Landau Singularities Revisited

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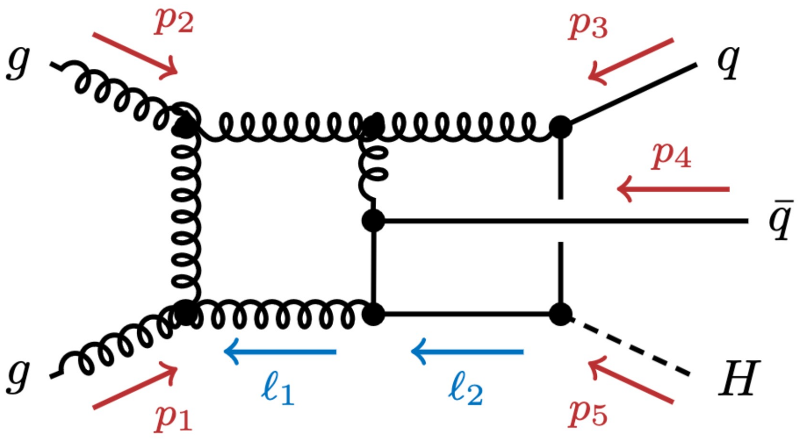
The topic of this talk will be the
simplest analytic property of scattering amplitudes:

kinematic singularities

We currently *do not* have a self-consistent algorithm that would predict kinematic singularities for a given Feynman integral:

Running example:



$$(s_{12}, s_{23}, s_{34}, s_{45}, s_{51}, p_5^2)$$

[Talks by Bechetti, Chicherin, ...]

Some are easy to predict

$$s_{12} = 0$$

$$\Delta_1(s_{ij}, p_i^2) = 0$$

$$\Delta_2(s_{ij}, p_i^2) = 0$$

$$\Delta_3(s_{ij}, p_i^2) = 0$$

$$\vdots$$

How to predict a complete list?

What could we do if we knew such an algorithm?

- **Differential equations**
  singular points and boundary conditions

- **Symbol alphabet**
  zeros and singularities of symbol letters

- **Numerical integration**
  analytic continuation and contour deformations

- **Bootstrapping Feynman integrals**
  constraints on the ansatz, discontinuities, …

…

Textbook story: Landau equations

[Bjorken, Landau, Nakanishi '54]

For every propagator:

$$\ell_i^2 = m_i^2$$

momenta masses

For every vertex:

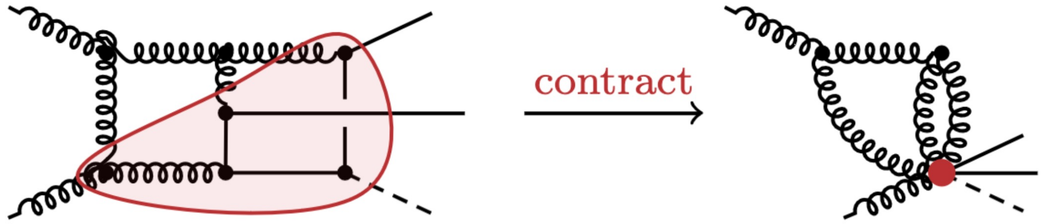
$$\sum_{i \in \text{vertex}} \pm \ell_i^\mu = 0$$

For every loop:

$$\sum_{i \in \text{loop}} \pm \alpha_i \ell_i^\mu = 0$$

Schwinger parameters

All*** singularities are obtained by studying reduced diagrams:



$$\alpha_i = 0$$

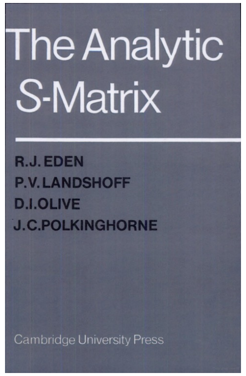
for subsets of edges

For the experts: There are other formulations

In the representation (2.2.4) the α are the only integration variables. The only surface S of singularity of the integrand is $D = 0$, while the boundaries of the hypercontour are again $\alpha_i = 0$. Hence the analogues of (2.2.9) and (2.2.10) are

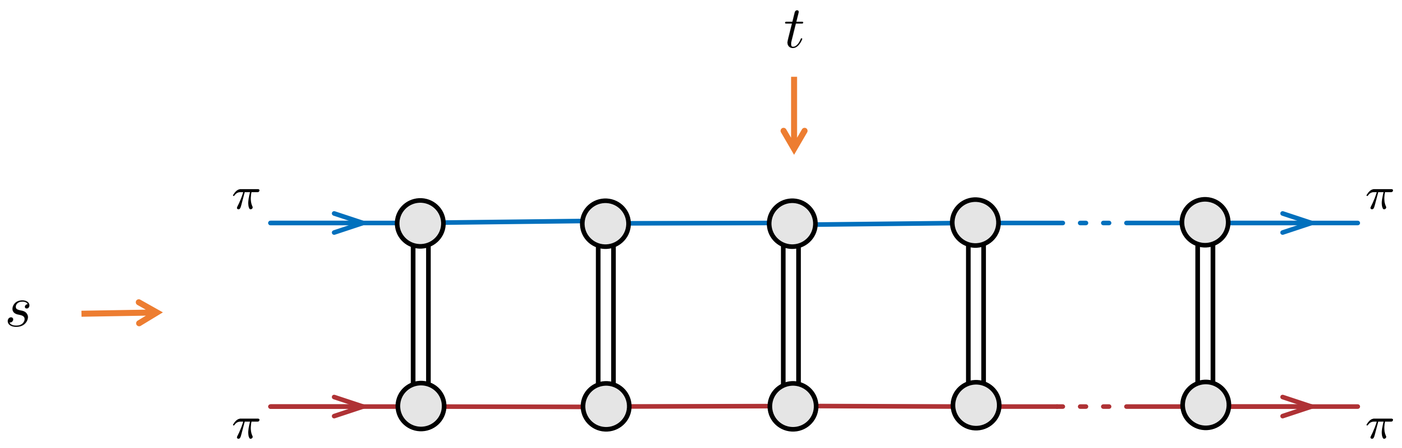
$$\left.\begin{array}{llll} & & D = 0, & \\ \text{and} & \textit{either} & \alpha_i = 0, & \\ & \textit{or} & \dfrac{\partial D}{\partial \alpha_i} = 0, & \text{for each } i. \end{array}\right\} \qquad (2.2.11)$$

That these equations are essentially equivalent to (2.2.9) and (2.2.10) can be seen from (1.5.26) and (1.5.27), except that further investigation is required when $C = 0$. This matter is taken up again in § 2.10.



Bottom line: Correct for <u>leading</u> Landau singularities,
but accounting for <u>all</u> singularities becomes much more intricate

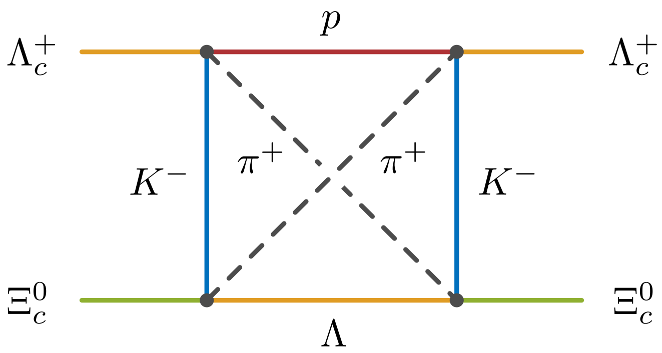
Many ways to find *leading* Landau singularities
(geometric methods, on-shell diagrams, Schubert calculus, elastic unitarity, …)
including examples known to all-loop orders, e.g.

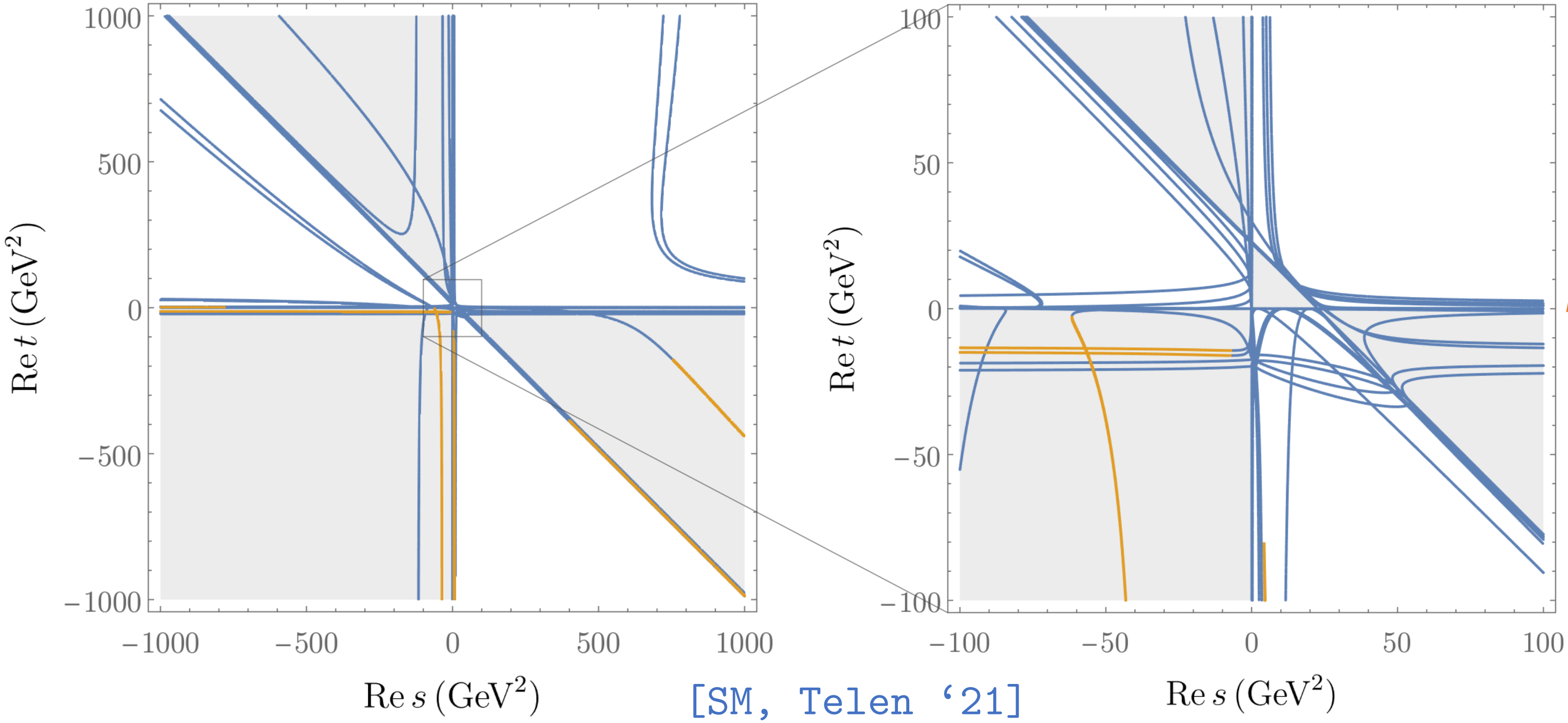


Singularities at $t = \frac{1}{2}(s - 4m_\pi^2)\left[T_{\frac{L+1}{2}}\left(1 + \frac{8m_\pi^2}{s - 4m_\pi^2}\right) - 1\right]$ [SM '22]

Chebyshev polynomials

Leading singularities can get quite wild, e.g.





Every curve is a branch surface

[SM, Telen ‘21]
[Hannesdottir, SM ‘22]

Except, there are a few asterisks…

* Second-type singularities

When all loop momenta blow up, $\ell_i \to \infty$

[Cutkosky ‘60, Fairlie, Landshoff, Nuttall, Polkinghorne ‘62]

** Mixed-type singularities

When a subset of loop momenta blows up

[Drummond ’63, Boyling ‘67]

*** Even more new classes (today)

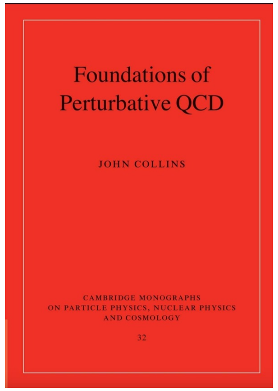
When loop momenta approach limits at *different rates*

Particularly important to understand when a diagram has <u>massless particles</u> or <u>UV/IR divergences</u>

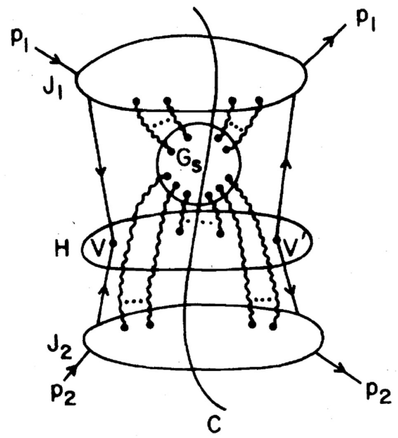
Well-known problem

5.3 (***) Find in the published literature, or construct for yourself, a proof that the Landau equations are actually necessary and sufficient for a PSS of a Feynman graph. To see that this is a non-trivial exercise, critically examine the accounts given in a standard textbook, e.g., Bogoliubov and Shirkov (1959); Eden *et al.* (1966); Itzykson and Zuber (1980); Peskin and Schroeder (1995); Sterman (1993). Are full proofs actually given? Do they actually work, and cover both necessity and sufficiency? Do they apply to the massless case, or do they make implicit assumptions only valid in the massive case?

[Exercise 5.3, Collins
“Foundations of Perturbative QCD”]



Two versions of the problem

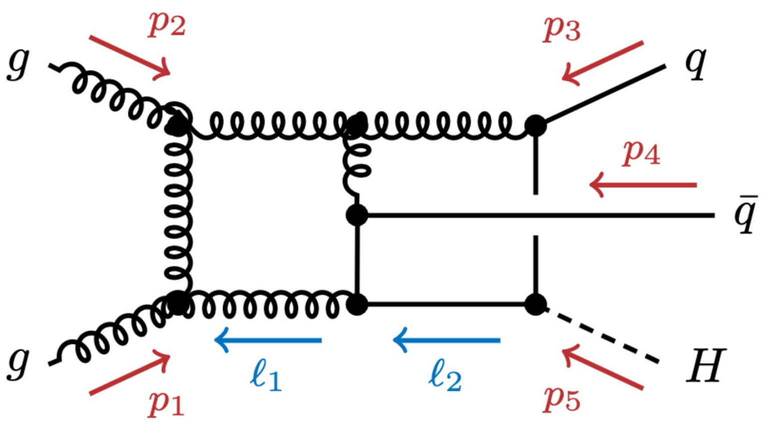


[Libby-Sterman '78]

Physical-region singularities

- Impose $\alpha_i \geqslant 0$
- Typically, already know the external kinematics, e.g., $p_i^2 \to 0$
- Related to SCET, expansion by regions, …

[Talks by Ma, Navichkov, Maheria, Sarkar, ...]



All singularities (any kinematics, any sheet)

- Any complex α_i, ℓ_i
- Want to find the singular kinematics

[This talk]

To make sure we're on the same page, simple example:

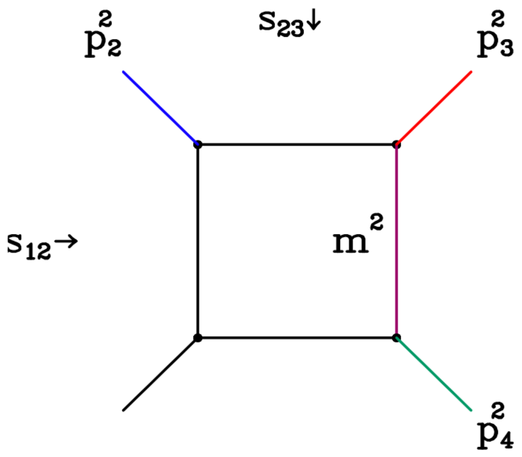
[QCDloop]

$$I_4^{\{D=4-2\epsilon\}}(0,p_2^2,p_3^2,p_4^2;s_{12},s_{23};0,0,0,m^2) = \frac{1}{(s_{12}s_{23}-m^2 s_{12}-p_2^2p_4^2+m^2p_2^2)}$$

$$\times \left[\frac{1}{\epsilon}\ln\left(\frac{(m^2-p_4^2)p_2^2}{(m^2-s_{23})s_{12}}\right) + \mathrm{Li}_2\left(1+\frac{(m^2-p_3^2)(m^2-s_{23})}{p_2^2m^2}\right) - \mathrm{Li}_2\left(1+\frac{(m^2-p_3^2)(m^2-p_4^2)}{s_{12}m^2}\right)\right.$$

$$+ 2\,\mathrm{Li}_2\left(1-\frac{m^2-s_{23}}{m^2-p_4^2}\right) - 2\,\mathrm{Li}_2\left(1-\frac{p_2^2}{s_{12}}\right) + 2\,\mathrm{Li}_2\left(1-\frac{p_2^2(m^2-p_4^2)}{s_{12}(m^2-s_{23})}\right)$$

$$\left. + 2\ln\left(\frac{\mu m}{m^2-s_{23}}\right)\ln\left(\frac{(m^2-p_4^2)p_2^2}{(m^2-s_{23})s_{12}}\right)\right] + \mathcal{O}(\epsilon)$$



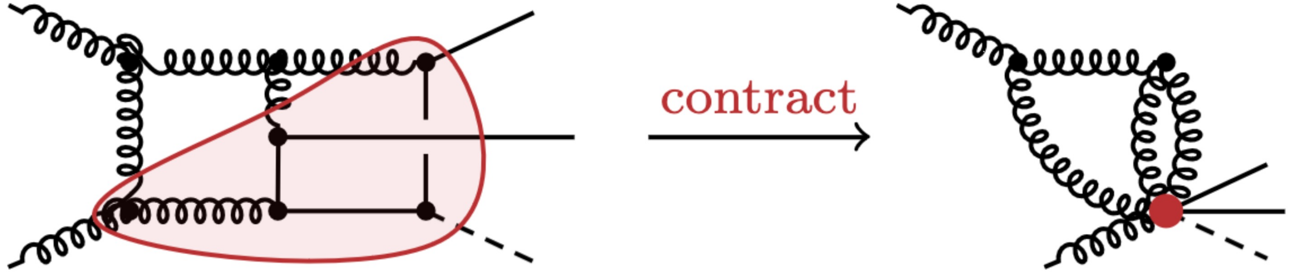
Total of 17 distinct singularities

(completely understood for any one-loop diagram)

There are two inconsistencies with the standard analysis

(well-known to anyone who tried
solving Landau equations in practice)

(I) Interplay with UV/IR divergences



- Need to be much more careful:

$$\frac{1}{\epsilon^k} f(s_{ij}, p_i^2)$$

UV/IR divergence (discard)

Kinematic singularities (keep)

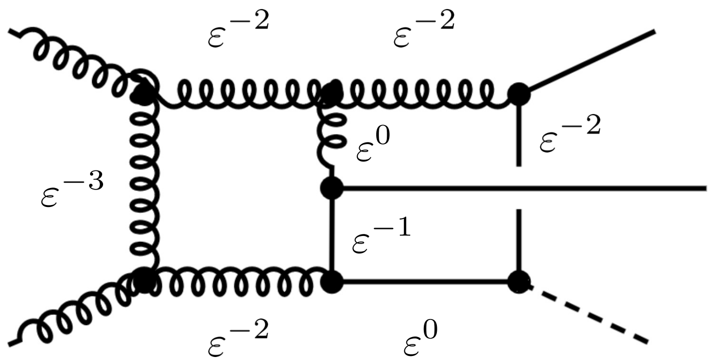
- Sits directly at an IR divergence
  - No mass scales/ill-defined
    - Regularize? Discard?

Need to find kinematic singularities "underneath" UV/IR divergences

(II) Beyond the standard classification

We also need to allow Schwinger parameters
(and loop momenta) to approach zero/infinity at different rates

$$\alpha_i \to \varepsilon^{w_i} \alpha_i \qquad \text{with} \qquad \epsilon \to 0 \qquad \text{and} \qquad w_i \in \mathbb{Z}$$



Scalings can get quite complicated

Not just reduced diagrams

(related to toric compactifications, blow-ups, tropical geometry, …)

The goal of this talk is to address these issues in a *practical* way

Q: Why does it come to light only now?
A: This is the first time we have computational *tools* to do it consistently

(Computational algebraic geometry, homotopy continuation, irreducible decomposition via monodromy, etc.)

Practical = be able to put it on a computer

```
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codim: 5, face: 1464/1469, weights: [2, 2, 2, 0, 1, 0, 0, 0], discriminant: 1
codim: 5, face: 1465/1469, weights: [1, 2, 2, 0, 0, -2, -1, -1], discriminant: 1
codim: 5, face: 1466/1469, weights: [2, 2, 2, 0, 0, 0, 0, 1], discriminant: 1
codim: 5, face: 1467/1469, weights: [2, 2, 2, 0, 0, 0, 1, 0], discriminant: 1
codim: 5, face: 1468/1469, weights: [3, 3, 3, 1, 1, 1, 1, 1], discriminant: 1
codim: 5, face: 1469/1469, weights: [2, 2, 2, 1, 0, 0, 0, 0], discriminant: 1
Unique discriminants after codim 5: -4*s12*M2 + s35^2 + 2*s35*s45 + s45^2, -4*s12^2*s23*M2 + s12^2*s45^2 + 2*s12^2*s45*s51 + s12^2*s51^2 - 4*s12*s23^2*M2 + 2*s12*s23*s35*s45
 - 2*s12*s23*s35*s51 + 4*s12*s23*s35*M2 + 2*s12*s23*s45^2 + 2*s12*s23*s45*s51 + 4*s12*s23*s45*M2 + 4*s12*s23*s51*M2 - 4*s12*s23*M2^2 - 2*s12*s35*s45^2 - 2*s12*s35*s45*s51 -
2*s12*s45^3 - 4*s12*s45^2*s51 + 2*s12*s45^2*M2 - 2*s12*s45*s51^2 + 2*s12*s45*s51*M2 + s23^2*s35^2 + 2*s23^2*s35*s45 + s23^2*s45^2 - 2*s23*s35^2*s45 - 4*s23*s35*s45^2 - 2*s23
*s35*s45*s51 + 2*s23*s35*s45*M2 - 2*s23*s45^3 - 2*s23*s45^2*s51 + 2*s23*s45^2*M2 + s35^2*s45^2 + 2*s35*s45^3 + 2*s35*s45^2*s51 - 2*s35*s45^2*M2 + s45^4 + 2*s45^3*s51 - 2*s45
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3*s35*s51 - 2*s12*s23*s35*M2 + 2*s12*s23*s45^2 + 2*s12*s23*s45*s51 - 4*s12*s23*s45*M2 - 2*s12*s23*s51*M2 + 2*s12*s23*M2^2 - 2*s12*s35*s45^2 - 2*s12*s35*s45*s51 + 4*s12*s35*s
45*M2 + 2*s12*s35*s51*M2 - 2*s12*s35*M2^2 - 2*s12*s45^3 - 4*s12*s45^2*s51 + 6*s12*s45^2*M2 - 2*s12*s45*s51^2 + 8*s12*s45*s51*M2 - 6*s12*s45*M2^2 + 2*s12*s51^2*M2 - 4*s12*s51
*M2^2 + 2*s12*M2^3 + s23^2*s35^2 + 2*s23^2*s35*s45 - 2*s23^2*s35*M2 + s23^2*s45^2 - 2*s23^2*s45*M2 + s23^2*M2^2 - 2*s23*s35^2*s45 + 2*s23*s35^2*M2 - 4*s23*s35*s45^2 - 2*s23*
s35*s45*s51 + 8*s23*s35*s45*M2 + 2*s23*s35*s51*M2 - 4*s23*s35*M2^2 - 2*s23*s45^3 - 2*s23*s45^2*s51 + 6*s23*s45^2*M2 + 4*s23*s45*s51*M2 - 6*s23*s45*M2^2 - 2*s23*s51*M2^2 + 2*
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^4 + 2*s45^3*s51 - 4*s45^3*M2 + s45^2*s51^2 - 6*s45^2*s51*M2 + 6*s45^2*M2^2 - 2*s45*s51^2*M2 + 6*s45*s51*M2^2 - 4*s45*M2^3 + s51^2*M2^2 - 2*s51*M2^3 + M2^4, s23, s23 - s35 -
 s45 - s51 + M2, s23 - s45, s23 - s45 - s51 + M2, s23 - s51, s35, s35 - M2, s45, s45 - M2, s51, s51 - M2
------- codim = 4, 1255 faces
codim: 4, face: 1/1255, weights: [-2, -2, -2, 0, 0, 0, 0, 0], discriminant: s12
codim: 4, face: 2/1255, weights: [-2, -1, -1, -1, -1, -1, -1, 0], discriminant: s12
codim: 4, face: 3/1255, weights: [-1, 0, 0, 1, 1, 1, 1, 2], discriminant: 1
codim: 4, face: 4/1255, weights: [-2, -3, -2, -1, -1, -1, -1, 0], discriminant: s12 + s23 - s45
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Our formulation is inspired by the work of Gelfand, Kapranov, Zelevinsky

Principal A-determinant

$\Leftrightarrow$

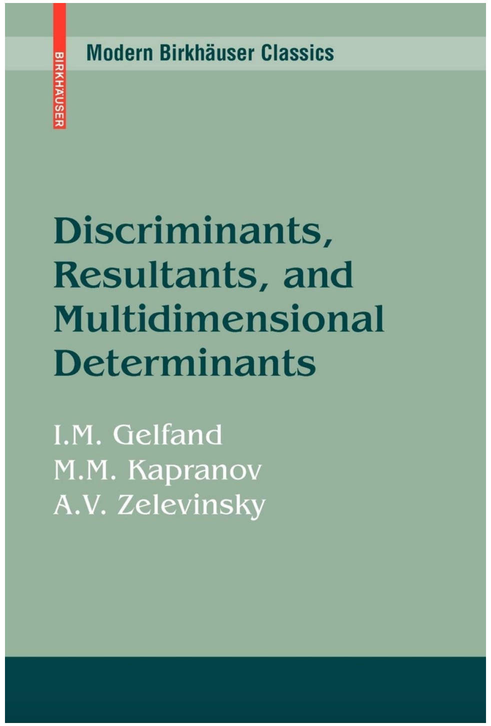
Singularity locus of *generic* generalized hypergeometric integrals

[see also Klausen '21, Dlapa, Helmer, Papathanasiou, Tellander '23]

- Can't use it directly: Singularities of Feynman integrals turn out to be much more complicated (UV/IR divergences)

- We introduce the ***principal Landau determinant*** to formalize Landau singularities

Ask me later for a rigorous definition



As usual, after integrating out the loop momenta we get:

$$\mathcal{I}_{\nu_1,\nu_2,\ldots,\nu_m} = \int \frac{\mathrm{d}^{\mathrm{D}}\ell_1\,\mathrm{d}^{\mathrm{D}}\ell_2\cdots\mathrm{d}^{\mathrm{D}}\ell_{\mathrm{L}}}{P_1^{\nu_1}P_2^{\nu_2}\cdots P_m^{\nu_m}}$$

Includes ISP's

Schwinger parametrization:

$$\sum_{i=1}^{m}\alpha_i P_i = \sum_{a,b=1}^{\mathrm{L}} \ell_a\cdot\ell_b\,\mathbf{Q}_{ab} + 2\sum_{a=1}^{\mathrm{L}}\ell_a\cdot\mathbf{L}_a + c$$

Symanzik polynomials:

$$\mathcal{U} := \det\mathbf{Q}, \qquad \mathcal{F} := (\mathbf{L}^{\intercal}\cdot\mathbf{Q}\mathbf{L} - c)\,\mathcal{U}$$

Depends on the kinematics

After integrating out the loop momenta:

$$\mathcal{I}_{\nu_1,\nu_2,\ldots,\nu_m} = \#\int_0^\infty \frac{\mathrm{d}^m\alpha}{(\mathcal{U}+\mathcal{F})^{\mathrm{D}/2}}\alpha_1^{\nu_1-1}\alpha_2^{\nu_2-1}\cdots\alpha_m^{\nu_m-1}$$

[Lee-Pomeransky '13]

Simplest singularity

Determines the
"pinch surface"
(incidence variety)

$$\mathcal{U} + \mathcal{F} = 0$$

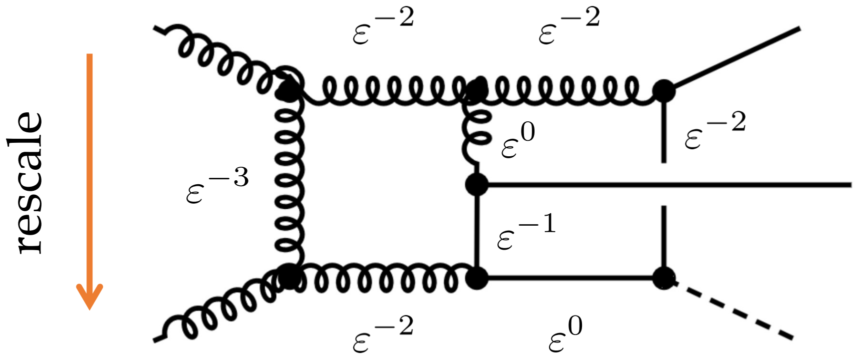
$$\partial_{\alpha_i}(\mathcal{U} + \mathcal{F}) = 0 \quad \text{for} \quad i = 1, 2, \ldots, m$$

Corresponds to the leading second-type singularity
in the standard classification

How to find all ways of rescaling $\alpha_i \to \varepsilon^{w_i}\alpha_i$
leading to all inequivalent systems of equations?

Example

$$\mathcal{U}+\mathcal{F} = \alpha_2\alpha_6\alpha_8 s_{14} + \alpha_3\alpha_5\alpha_7 s_{34} + \alpha_2\alpha_4\alpha_7 s_{24} + \alpha_1\alpha_3\alpha_4 s_{12} + \alpha_3\alpha_4\alpha_6 s_{35} + \alpha_1\alpha_3\alpha_5 s_{12} + \alpha_1\alpha_3\alpha_6 s_{12}$$
$$+ \alpha_1\alpha_3\alpha_7 s_{12} + \alpha_3\alpha_6\alpha_7 s_{12} + \alpha_1\alpha_3\alpha_8 s_{12} + \alpha_1\alpha_4\alpha_8 s_{12} + \alpha_2\alpha_5\alpha_7 s_{15} + \alpha_2\alpha_5\alpha_8 s_{23} + \alpha_1\alpha_4\alpha_6 s_{35}$$
$$+ \alpha_2\alpha_4\alpha_6 s_{35} + \alpha_1\alpha_4\alpha_7 s_{35} + \alpha_4\alpha_6\alpha_7 s_{35} + \alpha_3\alpha_6\alpha_8 s_{35} + \alpha_4\alpha_6\alpha_8 s_{35} + \alpha_1\alpha_5\alpha_8 s_{45} + \alpha_1\alpha_4 + \alpha_2\alpha_4$$
$$+ \alpha_3\alpha_4 + \alpha_1\alpha_5 + \alpha_2\alpha_5 + \alpha_3\alpha_5 + \alpha_1\alpha_6 + \alpha_2\alpha_6 + \alpha_3\alpha_6 + \alpha_1\alpha_7 + \alpha_2\alpha_7 + \alpha_3\alpha_7 + \alpha_4\alpha_7 + \alpha_5\alpha_7$$
$$+ \alpha_6\alpha_7 + \alpha_1\alpha_8 + \alpha_2\alpha_8 + \alpha_3\alpha_8 + \alpha_4\alpha_8 + \alpha_5\alpha_8 + \alpha_6\alpha_8 + \alpha_1\alpha_5\alpha_6 p_5^2 + \alpha_2\alpha_5\alpha_6 p_5^2 + \alpha_3\alpha_5\alpha_6 p_5^2$$
$$+ \alpha_1\alpha_5\alpha_7 p_5^2 + \alpha_5\alpha_6\alpha_7 p_5^2 + \alpha_5\alpha_6\alpha_8 p_5^2$$

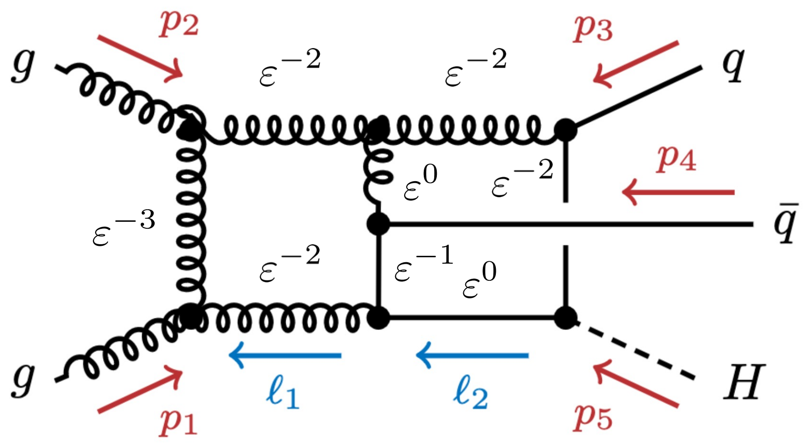


Not a reduced diagram

$$\mathcal{U}+\mathcal{F} \to \varepsilon^{-6}\left[\alpha_1\alpha_3(\alpha_4+\alpha_5)s_{12} + \alpha_2\alpha_5\alpha_7 s_{15} + \alpha_2\alpha_4\alpha_7 s_{24}\right]$$

The solution of $\partial_{\alpha_i}(\mathcal{U}+\mathcal{F}) = 0$ for this scaling is: $s_{24} - s_{15} = 0$

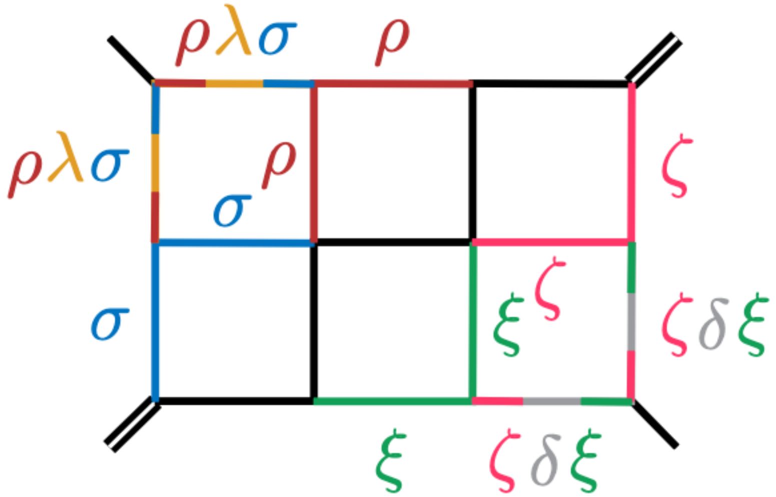
Back in the loop-momentum space



$$\ell_1 = p_1 + \mathcal{O}(\varepsilon)$$

$$\ell_2 = p_1 + p_2 + p_4 + p_3 \frac{\alpha_5}{\alpha_4 + \alpha_5} + \mathcal{O}(\varepsilon)$$

Related to the new perspective on soft/collinear divergences in terms of the Schwinger parameters



Edges expanding at different relative rates

$$\rho, \lambda, \sigma \to \infty$$

$$\zeta, \delta, \xi \to \infty$$

[Arkani-Hamed, Hillman, SM ‘22]

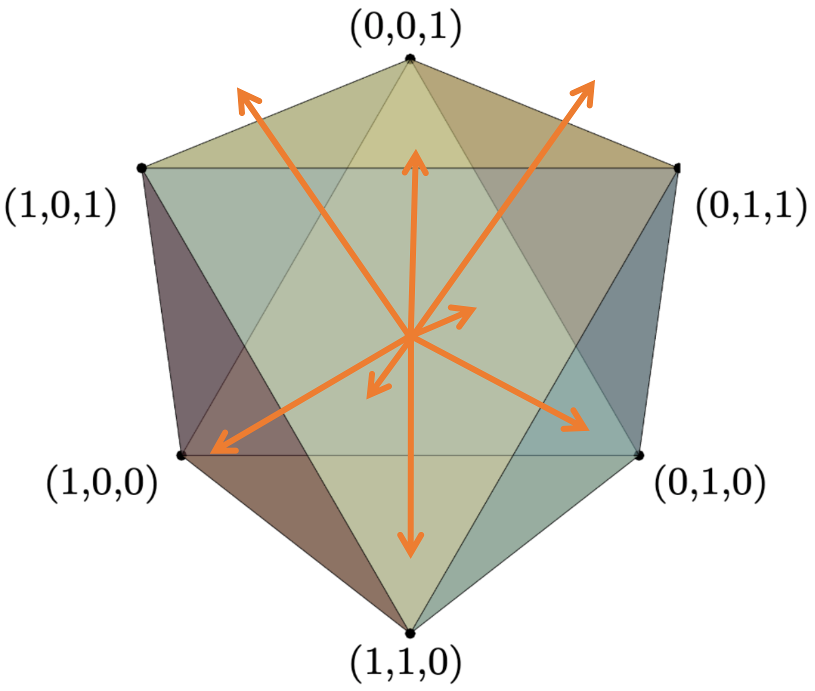
[see also Gardi, Herzog, Jones, Ma, Schlenk ‘22]

For the experts: Classification solved by polyhedral/tropical geometry

$$\mathbf{P} = \mathrm{Newt}\big(\mathcal{U} + \mathcal{F}\big)$$

Newton polytope

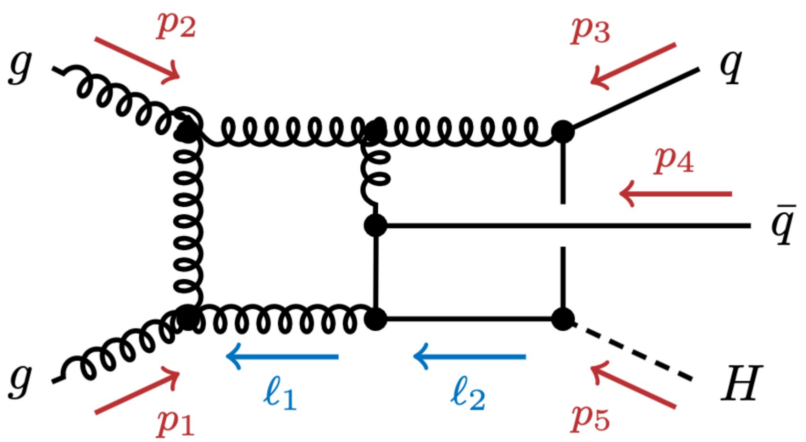
(dimension = number of propagators + ISP's)



- Classifies all ways of degenerating the system of equations
- Codimension-1 faces (facets) are used in sector decomposition

[FIESTA, pySecDec, ...]

Number of new systems of generalized "Landau equations"



Compared to just $2^8 = 256$ reduced diagrams

Without numerators: 4895

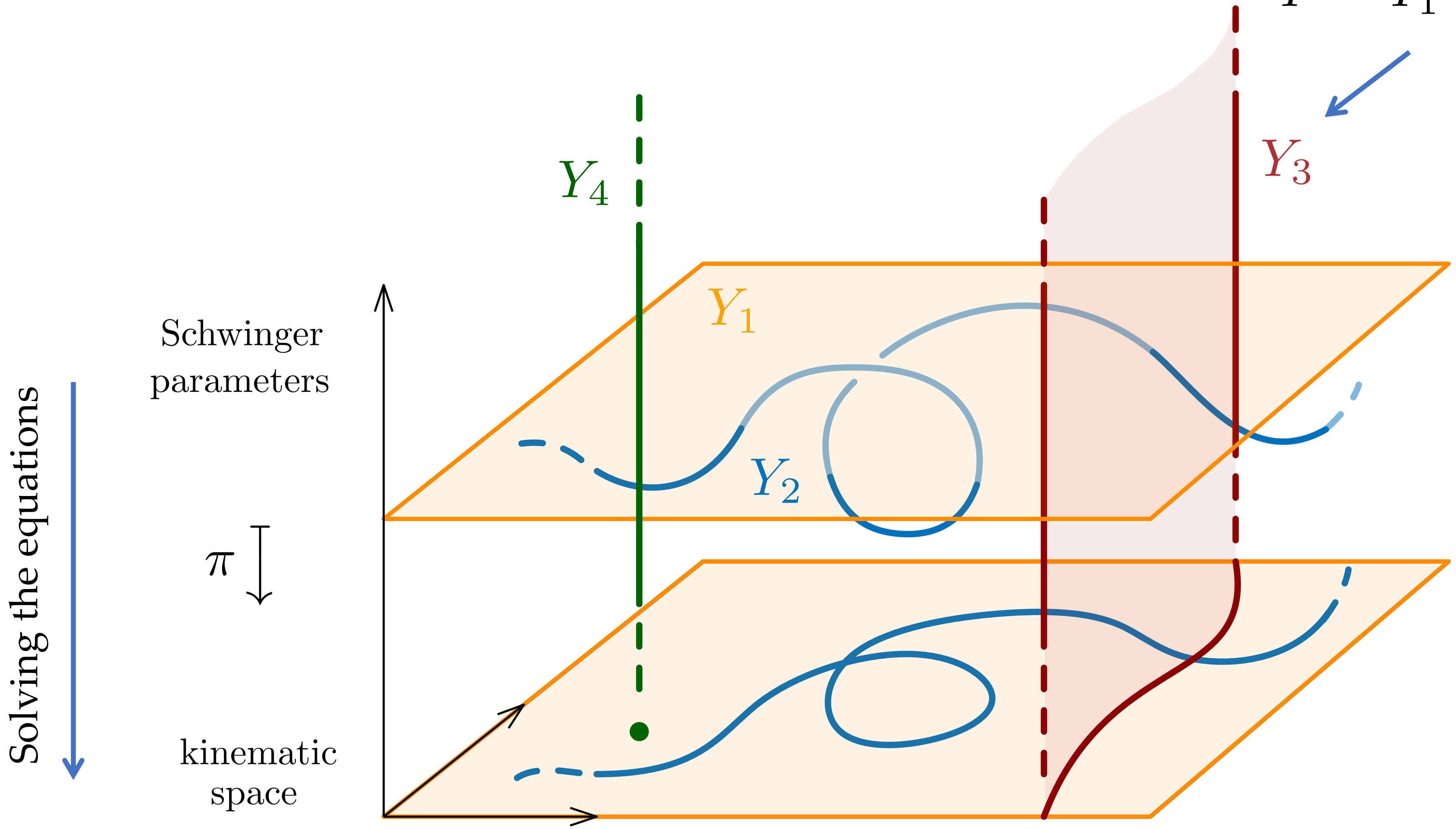
With numerators: 117097

These turn out to be impossible to solve using standard elimination theory tools such as Gröbner bases $\implies$ introduce a numerical algorithm

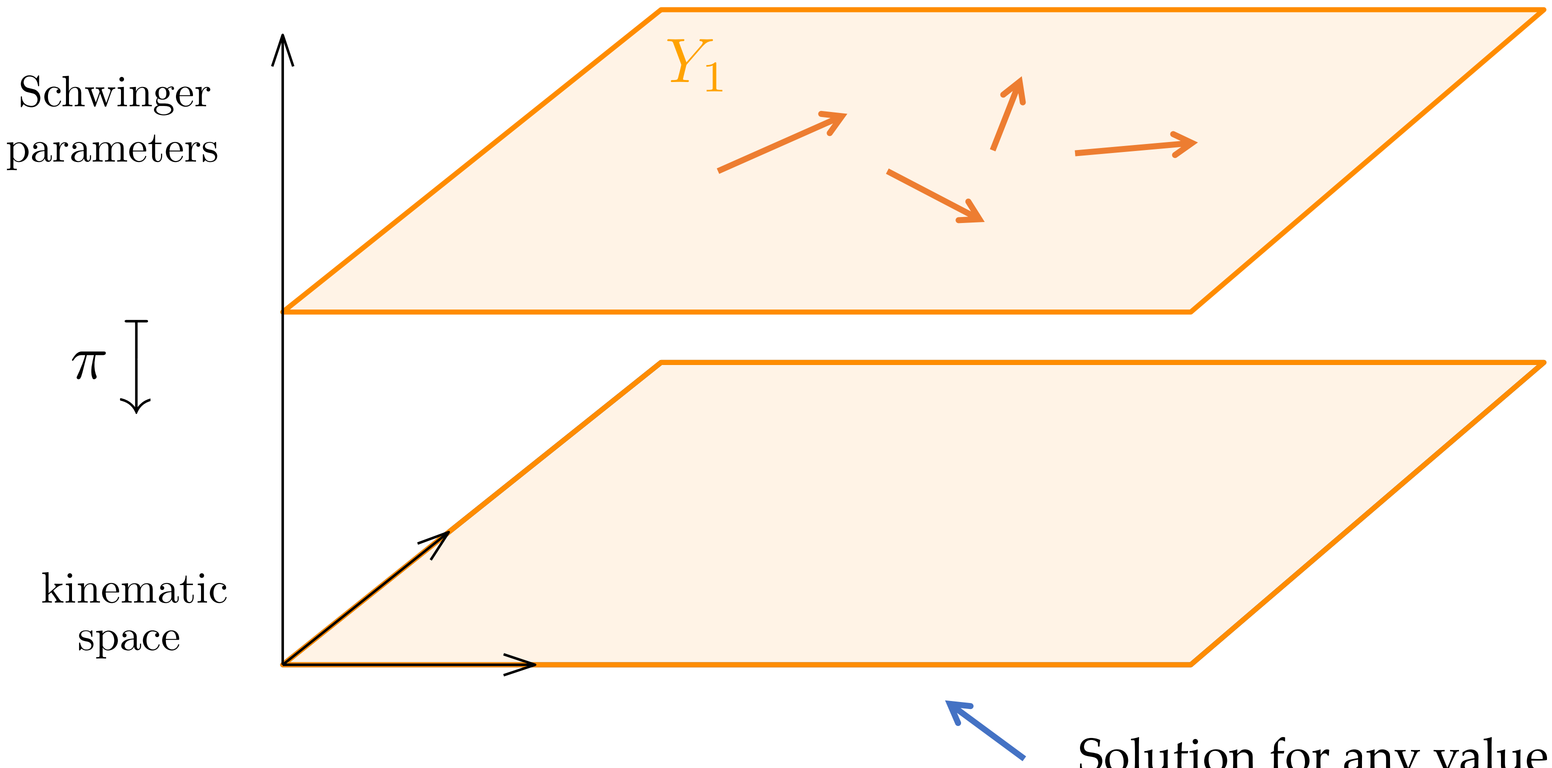
Geometry of singularities for one face

$$Y = Y_1 \cup Y_2 \cup Y_3 \cup Y_4 \cup \ldots$$

are components
of pinch surfaces



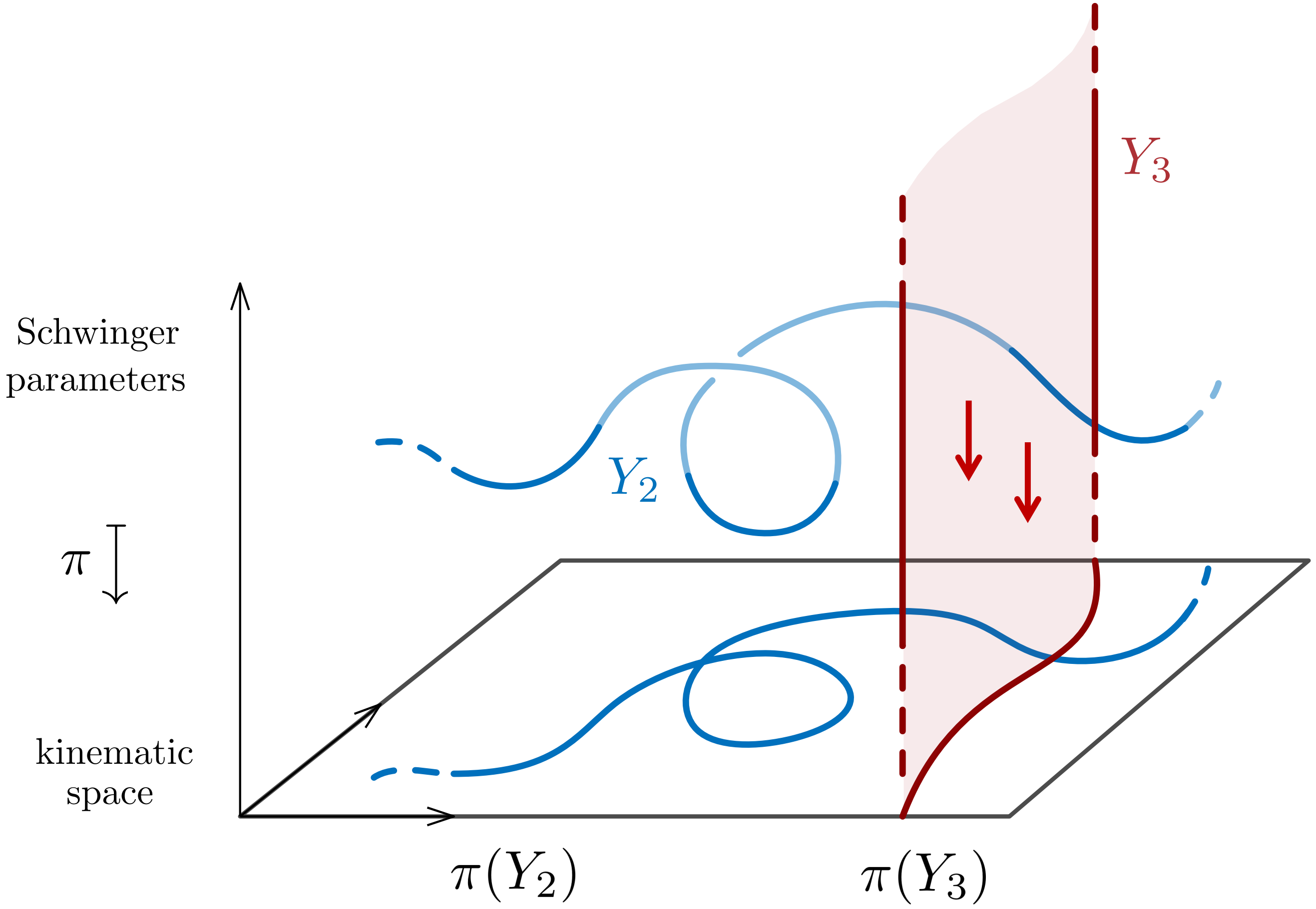
Discard dominant components (UV/IR divergences)



Solution for any value of kinematics
(regulated by dim. reg.)

$$\pi(Y_1) = \varnothing$$

We're only interested in codimension-1 singularities



For example: $\{s_{24} - s_{15} = 0\}$ $\{s_{35}s_{45} - s_{12}p_5^2 = 0\}$

Numerical strategy

- Intersect with random planes to collect samples
- Consistently filter out UV/IR divergences
- Gives the degree of the curve
- Write an ansatz

Ask me later for details

$$\sum_{a,b,c,d,e,f} C_{abcdef}\, s_{12}^a s_{23}^b s_{34}^c s_{45}^d s_{51}^e p_5^{2f} = 0$$

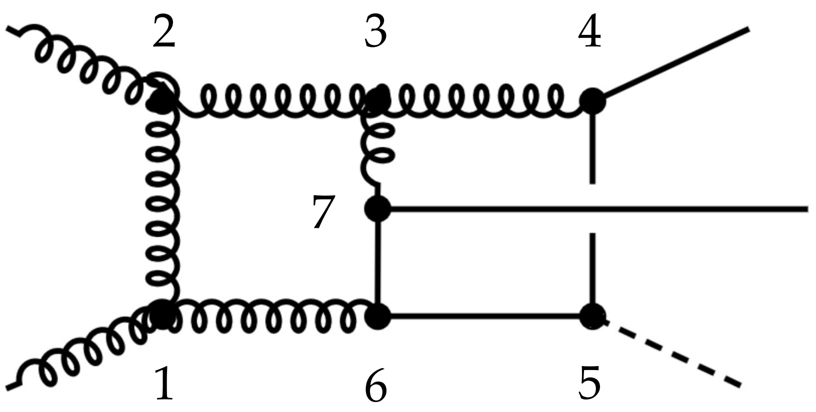
- Reconstruct the integer coefficients C_{abcdef} with the samples

… repeat the same for all the other 4894 faces (parallelizable)

Open-source implementation in `Julia`: `PrincipalLandauDeterminants.jl` (soon on `arXiv`)

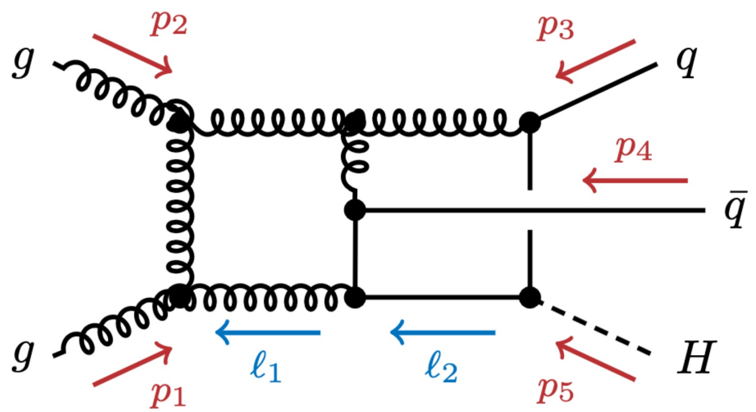
```
edges = [[6,1],[1,2],[2,3],[3,4],[4,5],[5,6],[6,7],[7,3]]
nodes = [1,2,4,7,5]
internal_masses = [0,0,0,0,0,0,0,0]
external_masses = [0,0,0,0,M2]

getPLD(edges, nodes, internal_masses, external_masses, method = :sym)
```



Example result

$$\{p_5^2 = 0\} \cup \{s_{ij} = 0 \text{ for all } ij \neq 25\}$$
$$\{s_{i5} - p_5^2 = 0 \text{ for } i = 1, 2, 3, 4\}$$
$$\{s_{12} - s_{45} = 0\} \cup \{s_{12} - s_{35} = 0\}$$
$$\{s_{24} - s_{15} = 0\} \cup \{s_{14} - s_{35} = 0\}$$
$$\{s_{24} - s_{35} = 0\} \cup \{s_{23} - s_{15} = 0\}$$
$$\{s_{23} - s_{45} = 0\} \cup \{s_{15} - s_{34} = 0\}$$
$$\{s_{35}s_{45} - s_{12}p_5^2 = 0\}$$
$$\{s_{15}s_{35} - s_{24}p_5^2 = 0\}$$
$$\{s_{15}s_{45} - s_{23}p_5^2 = 0\}$$
$$\{p_5^2 s_{12} - (s_{15} - p_5^2)(s_{25} - p_5^2) = 0\}$$
$$\{p_5^2 s_{23} - (s_{25} - p_5^2)(s_{35} - p_5^2) = 0\}$$



$$\{\det G(p_1{+}p_2, p_3{+}p_4) = 0\}$$
$$\{\det G(p_1{+}p_4, p_2{+}p_3) = 0\}$$
$$\{\Delta_5 := \det G(p_1, p_2, p_3, p_4) = 0\}$$

Gram determinants

$$\{\Sigma_5 := 16\Delta_5 + p_5^2 s_{24}[p_5^2(s_{15}{+}s_{23}{-}s_{34}) + 2s_{12}s_{15} -2s_{45}s_{15} - 2s_{12}s_{23} - 2s_{23}s_{34} + 2s_{34}s_{45}] = 0\}$$

[matches with Abreu, Chicherin, Ita, Page, Sotnikov, Tschernow, Zoia]

Including inverse powers of numerators can also introduce new singularities

$$\{2s_{12} - s_{35} = 0\}$$

$$\{s_{12} - s_{24} = 0\}$$

$$\{s_{12} - s_{14} = 0\}$$

$$\{p_5^2(s_{35} - 2s_{12}) + s_{12}s_{45} = 0\}$$

$$\{s_{12}s_{15} - s_{24}s_{45} = 0\}$$

$$\{(s_{34} - s_{15})\, s_{45} + s_{23}\,(s_{35} + s_{45}) = 0\}$$

$$\{p_5^2\,(s_{34} - s_{15}) + s_{12}s_{15} + s_{23}s_{34} + (s_{15} - s_{34})\, s_{45} = 0\}$$

$$\{s_{23}\left(s_{12} + s_{34} - p_5^2\right) + (s_{15} - s_{34})\, s_{45} = 0\}$$

$$\vdots$$

One-slide summary

- Predicting singularities of cutting-edge Feynman integrals forces us to revisit the standard formulation of Landau equations

- Classification of Landau singularities much richer than just reduced diagrams, particularly when massless particles are present

- New tool for the community: `PrincipalLandauDeterminants.jl` (soon on `arXiv`)

Thank you