Four-loop large-n_f contributions to the non-singlet structure functions

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[2211.16485]

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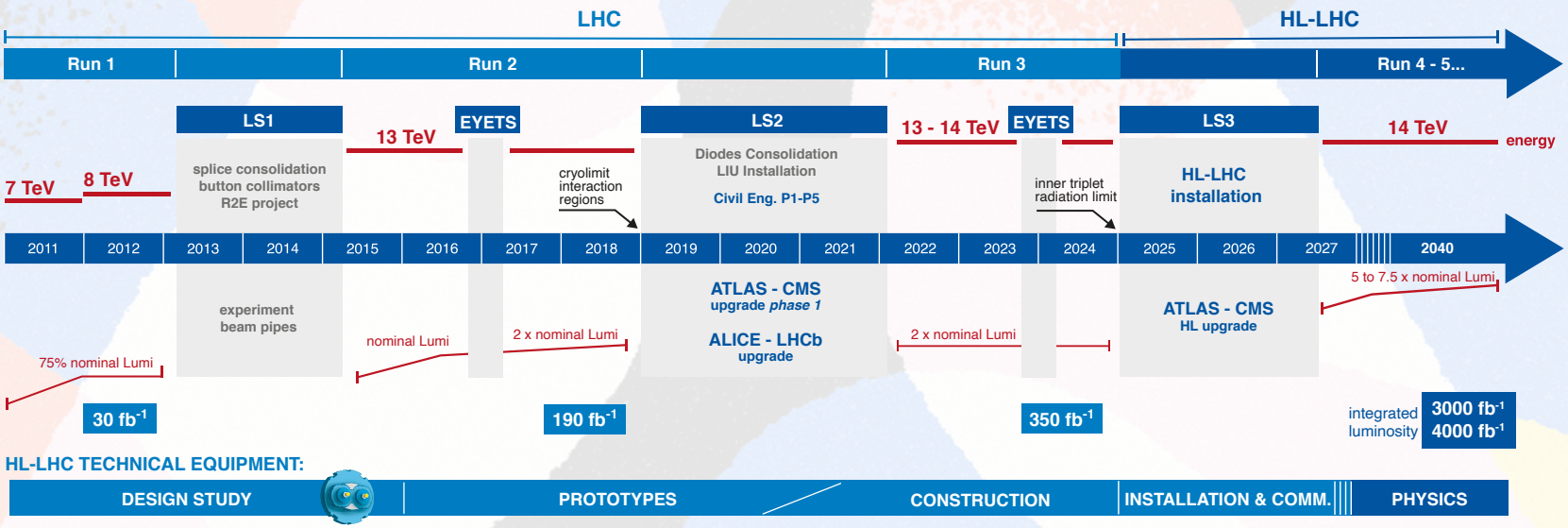
Nikhef

Centre for Theoretical Physics · Higgs

The University of Edinburgh

Motivation

- The computation of contributions to the DIS coefficient functions represents the ideal testing ground for the possible applications for computing splitting functions.
- **Theoretical predictions** need to keep up with the ever-increasing precision of **experimental measurements**
- Need to understand the SM background in order to resolve **new physics**



- **Example:** Higgs inclusive: $8\% \to 3\%$ expected experimental uncertainty at $3000\ \text{fb}^{-1}$. The PDF uncertainty on the theoretical prediction cannot be neglected anymore.

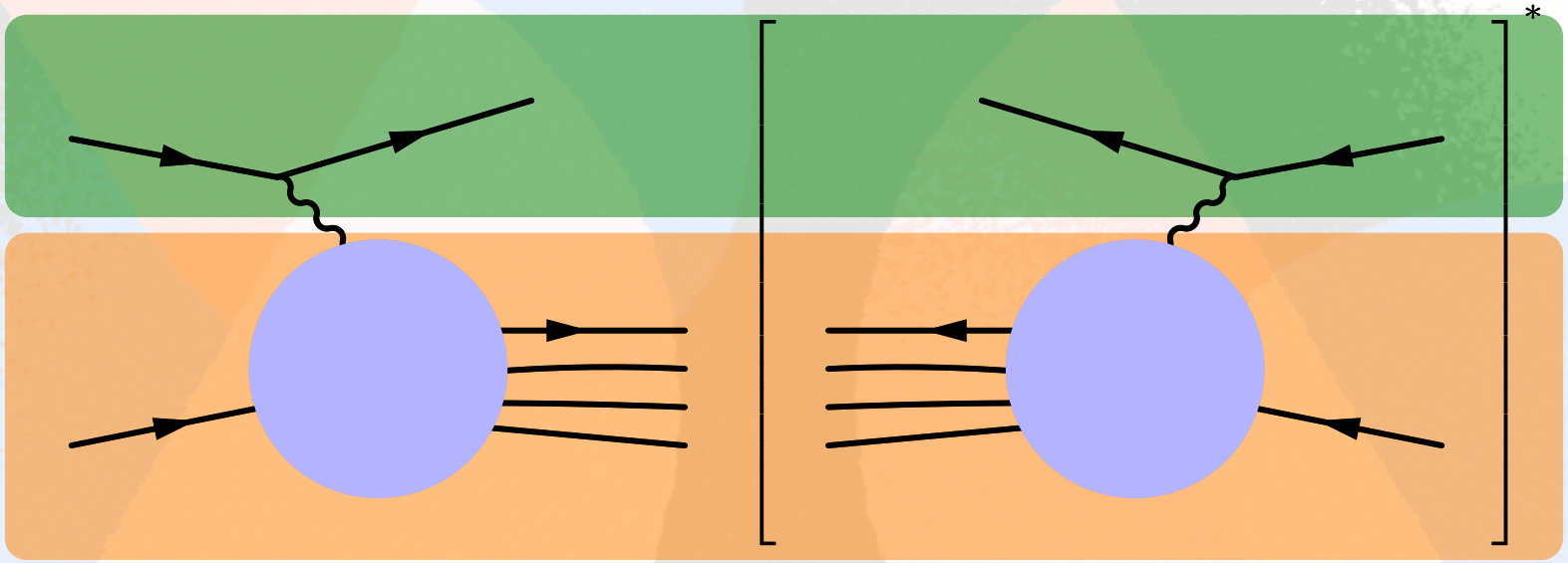
Motivation

	Q [GeV]	$\delta\sigma^{\mathrm{N^3LO}}$	$\delta\sigma^{\mathrm{NNLO}}$	$\delta(\mathrm{scale})$	$\delta(\mathrm{PDF}+\alpha_S)$	$\delta(\mathrm{PDF\text{-}TH})$
$gg \to$ Higgs	m_H	3.5%	30%	$^{+0.21\%}_{-2.37\%}$	$\pm 3.2\%$	$\pm 1.2\%$
$b\bar{b} \to$ Higgs	m_H	-2.3%	2.1%	$^{+3.0\%}_{-4.8\%}$	$\pm 8.4\%$	$\pm 2.5\%$
NCDY	30	-4.8%	-0.34%	$^{+1.53\%}_{-2.54\%}$	$^{+3.7\%}_{-3.8\%}$	$\pm 2.8\%$
	100	-2.1%	-2.3%	$^{+0.66\%}_{-0.79\%}$	$^{+1.8\%}_{-1.9\%}$	$\pm 2.5\%$
CCDY(W^+)	30	-4.7%	-0.1%	$^{+2.5\%}_{-1.7\%}$	$\pm 3.95\%$	$\pm 3.2\%$
	150	-2.0%	-0.1%	$^{+0.5\%}_{-0.5\%}$	$\pm 1.9\%$	$\pm 2.1\%$
CCDY(W^-)	30	-5.0%	-0.1%	$^{+2.6\%}_{-1.6\%}$	$\pm 3.7\%$	$\pm 3.2\%$
	150	-2.1%	-0.6%	$^{+0.6\%}_{-0.5\%}$	$\pm 2\%$	$\pm 2.13\%$

J. Baglio, C. Duhr, B. Mistlberger, R. Szafron JHEP 66 (2022)

Deep Inelastic Scattering

Probing the **hadron** structure by mean of high energetic **leptons** :



- Factorize the **leptonic** from the **hadronic** part in the cross-section

$$\frac{\sigma_{DIS}}{dxdy} = \frac{2\pi\alpha_{em}^2}{Q^2} L^{\mu\nu} W_{\mu\nu}$$

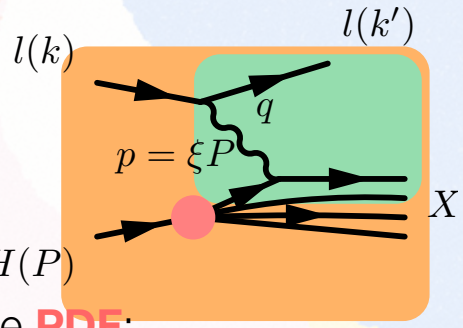
$$Q = -q^2, \qquad x = \frac{Q^2}{2P\cdot q}, \qquad y = \frac{P\cdot q}{P\cdot k}$$

Structure Functions

- The computation of the corresponding 4-loop Wilson coefficients is extremely challenging from the theoretical point of view

$$\frac{\sigma_{DIS}}{dxdy} = \frac{2\pi\alpha_{em}^2}{Q^2} L^{\mu\nu} W_{\mu\nu}$$

$$W_{\mu\nu} = \left(P^\mu - \frac{(P\cdot q)q_\mu}{q^2}\right)\left(P^\nu - \frac{(P\cdot q)q_\nu}{q^2}\right)\frac{F_1(x,Q^2)}{P\cdot q} + \left(-g_{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) F_2(x,Q^2) + i\epsilon_{\mu\nu\rho\sigma}\frac{P^\rho q^\sigma}{2P\cdot q} F_3(x,Q^2)$$



- Where the **hadronic** and **partonic** quantities are related via the **PDF**:

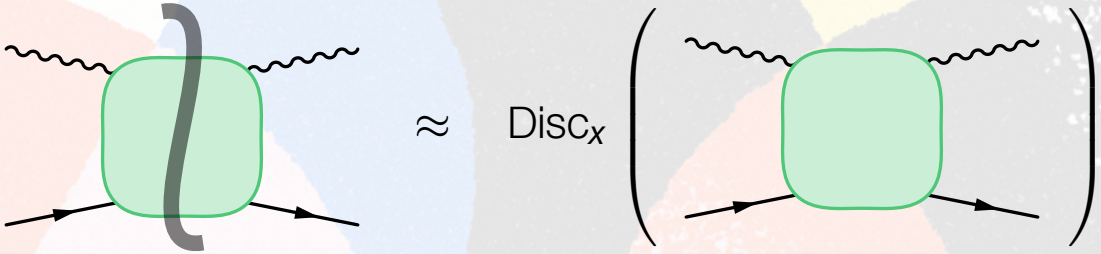
$$F_a(x,Q^2) = \sum_i \left[f_i(\xi) \otimes \mathcal{C}_{a,i}(\xi,Q^2) \right](x)$$

- The problem can be simplified by using the optical theorem for extracting the **Mellin moments** of the process.

Mellin Moments

Objective:

- Compute the hadronic cross-section $\hat{W}_{\mu\nu}$ using the forward scatting $\hat{T}_{\mu\nu}$



How:

- Compute the **Mellin moments** of the structure functions:

$$F_a(x, Q^2) = \sum_i \left[f_i(\xi) \otimes C_{a,i}(\xi, Q^2) \right](x)$$

with the **Mellin transform** defined by

$$M[f(x)](N) = \int_0^1 \mathrm{d}x\, x^{N-1} f(x)$$

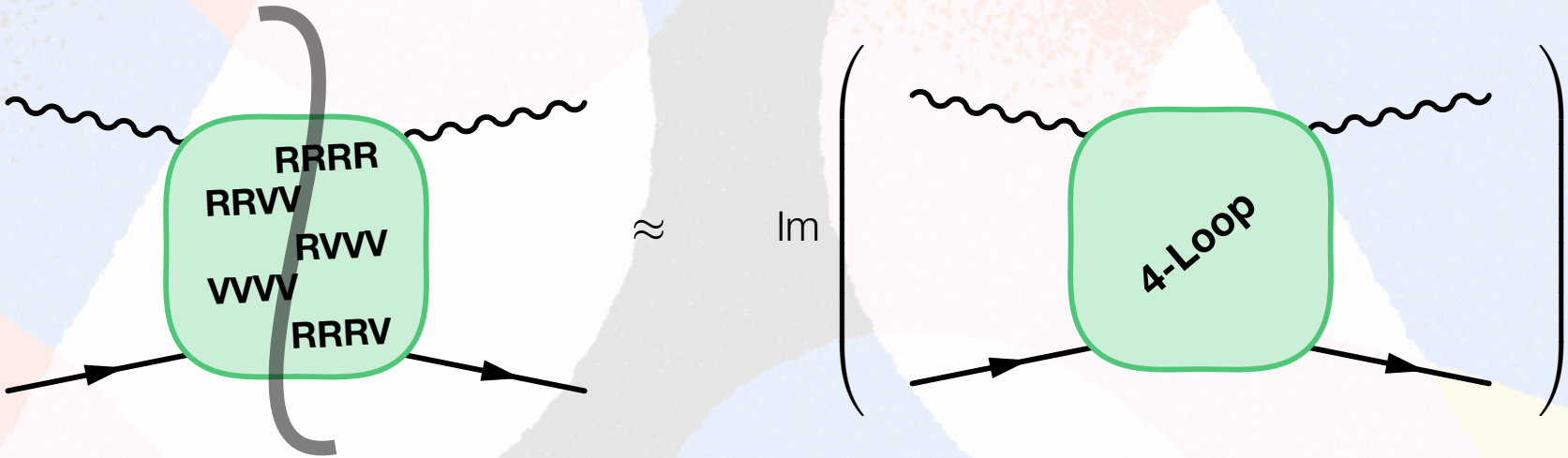
- The Mellin moments of **cross-section** correspond to the expansion coefficients around $\omega = \frac{1}{x} = 0$ of the **Forward Scattering Amplitude** :

$$M[\hat{W}_{\mu\nu}](N) = \frac{1}{N!} \left[\frac{\mathrm{d}^N T_{\mu\nu}}{\omega^N} \right]_{\omega=0}$$

Optical Theorem

- The optical theorem allows us to relate the **cross-section** to the imaginary part of the **Forward Compton Amplitude**

$$\hat{W}_a = \frac{1}{\pi}\mathrm{Im}\hat{T}_a$$



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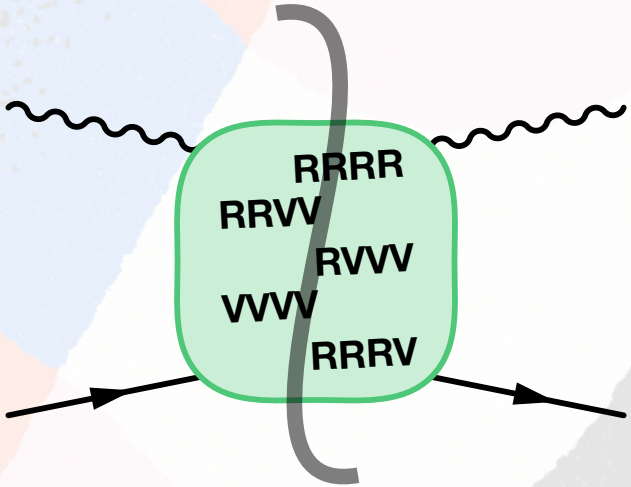
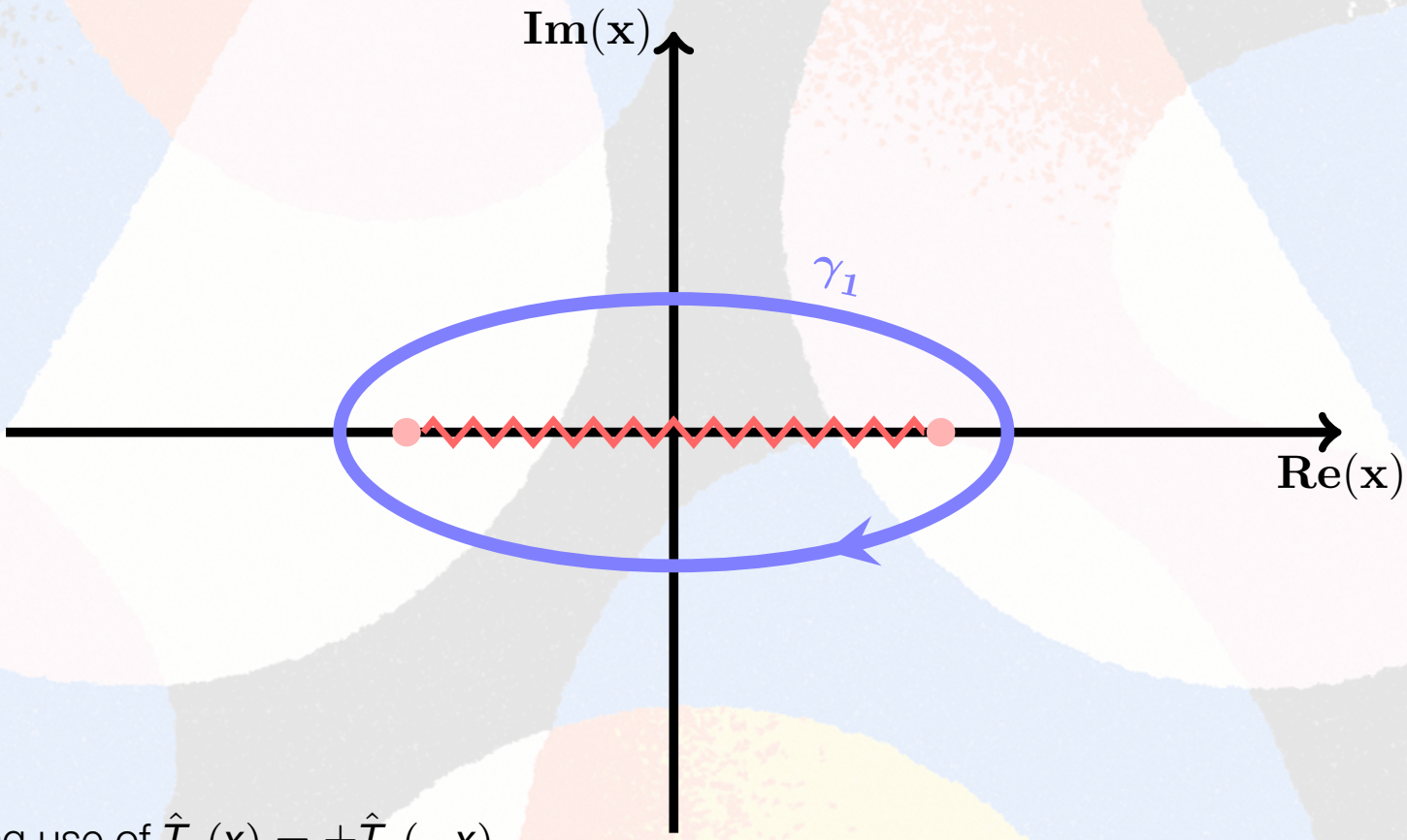


$\approx$ Disc_x

Optical Theorem

We are interested in the **Mellin moments** of the DIS cross-section:

$$M[\hat{W}_a(x)](N) = \int_0^1 \mathrm{d}x\, x^{N-1} \hat{W}_a(x) \approx \int_0^1 \mathrm{d}x\, x^{N-1} Disc_x \left[\hat{T}_a \left(\frac{1}{x} \right) \right]$$

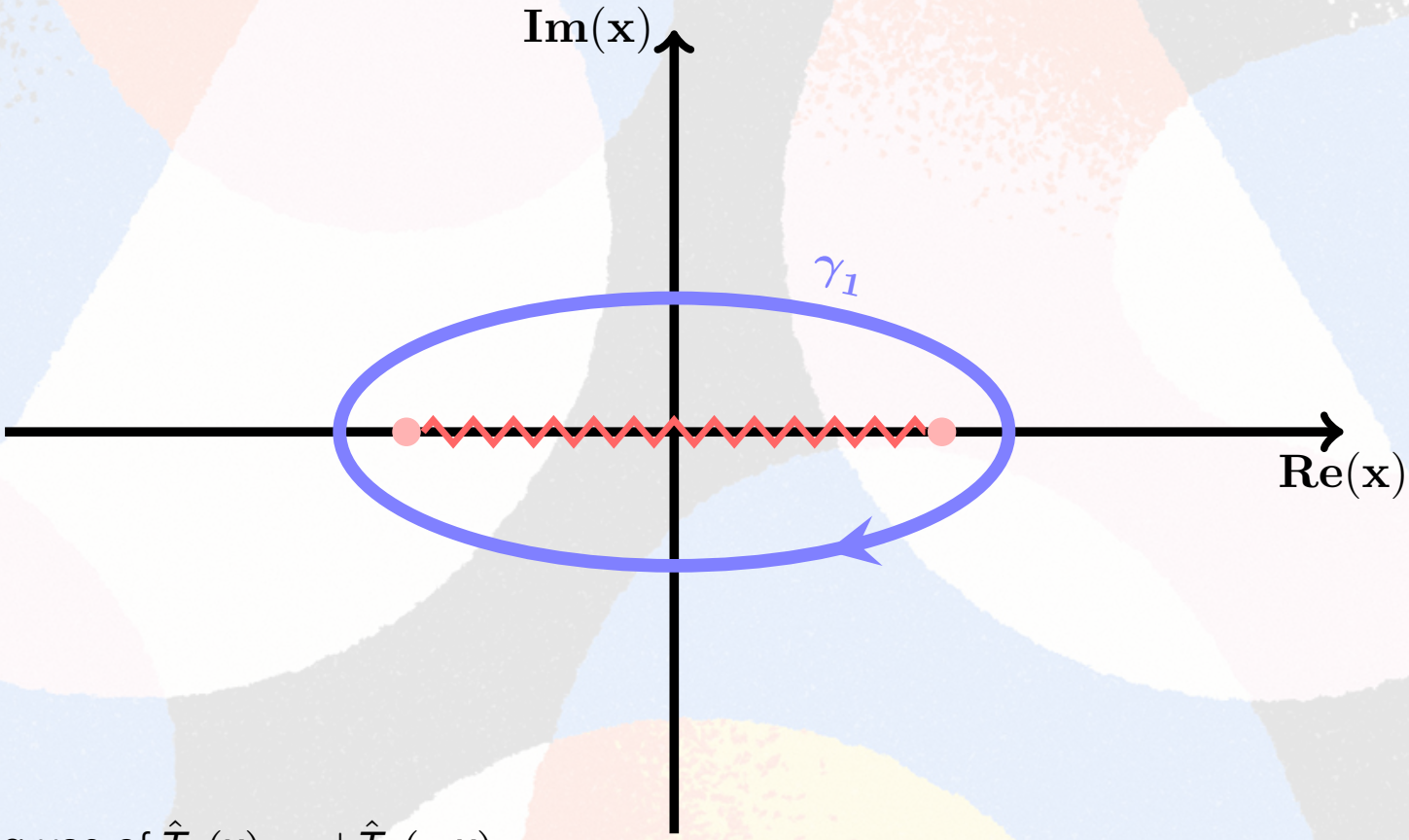


Making use of $\hat{T}_a(x) = \pm \hat{T}_a(-x)$

Optical Theorem

We are interested in the **Mellin moments** of the DIS cross-section:

$$M[\hat{W}_a(x)](N) \approx \int_{\gamma_1} d\omega\, \omega^{-1-N} \hat{T}_a(\omega)$$

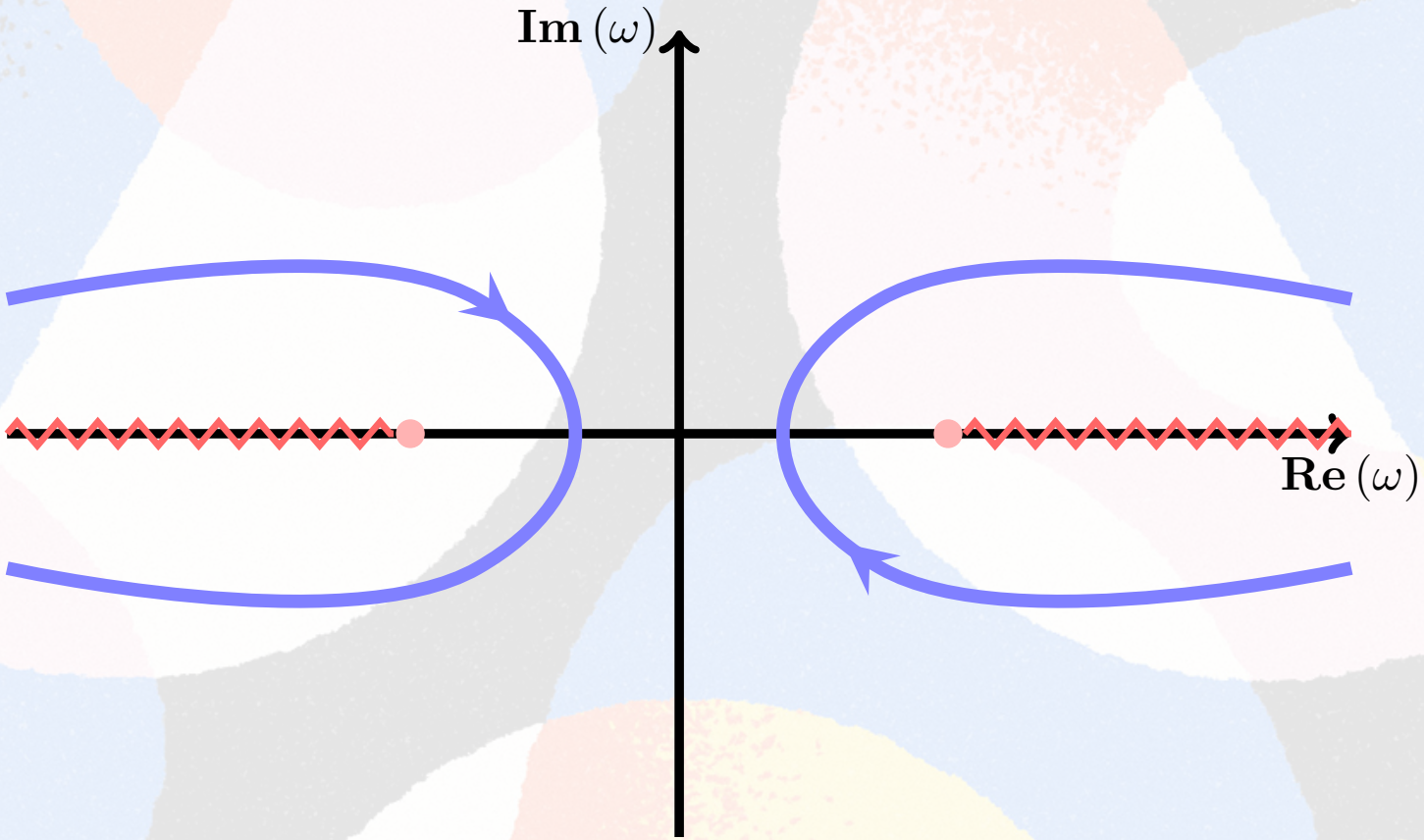


Making use of $\hat{T}_a(x) = \pm \hat{T}_a(-x)$

Optical Theorem

We are interested in the **Mellin moments** of the DIS cross-section:

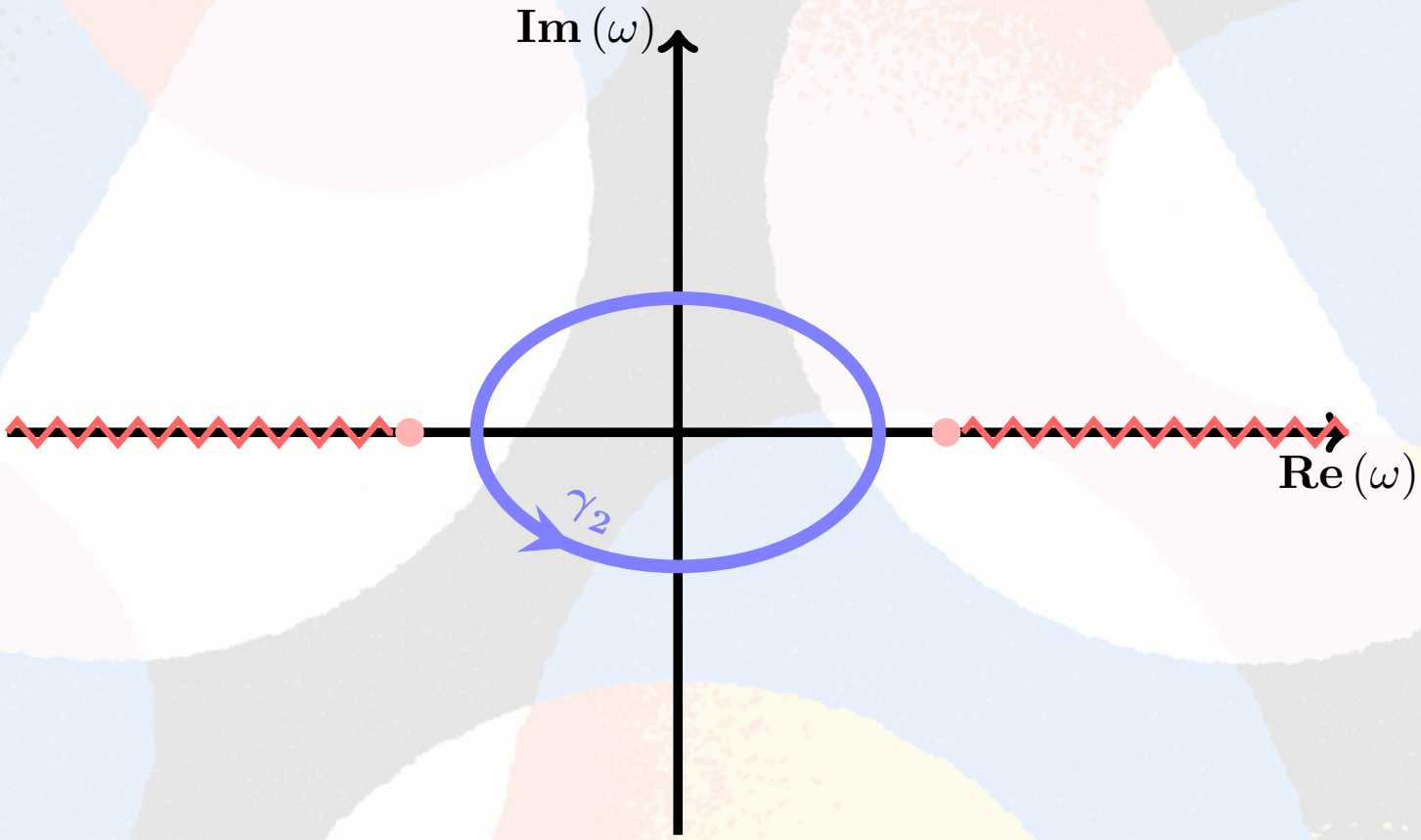
$$M[\hat{W}_a(x)](N) \approx \int_{\gamma_2} d\omega\, \omega^{-1-N} \hat{T}_a(\omega)$$



Optical Theorem

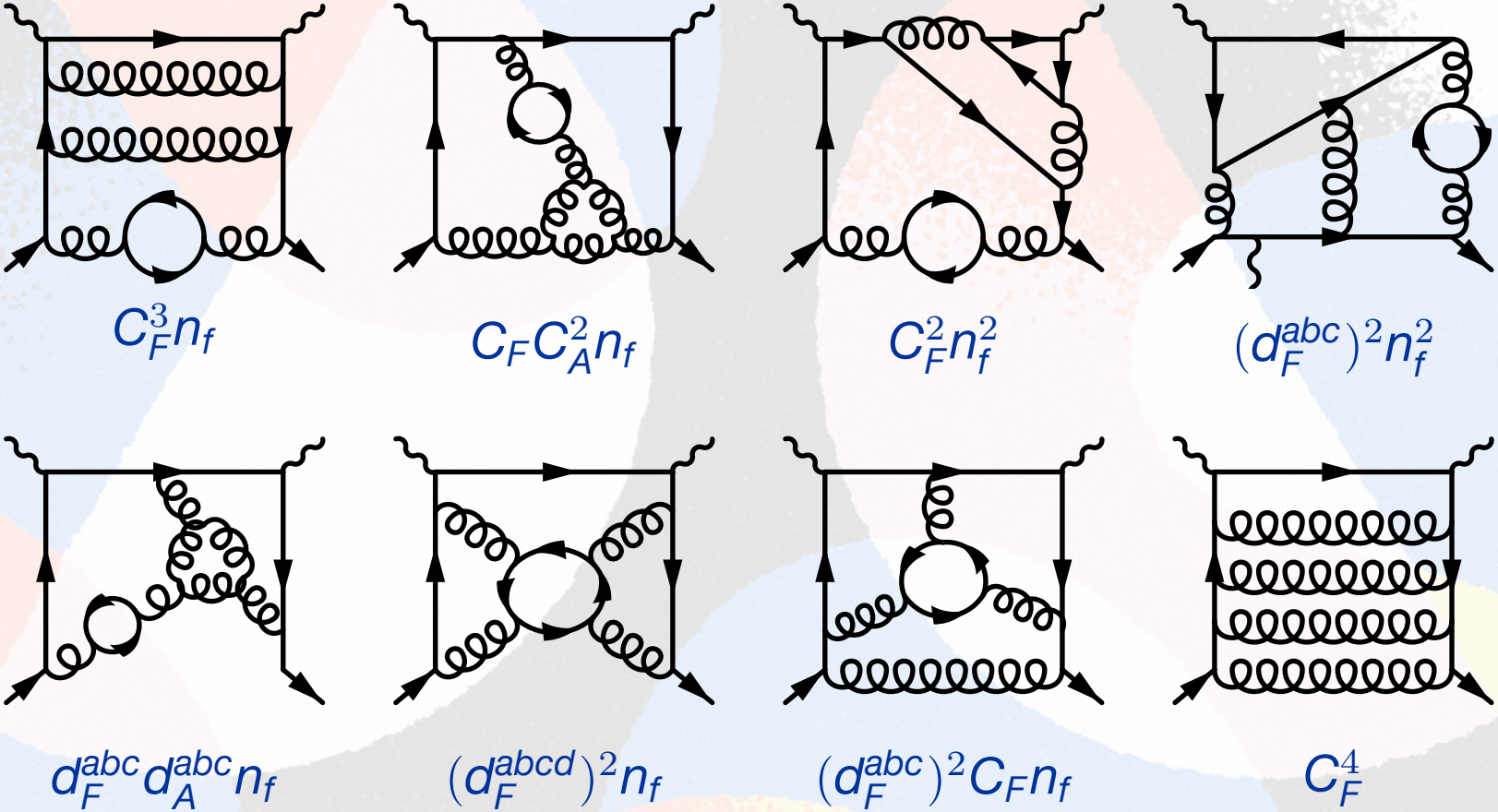
We are interested in the **Mellin moments** of the DIS cross-section:

$$M[\hat{W}_a(x)](N) \approx \frac{1}{N!}\left[\frac{\mathrm{d}^N T_a}{\omega^N}\right]_{\omega=0}$$



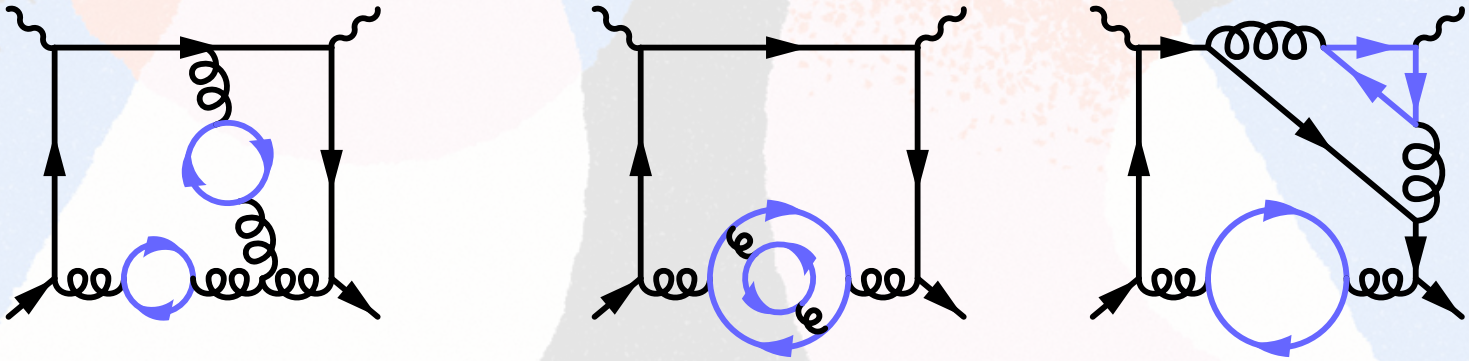
System of differential equations

- Contributions to the non-singlet forward amplitude can be separated by **color structure** and n_f order:



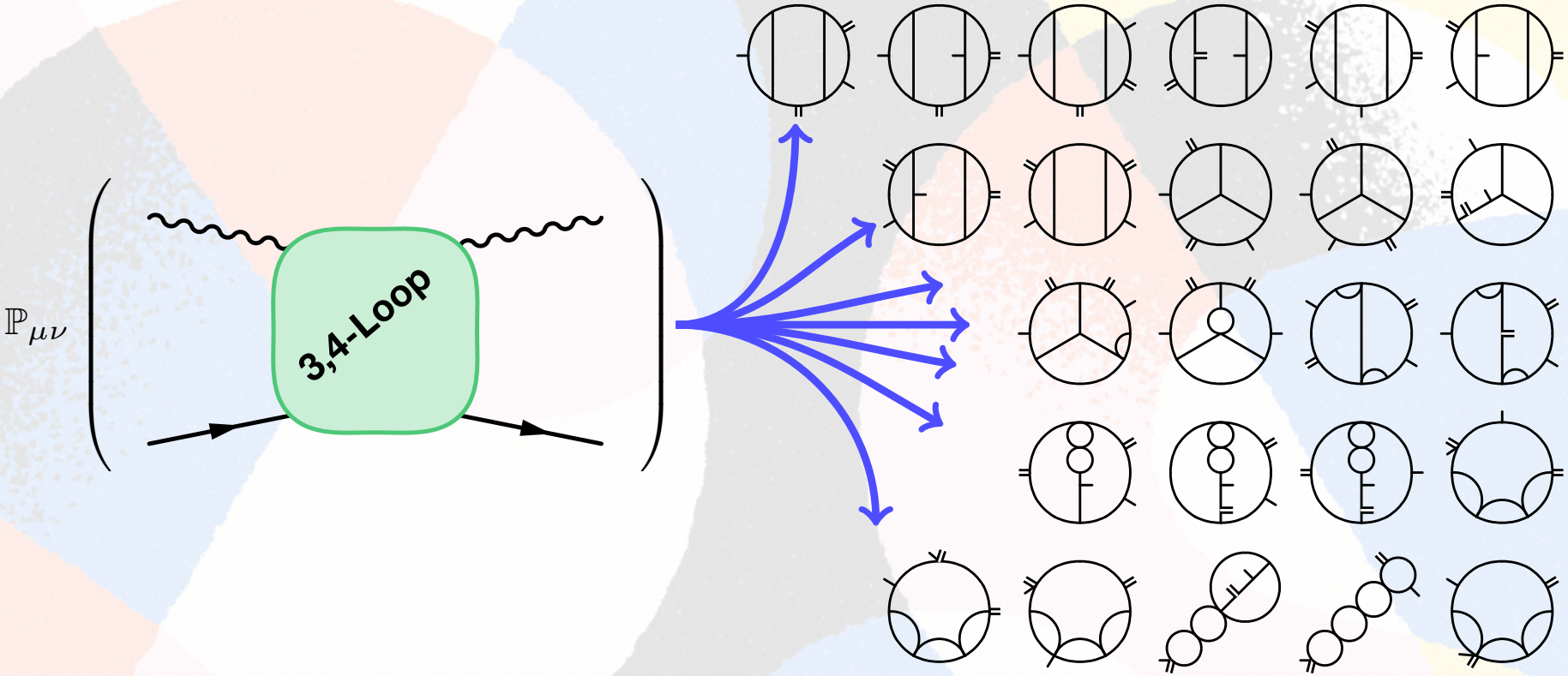
System of differential equations

- We focus our attention on the non-singlet coefficient functions contribution of order $[n_f^3, n_f^2]$ for $q + \gamma \to q + \gamma$:



SYSTEM OF DIFFERENTIAL EQUATIONS:

We process all the diagrams for the relevant process and cast them into 24 different **topologies** :



Still left with $\approx 10^5$ different integrals to be computed!

- We can resize the problem by using **IBP** relations among these integrals.
- Many publicly available programs to perform **reductions to master integrals** each with its strengths and weaknesses.

FIRE [1901.07808]
Reduze [1201.4330]
Kira [1705.05610]

System of differential equations

- Within each **topology** we perform a reduction to **master integrals** :

$$I^{(n)}(\omega,\epsilon) = \sum_i c_i(\omega,\epsilon)\cdot \boxed{M_i^{(n)}(\omega,\epsilon)}^{\text{Expand in } \omega}, \qquad \omega = \frac{1}{x}$$

- The master integrals allow to construct a closed system of differential equations:

$$\frac{\partial}{\partial\omega}\vec{M}(\omega,\epsilon) = A(\omega,\epsilon)\cdot\vec{M}(\omega,\epsilon).$$

Assuming:

- The DE matrix has at most a simple pole in ω:

$$A = \frac{A_{-1}}{\omega} + \sum_{k=0}^{\infty} A_k \omega^k$$

Note: Can always be done by applying a linear transformation T for system with **regular singularities** :

$$\vec{M} \to T\cdot\vec{M}, \qquad A \to \frac{\partial T}{\partial\omega}T^{-1} + T\cdot A\cdot T^{-1}$$

J. Moser 1959, J. Henn [1412.2296], Epsilon [1701.00725], Fuchsia [1701.04269], Libra [2012.00279]

System of differential equations

- Within each **topology** we perform a reduction to **master integrals** :

$$I^{(n)}(\omega,\epsilon) = \sum_i c_i(\omega,\epsilon)\cdot \boxed{M_i^{(n)}(\omega,\epsilon)} \quad , \qquad \omega = \frac{1}{x}$$

Expand in ω

Mellin moments generation

$$\frac{\partial}{\partial\omega}\vec{M}(\omega,\epsilon) = A(\omega,\epsilon)\cdot\vec{M}(\omega,\epsilon).$$

$$A = \frac{A_{-1}}{\omega} + \sum_{k=0}^{\infty} A_k\omega^k \qquad \vec{M} = \sum_{k=0}^{\infty}\vec{m}_k\omega^k$$

$$\underbrace{((k+1)\mathbb{1} - A_{-1})}_{:=B_k}\cdot\vec{m}_{k+1} = \sum_{j=0}^{k} A_j\vec{m}_{k-j}$$

$\det(B_k) \neq 0$ $\qquad$ $\det(B_k) = 0$

Recursive Expression:

$$\vec{m}_{k+1} = B_k^{-1}\cdot\left(\sum_{j=0}^{k} A_j\vec{m}_{k-j}\right)$$

Gaussian Elimination:

Required by a **finite number** of k

EXPANSION

- We need to fix the **boundary** condition for $p \to 0$ **FORCER [1704.06650]**
- For all $n_f^{2,3}$ cases the **transformation** matrix T consists of a simple rescaling of the master integrals

$$T = \text{diag}\left(\omega^{\vec{a}}\right), \qquad \vec{a} \in \mathbb{N}_0^{\#\text{ of masters}}$$

- We can perform a simultaneous expansion in the **dimensional regulator** ϵ in order to speed up the computation provide the two limits independence of the ϵ and ω poles

$$\frac{1}{f(\omega) + g(\epsilon)}, \qquad f(0) + g(0) \neq 0 \textbf{ if } f(\omega), g(\epsilon) \neq const.$$

Smirnov^2 [2002.08042], J. Usovitsch [2002.08173]

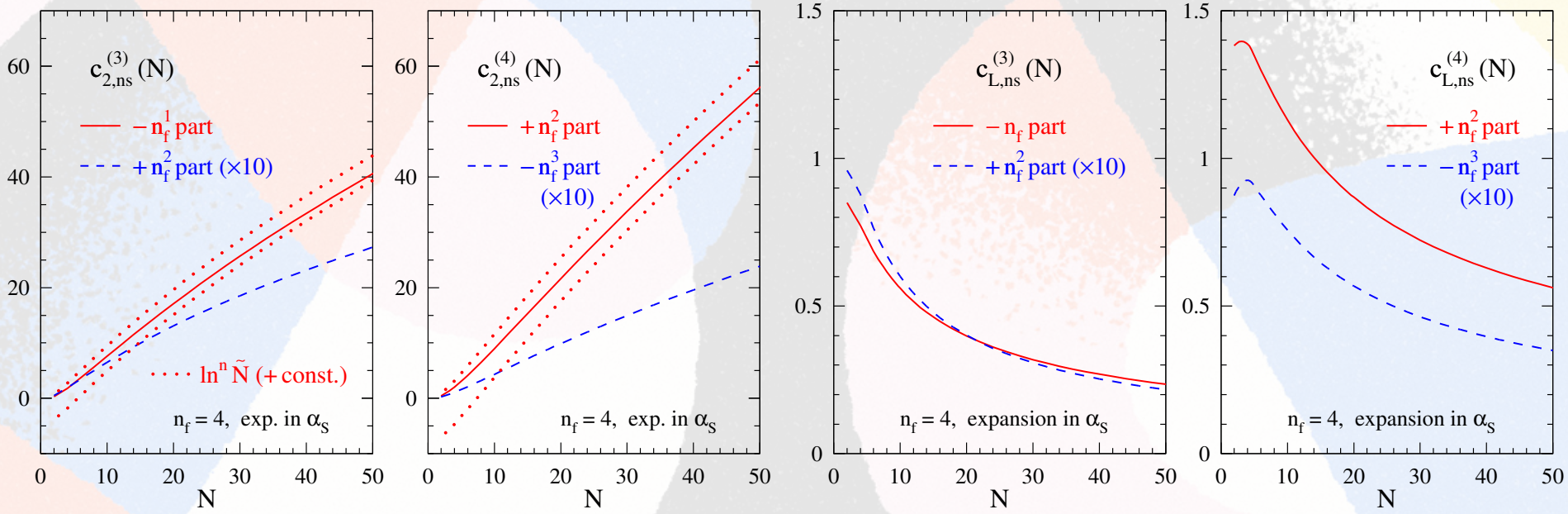
- The result can be efficiently expanded to high order in **Mellin moments** once we reach high enough in the series expansion to the regioin of validity of the recursive expression ($\det(B_k) \neq 0$).
- The **reductions** to master integrals remain the main bottleneck of the computation

FIRE [1901.07808]

Mellin moments at 4-loop

$c_{3,ns}$ Coming Soon!

- Starting to explore the DIS expression for a simple subgroup $[n_f^3, n_f^2]$ for $q + \gamma \to q + \gamma$:



Plots from: [2211.16485]

- **Expansion** in ω is possible to **high orders** within a day:

$$\mathcal{O}(\omega^{1500})$$

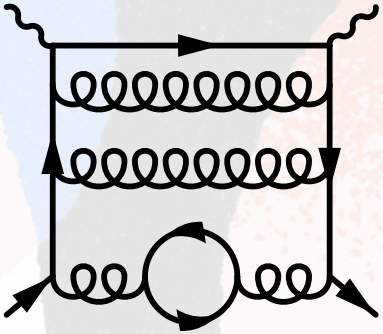
- Allows for the reconstruction of the **structure functions** in **x**-space at **all orders**

Harmonic series: $S_{\vec{m}}(N)$ $\quad\to\quad$ Harmonic Polylogarithms: $H_{\vec{n}}(x)$

Adventuring further into 4-loop

The **new** goal is to push the same technique to new horizons:

- Consider the diagrams contributing to $C_F^3 n_f$



- Effectively 3-loop topologies with **bubble insertions**!
- The added degrees of freedom start to become a real problem for the reduction to **master integrals:**
  - Moving from 11 to 12 propagators out of 18 degrees of freedom
  - Higher powers in the numerator
- Implement a tailored reduction routine for this specific problem

Tackling the Problems

General Reduction programs :

- Reliable on a wide range of problems
- Thoroughly checked through years of usage and feedbacks
- Parallelization of the reduction problem
- Multiple ways to solve the problem and implementation of general optimization
- Already too slow for the integrals we are dealing with

Problem we are facing :

- Relatively contained number of integrals to be reduced (compared with the d.o.f)
- Very few integrals have the highest complexity (numerator/denominator powers)

Reduction Overview

We have implemented our own reduction procedure to try to improve the main bottleneck of our approach. The reductions is organized into three levels:

Finite Fields

- Numerical Gaussian Elimination

Solve selected **IBP** relations using **Finite Fields** by evaluating the variables at some arbitrary points

- Generate instructions table

Create a **logfile** to keep track of all the arithmetic operations performed

Algebraic

- Optimize logfile

Keep only the instructions relevant to the **reduction coefficients**

- Algebraic evaluation

Read out the **exact coefficients** from the logfile by using the un-replaced variables

Reconstruction

If the algebraic evaluation **fails** than we use rational reconstruction

- Interpolate
- Reconstruct

Tackling the Problems

Problem :

- With the current available reduction programs it would be impossible to obtain the necessary DEs in a reasonable ammount of time

Idea :

- We want to use out taylor reduction as a **complementary** tool of the full reduction

How:

Obtain a fast partial reduction (not necessarily master integrals) to be able to give simpler problems to the public reduction programs

We then turn to FIRE for eventually refine the reduction and obtain a factorized base for the system of DEs

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Necessary and successful for building the **DEs** for $C_F^3 n_f$ and for C_3 and n_f^2

Summary

- Use IBP identities for a **reduction to master integrals** and build a system of differential equations
- **Transform** the system to allow for an efficient recursive expression for the extraction of the series coefficients
- Tested the method by computing high numbers of Mellin moments for the DIS **Wilson coefficients** $\mathcal{C}_L$, $\mathcal{C}_2$ and $\mathcal{C}_3$ at **3-loop**
- Generated 1500 Mellin moments for the non-singlet n_f^2 contribution at **4-loop** to obtain for the first time the corresponding **Wilson coefficients**
- Implemented a taylored **reduction** program to be usued together with publicly available tools

Upcoming:

- Reconstruct the expression for $\mathcal{C}_3$ (11 extra topologies )

Future:

- Apply the same method for extracting Mellin moments at **4-loop** for the $C_F^3 n_f$ to obtain the **splitting functions** with the newly obtain DEs (~ 80 new topologies)

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Thank you!