UT candidates MIs
- More in depth analysis performed exploiting Leading Singularities, Loop-by-Loop…

A cooking recipe for Canonical Basis



Fast pipeline to generate and tests possible candidates integrals for Canonical Basis

New Five-Point MIs sectors

- **Pentagon-Box:**

$$\mathcal{N}_1 = (k_1 + p_5)^2, \quad \mathcal{N}_2 = \mu_{11}, \quad \mathcal{N}_3 = \mu_{12}$$

- **Double-Boxes:**

$$\mathcal{N}_4 = (k_1 + p_5)^2, \quad \mathcal{N}_5 = (k_2 + p_1)^2, \quad \mathcal{N}_6 = \mu_{12}$$

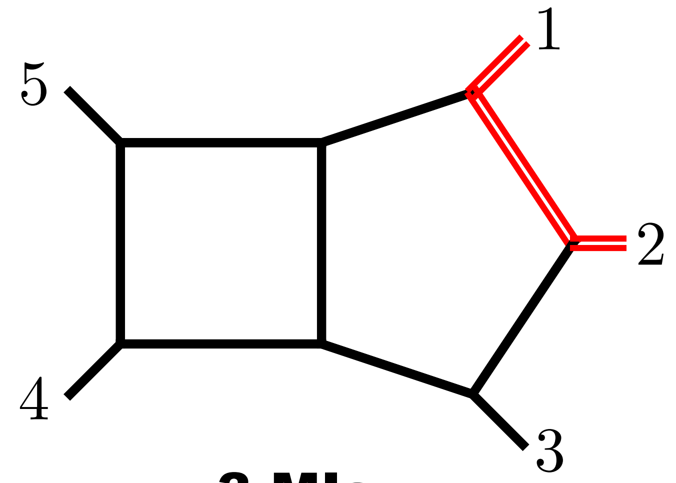
- **Box-Triangles:**

$$\mathcal{N}_7 = (k_2 + p_1 + p2)^2, \quad \mathcal{N}_8 = \frac{\mu_{11}}{D_8}, \quad \mathcal{N}_9 = \frac{\mu_{12}}{D_8}, \quad \mathcal{N}_{10} = \frac{\mu_{22}}{D_8}$$

- **Pentagon-Bubble:**

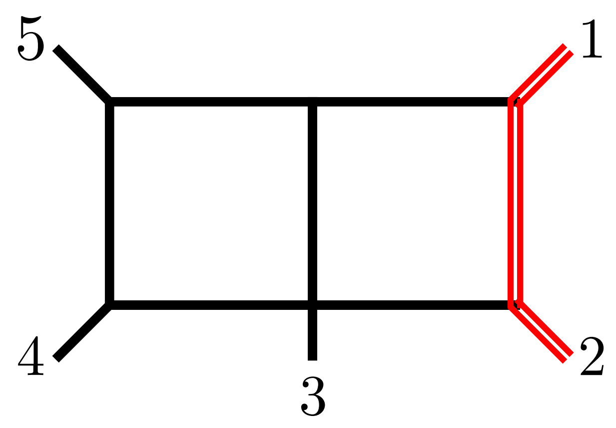
$$\mathcal{N}_{11} = \frac{\mu_{11}}{D_8}$$

Pentagon-Box



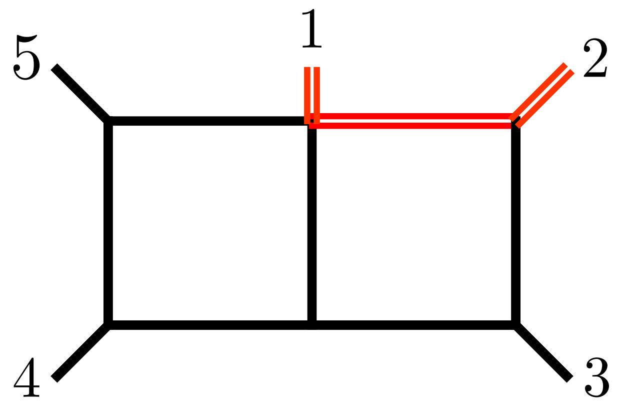
3 MIs

Double-Box



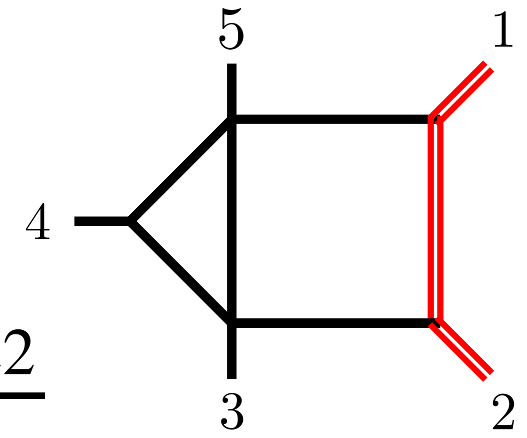
4 MIs

Double-Box



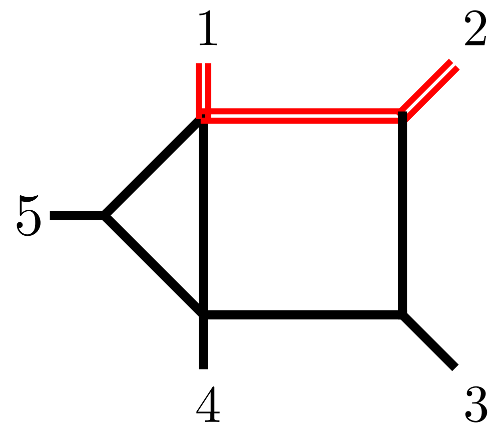
3 MIs

Box-Triangle



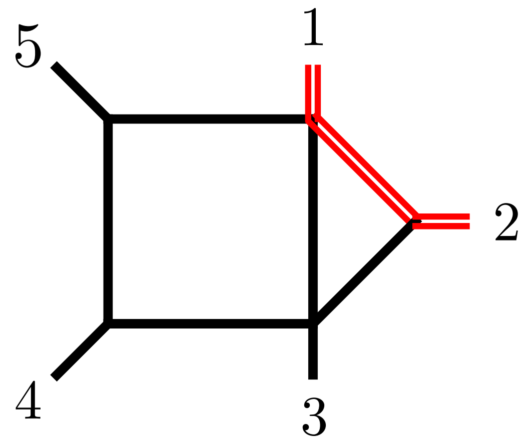
2 MIs

Box-Triangle



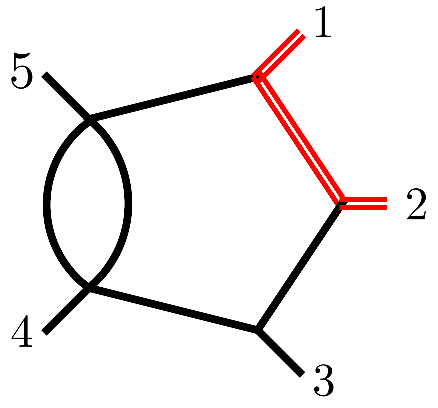
2 MIs

Box-Triangle



5 MIs

Pentagon-Bubble



2 MIs

Generalised Power Series Evaluation: A proof of concept

- We exploit the Generalised Power Series method as implemented in DiffExp

Series Solution around singular points of DEQs

$$\vec{f}(t,\epsilon) = \sum_{k=0}^{\infty} \epsilon^k \sum_{i=0}^{N-1} \rho_i(t) \vec{f}_i^{(k)}(t), \quad \rho(t) = \begin{cases} 1, & t \in [t_i - r_i, t_i + r_i) \\ 0, & t \notin [t_i - r_i, t_i + r_i) \end{cases}, \qquad \vec{f}_i^{(k)}(t) = \sum_{l_1=0}^{\infty} \sum_{l_2=0}^{N_{i,k}} c_k^{(i,l_1,l_2)} \left(t - t_i\right)^{\frac{l_1}{2}} \log(t - t_i)^{l_2}$$

- Numerical boundary values obtained with AMFlow
- Numerical evaluation of MIs in whole phase-space
- Proof of Concept for this computation
- Optimisation could lead to phenomenological applications

Summary

- First computation for two-loop five-point Feynman Integrals with Internal Massive Propagators
- MIs computation for two-loop planar topology represents the first ingredient for a NNLO QCD corrections to ttj
- UT structure follows emerging pattern amongst five-point processes

Outlook

- Computation of all the MIs relevant for the NNLO ttj production in the planar leading color limit
- In the planar leading color limit the functional space of MIs should NOT involve elliptic integrals
- Efficient numerical evaluation: Construction of a Pentagon Functions basis for ttj at NNLO in the planar limit

Thank you for your attention!