DIFFERENTIAL EQUATIONS FOR PARAMETRISED FEYNMAN INTEGRALS

Lorenzo Magnea

Università di Torino - INFN Torino

RadCor 2023 - Crieff - 31/05/2023

Outline

- A memorial
- Ancient history
- Feynman integrals and projective forms
- Differential equations in parameter space
- Examples at one and two loops

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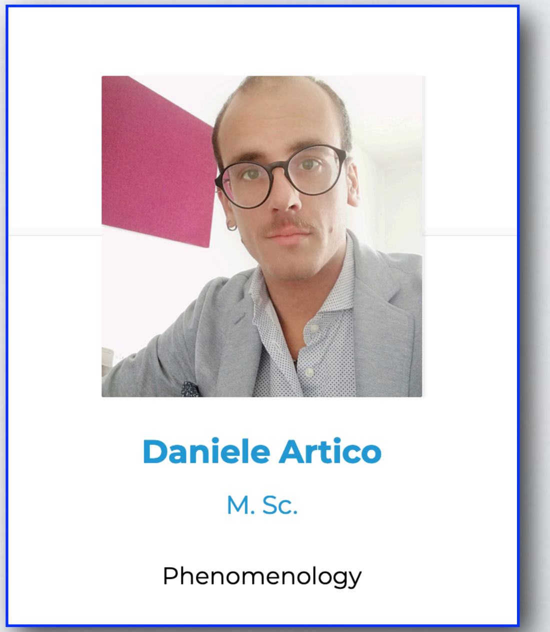
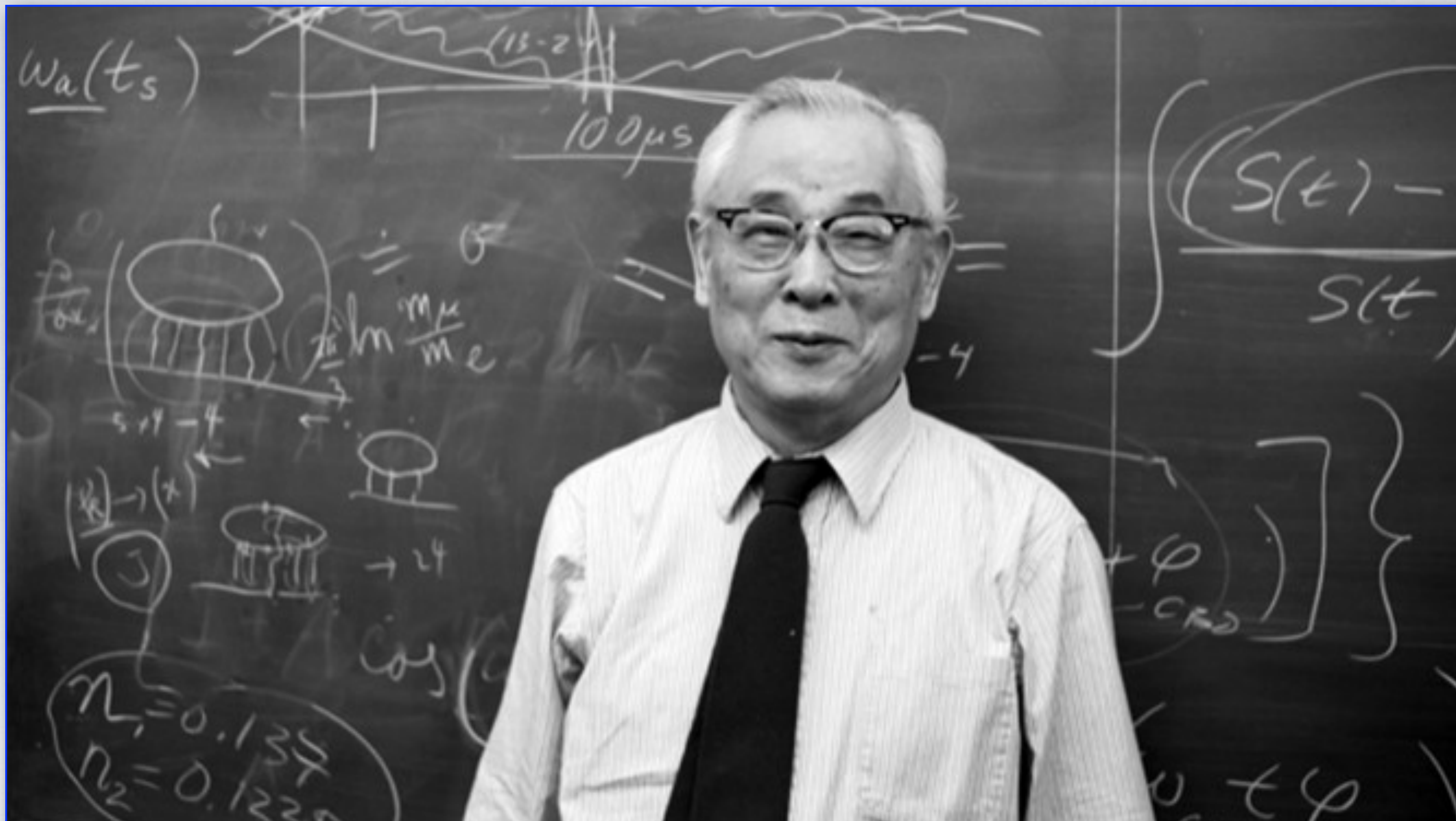


A MEMORIAL



TOICHIRO KINOSHITA (23/1/1925 - 23/3/2023)

A young Kinoshita, USA, ca. 1953

A degree in Physics: Tokyo 1944 - 1947

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https://www.aip.org/history-programs/niels-bohr-library/oral-histories/47264

A degree in Physics: Tokyo 1944 - 1947

Yeah. And then went into Tokyo University. And then as I probably wrote somewhere, Tokyo University should take three years, too. But because of the war situation, it was compressed into essentially one and a half. And then you would be drafted. Actually, mostly physics graduates were not really bad off, because they are drafted to work in some lab in the army or navy or something of that sort. But my classmates in the other parts of the university or Daiichi High School, specializing in literature and other non-scientific stuff, most of them were drafted into the service, and many of them passed away during the war.

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Crease:

Wait. First, when the atomic bombs went off in Hiroshima and Nagasaki, how far away were you?

Kinoshita:

Actually, as I said, my parents lived in Yonago, which is right north of Hiroshima, but beyond the mountain range. So, we didn't see anything. Only on the radio or newspaper we heard that Hiroshima was flattened.

Crease:

And did you understand what had flattened it?

Kinoshita:

Sure.

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Crease:

And did you understand what had flattened it?

Kinoshita:

Sure.

Crease:

And what did it say, the broadcast?

Kinoshita:

Well, he said, "*Japan surrender, but don't disturb the order*" and so on. And "*Accept occupation force*" and so on. Anyway, my feeling is that, "Wow, that's good." [laugh] So, I don't have to die. [laugh] And, of course, in Shinjuku Station, it's full of people, and most people seems to be relaxed after hearing that.

https://www.aip.org/history-programs/niels-bohr-library/oral-histories/47264

Meeting the S matrix: 1946

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Meeting the S matrix: 1946

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Crease:

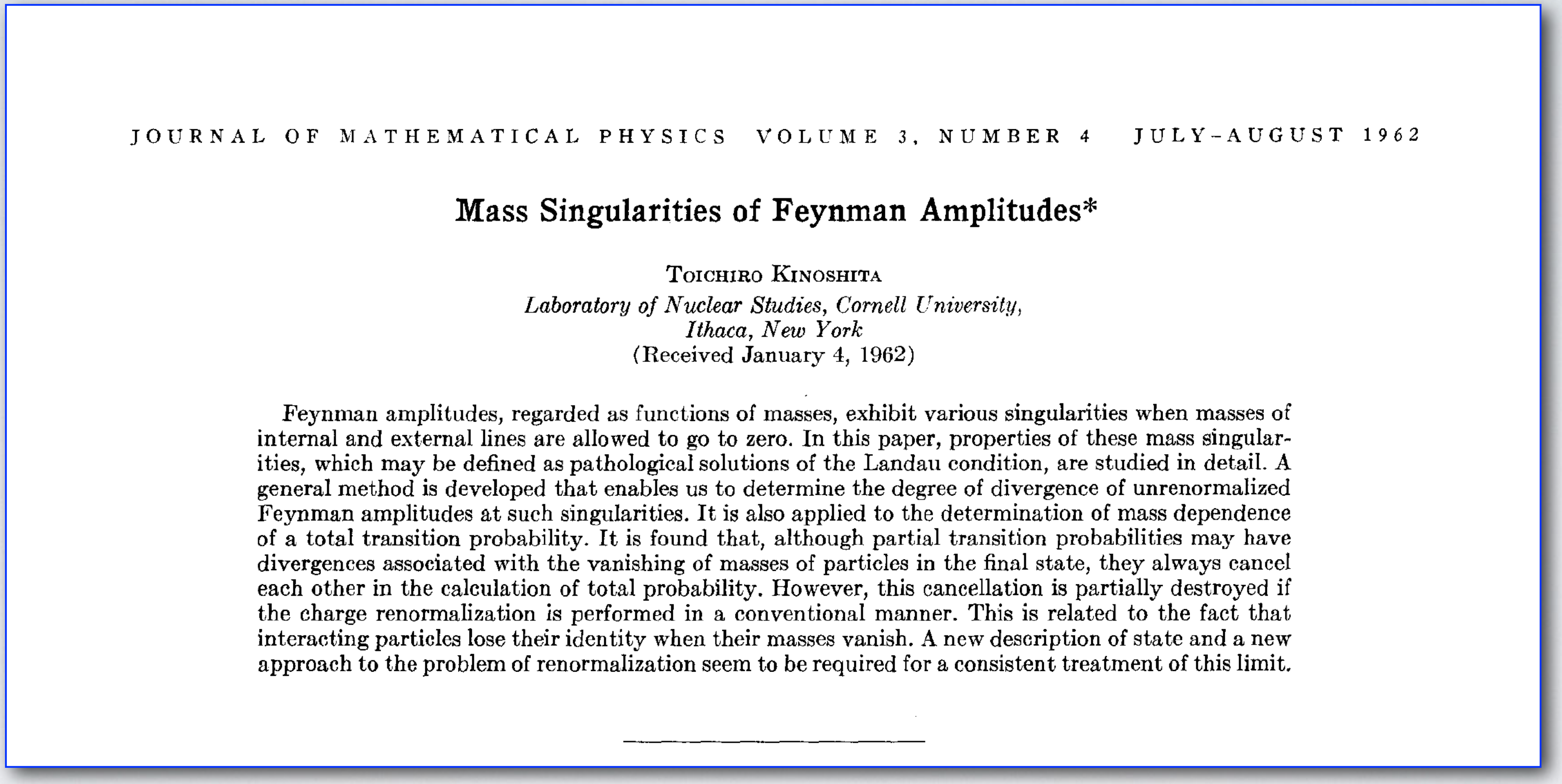
You said something about them being stamped top secret.

Kinoshita:

Oh, that's a different story. While reading these Dirac papers, I'm not exactly sure we found a statement, but we found that the Pauli mentioned that Dirac's—no, Heisenberg's S-matrix theory is a picture frame without a picture in it. This statement is actually made and recorded in some part—I can find it—but in a letter to Dirac, I think, from Pauli, 1943 or something of that sort. There is a collection of letters to Dirac which has this. But anyway, I don't exactly know how I got hold of this information about Heisenberg's S-matrix, which was first—I didn't know that until that time. And we start looking for Heisenberg's S-matrix theory, which was not in Todai's library or physics department. But my friend Yamaguchi found out that Tomonaga's lab has a copy at this time. And this paper, two papers actually, were in one of these books which was smuggled essentially into Japan by a German submarine. Actually, it was brought from Germany to someplace in the Indian Ocean, and then there it was transferred to a Japanese submarine and carried back to Japan. [laugh]

https://www.aip.org/history-programs/niels-bohr-library/oral-histories/47264

The Kinoshita-Lee-Nauenberg Theorem (1962)



JOURNAL OF MATHEMATICAL PHYSICS VOLUME 3, NUMBER 4 JULY-AUGUST 1962

Mass Singularities of Feynman Amplitudes*

TOICHIRO KINOSHITA
Laboratory of Nuclear Studies, Cornell University, Ithaca, New York
(Received January 4, 1962)

Feynman amplitudes, regarded as functions of masses, exhibit various singularities when masses of internal and external lines are allowed to go to zero. In this paper, properties of these mass singularities, which may be defined as pathological solutions of the Landau condition, are studied in detail. A general method is developed that enables us to determine the degree of divergence of unrenormalized Feynman amplitudes at such singularities. It is also applied to the determination of mass dependence of a total transition probability. It is found that, although partial transition probabilities may have divergences associated with the vanishing of masses of particles in the final state, they always cancel each other in the calculation of total probability. However, this cancellation is partially destroyed if the charge renormalization is performed in a conventional manner. This is related to the fact that interacting particles lose their identity when their masses vanish. A new description of state and a new approach to the problem of renormalization seem to be required for a consistent treatment of this limit.

The lepton anomalous magnetic moment
(1967-2018)

RIKEN-QHP-25

Tenth-Order QED Contribution to the Electron $g-2$ and an Improved Value of the Fine Structure Constant

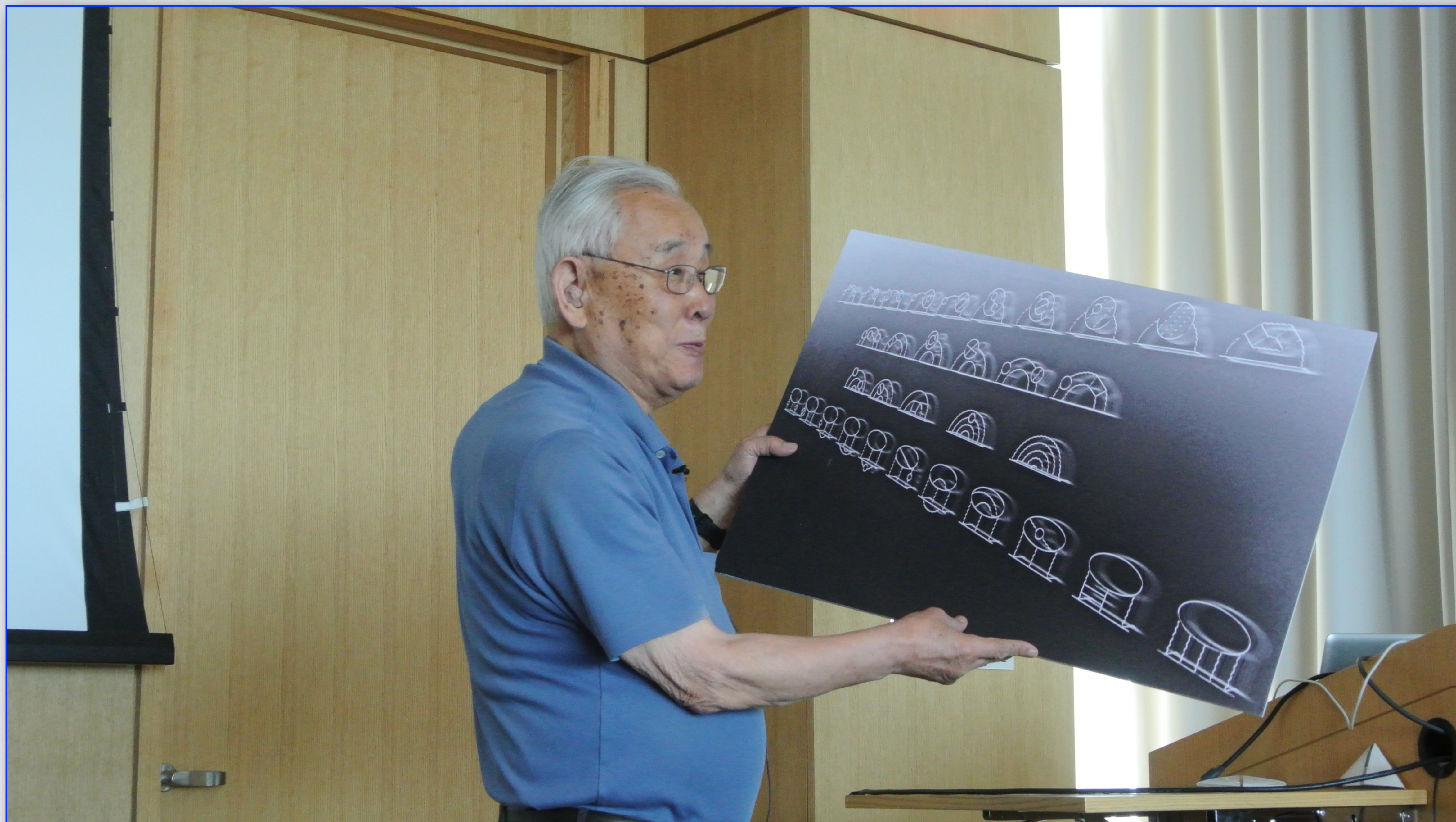
Tatsumi Aoyama,[1,2] Masashi Hayakawa,[3,2] Toichiro Kinoshita,[4,2] and Makiko Nio[2]

[1] *Kobayashi-Maskawa Institute for the Origin of Particles and the Universe (KMI), Nagoya University, Nagoya, 464-8602, Japan*
[2] *Nishina Center, RIKEN, Wako, Japan 351-0198*
[3] *Department of Physics, Nagoya University, Nagoya, Japan 464-8602*
[4] *Laboratory for Elementary Particle Physics, Cornell University, Ithaca, New York, 14853, U.S.A*

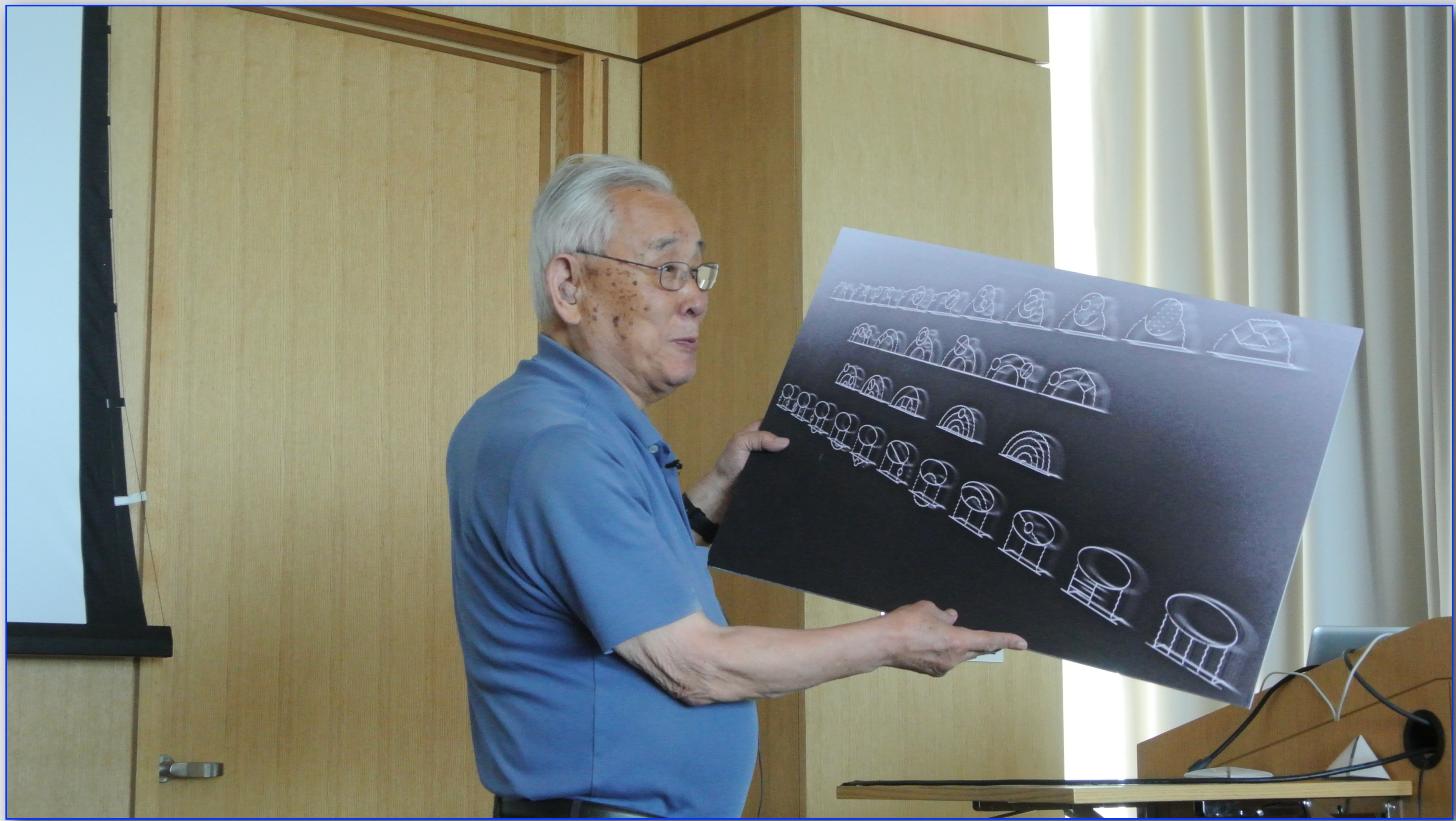
(Dated: August 21, 2012)

This paper presents the complete QED contribution to the electron $g-2$ up to the tenth order. With the help of the automatic code generator, we have evaluated all 12672 diagrams of the tenth-order diagrams and obtained 9.16 (58)$(\alpha/\pi)^5$. We have also improved the eighth-order contribution obtaining -1.9097 (20)$(\alpha/\pi)^4$, which includes the mass-dependent contributions. These results lead to $a_e(\text{theory}) = 1\ 159\ 652\ 181.78\ (77) \times 10^{-12}$. The improved value of the fine-structure constant $\alpha^{-1} = 137.035\ 999\ 174\ (35)$ [0.25ppb] is also derived from the theory and measurement of a_e.

PACS numbers: 13.40.Em, 14.60.Cd, 06.20.Jr, 12.20.Ds

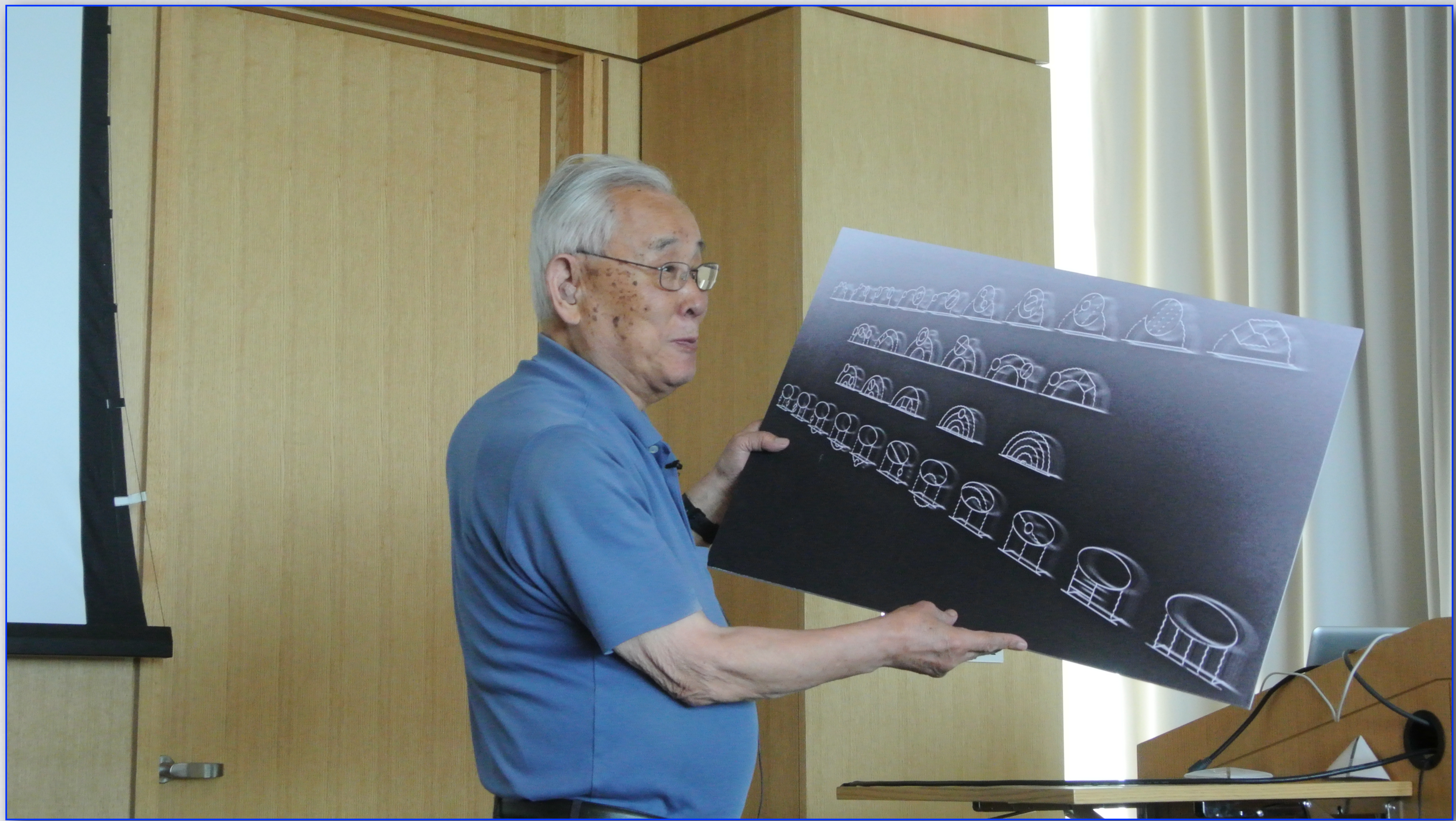


Some years later, a retirement celebration at Cornell - July 2014



Some years later, a retirement celebration at Cornell - July 2014

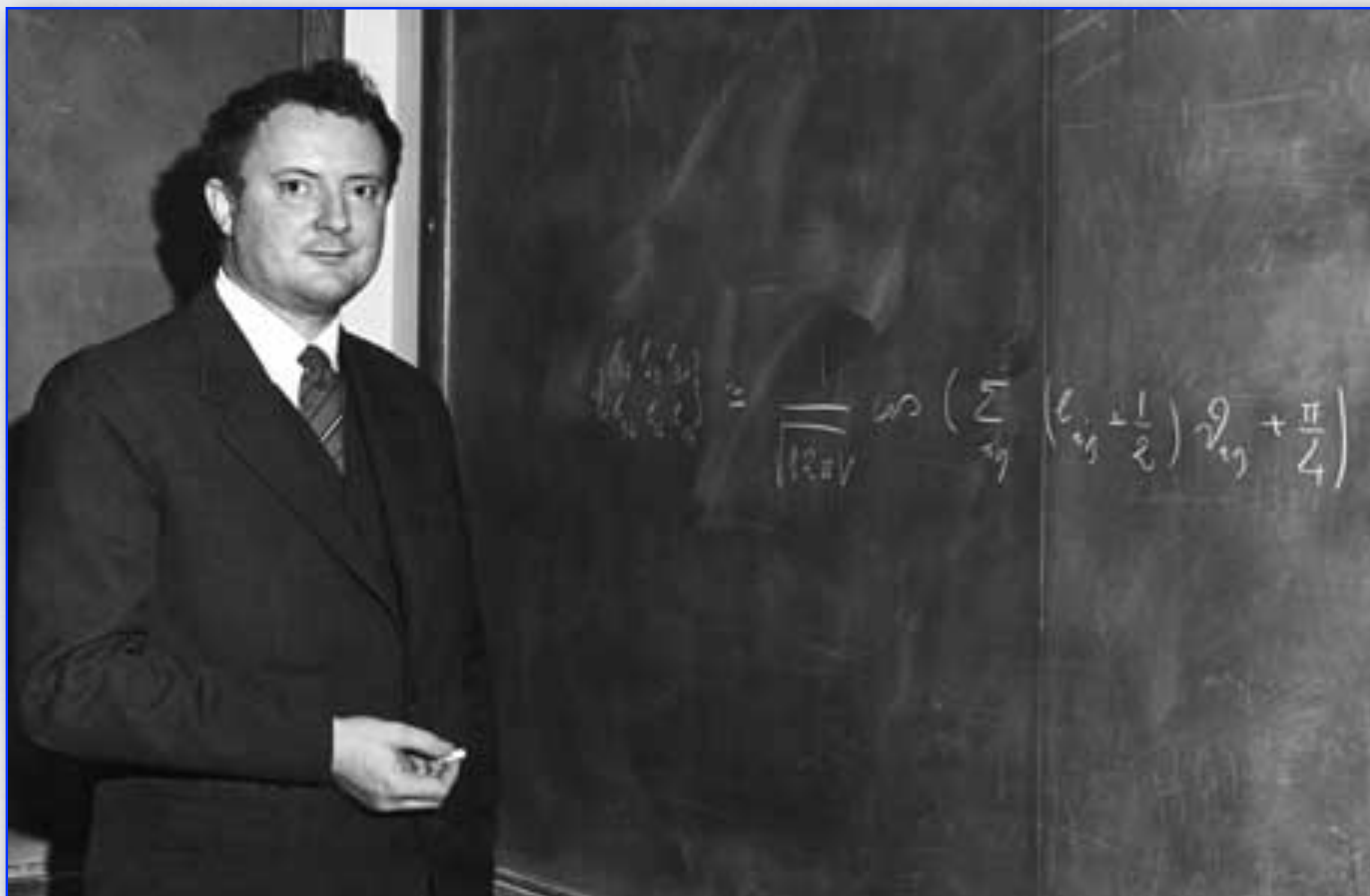
"Professor Toichiro Kinoshita was a giant in precision calculations in quantum electrodynamics," said Tung-Mow Yan, professor of physics emeritus (A&S). "The results of Tom and his team are still the best in the world."



Some years later, a retirement celebration at Cornell - July 2014

"His focus on the integrity of his work on high-precision QED was off-scale," Lepage said. "It's a masterpiece of science. I'm amazed by Tom's tenacity and extraordinary diligence."

ANCIENT HISTORY



XVI

Algebraic Topology Methods in the Theory of Feynman Relativistic Amplitudes[1]

TULLIO REGGE

PART I

In this lecture I would like to introduce you to the spirit and to some of the technicalities of the work currently being carried out by Lascoux and myself on Feynman relativistic amplitudes (FRA).

FRA are analytic functions of a very special kind which have been defined in connection with relativistic field theory. It is not my job here to discuss at length their definition, as this has been done already by Lascoux and Hepp in previous lectures. The reason why these functions are interesting to so many people could be summarized as follows:

1. The scattering amplitude for any physical process can be in principle expressed, if field theory is right, as a power expansion in the coupling constants, each coefficient being a Feynman relativistic amplitude.

2. In general we cannot compute explicitly these amplitudes since they are given by rather complicated integrals. We are therefore happy to gather whatever information is available on them, including analytic properties, which may yield, hopefully, dispersion relations.

3. Even supposing that we know each amplitude, the following troubles may arise: the power expansion may not converge for some values of the parameters involved; the power expansion never converges and it should be interpreted as an asymptotic series; some of the coefficients are infinite.

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[1] See also: *The Analytic S-Matrix*, R. J. Eden, P. V. Landshoff, D. I. Olive, and J. C. Polkinghorne, Cambridge University Press, 1966.

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HYPERGEOMETRIC FEYNMAN INTEGRALS

René Pascal Klausen, 25th February 2023

This is a minor updated version of my PhD thesis (date of the original version 3rd August 2022), which I defend at the Institute of Physics at Johannes Gutenberg University Mainz on the 24th of November 2022. The referees were Christian Bogner (Johannes Gutenberg University Mainz) and Dirk Kreimer (Humboldt University of Berlin).

Abstract

In this thesis we will study Feynman integrals from the perspective of $\mathcal{A}$-hypergeometric functions, a generalization of hypergeometric functions which goes back to Gelfand, Kapranov, Zelevinsky (GKZ) and their collaborators. This point of view was recently initiated by the works [74] and [150]. Inter alia, we want to provide here a concise summary of the mathematical foundations of $\mathcal{A}$-hypergeometric theory in order to substantiate this viewpoint. This overview will concern aspects of polytopal geometry, multivariate discriminants as well as holonomic D-modules.

As we will subsequently show, every scalar Feynman integral is an $\mathcal{A}$-hypergeometric function. Furthermore, all coefficients of the Laurent expansion as appearing in dimensional and analytical regularization can be expressed by $\mathcal{A}$-hypergeometric functions as well. By applying the results of GKZ we derive an explicit formula for series representations of Feynman integrals. Those series representations take the form of Horn hypergeometric functions and can be obtained for every regular triangulation of the Newton polytope $\operatorname{Newt}(\mathcal{U}+\mathcal{F})$ of the sum of Symanzik polynomials. Those series can be of higher dimension, but converge fast for certain kinematical regions, which also allows an efficient numerical application. We will sketch an algorithmic approach which evaluates Feynman integrals numerically by means of these series representations. Further, we will examine possible issues which can arise in a practical usage of this approach and provide strategies to solve them. As an illustrative example we will present series representations for the fully massive sunset Feynman integral.

Moreover, the $\mathcal{A}$-hypergeometric theory enables us to give a mathematically rigorous description of the analytic structure of Feynman integrals (also known as Landau variety) by means of principal A-determinants and A-discriminants. This description of the singular locus will also comprise the various second-type singularities. Furthermore, we will find contributions to the singular locus occurring in higher loop diagrams, which seem to have been overlooked in previous approaches. By means of the Horn-Kapranov-parameterization we also provide a very efficient way to determine parameterizations of Landau varieties. We will illustrate those methods by determining the Landau variety of the dunce's cap graph. We furthermore present a new approach to study the sheet structure of multivalued Feynman integrals by use of coamoebas.

Klausen 2302.13184

Commun. math. Phys. 15, 83—132 (1969)

The Monodromy Rings of a Class of Self-Energy Graphs

G. PONZANO, T. REGGE, E. R. SPEER*, and M. J. WESTWATER*
Institute for Advanced Study, Princeton, New Jersey

Received April 18, 1969

Abstract. The monodromy rings of self-energy graphs, with two vertices and an arbitrary number of connecting lines, are determined.

§ 1. Introduction

This paper is the first of a series of publications in which we hope to elucidate in a systematic way the properties of Feynman integrals. The motivation for this work is clear: we hope to develop sufficiently the methods of investigating functions of several complex variables defined by integrals to give a basis for the determination of the analytic structure of the *S*-matrix itself. This is admittedly not an easy task and one whose outcome we cannot guarantee. An ideal research program should be carried out in three steps:

1969 - Bananas

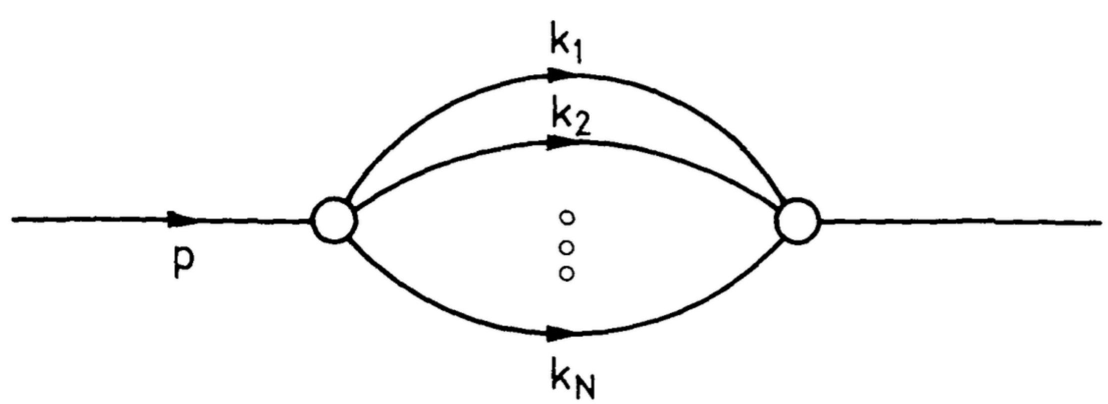


Fig. 9. The self-energy graph G_N

Commun. math. Phys. 18, 1—64 (1970)

The Monodromy Rings of One Loop Feynman Integrals

G. PONZANO, T. REGGE, E. R. SPEER*, and M. J. WESTWATER*
Institute for Advanced Study, Princeton, New Jersey

Received January 20, 1970

Abstract. The monodromy rings of Feynman integrals for one loop graphs with an arbitrary number of lines are determined.

§ 1. Introduction

This paper is the second of a series whose general aims were outlined in the introduction to the first paper [1]. In this paper we make a systematic study of the Feynman integral for a general one loop graph in an arbitrary space-time dimension; we classify the possible paths of analytic continuation, label the determinations of the function over a fixed base point, and obtain explicit formulae for the action of analytic continuation on the vector space spanned by these determinations.

1970 - One Loop

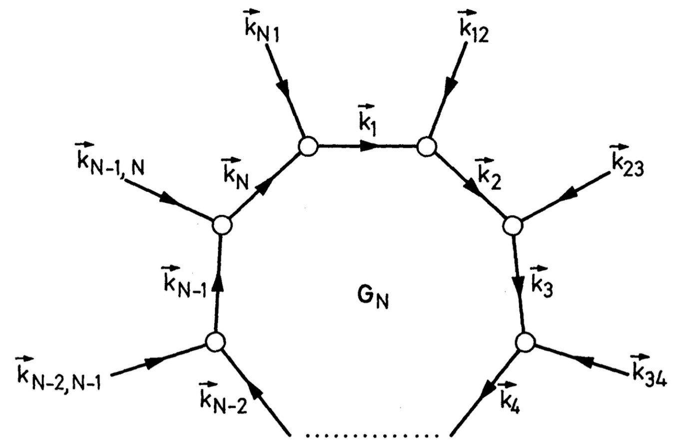


Fig. 1. The single loop graph G_N

Fortschritte der Physik 20, 365—420 (1972)

The Monodromy Rings of the Necklace Graphs

TULLIO REGGE

Institute for Advanced Study, Princeton, New Jersey 08540

EUGENE R. SPEER

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

and

MICHAEL J. WESTWATER

Department of Mathematics, University of Washington, Seattle, Washington 98105

Abstract

A *necklace graph* is a Feynman graph obtained from a single loop graph by replacing each internal line by a multiplet (i.e. a set of one or more internal lines joining the same two vertices). In this paper the monodromy rings of the necklace graphs are determined.

1972 - Necklaces

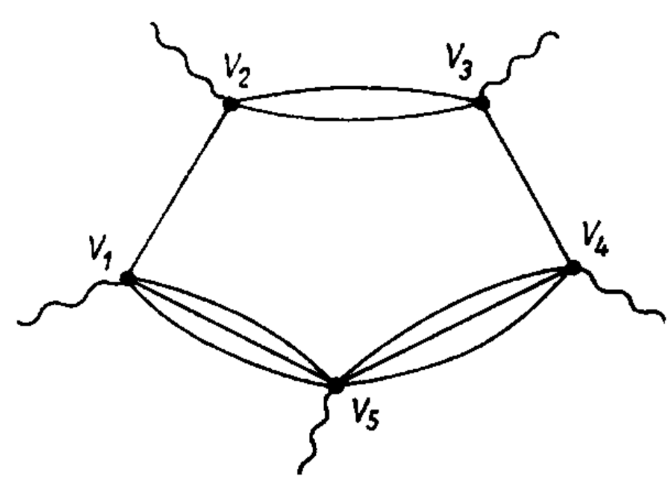


Fig. 1. The necklace graph G (5; 1, 2, 1, 3, 3)

Differential equations for one-loop generalized Feynman integrals

G. Barucchi

Istituto di Fisica dell'Università, Torino, Italy

Istituto Nazionale di Fisica Nucleare, Sezione di Torino, Italy.

Istituto di Fisica Matematica dell'Università, Torino, Italy

G. Ponzano

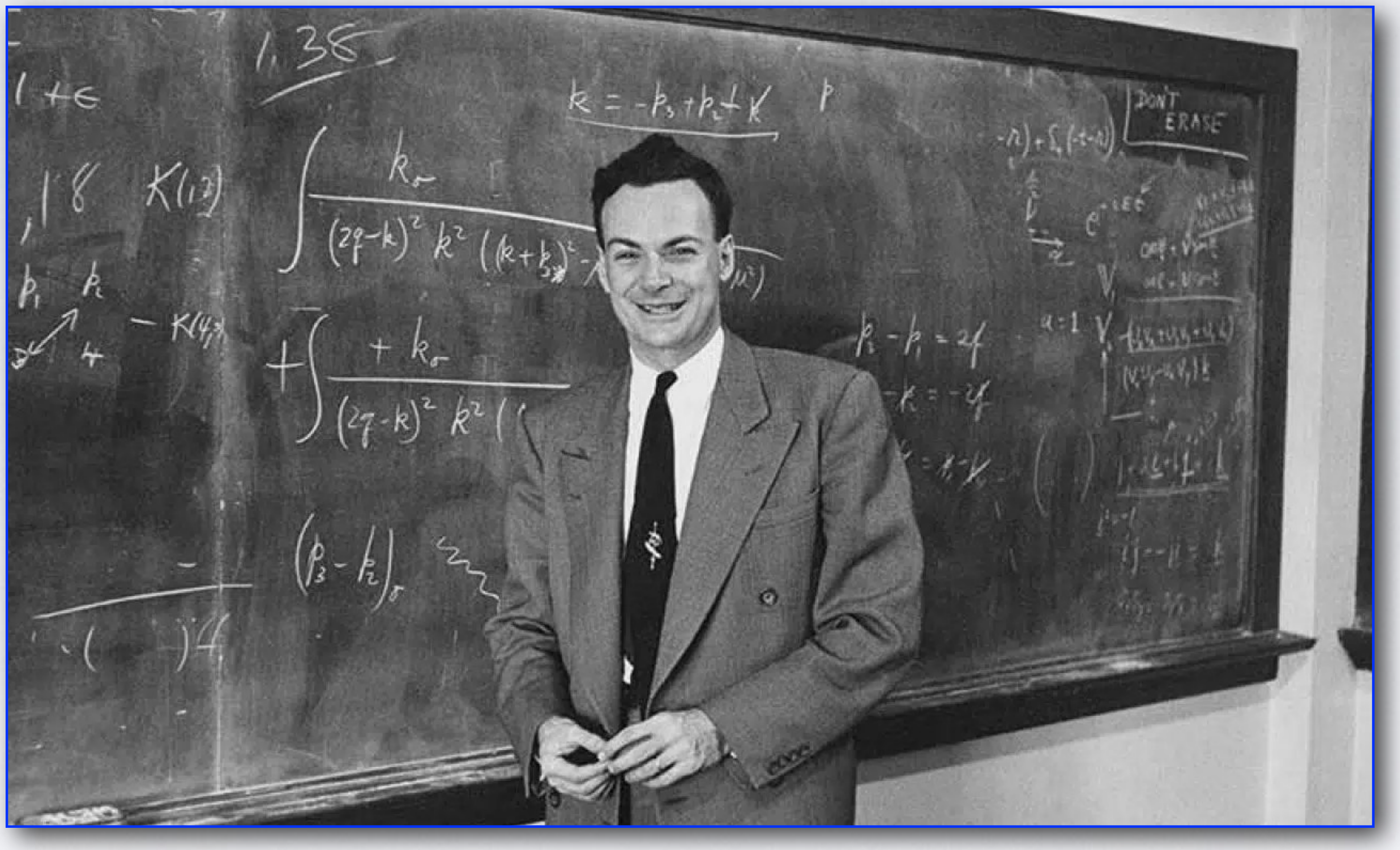
Istituto Nazionale di Fisica Nucleare, Sezione di Torino, Italy

Istituto di Fisica dell'Universitá, Torino, Italy
(Receive 21 July 1972)

A system of $(2^N - 1)$ first-order linear homogeneous differential equations in each variable is derived for the generalized (with Speer λ parameters) Feynman integrals corresponding to the one-loop graph with N external lines. This system of differential equations is shown to belong to the class studied by Lappo-Danilevsky. A connection with the matrix representation of the monodromy group in all variables is pointed out.

1972 - A Theorem

PARAMETER SPACE



Weinzierl 2201.03593

Feynman parameters

Scalar Feynman integrals in momentum space are defined by

$$I_G(\nu_i, d) = (\mu^2)^{\nu - ld/2} \int \prod_{r=1}^{l} \frac{d^d k_r}{i\pi^{d/2}} \prod_{i=1}^{n} \frac{1}{(-q_i^2 + m_i^2)^{\nu_i}},$$

$$q_i = \sum_{r=1}^{l} \alpha_{ir} k_r + \sum_{j=1}^{m} \beta_{ij} p_j,$$

Feynman's trick always allow to perform the integration over loop momenta

$$I_G(\nu_i, d) = \frac{\Gamma(\nu - ld/2)}{\prod_{j=1}^{n} \Gamma(\nu_j)} \int_{z_j \geq 0} d^n z\, \delta\left(1 - \sum_{j=1}^{n} z_j\right) \left(\prod_{j=1}^{n} z_j^{\nu_j - 1}\right) \frac{\mathcal{U}^{\nu - (l+1)d/2}}{\mathcal{F}^{\nu - ld/2}},$$

Where the Symanzik polynomials are defined purely as graph-theoretical quantities, as

$$\mathcal{U} = \sum_{\mathcal{T}_G} \prod_{i \in \mathcal{T}_G} z_i\,.$$

$$\mathcal{F} = \sum_{\mathcal{C}_G} \frac{\hat{s}(\mathcal{C}_G)}{\mu^2} \prod_{i \in \mathcal{C}_G} z_i - \mathcal{U} \sum_{i \in \mathcal{I}_G} \frac{m_i^2}{\mu^2} z_i\,.$$

- $\mathcal{T}_G$ is a co-tree: its complement is a spanning tree of the graph G.
- The complement of $\mathcal{C}_G$ is a spanning 2-forest of the graph G.
- U and F are homogeneous in the parameters, of degree ℓ and $\ell+1$, respectively.
- The integrand (including the measure) is homogeneous of degree zero.

Weinzierl 2201.03593

Projective forms

Parametrised Feynman integrands are naturally interpreted as projective forms

For any subset A, |A| = a, of D = {1, …, n}, define the ordered (volume) a-form

$$\omega_A = dz_{i_1} \wedge \ldots \wedge dz_{i_a} \, ,$$

The a-form can be `integrated' to an (a-1)-form defining

$$\eta_A = \sum_{i \in A} \epsilon_{i, A-i} \, z_i \, \omega_{A-i}$$

$$\epsilon_{k,B} = (-1)^{|B_k|} \, , \qquad B_k = \{ i \in B, i < k \} \, ,$$

The (a-1)-form η differentiates to ω, and vanishes on the boundary sub-simplexes

$$d\eta_A = a \, \omega_A \, .$$

for example

$$\eta_{\{1,2,3\}} = z_1 \, dz_2 \wedge dz_3 - z_2 \, dz_1 \wedge dz_3 + z_3 \, dz_1 \wedge dz_2 \, ,$$

One then defines affine and projective q-forms as

$$\psi_q = \sum_{|A|=q} R_A(z_i) \, \omega_A \, ,$$

affine ⟶ projective

$$\psi_q = \sum_{|B|=q+1} T_B(z_i) \, \eta_B \, .$$

- R_A `rational' and homogeneous of degree -q
- T_B `rational' and homogeneous of degree -(q+1)

for example

$$\psi_3(\lambda, r) = \frac{(z_1 + z_2 + z_3 + z_4)^\lambda}{(r \, z_1 z_3 + z_2 z_4)^{2+\lambda/2}} \, \eta_{\{1,2,3,4\}} \, ,$$

Weinzierl 2201.03593

Projective forms

Parameter integrands belong to the class of projective (n-1)-forms

$$\alpha_{n-1} = \eta_{n-1} \frac{Q(\{z_i\})}{D^P(\{z_i\})},$$

with $P = \nu - \ell\, d/2$, and Q homogeneous as appropriate

Two theorems then hold

Theorem 1. *The boundary of a projective form is itself projective.* → IBP identities!

$$p : \sum_{|A|=q} R_A(z_i)\, \omega_A \quad \to \quad \sum_{|A|=q} R_A(z_i)\, \eta_A \,.$$

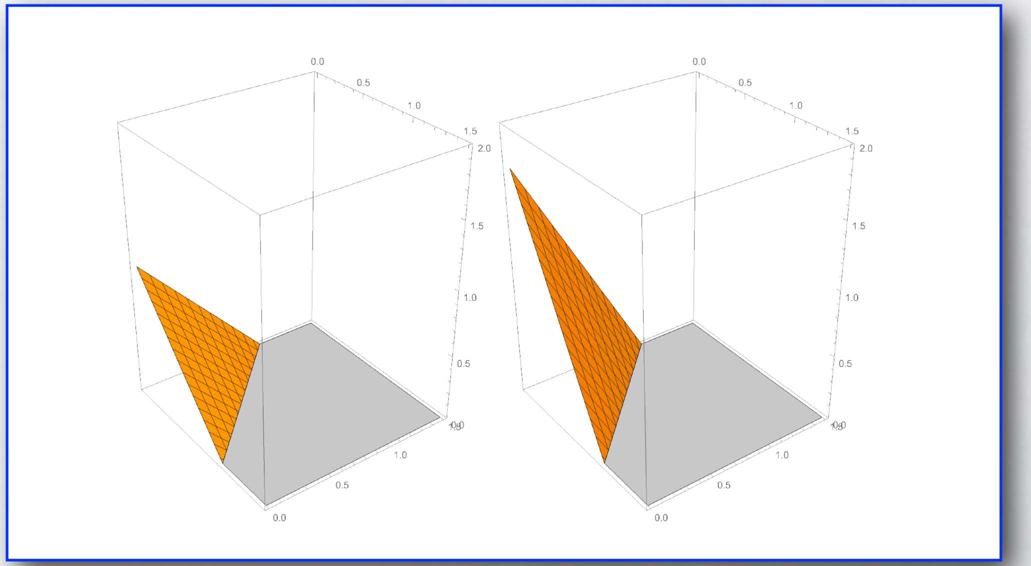
$$p^2 = 0\,,$$

$$d \circ p + p \circ d = 0\,,$$

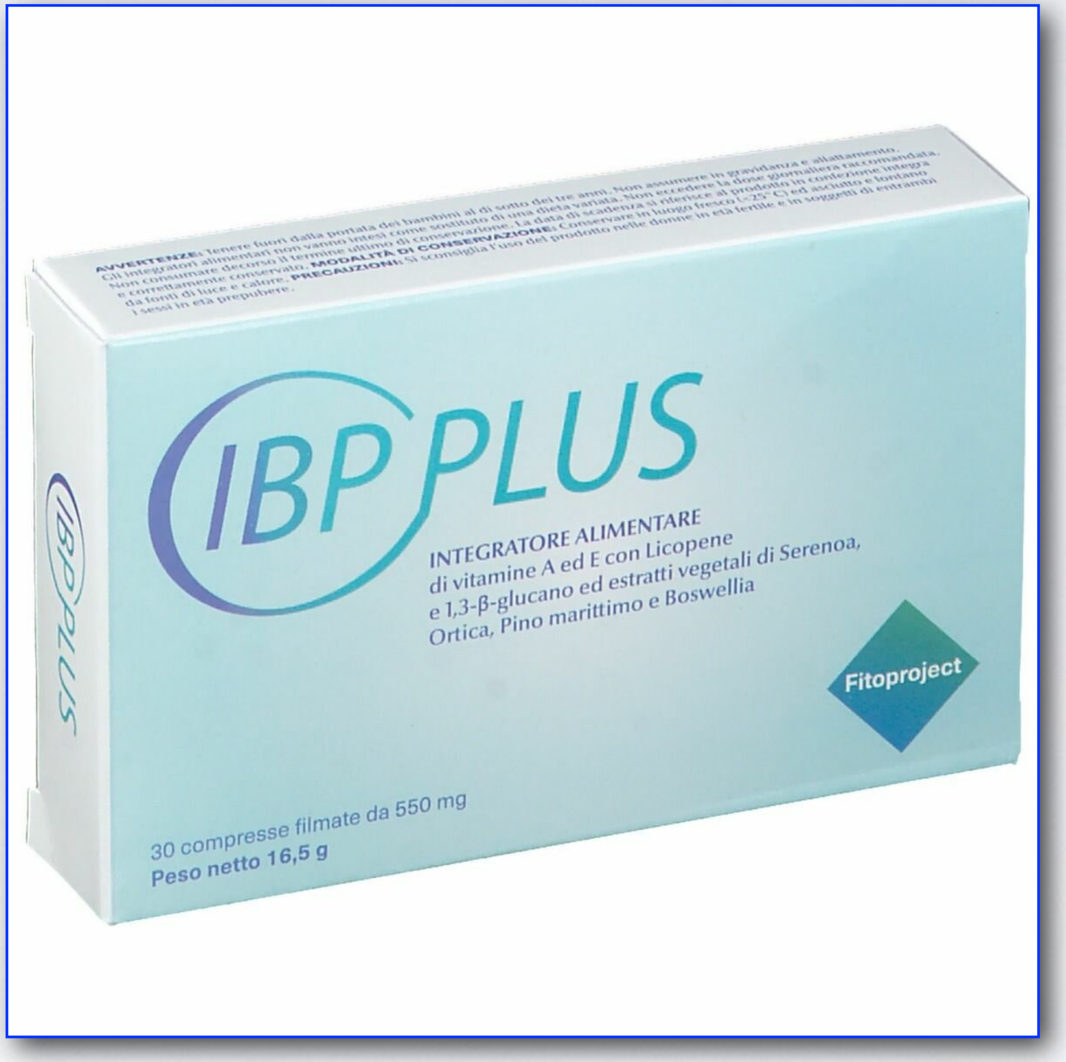
Theorem 2. *Given two integration domains, $O, O' \in \mathbb{C}^n$, if their image in $\mathbb{PC}^{n-1}$ is the same simplex, then $\int_O \alpha_{n-1} = \int_{O'} \alpha_{n-1}$.*

- α_{n-1} is a closed form, Stokes theorem applies.
- η_{n-1} vanishes on the boundary cone.

→ `Cheng-Wu' theorem: use any partial sum of parameters in the δ-function.



IBP IDENTITIES



A general identity

Choosing the standard simplex as integration domain

$$dz_n = -\sum_{i=1}^{n-1} dz_i \,,$$

$$\int_{S_{n-1}} \eta_{n-1} \frac{Q(z)}{D^P(z)} = \int_{z_i \geq 0} dz_1 \ldots dz_n \, \delta\left(1 - \sum_{i=1}^{n} z_i\right) \frac{Q(z)}{D^P(z)} \,,$$

With appropriate Q(z), D(z) and P, this defines a large class of generalised Feynman integrals

We can now move around the set of GFIs, exploiting the closure of the set of projective forms under exterior differentiation (Theorem 1).

Define for example

$$\omega_{n-2} \equiv \sum_{i=1}^{n} (-1)^i \, \eta_{\{z\}-z_i} \frac{H_i(z)}{(P-1)\,\big(D(z)\big)^{P-1}} \,,$$

so that it be projective

ω_{n-2} provides a possible integrand for a Feynman integral: differentiation gives another one

$$d\omega_{n-2} = \frac{1}{(P-1)\,\big(D(z)\big)^{P-1}} \, \eta_{\{z\}} \sum_{i=1}^{n} \frac{\partial H_i(z)}{\partial z_i} - \frac{\eta_{\{z\}}}{\big(D(z)\big)^P} \sum_{i=1}^{n} H_i \frac{\partial D(z)}{\partial z_i} \,.$$

This identity applies at any loop order, encompasses IBP identities and dimensional shifts, and feeds into a hierarchy of differential equations directly in parameter space.

One-loop identities

One-loop parameter integrals can be written in a very compact form

$$I_G(\nu_i, d) = \frac{\Gamma(\nu - d/2)}{\prod_{j=1}^{n} \Gamma(\nu_j)} \int_{z_j \geq 0} d^n z \, \delta\left(1 - z_{n+1}\right) \frac{\prod_{j=1}^{n+1} z_j^{\nu_j - 1}}{\left[\sum_{i=1}^{n+1} \sum_{j=1}^{i-1} s_{ij} z_i z_j\right]^{\nu - d/2}},$$

where the first Symanzik polynomial has been labelled as a new variable with a new index

$$z_{n+1} \equiv \sum_{i=1}^{n} z_i, \qquad \nu_{n+1} \equiv \nu - d + 1.$$

and the extended Cayley matrix has been introduced

$$s_{ij} = \frac{(q_j - q_i)^2}{\mu^2} \quad (i, j = 1, \ldots, n), \qquad s_{i,n+1} = s_{n+1,i} \equiv -\frac{m_i^2}{\mu^2},$$

We now pick as $H_i(z)$ the numerator of the one-loop integrand, treat each term in the sum defining ω_{n-2} independently, and properly adjust the value of P. Thus

$$H_i = \delta_{ih} \left(\prod_{j=1}^{n} z_j^{\nu_j - 1}\right) \left(\sum_{k=1}^{n} z_k\right)^{\nu - d} = \delta_{ih} \prod_{j=1}^{n+1} z_j^{\nu_j - 1},$$

Differentiation with respect to each z_i lowers the value of the index ν_i while differentiating the denominator effectively shifts the value of d. A set of IBP identities follows.

One-loop identities

The one-loop IBP identities can be described in terms of raising and lowering operators. Denoting the index set by $\mathcal{R}$, and picking three subsets $\mathcal{I}$, $\mathcal{J}$, and $\mathcal{K}$, define

$$f\big(\{\nu_1,\ldots,\nu_{n+1}\}\big) \equiv f\big(\{\mathcal{R}\}\big) = \eta_{\{z\}} \frac{\prod_{j=1}^{n+1} z_j^{\nu_j-1}}{\left(\sum_{i=1}^{n+1}\sum_{j=1}^{i-1} s_{ij} z_i z_j\right)^{\nu-d/2}} .$$

$$\longrightarrow \quad f\big(\{\mathcal{I}\}_{-1}, \{\mathcal{J}\}_0, \{\mathcal{K}\}_1\big)$$

The simple form of the first Symanzik polynomial leads to the identity

$$\sum_{i=1}^{n} f\big(\{\mathcal{R}-i\}_0, \{i\}_1\big) = f\big(\{\mathcal{R}-\{n+1\}\}_0, \{n+1\}_1\big)$$

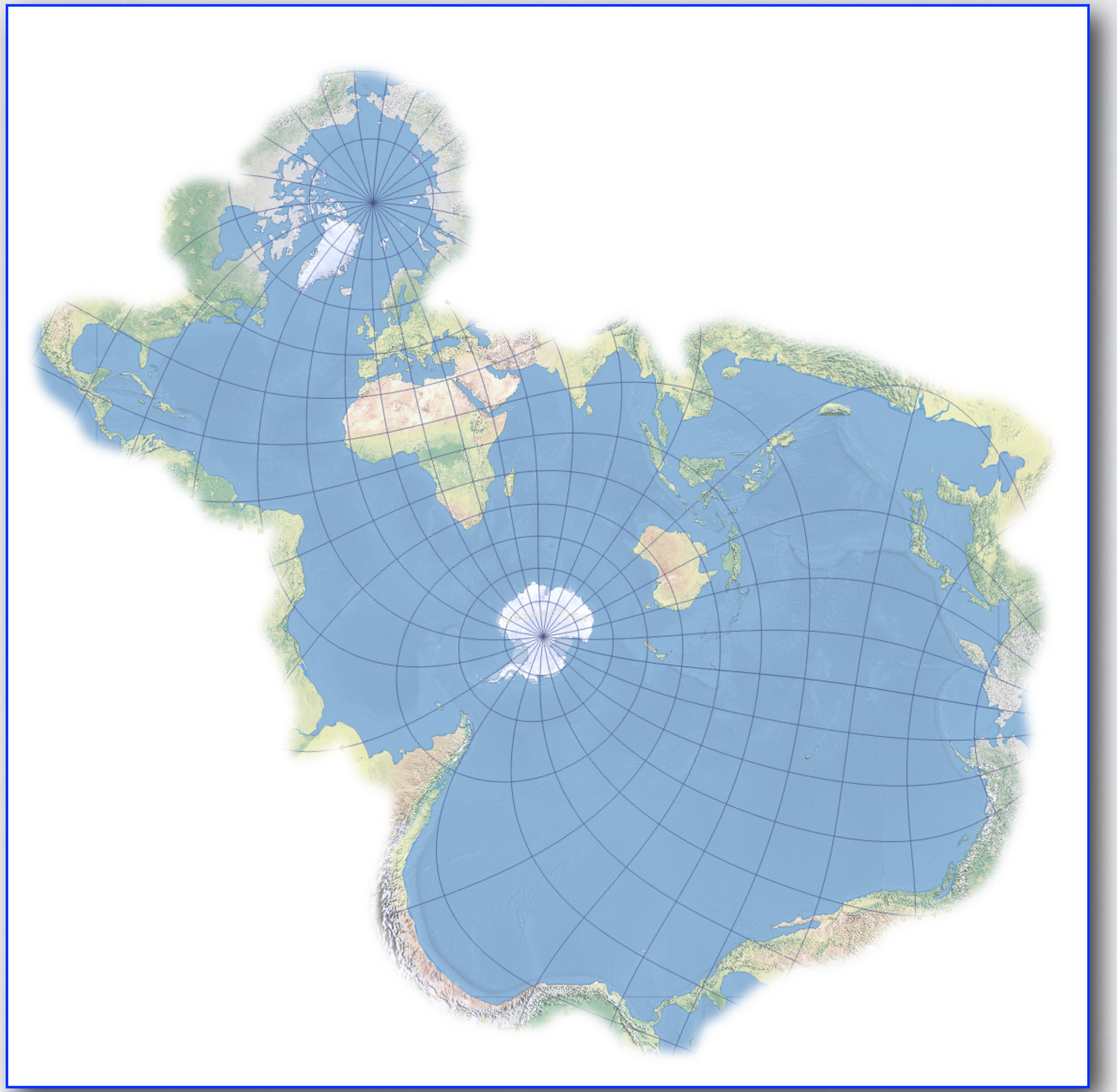
The general IBP identity can then be specialised at one loop. At differential level one finds

$$d\omega_{n-2} + \sum_{k=1}^{n+1}(s_{kh}+s_{k,n+1})\, f\big(\{\mathcal{R}-k\}_0, \{k\}_1\big) = \frac{\nu_h-1}{\nu-(d+1)/2} f\big(\{h\}_{-1}, \{\mathcal{R}-h\}_0\big) + \frac{\nu-d}{\nu-(d+1)/2} f\big(\{n+1\}_{-1}, \{\mathcal{R}-\{n+1\}\}_0\big) .$$

- There are n equations: as kinematic derivatives raise ν_i's , IBP identities lower them.
- At one loop, a Barucchi-Ponzano-Regge (BPR) theorem holds:

→ Master integrals obey a closed system of (at most) $(2^n - 1)$ first-order differential equations.

EXAMPLES AT LOW ORDERS



Massless box

As a simple example we consider the one-loop massless box

$$I_{\rm box} = \Gamma(2+\epsilon) \int_{S_{n-1}} \eta_{\{z\}} \frac{(z_1+z_2+z_3+z_4)^{2\epsilon}}{(r z_1 z_3 + z_2 z_4)^{2+\epsilon}} \equiv \Gamma(2+\epsilon)\, I(0,0,0,0;2\epsilon)\,,$$

where r = t/s and for the box family we use the notation $I(\nu_1-1,\nu_2-1,\nu_3-1,\nu_4-1;\nu_5).$

The generalised Cayley matrix is very simple

$$s_{ij} = \begin{pmatrix} 0 & 0 & r & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ r & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Taking derivatives with respect to Mandelstam invariants raises the values of ν_i's in pairs

$$\partial_r I(0,0,0,0;2\epsilon) = -(2+\epsilon)\, I(1,0,1,0;2\epsilon)$$

$$\partial_r I(1,0,1,0;2\epsilon) = -(3+\epsilon)\, I(2,0,2,0;2\epsilon)\,.$$

IBP identities and the BPR theorem suggest a (redundant) basis of four master integrals

$$\Big\{ I(0,0,0,0;2\epsilon), I(1,0,1,0;2\epsilon), I(0,1,0,1;2\epsilon), I(1,1,1,1;2\epsilon) \Big\}\,.$$

Massless box

We use the freedom to initialise the recursion by picking $\nu_1=3$, $\nu_2=2$, $\nu_3=\nu_4=1$. Then, for $h = 1$

$$r I(2,0,2,0;2\epsilon) + \int d\omega_{n-2} = \frac{2}{3+\epsilon} I(1,0,1,0;2\epsilon) + \frac{2\epsilon}{3+\epsilon} I(2,0,1,0;-1+2\epsilon)\,.$$

With these choices, the boundary term in the recursion vanishes

$$\frac{z_1^2 z_3 \left(z_1+z_2+z_3+z_4\right)^{2\epsilon}}{(3+\epsilon)(r z_1 z_3 + z_2 z_4)^{3+\epsilon}} \left(z_2 dz_3 \wedge dz_4 - z_3 dz_2 \wedge dz_4 + z_4 dz_2 \wedge dz_3\right)\Big|_{\partial S_{n-1}} = 0\,,$$

Closing the system of differential equations on the chosen basis is not difficult and one gets

$$\partial_r \mathbf{b} \equiv \partial_r \begin{pmatrix} I(0,0,0,0;2\epsilon) \\ I(1,0,1,0;2\epsilon) \\ I(0,1,0,1;2\epsilon) \\ I(1,1,1,1;2\epsilon) \end{pmatrix} = \begin{pmatrix} 0 & -(2+\epsilon) & 0 & 0 \\ 0 & -\frac{3+\epsilon}{r} & 0 & -\frac{3+\epsilon}{r} \\ 0 & 0 & 0 & -(3+\epsilon) \\ 0 & -\frac{1}{(3+\epsilon)r(1+r)} & \frac{1}{(3+\epsilon)r(1+r)} & -\frac{1+\epsilon+3r}{(3+\epsilon)r(1+r)} \end{pmatrix} \mathbf{b}\,.$$

The result is not in canonical form, but it can be brought to it, for example with the technique of Magnus exponentiation. The system can then be solved iteratively in ε as usual, which gives

$$\begin{aligned} I_{\text{box}} = \frac{k(\epsilon)}{r} \Bigg[\frac{1}{\epsilon^2} - \frac{\log r}{2\epsilon} - \frac{\pi^2}{4} + \epsilon \Bigg(\frac{1}{2}\,\text{Li}_3(-r) - \frac{1}{2}\,\text{Li}_2(-r)\log r + \frac{1}{12}\log^3 r \\ - \frac{1}{4}\log(1+r)\left(\log^2 r + \pi^2\right) + \frac{1}{4}\pi^2 \log r + \frac{1}{2}\zeta(3) \Bigg) + \mathcal{O}(\epsilon^2) \Bigg], \end{aligned}$$

Matching known results (see for example Henn 2201.03593).

Bern, Dixon, Kosower hep-ph/9306240

Massless pentagon

The massless pentagon is well-known to reduce to a sum of boxes with a massive leg in d=4.

The reduction of propagators happens in our framework through boundary terms: thus we consider low values of the indices, starting with $\nu_i = 1$. Picking for example $h = 1$, we get

$$\int_{S_{\{1,2,3,4,5\}}} d\omega_3 + s_{13}\, I(0,0,1,0,0;2\epsilon) + s_{14}\, I(0,0,0,1,0;2\epsilon) = \frac{2\epsilon}{2+\epsilon} I(0,0,0,0,0;-1+2\epsilon)\,,$$

The 3-form ω_3 for $h = 1$ does not vanish on the $z_1 = 0$ boundary, which contributes

$$\int_{S_{\{2,3,4,5\}}} \eta_{\{2,3,4,5\}} \frac{(z_2+z_3+z_4+z_5)^{2\epsilon}}{(s_{24}z_2z_4 + s_{25}z_2z_5 + s_{35}z_3z_5)^{2+\epsilon}} = I^{(1)}_{\rm box}(s_{25})\,,$$

The exercise repeats for all values of h, generating permutations of s_{ij}. The resulting system is algebraic and can be solved for the original pentagon integral in $d = 4 - 2\varepsilon$, yielding

$$\begin{pmatrix} 0 & 0 & s_{13} & s_{14} & 0 \\ 0 & 0 & 0 & s_{24} & s_{25} \\ s_{13} & 0 & 0 & 0 & s_{35} \\ s_{14} & s_{24} & 0 & 0 & 0 \\ 0 & s_{25} & s_{35} & 0 & 0 \end{pmatrix}$$

Coeffcent matrix

$$\begin{aligned} 2(2+\epsilon)\, I(0,0,0,0,0;1+2\epsilon) = \Bigg\{ & \frac{s_{13}s_{24} - s_{13}s_{25} - s_{14}s_{25} + s_{14}s_{35} - s_{24}s_{35}}{s_{13}s_{14}s_{25}} I^{(1)}_{\rm box} \\ & - \frac{s_{13}s_{24} + s_{13}s_{25} - s_{14}s_{25} + s_{14}s_{35} - s_{24}s_{35}}{s_{13}s_{24}s_{25}} I^{(2)}_{\rm box} \\ & - \frac{s_{13}s_{24} - s_{13}s_{25} + s_{14}s_{25} - s_{14}s_{35} + s_{24}s_{35}}{s_{13}s_{24}s_{35}} I^{(3)}_{\rm box} \\ & + \frac{s_{13}s_{24} - s_{13}s_{25} + s_{14}s_{25} - s_{14}s_{35} - s_{24}s_{35}}{s_{14}s_{24}s_{35}} I^{(4)}_{\rm box} \\ & - \frac{s_{13}s_{24} - s_{13}s_{25} + s_{14}s_{25} + s_{14}s_{35} - s_{24}s_{35})}{s_{14}s_{25}s_{35}} I^{(5)}_{\rm box} \Bigg\} \\ & + 2\epsilon\, I(0,0,0,0,0;-1+2\epsilon)\,. \end{aligned}$$

BDK solution

Laporta, Remiddi hep-ph/0406160

Equal-mass sunrise

"Banana" integrals are very interesting: two-point, elliptics, Calabi-Yau's; were studied by Regge. The first Symanzik polynomial is very simple and symmetric

$$\mathcal{U} = \sum_{i=1}^{l+1} z_1 ... \hat{z}_i ... z_{l+1}$$

→ Raising and lowering operators obey sum rules similar to the case of one-loop n-legs

Consider the family (d = 2 - 2ε, z = p²/m²)

$$I(\nu_1-1,\nu_2-1,\nu_3-1;\lambda_4) = \int_{S_{\{1,2,3\}}} \eta_3 \frac{z_1^{\nu_1-1} z_2^{\nu_2-1} z_3^{\nu_3-1} (z_1 z_2 + z_2 z_3 + z_3 z_1)^{\lambda_4}}{\left[z z_1 z_2 z_3 - (z_1 z_2 + z_2 z_3 + z_3 z_1)(z_1+z_2+z_3)\right]^{\frac{2\lambda_4+\nu}{3}}},$$

Once again we use the numerator for IBPs

$$H = z_1^{\nu_1-1} z_2^{\nu_2-1} z_3^{\nu_3-1} (z_1 z_2 + z_2 z_3 + z_3 z_1)^{\lambda_4}$$

The recursion gives non-vanishing boundary terms, and they are simple `bubble' integrals

$$\int d\omega_1 = \frac{1}{2(1+\epsilon)} \int_{S_{\{1,2\}}} \eta_{\{1,2\}} \frac{z_1 z_2 (z_1 z_2)^{1+3\epsilon}}{(-z_1 z_2 (z_1+z_2))^{2+2\epsilon}} = \frac{(-1)^{2\epsilon}}{2+2\epsilon} \frac{\Gamma(1+\epsilon)\Gamma(1+\epsilon)}{\Gamma(2+2\epsilon)}.$$

Through elimination, one finds a system of two first-order differential equations for the I(0,0,0, 3ε) and I(1,0,0;1 + 3ε). It turns into the known 2^d-order equation for the sunrise

$$\frac{z}{3}\frac{d^2}{dz^2} I(000,3\epsilon) + \left(\frac{1}{3} + \frac{3}{z-9} + \frac{1}{3(z-1)}\right)\frac{d}{dz} I(000,3\epsilon) - \left(\frac{1}{4(z-9)} + \frac{1}{12(z-1)}\right) I(000,3\epsilon) = \frac{2}{(z-1)(z-9)},$$

OUTLOOK

Outlook

- Digging back into ancient history may deliver hidden treasures.
- Some of the old mathematical results may be worth translating into modern language.
- Parameter space may provide a new way to walk through the integral woods.
- Symmetries and graphical properties are best encoded in parameter language.
- Parameter-space differential equations are closely related to monodromies.
- It is worthwhile to study more complex examples and explore systematics.

The parameter-space road to differential equations is worth exploring!


merci
thank you
gracias
danke
obrigado
go raibh maith agat
спасибо
grazie
bedankt
arigatô
ngiyabonga
sukriya
dakujem
mochchakkeram
teşekkür ederim
terima kasih
dank je
sagolun
chokrane
dziękuję
謝謝
tapadh
takk
mauruuru
vinaka
Баярлалаа
ευχαριστώ
kiitos
barka
tau
kia ora
ありがとう
grazzi
мерси
diolch
mersi
tanemirt
dhanyavad
shukriya
spas
welalin
gratias ago
misaotra
matondo
dhanyavadagalu
hvala
taiku
trugarez
kam sah hamnida
faafetai lava
paldies
kop khun
najis tuke
mamnun
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akun
aciū
rahmet
kop khun krap
mercé
leat
nandri
mahalo
didi madloba
blagodaram
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