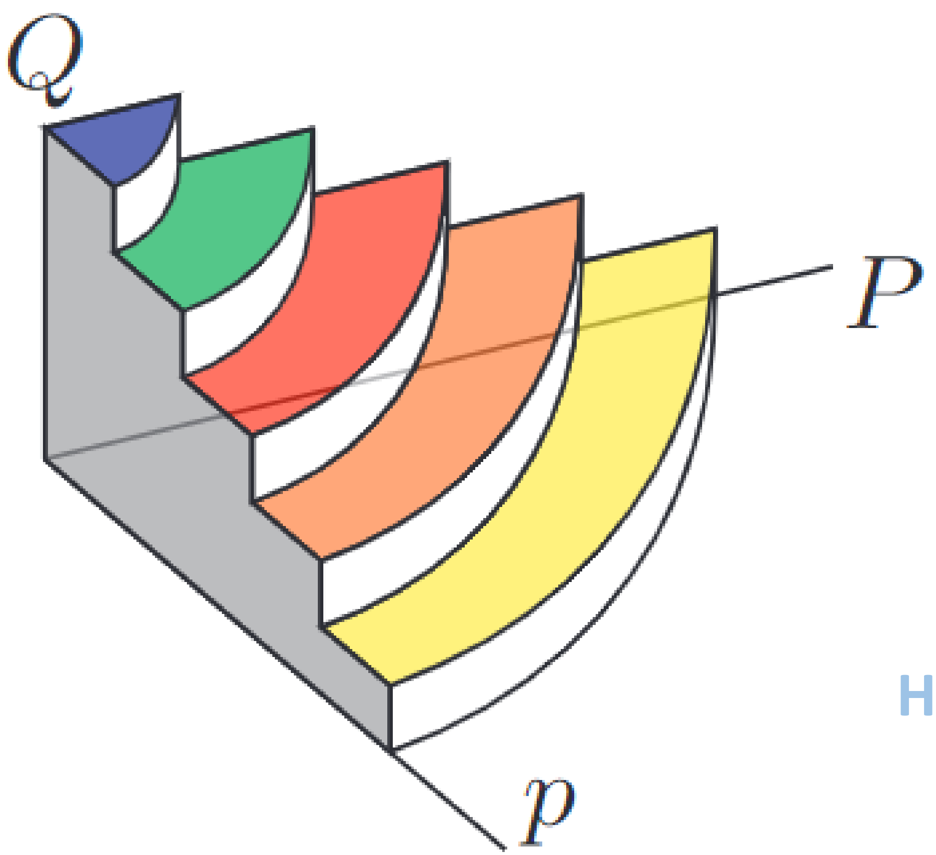


A graph-theoretical approach to the Method of Regions

Yao Ma

Higgs Centre for Theoretical Physics
University of Edinburgh

The method of regions (MoR)

$$\text{entire space} = R_1 \cup R_2 \cup \cdots \cup R_n$$

$$\mathcal{I} = \mathcal{I}^{(R_1)} + \mathcal{I}^{(R_2)} + \cdots + \mathcal{I}^{(R_n)}.$$

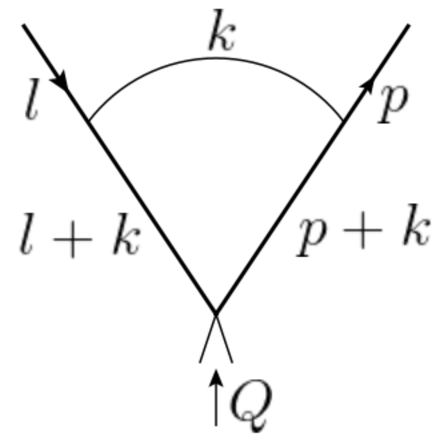
- **The original integral $\mathcal{I}$ can be approximated, or even restored, by the sum over contributions from each region.**

- **The integration measure is the entire space for each term.**

- **The regions are chosen using heuristic methods based on examples and experience.**

The method of regions (MoR)

Example: one-loop Sudakov form factor



Kinematic limit (on-shell limit)

$$p^\mu \sim Q(\underset{+}{\lambda}, \underset{-}{1}, \underset{\perp}{\lambda^{\frac{1}{2}}}), \quad l^\mu \sim Q(\underset{+}{1}, \underset{-}{\lambda}, \underset{\perp}{\lambda^{\frac{1}{2}}})$$

$$p^2/Q^2 \sim l^2/Q^2 \sim \lambda \ll 1$$

$$\mathcal{I} = i\pi^{-D/2}\mu^{4-D} \int d^D k \frac{1}{(k^2 + i0)\ [(k+l)^2 + i0]\ [(k+p)^2 + i0]}$$

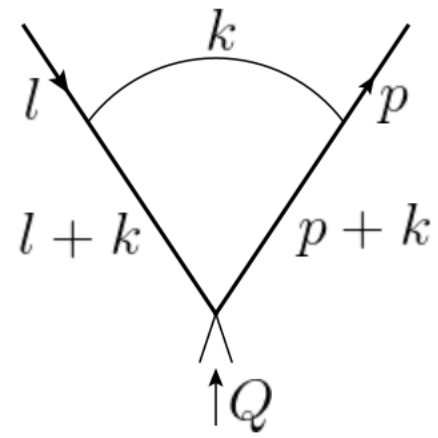
This integral can be evaluated directly.

Or, we can apply the method of regions.

The method of regions (MoR)

Four regions in total:

$$\text{Hard region}: \ k^\mu \sim (1,1,1)Q$$
$$\text{Collinear region to } p: k^\mu \sim (\lambda, 1, \lambda^{\frac{1}{2}})Q$$
$$\text{Collinear region to } l: k^\mu \sim (1, \lambda, \lambda^{\frac{1}{2}})Q$$
$$\text{Soft region}: k^\mu \sim (\lambda, \lambda, \lambda)Q$$



Approximate the original integral w.r.t. each region:

$$I_h = i\pi^{-D/2}\mu^{4-D} \int d^D k \frac{1}{(k^2+i0)(k^2+2k_- \cdot l_+ + i0)(k^2+2k_+ \cdot p_- + i0)} + \cdots$$

$$I_{c1} = i\pi^{-D/2}\mu^{4-D} \int d^D k \frac{1}{(k^2+i0)(2k_- \cdot l_+ + i0)((k+p)^2 + i0)} + \cdots$$

$$I_{c2} = i\pi^{-D/2}\mu^{4-D} \int d^D k \frac{1}{(k^2+i0)((k+l)^2 + i0)(2k_+ \cdot p_- + i0)} + \cdots$$

$$I_s = i\pi^{-D/2}\mu^{4-D} \int d^D k \frac{1}{(k^2+i0)(2k_- \cdot l_+ + l^2 + i0)(2k_+ \cdot p_- + p^2 + i0)} + \cdots$$

The method of regions (MoR)

Four regions in total:

$$\text{Hard region}: \ k^\mu \sim (1,1,1)Q$$
$$\text{Collinear region to } p: k^\mu \sim (\lambda, 1, \lambda^{\frac{1}{2}})Q$$
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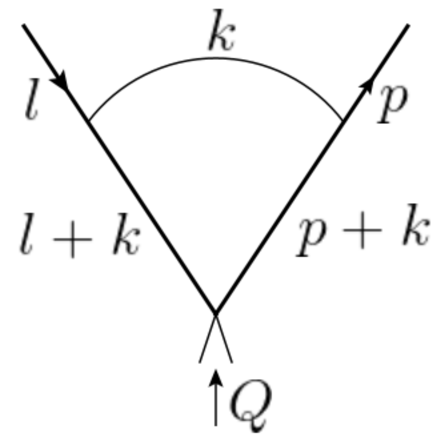

The original integral is reproduced.

$$I = I_h + I_l + I_p + I_s = \frac{1}{Q^2}\left(\ln\frac{Q^2}{(-l^2)}\ln\frac{Q^2}{(-p^2)} + \frac{\pi^2}{3} + \mathcal{O}(\lambda)\right)$$

This equality holds up to ALL orders of λ!

(More examples are presented in Smirnov's book "Applied Asymptotic Expansions in Momenta and Masses".)

The method of regions (MoR)

One can also use the Lee-Pomeransky representation.

$$\mathcal{I}(\boldsymbol{x};\boldsymbol{s}) = \mathcal{C}\int_0^\infty \left(\prod_{e\in G} dx_e\, x_e^{\nu_e-1}\right)\left(\mathcal{P}(\boldsymbol{x};\boldsymbol{s})\right)^{-D/2}$$

edge

$$\mathcal{P}(\boldsymbol{x},\boldsymbol{s}) \equiv \mathcal{U}(\boldsymbol{x}) + \mathcal{F}(\boldsymbol{x},\boldsymbol{s}),$$

$$\mathcal{U}(\boldsymbol{x}) = \sum_{T^1}\prod_{e\notin T^1} x_e, \qquad \mathcal{F}(\boldsymbol{x},\boldsymbol{s}) = -\sum_{T^2} s_{T^2}\prod_{e\notin T^2} x_e + \mathcal{U}(\boldsymbol{x})\sum_e m_e^2 x_e\ .$$

The method of regions (MoR)

One can also use the Lee-Pomeransky representation.

$$\mathcal{I}(\boldsymbol{x};\boldsymbol{s}) = \mathcal{C}\int_0^\infty \left(\prod_{e\in G} dx_e x_e^{\nu_e-1}\right)\left(\mathcal{P}(\boldsymbol{x};\boldsymbol{s})\right)^{-D/2}$$

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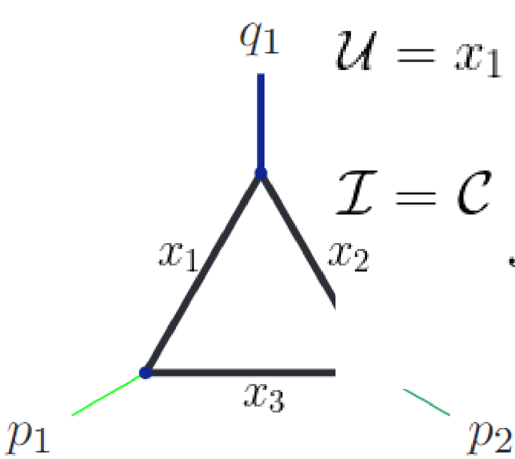


$$\mathcal{U} = x_1 + x_2 + x_3, \quad \mathcal{F} = (-p_1^2)x_1x_3 + (-p_2^2)x_2x_3 + (-q_1^2)x_1x_2$$

$$\mathcal{I} = \mathcal{C}\int_0^\infty dx_1 dx_2 dx_3 x_1^{\nu_1-1} x_2^{\nu_2-1} x_3^{\nu_3-1}$$

$$\cdot \left(x_1 + x_2 + x_3 - p_1^2 x_1 x_3 - p_2^2 x_2 x_3 - q_1^2 x_1 x_2\right)^{D/2}$$

The method of regions (MoR)

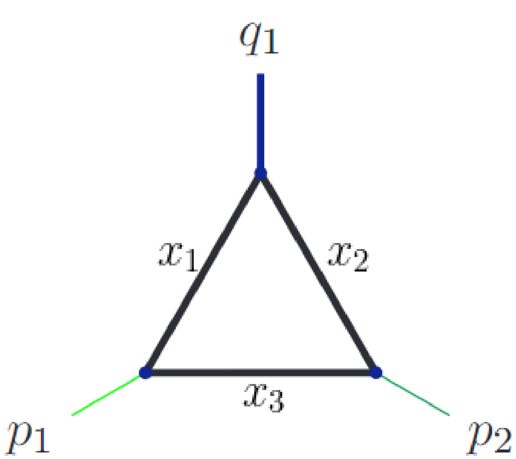
One can also use the Lee-Pomeransky representation.

$$\mathcal{I}(\boldsymbol{x};\boldsymbol{s}) = \mathcal{C}\int_0^\infty \left(\prod_{e\in G} dx_e x_e^{\nu_e-1}\right)\left(\mathcal{P}(\boldsymbol{x};\boldsymbol{s})\right)^{-D/2}$$

edge

$$\mathcal{P}(\boldsymbol{x},\boldsymbol{s}) \equiv \mathcal{U}(\boldsymbol{x}) + \mathcal{F}(\boldsymbol{x},\boldsymbol{s}),$$

$$\mathcal{U}(\boldsymbol{x}) = \sum_{T^1}\prod_{e\notin T^1} x_e, \qquad \mathcal{F}(\boldsymbol{x},\boldsymbol{s}) = -\sum_{T^2} s_{T^2}\prod_{e\notin T^2} x_e + \mathcal{U}(\boldsymbol{x})\sum_e m_e^2 x_e \ .$$



Applying the method of regions

$$\text{Hard region}: \ \boldsymbol{x_1, x_2, x_3 \sim \lambda^0}$$

$$\text{Collinear region to } \boldsymbol{p_1}: \boldsymbol{x_1, x_3 \sim \lambda^{-1}}, \ \boldsymbol{x_2 \sim \lambda^0}$$

$$\text{Collinear region to } \boldsymbol{p_2}: \boldsymbol{x_1 \sim \lambda^0, x_2, x_3 \sim \lambda^{-1}}$$

$$\text{Soft region}: \boldsymbol{x_1, x_2 \sim \lambda^{-1}}, \ \boldsymbol{x_3 \sim \lambda^{-2}}$$

The method of regions (MoR)

Momentum representation

- Divide the entire integration measure into regions
- Approximate the integrand only
- Sum over the contributions

Parameter representation

- Divide the entire integration measure into regions
- Approximate the integrand only
- Sum over the contributions

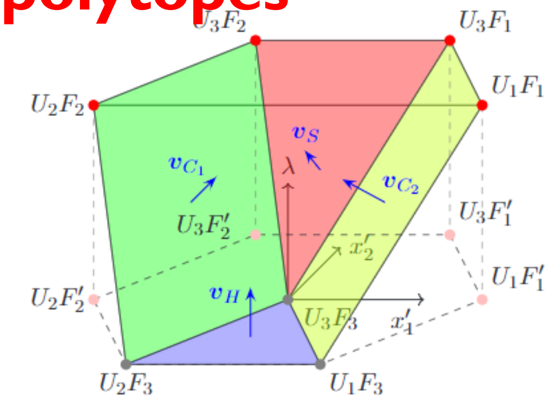
The method of regions (MoR)

Momentum representation

- Divide the entire integration measure into regions
- Approximate the integrand only
- Sum over the contributions
- **Regions are chosen based on examples and experience.**

Parameter representation

- Divide the entire integration measure into regions
- Approximate the integrand only
- Sum over the contributions
- **Regions are given by Newton polytopes**

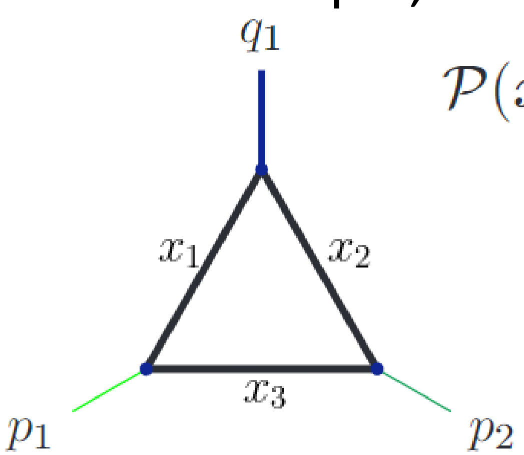


The Newton polytope approach

- **This is a systematic way to find the regions.**

- **For a given Lee-Pomeransky polynomial, we construct the associated Newton polytope: it is the convex hull of the exponents of the Lee-Pomeransky polynomial.**

- For example,



$$\mathcal{P}(\boldsymbol{x},\boldsymbol{s}) = x_1 + x_2 + x_3 - p_1^2 x_1 x_3 - p_2^2 x_2 x_3 - q_1^2 x_1 x_2$$

(1,0,0;0) **(0,1,0;0)** **(0,0,1;0)** **(1,0,1;1)** **(0,1,1;1)** **(1,1,0;0)**

The Newton polytope approach

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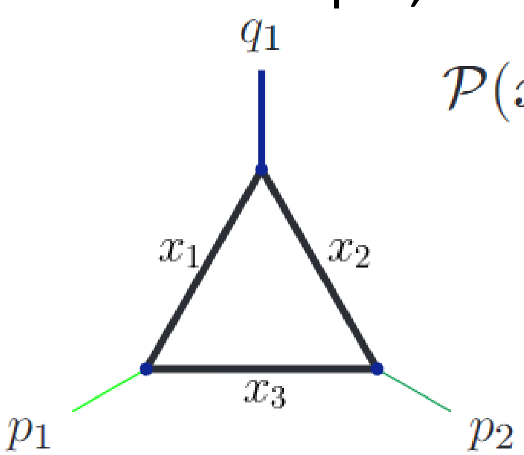


$$\mathcal{P}(\boldsymbol{x},\boldsymbol{s}) = x_1 + x_2 + x_3 - p_1^2 x_1 x_3 - p_2^2 x_2 x_3 - q_1^2 x_1 x_2$$

(1,0,0;0) **(0,1,0;0)** **(0,0,1;0)** **(1,0,1;1)** **(0,1,1;1)** **(1,1,0;0)**

- **The regions are the lower facets of this Newton polytope!**

The Newton polytope approach

Back to our example:

Each region (hard, collinear-1, collinear-2, soft) corresponds to a specific facet containing certain points.

$$\mathcal{P}(\boldsymbol{x},\boldsymbol{s}) = x_1 + x_2 + x_3 - p_1^2 x_1 x_3 - p_2^2 x_2 x_3 - q_1^2 x_1 x_2$$

(1,0,0;0) **(0,1,0;0)** **(0,0,1;0)** **(1,0,1;1)** **(0,1,1;1)** **(1,1,0;0)**

These points are in the **hard facet**.

$$\textbf{Hard region}:\ \boldsymbol{x_1, x_2, x_3 \sim \lambda^0}$$

$$\mathcal{I}_h = \mathcal{C}\int_0^\infty dx_1 dx_2 dx_3 x_1^{\nu_1-1} x_2^{\nu_2-1} x_3^{\nu_3-1} \cdot (x_1 + x_2 + x_3 - q_1^2 x_1 x_2)^{-D/2},$$
$$+\cdots$$

The Newton polytope approach

Back to our example:

Each region (hard, collinear-1, collinear-2, soft) corresponds to a specific facet containing certain points.

$$\mathcal{P}(\boldsymbol{x}, \boldsymbol{s}) = x_1 + x_2 + x_3 - p_1^2 x_1 x_3 - p_2^2 x_2 x_3 - q_1^2 x_1 x_2$$

(1,0,0;0) **(0,0,1;0)** **(1,0,1;1)** **(1,1,0;0)**

These points are in the **collinear-1 facet**.

The Newton polytope approach

Back to our example:

Each region (hard, collinear-1, collinear-2, soft) corresponds to a specific facet containing certain points.

$$\mathcal{P}(\boldsymbol{x},\boldsymbol{s}) = x_1 + x_2 + x_3 - p_1^2 x_1 x_3 - p_2^2 x_2 x_3 - q_1^2 x_1 x_2$$

(0,0,1;0) **(1,1,0;0)**

(0,1,0;0) **(0,1,1;1)**

These points are on the **collinear-2 facet**:

The Newton polytope approach

Back to our example:

Each region (hard, collinear-1, collinear-2, soft) corresponds to a specific facet containing certain points.

$$\mathcal{P}(\boldsymbol{x}, \boldsymbol{s}) = x_1 + x_2 + x_3 - p_1^2 x_1 x_3 - p_2^2 x_2 x_3 - q_1^2 x_1 x_2$$

(0,0,1;0) **(1,0,1;1)** **(0,1,1;1)** **(1,1,0;0)**

These points are on the **soft facet**.

The Newton polytope approach

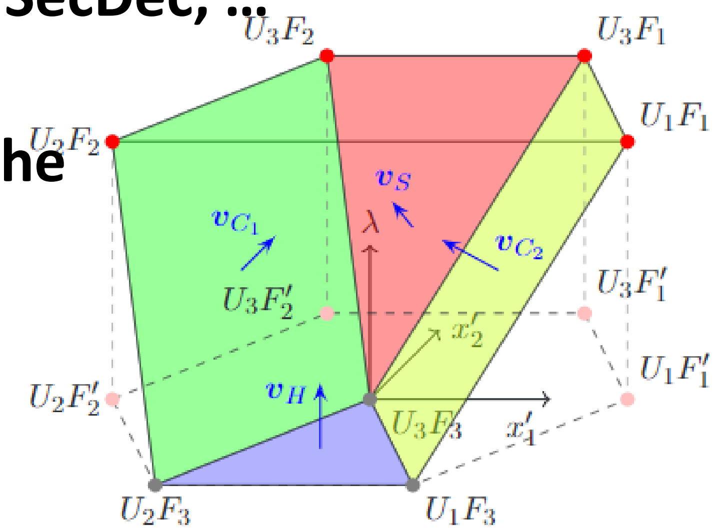
- **Newton polytope = convex hull of the exponents**

  If a graph has N propagators, then the Newton polytope is (N+1)-dimensional.

- **Regions: the lower facets of the Newton polytope!**

- **Some computer codes: Asy2, ASPIRE, pySecDec, …**

- **Our aim: an analytic way to determine the regions.**



Relating regions to Landau equations

- **The Landau equations**

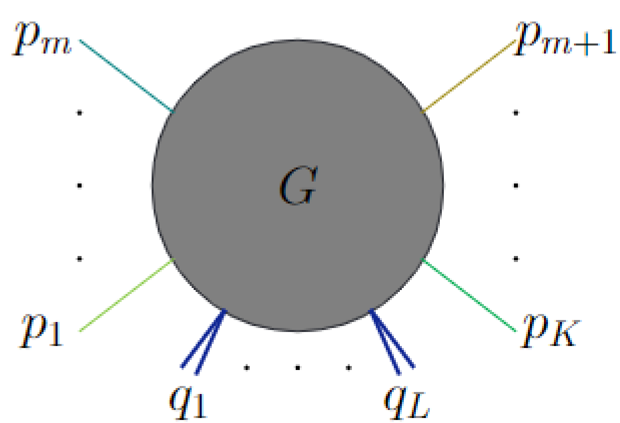
$$\alpha_e l_e^2(k,p,q) = 0 \qquad \forall e \in G$$
$$\frac{\partial}{\partial k_a}\mathcal{D}\left(k,p,q;\alpha\right) = 0 \qquad \forall a \in \{1,\ldots,L\}\,.$$

are necessary conditions for infrared singularity. The solutions of the Landau equations are called pinch surfaces.

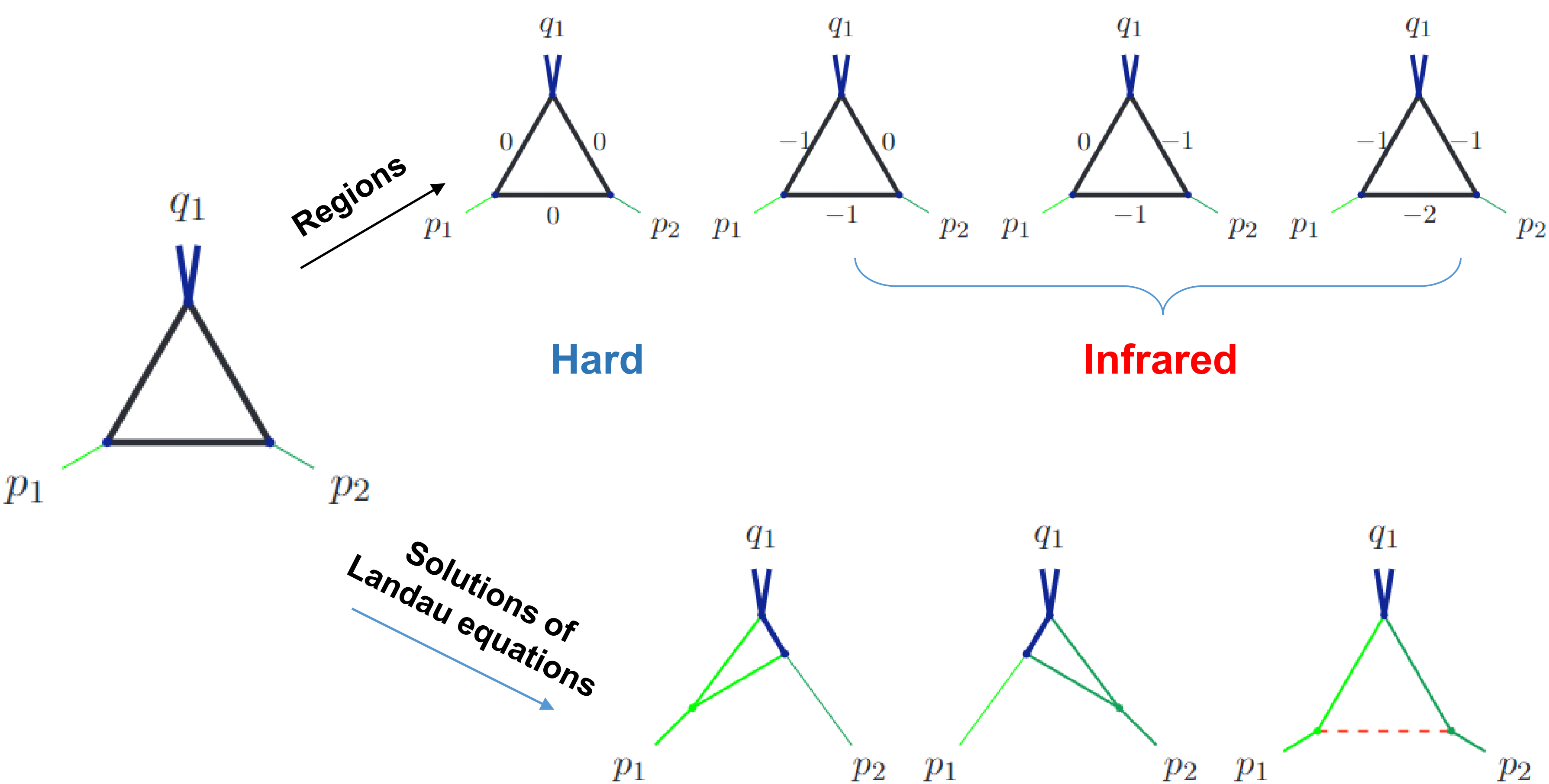
- **The pinch surfaces of hard processes has been studied in detail in the past decades.**

- **Motivation: it looks that the infrared regions are in one-to-one correspondence with the pinch surfaces!**

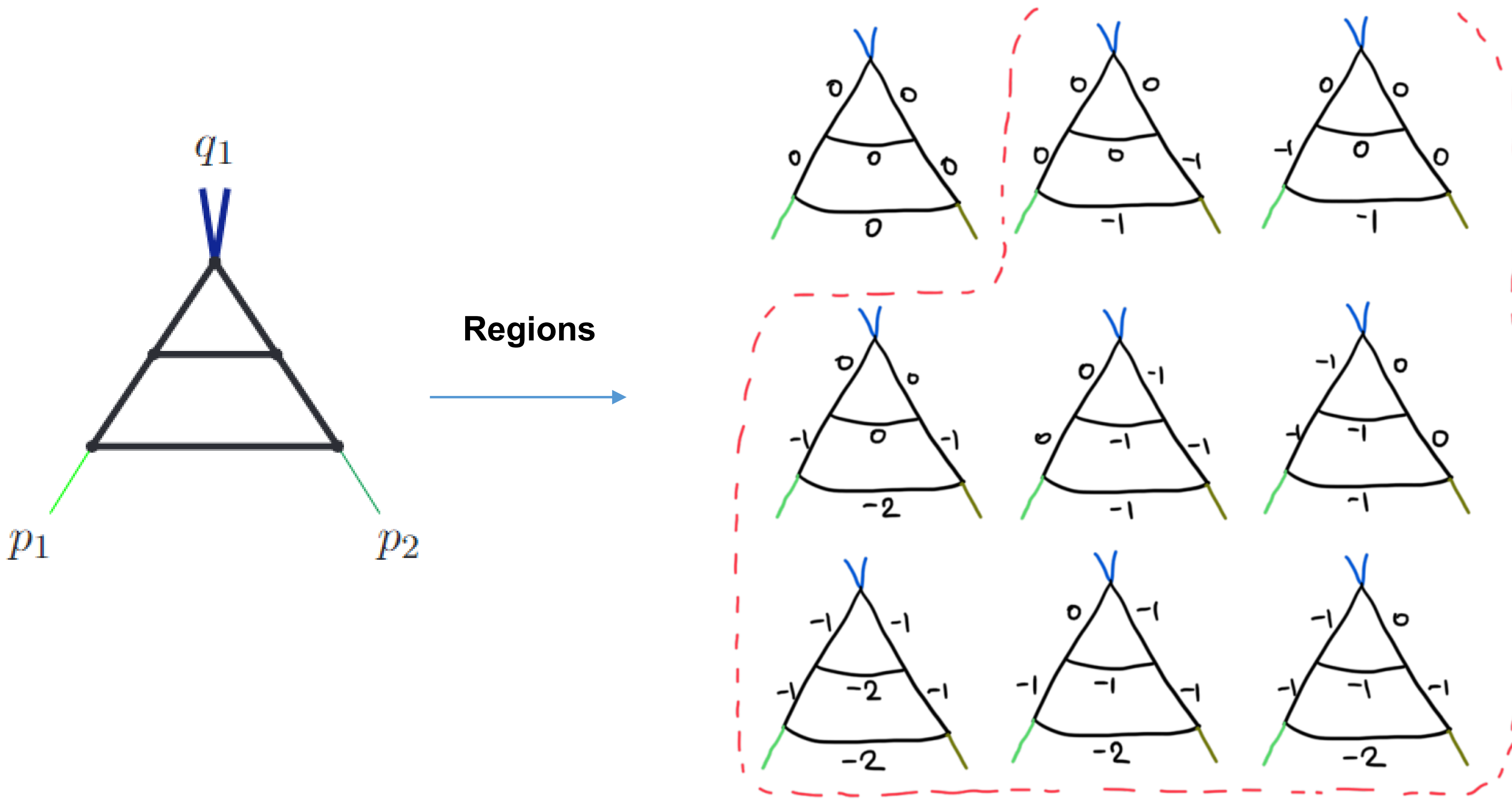
On-shell expansion



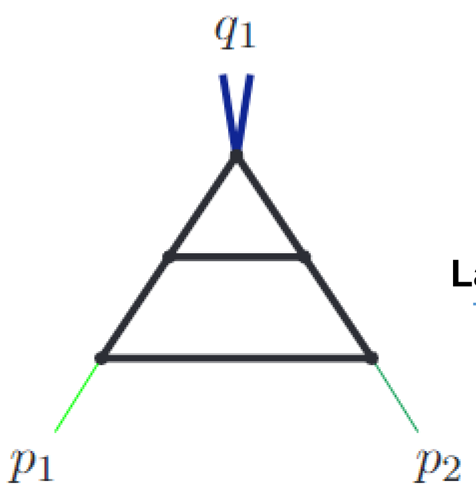
Relating regions to Landau equations



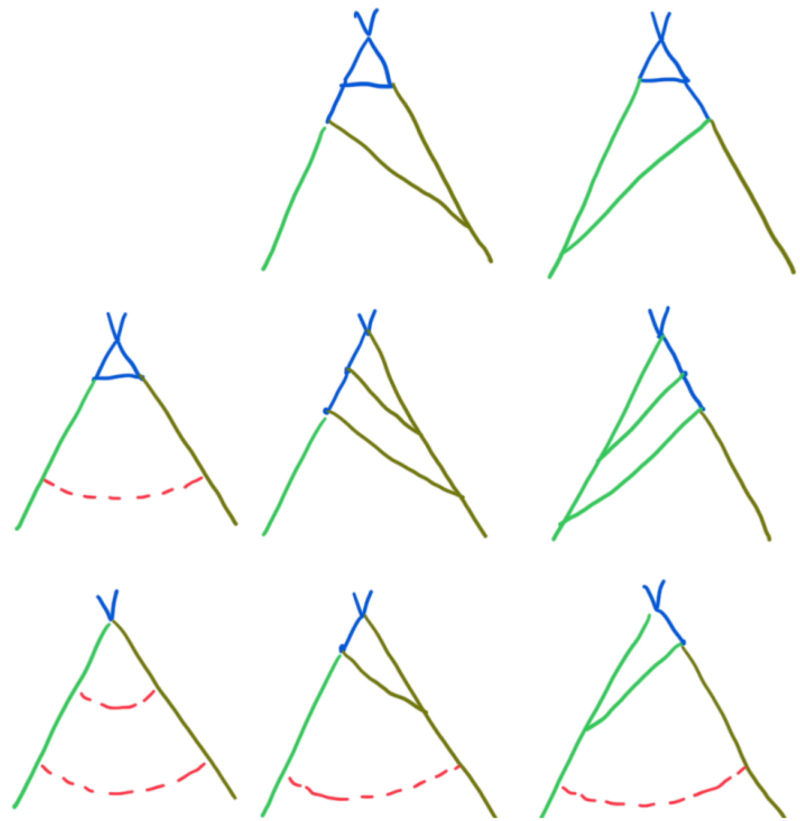
Relating regions to Landau equations



Relating regions to Landau equations



Solutions of Landau equations

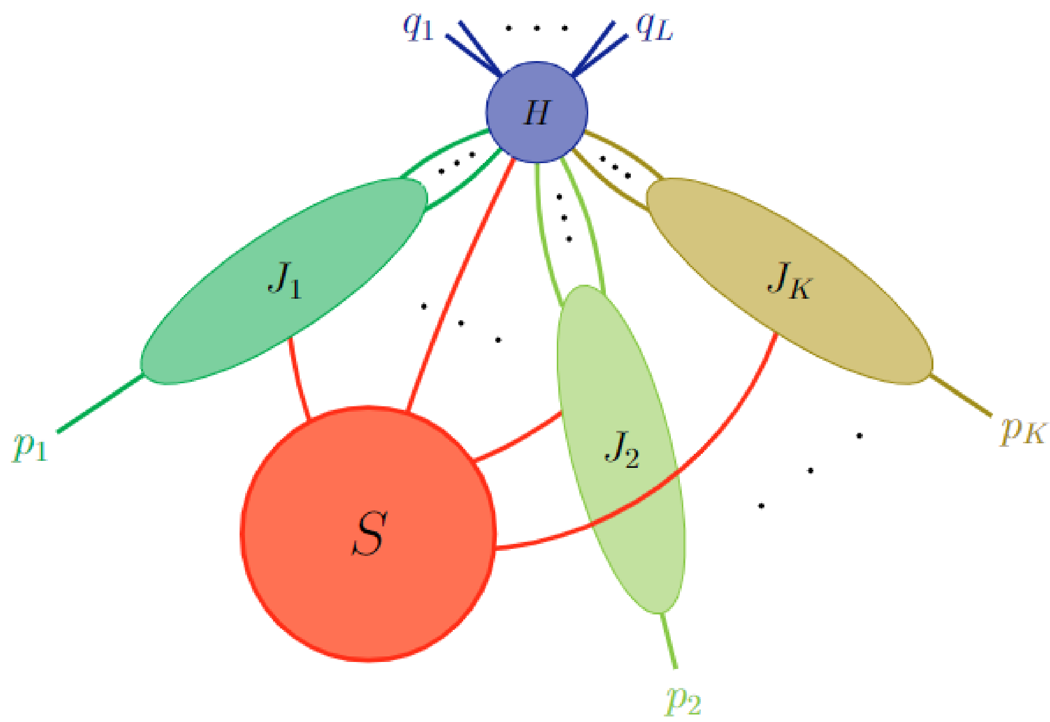


Relating regions to Landau equations

- **This correspondence can be extended to ALL orders.**

- E.Gardi, F.Herzog, S.Jones, YM, J.Schlenk '22

Each solution of the Landau equations corresponds to a region, provided that some requirements of H,J,S are satisfied.



Relating regions to Landau equations

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- E.Gardi, F.Herzog, S.Jones, YM, J.Schlenk '22

Each solution of the Landau equations corresponds to a region, provided that some requirements of H,J,S are satisfied.

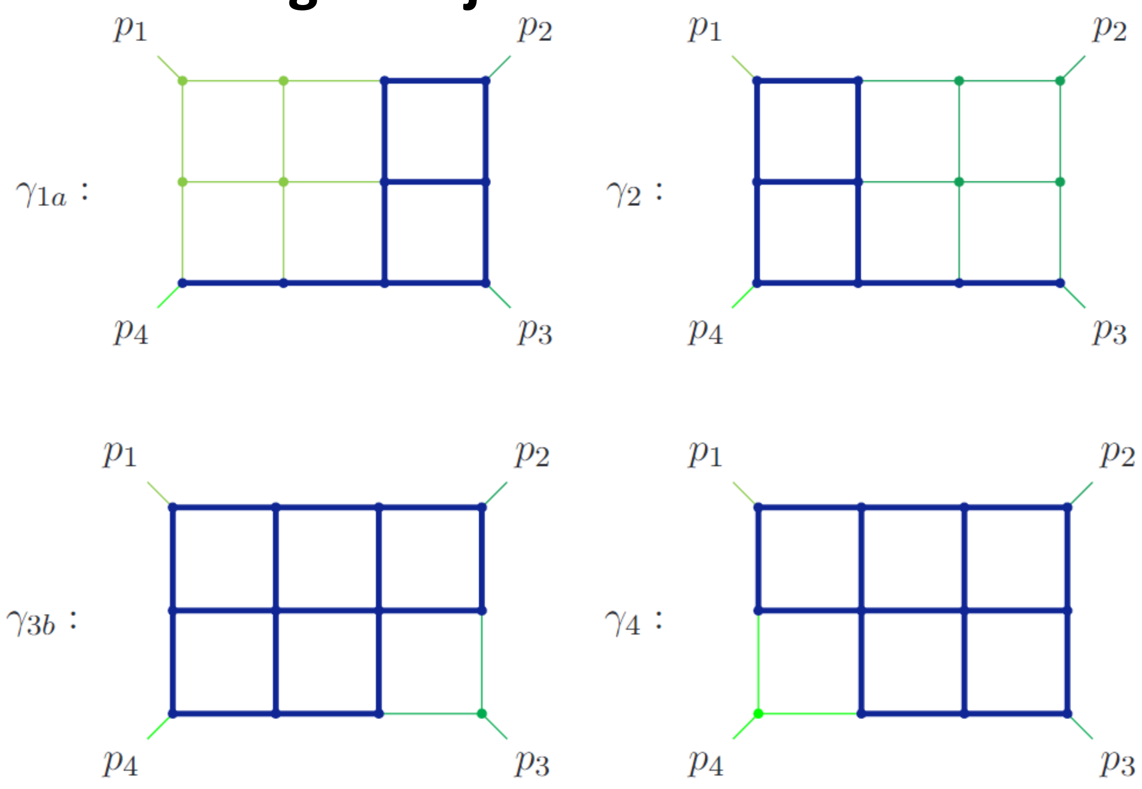
- *Requirement of H: all the internal propagators of H_{red}, which is the reduced form of H, are off-shell.*
- *Requirement of J: all the internal propagators of $\widetilde{J}_{i,\mathrm{red}}$, which is the reduced form of the contracted graph $\widetilde{J}_i$, carry exactly the momentum p_i^μ.*
- *Requirement of S: every connected component of S must connect at least two different jet subgraphs J_i and J_j.*

Landau equations -> Regions!

Relating regions to Landau equations

- Based on this conclusion, we can construct a **graph-finding algorithm** to unveil all the regions.
- A fishnet example

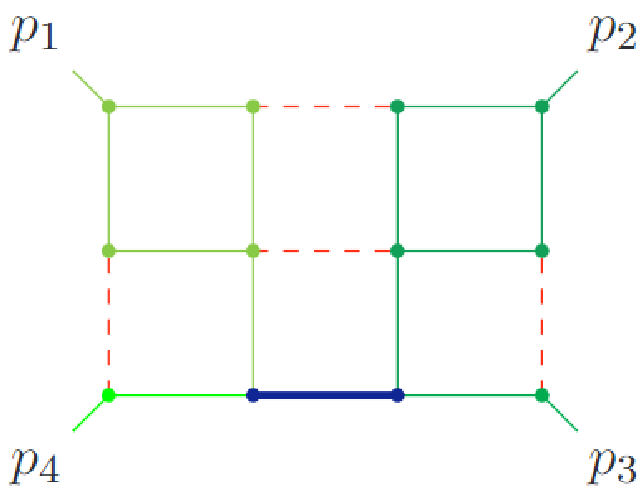
Step 1: constructing the “jets”:



Relating regions to Landau equations

- Based on this conclusion, we can construct a **graph-finding algorithm** to unveil all the regions.
- A fishnet example

Step 2: overlaying the "jets":



This algorithm does not involve constructing Newton polytopes, and can be much faster.

Relating regions to Landau equations

- E.Gardi, F.Herzog, S.Jones, YM, J.Schlenk '22

In addition, one can use this knowledge to study the analytic structure of wide-angle scattering, which further leads to properties regarding the commutativity of multiple on-shell expansions.

Theorem 4. *If R is a jet-pairing soft region that appears in the on-shell expansion of a wide-angle scattering graph G, then the all-order expansion of $\mathcal{I}(G)$ in this region can be written as follows:*

$$\mathcal{T}_{\boldsymbol{t}}^{(R)}\mathcal{I}(\boldsymbol{s}) = \Bigg(\prod_{p_i^2\in\boldsymbol{t}} (p_i^2)^{\rho_{R,i}(\epsilon)}\Bigg) \cdot \sum_{k_1,\dots,k_{|\boldsymbol{t}|}\geqslant 0} \Bigg(\prod_{p_i^2\in\boldsymbol{t}} (-p_i^2)^{k_i}\Bigg) \cdot \overline{\mathcal{I}}_{\{k\}}^{(R)}(\boldsymbol{s}\setminus\boldsymbol{t})\,, \tag{5.8}$$

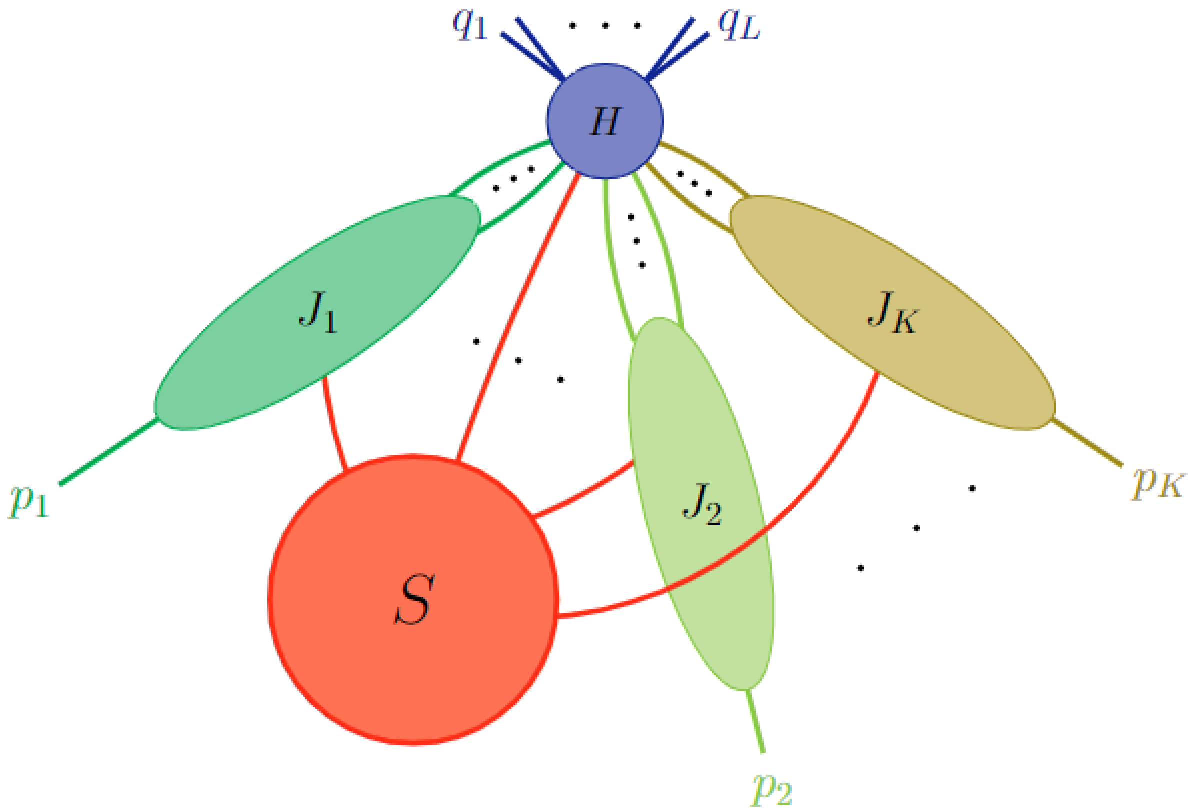
where $\rho_{R,i}(\epsilon)$ is a linear function of ϵ, k_i are non-negative integer powers and $\overline{\mathcal{I}}_{\{k\}}^{(R)}(\boldsymbol{s}\setminus\boldsymbol{t})$ is a function of the off-shell kinematics, independent of any $p_i^2\in\boldsymbol{t}$.

Do all the regions correspond to the solutions of the Landau equations?

Relating regions to Landau equations

- YM '23, to appear

For on-shell expansions, all the regions must correspond to the solutions of the Landau equations!



Relating regions to Landau equations

- YM '23, to appear

  Regions <-> Landau equations!

  The proof is based on graph-theoretical approaches.

$$\mathcal{U}(\boldsymbol{x}) = \underbrace{\sum_{T^1} \prod_{e \notin T^1} x_e}_{\text{Spanning trees}}, \qquad \mathcal{F}(\boldsymbol{x}, \boldsymbol{s}) = -\underbrace{\sum_{T^2} s_{T^2} \prod_{e \notin T^2} x_e}_{\text{Spanning 2-trees}} + \underbrace{\mathcal{U}(\boldsymbol{x})}_{\text{Spanning trees}} \sum_e m_e^2 x_e \ .$$

The problem of verifying **regions** is translated into the problem of finding certain **minimum spanning (2-)trees**!

Relating regions to Landau equations

- YM '23, to appear

  Result: "the on-shell region theorem"

On-shell region theorem

The region vectors appearing in the on-shell expansion of wide-angle scattering are all of the form of $\boldsymbol{v}_R = (u_{R,1}, u_{R,2}, \ldots, u_{R,N}; 1)$, such that for each edge e,

- $u_{R,e} = 0 \quad \Leftrightarrow \quad e \in H;$
- $u_{R,e} = -1 \quad \Leftrightarrow \quad e \in J;$
- $u_{R,e} = -2 \quad \Leftrightarrow \quad e \in S.$

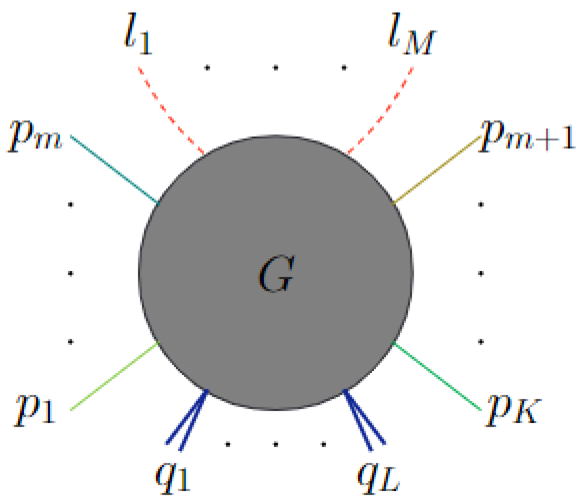
The subgraphs H, J, S are shown in the figure 24, which further satisfy:

(1) for each 1VI component of H, the total momentum flowing into it is off-shell;

(2) for each 1VI component of $\widetilde{J}_i$, the total momentum flowing into it is p_i^μ;

(3) every connected component of S connects at least two different jets.

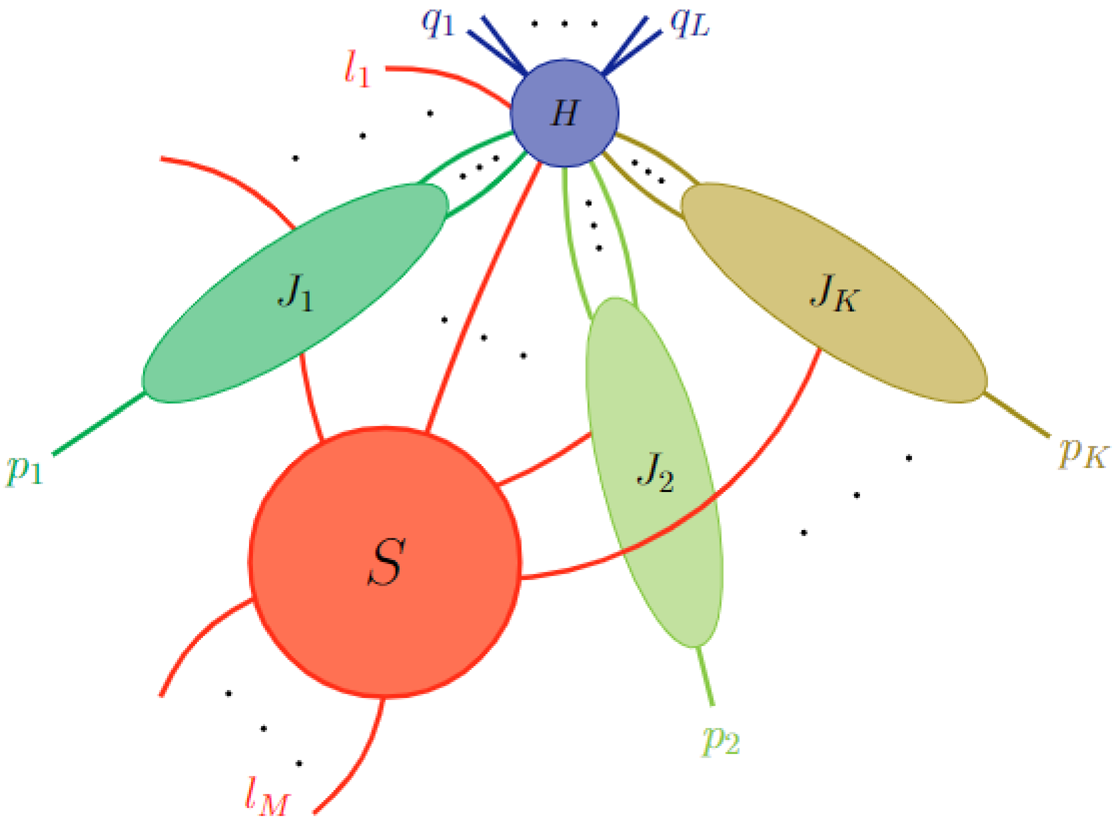
Threshold expansion



Relating regions to Landau equations

- YM '23, to appear

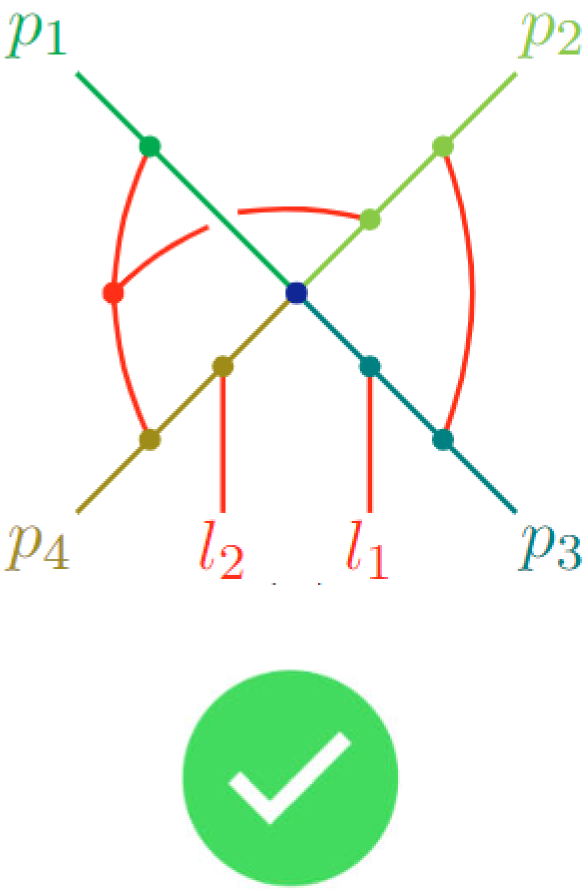
  Generic picture

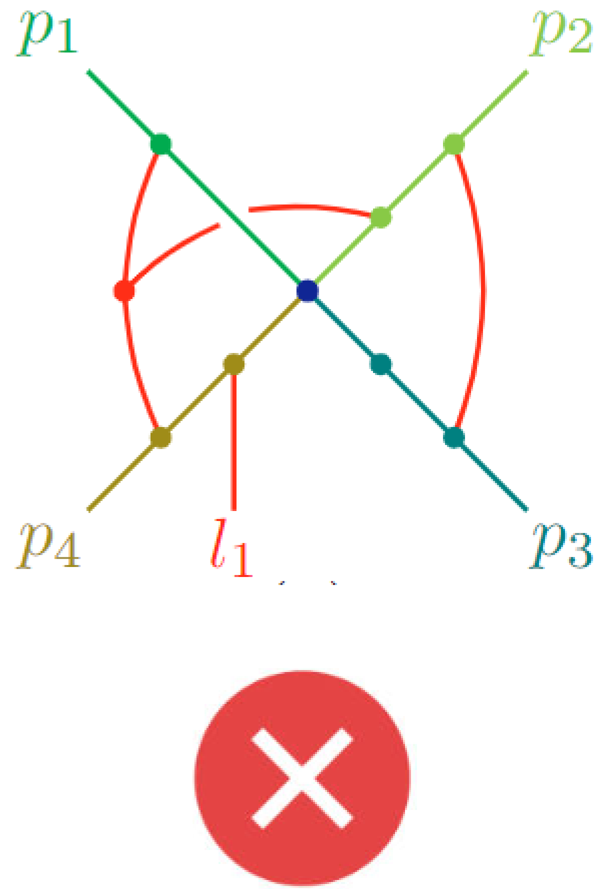


Relating regions to Landau equations

- YM '23, to appear

  Examples





Relating regions to Landau equations

- YM '23, to appear

Result: the "threshold region theorem"

Threshold region theorem

The region vectors appearing in the threshold expansion of wide-angle scattering are all of the form of $\boldsymbol{v}_R = (u_{R,1}, u_{R,2}, \ldots, u_{R,N}; 1)$, such that for each edge e,

- $u_{R,e} = 0 \quad \Leftrightarrow \quad e \in H$;
- $u_{R,e} = -1 \quad \Leftrightarrow \quad e \in J$;
- $u_{R,e} = -2 \quad \Leftrightarrow \quad e \in S$.

The subgraphs H, J, S are shown in the figure 26, which further satisfy the following.

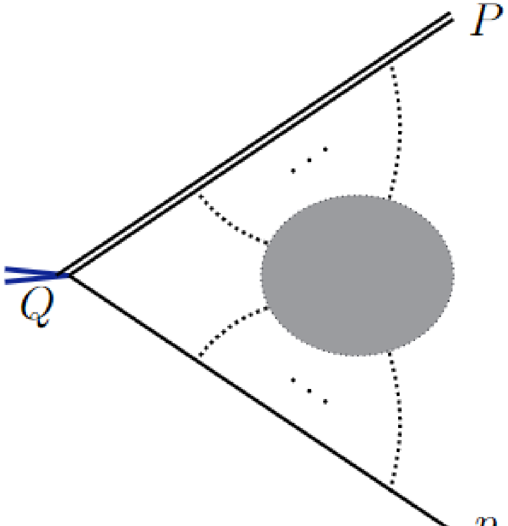
(1) Requirement of H: the momentum flowing into each 1VI component of H is off-shell.

(2) Requirement of J:

- the total momentum flowing into each 1VI component of $\widetilde{J}_i$ is p_i^μ;
- the 1VI component of $\widetilde{J}_i$, which is attached by the external momentum p_i^μ, must contain a vertex v where a soft momentum enters, and v cannot be shared by another 1VI component of $\widetilde{J}_i$;
- all the jets are infrared-compatible.

(3) Requirement of S: every component of S connects two or more jets.

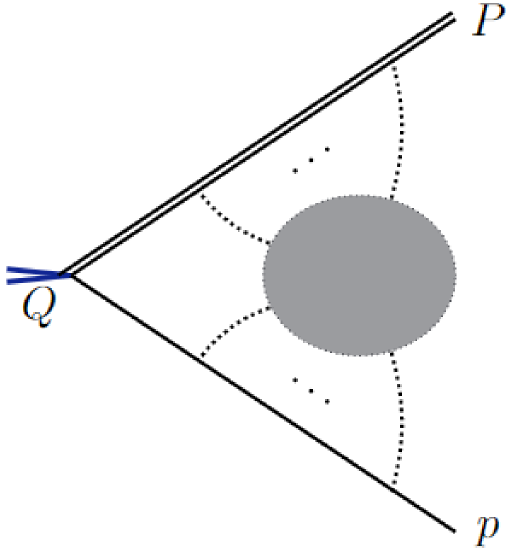
Mass expansion



Relating regions to Landau equations

- YM '23, to appear

  Graphs of the following configuration are considered



$$P^2 = M^2, \qquad p^2 = m^2, \qquad m^2 \ll M^2.$$

More modes are included:

hard, collinear, soft,

semi-hard,

soft·collinear, soft^2·collinear, soft^3·collinear, ….

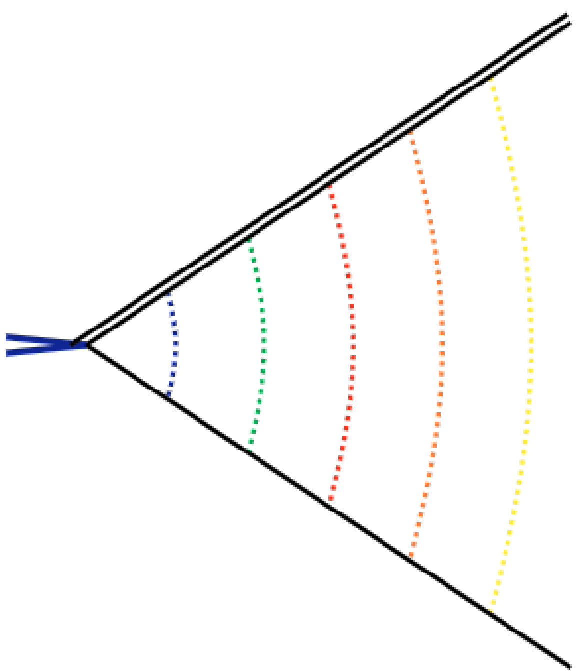
Relating regions to Landau equations

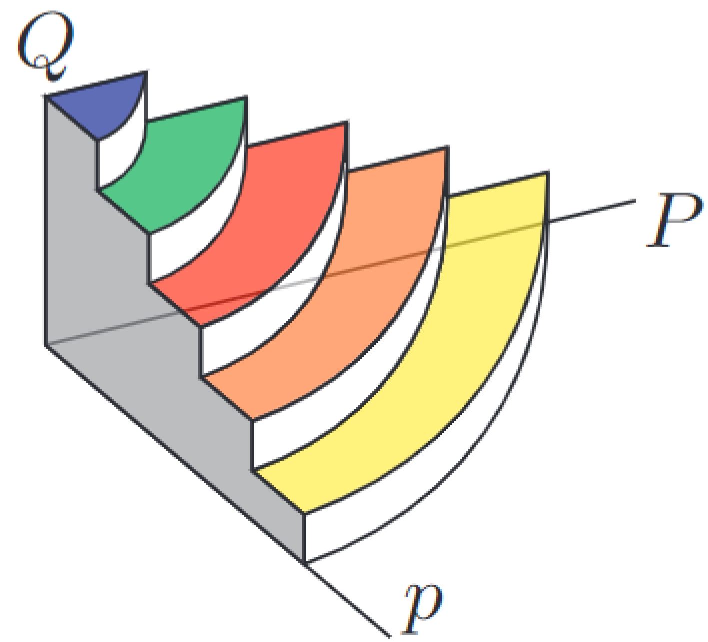
- YM '23, to appear

  More modes are included:

  hard, collinear, soft, semi-hard, soft·collinear, soft^2·collinear, soft^3·collinear, ….

- To characterize these regions: a "**terrace formalism**"





Relating regions to Landau equations

- YM ’23, to appear

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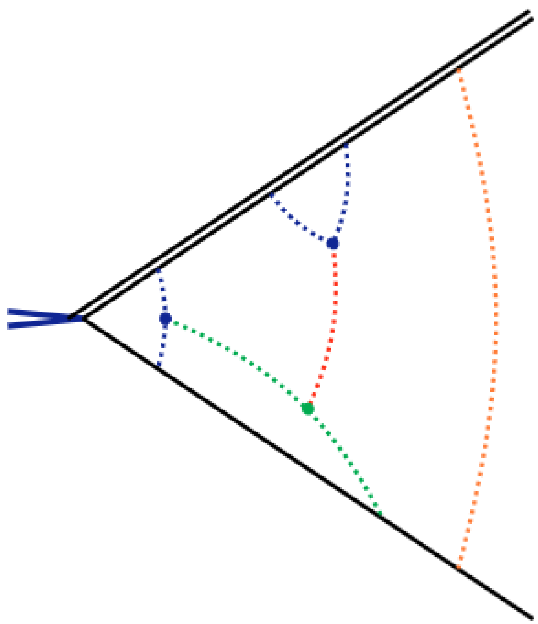
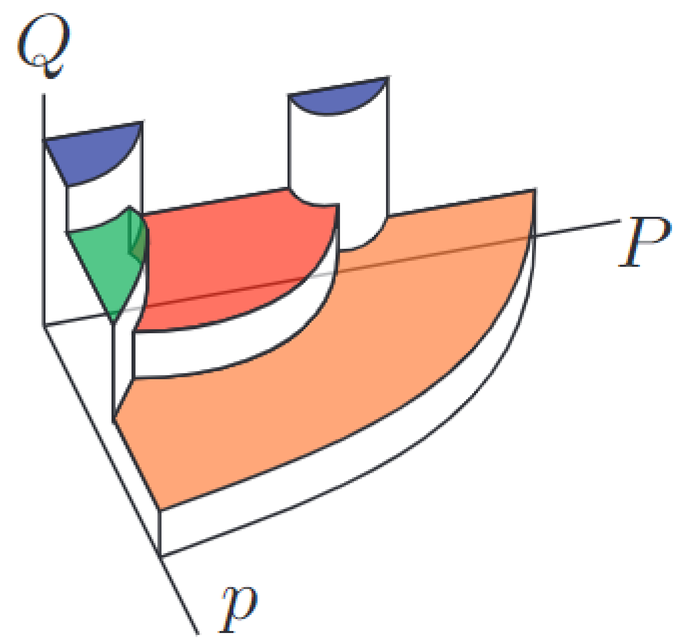




The “terrace formalism”

(some terrace fields in Vietnam)

Main conclusion

The regions corresponding to a given graph can be predicted from the infrared picture!

- on-shell expansion: hard, collinear, soft.

- threshold expansion: hard, collinear, soft.

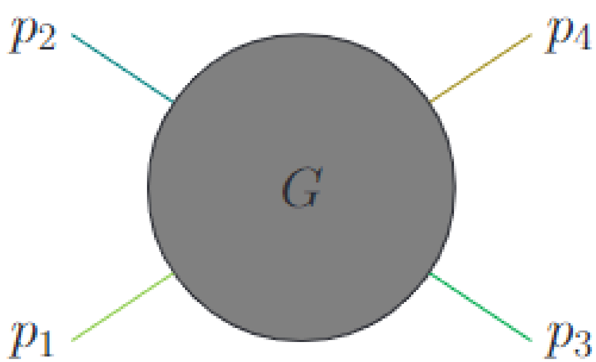
- mass expansion: hard, collinear, soft, semi-hard, soft·collinear, soft^2·collinear…

Outlook 1/2

E.Gardi, F.Herzog, S.Jones, YM [in preparation]

Regions in 2-to-2 high energy limit:

hard, collinear, soft, Glauber, soft·collinear, collinear^3, …



$$(p_1+p_2)^2 = s,$$
$$(p_1+p_3)^2 = t,$$
$$(p_1+p_4)^2 = u,$$

kinematic limit:

$$p_1^2 = p_2^2 = p_3^2 = p_4^2 = 0,$$
$$|t| \ll s \sim |u|,$$

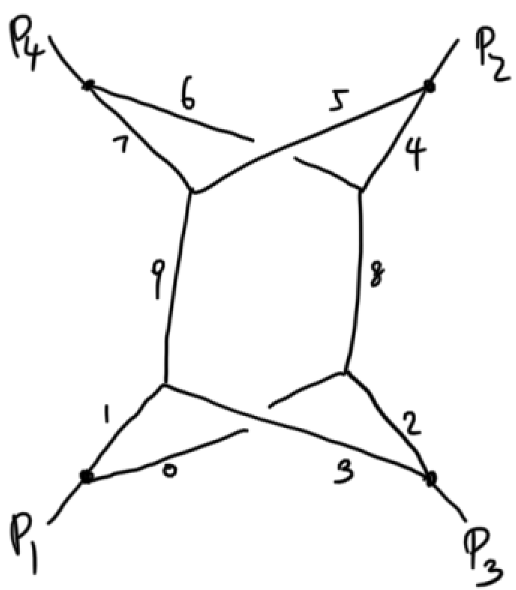
Outlook 1/2

E.Gardi, F.Herzog, S.Jones, YM [in preparation]

Regions in 2-to-2 high energy limit:

hard, collinear, soft, Glauber, soft·collinear, collinear^3, …

Not facets of the (original) Newton polytope



Cancellations occur within the Lee-Pomeransky polynomial, such as

$$(x_i - x_j)\cdot(\ldots)$$
$$(x_i x_j - x_k x_l)\cdot(\ldots)$$
$$\ldots$$

Much more to explore!

Outlook 2/2

1. What will the conclusions be in some other expansions and/or processes?

2. Can one even justify the method of regions with the help of our results?

3. These results can be useful to investigate the infrared structure of gauge theories (e.g. the IR forest formula, etc.).

4. Application to other studies involving Newton polytopes, resummation, SCET?

5. …

Thank you!