Gradient flow renormalon subtraction and the hadronic tau decay series

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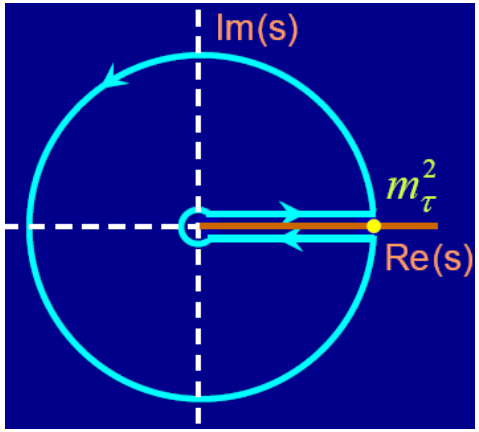
MB, H. Takaura, work in progress



α_s determinations and τ decay

Summary of α_s determinations (2203.08271, PDG update)

category	$\alpha_S(m_Z^2)$	relative $\alpha_S(m_Z^2)$ uncertainty
τ decays and low Q^2	0.1178 ± 0.0019	1.6%
$Q\overline{Q}$ bound states	0.1181 ± 0.0037	3.1%
PDF fits	0.1162 ± 0.0020	1.7%
e^+e^- jets & shapes	0.1171 ± 0.0031	2.6%
electroweak	0.1208 ± 0.0028	2.3%
hadron colliders	0.1165 ± 0.0028	2.4%
lattice	0.1182 ± 0.0008	0.7%
world average (without lattice)	0.1176 ± 0.0010	0.9%
world average (with lattice)	0.1179 ± 0.0009	0.8%

Hadronic τ decay width

- QCD PT + OPE condensates [Braaten (1989), Braaten, Narison, Pich (1992)]
- $\frac{\delta\alpha_s(M_Z)}{\alpha_s(M_Z)} \approx \frac{\alpha_s(M_Z)}{\alpha_s(M_\tau)} \times \frac{\delta\alpha_s(M_\tau)}{\alpha_s(M_\tau)}$
- Accuracy limited by a systematic discrepancy within *perturbation theory* – FOPT vs CIPT [MB, Jamin, 2008]

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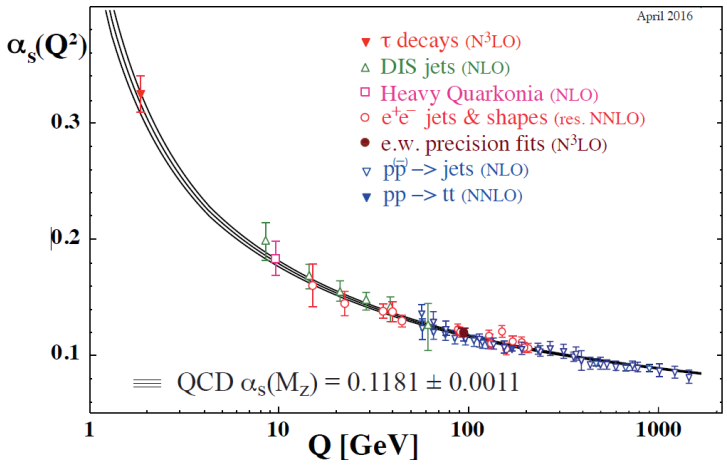
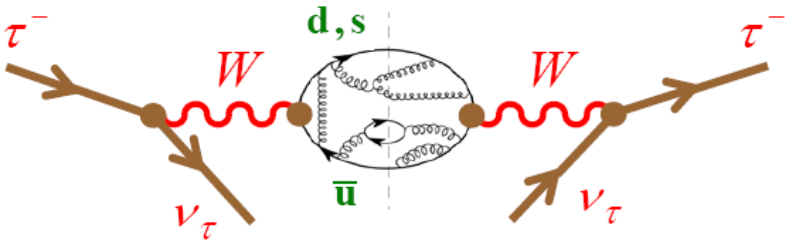


The τ hadronic width

$$R_\tau \equiv \frac{\Gamma[\tau^- \to \text{hadrons}\,\nu_\tau(\gamma)]}{\Gamma[\tau^- \to e^-\overline{\nu}_e\nu_\tau(\gamma)]} = \frac{1-\mathcal{B}_e-\mathcal{B}_\mu}{\mathcal{B}_e} = R_{\tau,V}+R_{\tau,A}+R_{\tau,S} = 3.6381 \pm 0.0075$$

[HFLAV, 2206.07501]

$\Gamma_{\tau\to\nu_\tau+\text{had}} \sim \text{Im}\Big\{$



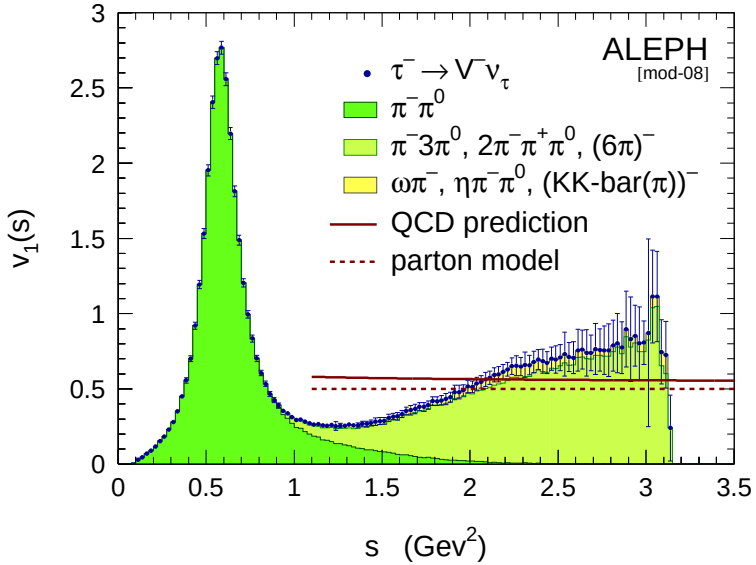
$\Big\}$

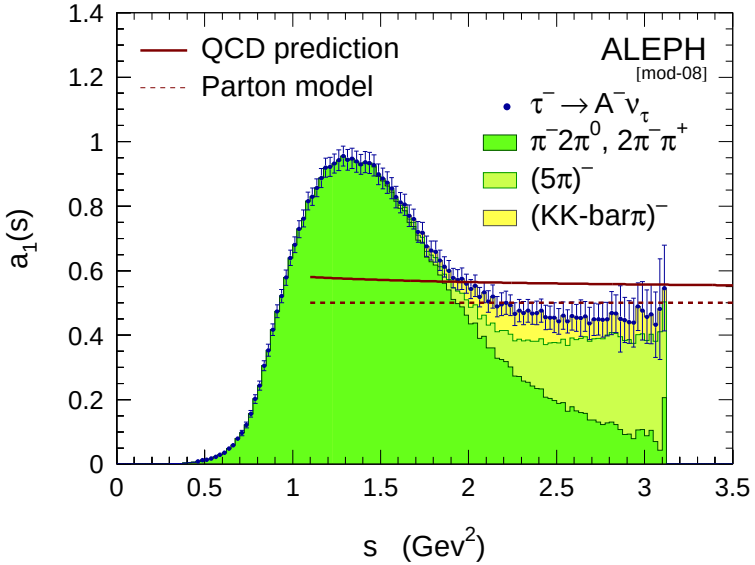
Focus on non-strange final states

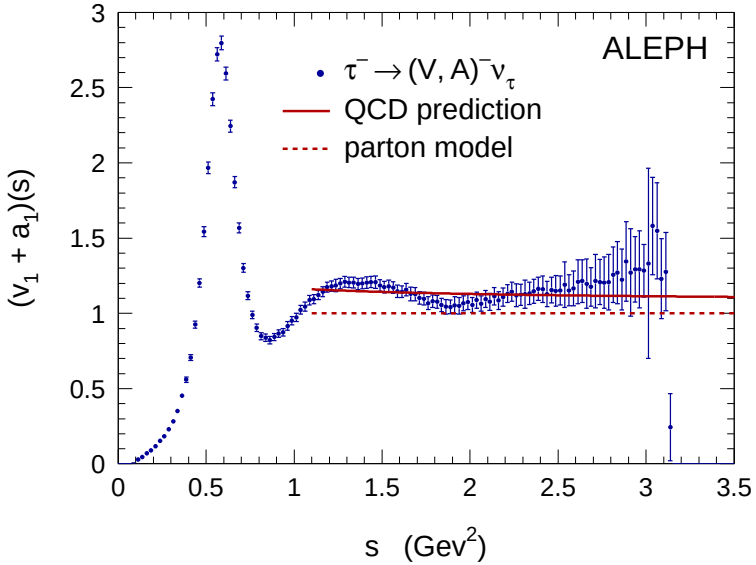
$$R_\tau = 12\pi \int_0^{M_\tau^2} \frac{ds}{M_\tau^2}\left(1-\frac{s}{M_\tau^2}\right)^2\left[\left(1+2\frac{s}{M_\tau^2}\right)\text{Im}\,\Pi^{(1)}(s)+\text{Im}\,\Pi^{(0)}(s)\right]$$

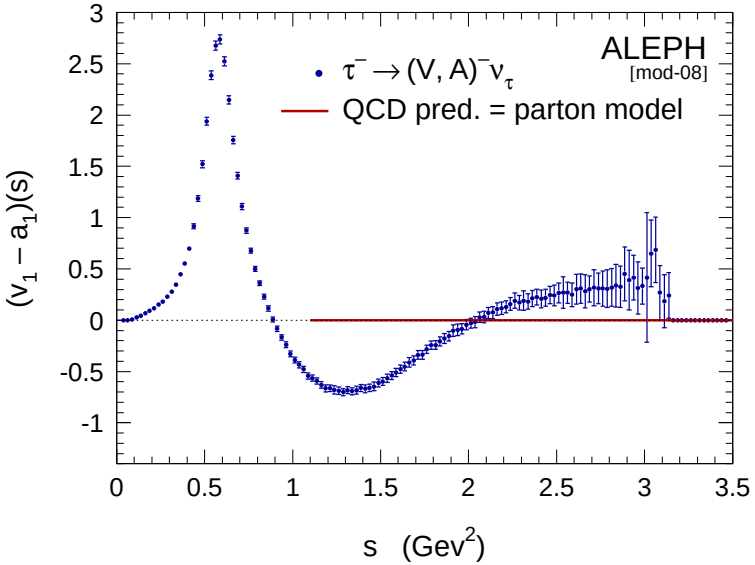
$$\Pi^{V/A}_{\mu\nu}(p) \equiv i\int dx\, e^{ipx}\,\langle\Omega|\,T\{J^{V/A}_\mu(x)\,J^{V/A}_\nu(0)^\dagger\}|\Omega\rangle = (p_\mu p_\nu - g_{\mu\nu}p^2)\,\Pi^{V/A,(1)} + p_\mu p_\nu\,\Pi^{V/A,(0)}$$

ALEPH spectral functions

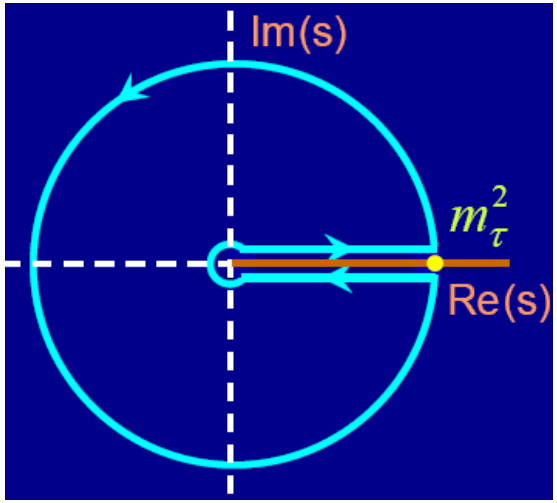








The τ hadronic width in QCD



- Analyticity
- Condensate expansion
- Slightly Euclidean [$(1-x)^3$ suppression]
- $D^{(1+0)}(s) \equiv -s\,\frac{d}{ds}\left[\Pi^{(1+0)}(s)\right]$ (Adler fn)

$$
\begin{aligned}
R_\tau &= 6\pi i \oint\limits_{|s|=M_\tau^2} \frac{ds}{M_\tau^2}\left(1-\frac{s}{M_\tau^2}\right)^2\left[\left(1+2\frac{s}{M_\tau^2}\right)\Pi^{(1)}(s)+\Pi^{(0)}(s)\right] \\
&= -i\pi \oint\limits_{|x|=1} \frac{dx}{x}(1-x)^3\left[3\,(1+x)\,D^{(1+0)}(M_\tau^2 x)+4\,D^{(0)}(M_\tau^2 x)\right] \\
&= N_c\,S_{\rm EW}\,|V_{ud}|^2\left[1+\delta^{(0)}+\delta'_{\rm EW}+\sum_{D\geq 2}\frac{C_D(s,\mu)\langle O_D(\mu)\rangle}{(-s)^{D/2}}\right]
\end{aligned}
$$

[Braaten, Narison, Pich, 1992]

Condensate expansion

- D=2 m_q^2
  $\longrightarrow (3.1 \pm 8.6) \cdot 10^{-5}$
- D=4 $m_q^4, m_q\langle\bar{q}q\rangle, \langle\frac{\alpha_s}{\pi}GG\rangle$
  $\longrightarrow (6.3 \pm 3.3) \cdot 10^{-4}$

  Suppression of the D=4 contribution due to the kinematic weight function
  $(1-x)^3(1+2x) = 1 - 2x + 2x^3 - x^4$
- D=6 $\langle\bar{q}q\bar{q}q\rangle, \langle\alpha_s G^3\rangle$
  $\longrightarrow (-4.8 \pm 2.9) \cdot 10^{-3}$ – dominant

  Factor 10 cancellation between V and A. This explains $R_{\tau,V-A} \approx 0.08$ ($\rightarrow$ Fig.)
- S+P $D^{(0)}(s)$ contribution dominated by the calculable pion pole contribution
  $\longrightarrow (-2.64 \pm 0.05) \cdot 10^{-3}$

Non-perturbative terms very small [3.5% of perturbative contribution!] due to V+A cancellation and kinematic suppression

$$\delta_{\rm PC} = (-6.8 \pm 3.5) \cdot 10^{-3}$$

Nevertheless, the gluon condensate will play an important role in the following.

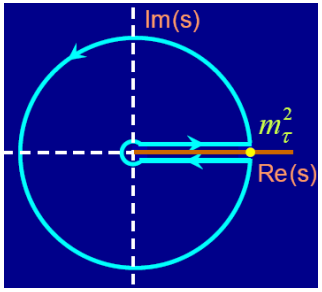
FO and CI perturbation theory

$$D_{V,A}^{(1+0)}(s) = \frac{N_c}{12\pi^2}\sum_{n=0}^{\infty} a_\mu^n \sum_{k=1}^{n+1} k\, c_{n,k}\, \ln^{k-1}\frac{-s}{\mu^2} = \frac{N_c}{12\pi^2}\sum_{n=0}^{\infty} c_{n,1}\, a_Q^n$$

$$= \frac{N_c}{12\pi^2}\left[1 + a_Q + 1.64 a_Q^2 + 6.37 a_Q^3 + 49.08 a_Q^4 + \ldots\right] \qquad (a_Q = \alpha_s(Q)/\pi)$$

[5-loop $c_{4,1}$: Baikov, Chetyrkin, Kühn, 2008]

$$R_\tau = -i\pi \oint_{|x|=1} \frac{dx}{x}(1-x)^3\left[3(1+x)\, D^{(1+0)}(M_\tau^2 x) + 4\, D^{(0)}(M_\tau^2 x)\right]$$



FOPT $\quad \delta_{\rm FO}^{(0)} = \sum_{n=1}^{\infty} a(M_\tau^2)^n \sum_{k=1}^{n} k\, c_{n,k}\, J_{k-1} \qquad J_l \equiv \frac{1}{2\pi i}\oint_{|x|=1}\frac{dx}{x}(1-x)^3(1+x)\ln^l(-x)$

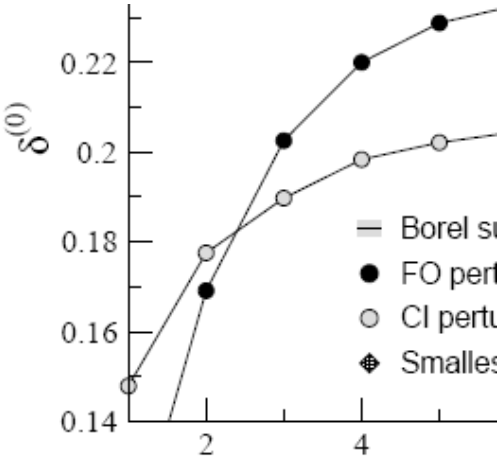
CIPT $\quad \delta_{\rm CI}^{(0)} = \sum_{n=1}^{\infty} c_{n,1}\, J_n^a(M_\tau^2) \qquad J_n^a(M_\tau^2) \equiv \frac{1}{2\pi i}\oint_{|x|=1}\frac{dx}{x}(1-x)^3(1+x)\, a^n(-M_\tau^2 x)$

The problem [MB, Jamin, 2008]

Numerical series expansions for $\alpha_s(M_\tau^2) = 0.34$.
(We will often use the estimate $c_{5,1} = 283 \pm 283$.)

$$\begin{array}{rcl} & & \alpha_s^1 \quad \alpha_s^2 \quad \alpha_s^3 \quad \alpha_s^4 \quad \alpha_s^5 \\ \delta_{\rm FO}^{(0)} & = & 0.1082 + 0.0609 + 0.0334 + 0.0174\,(+\,0.0088\,) \;=\; 0.2200\;(0.2288) \\ \delta_{\rm CI}^{(0)} & = & 0.1479 + 0.0297 + 0.0122 + 0.0086\,(+\,0.0038\,) \;=\; 0.1984\;(0.2021) \end{array}$$

- FO/CI difference *increases* by adding more orders. Systematic problem.
- Difference in α_s value is larger than the error of each individual method.



Large-order asymptotics of PT

Problem is connected with systematic pattern of coefficients in higher perturbative orders.

General structure of large-order behaviour is (believed to be) known.
Several components of factorial divergence of form

$$c_{n,1} \overset{n\gg 1}{=} \alpha_s^{n+1} K (a\beta_0)^n n!\, n^b \left(1 + \frac{s_1}{n} + O(1/n^2)\right)$$

Borel transform

$$F \sim \sum_{n=0}^{\infty} c_{n,1} \alpha_s^{n+1} \implies B[F](t) = \sum_{n=0}^{\infty} c_{n,1} \frac{t^n}{n!} \implies F(\alpha) = \int_0^{\infty} dt\, \mathrm{e}^{-t/\alpha} B[F](t)$$

$$c_{n,1} = K a^n \Gamma(n+1+b) \iff B[F](t) = \frac{K\Gamma(1+b)}{(1-at)^{1+b}}.$$

Minimal term at $n \approx 1/(|a\beta_0|\alpha_s(Q))$ of size

$$\Delta \approx e^{-1/(|a\beta_0|\alpha_s(Q))} \approx \left(\frac{\Lambda^2}{Q^2}\right)^{1/a} \approx \text{size of ambiguity}$$

IR renormalons and condensates

- IR renormalons – from small loop momentum
  Fixed-sign, singularity structure related to higher-dim operators in the OPE [David, 1984; Mueller, 1985; Zakharov, 1992; MB, 1993]

OPE $$\Pi(Q) = C_0(\alpha_s, Q/\mu) + \frac{1}{Q^d}\, C_d(\alpha_s, Q/\mu)\, \langle \frac{\alpha_s}{\pi} G^2 \rangle(\mu) + O(1/Q^6)$$

from $k < \mu \ll Q$ $$C_0^{\rm IR}(\alpha_s, Q/\mu) = \frac{1}{Q^d}\, C_d(\alpha_s, Q/\mu)\, \mu^d M(\alpha_s) + O(1/Q^6)$$

Ansatz $$M(\alpha_s) = \sum_n \alpha_s^{n+1} K (a\beta_0)^n n!\, n^b \left(1 + \frac{s_1}{n} + O(1/n^2)\right)$$

- Location of singularity related to operator dimension, $a = d/2$
- Nature of singularity related to operator anomalous dimension and β function including susb-leading $1/n^k$ terms:
$$b = \frac{d\beta_1}{2\beta_0^2} - \frac{\gamma_0}{2\beta_0}, \qquad s_1 = f(\beta_2, \gamma_1, \ldots)$$
- Normalization = Stokes constant is the only unknown (non-perturbative)

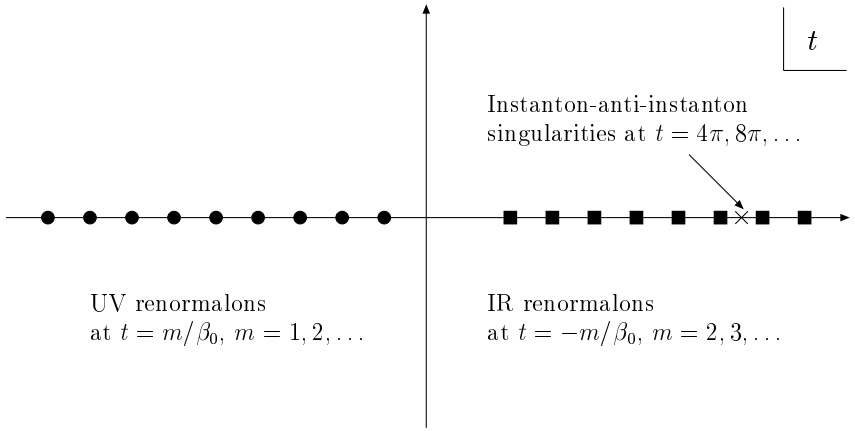
Borel plane singularities of the Adler function

IR renormalons

- Leading singularity at $u = 2$, since no D=2 operator.

  Especially simple structure, only one operator (GG).

- Sub-leading $u = 3$ related to many D=6 operators.



UV renormalons

- $u = -1$, leading singularity for Adler function and R_τ for very large orders (sign alternation)
- No sign-alternation seen in the real Adler function and R_τ series.
  c_{-1} must be small in the $\overline{\text{MS}}$ scheme – true in the bubble diagram ("large-β_0") approximation.

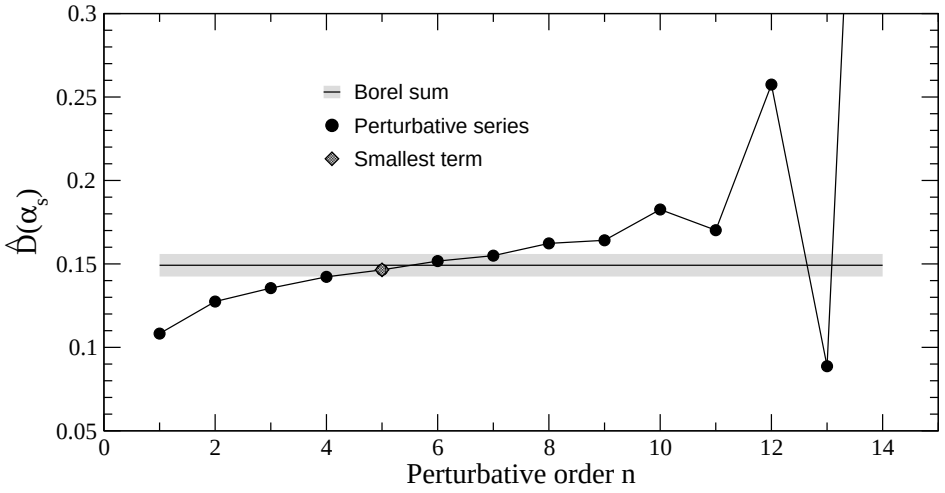
Expect fixed sign series in intermediate orders and sign-alternation only asymptotically – in the $\overline{\text{MS}}$ scheme.

Borel function model [MB, Jamin, 2008]

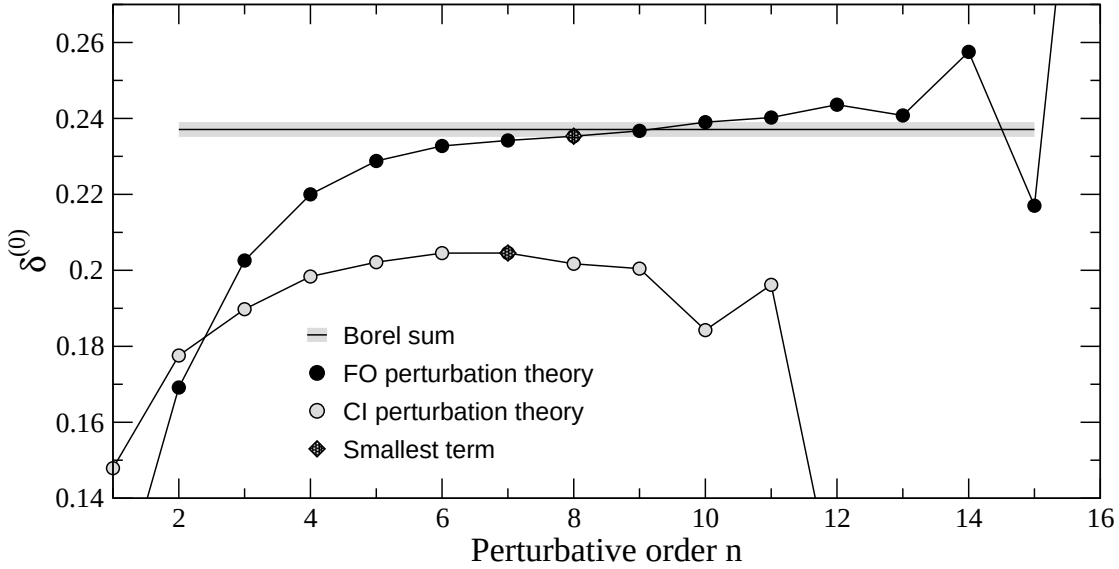
Incorporate the knowledge of asmyptotic behaviourinto an Ansatz for the Adler function that reproduces known $c_{n,1}$ to $n = 4$ and $c_{5,1} = 283$.
Then compare FOPT/CIPT.

$$B[D](u) \;=\; B[D_1^{\rm UV}](u) + B[D_2^{\rm IR}](u) + B[D_3^{\rm IR}](u) + d_0^{\rm PO} + d_1^{\rm PO} u$$

- Fit Stokes constants K_p for $u = -1, 2, 3$ to $c_{3,1}$, $c_{4,1}$ and $c_{5,1}$, and adjust $d_{0,1}^{\rm PO}$ to reproduce $c_{1,1}$ and $c_{2,1}$.
- Pole ansatz works well already at $n = 2$ ($d_1^{\rm PO}$ small). Apparently the series is very regular.



FO vs CI for R_τ



- FO converges to Borel sum
- CI converges more quickly than FO at low orders, but never reaches the Borel sum.
- At $n = 4, 5$ FO is close to the true result, CI too small $\Rightarrow$ α_s from CI too large.
  (A similar observation has been made in the large-β_0 approximation [Ball, MB, Braun, 1995].)

After 2010 the quoted value of α_s from τ decay in α_s compilations has been taken to be the average of FOPT/CIPT.

Inconsistency of CIPT

$$\delta_{\rm FO}^{(0)} = \sum_{n=1}^{\infty} \left[c_{n,1} + g_n\right] a(M_\tau^2)^n \qquad g_n = \sum_{k=2}^{n} k\, c_{n,k} J_{k-1}$$

- For the leading IR contribution ($u = 2$) there are *large cancellations* in going from Adler function to R_τ related to suppression of the gluon condensate contribution: [MB, 1993]

$$\frac{c_{n,1} + g_n}{c_{n,1}} \propto 1/n^2$$

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$$\frac{c_{n,1} + g_n}{c_{n,1}} \propto 1/n^2$$

- [MB, Jamin 2008; MB, Boito, Jamin, 2012] CIPT does not respect these cancellations, problem with OPE. For general spectral function moments, the poor behaviour of CIPT disappears when one assumes that there is no $u = 2$ singularity.

- ... many papers ... [Caprini, Fischer, 2009 –, Cvetic et al., 2010, ...]

- [Hoang, Regner, 2021; Gracia, Hoang, Mateu, 2023] Asymptotic behaviour of the CIPT series is inconsistent with the OPE.

- [Benitez-Rathgeb, Boito, Hoang, Jamin, 2022] Solution to the CIPT problem: Applying renormalon subtraction to the $u = 2$ singularity similar to the Adler function makes CIPT and FOPT agree.

Renormalon subtraction of the gluon condensate

$$\Pi(Q) = C_0(\alpha_s, Q/\mu) + C_{GG}(\alpha_s, Q/\mu)\,\frac{1}{Q^4}\,\langle\frac{\alpha_s}{\pi}G^2\rangle(\mu) + O(1/Q^6)$$

Any definition / subtraction of the divergent perturbative series implies a renormalization scheme for the quartic power-divergences of the operator $\langle\frac{\alpha_s}{\pi}G^2\rangle(\mu)$.

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- $\overline{\text{MS}}$-like schemes and principal value prescription [Lee, 2010; Bali et al., 2014; Ayala et al., 2020; Benitez-Rathgeb, Boito, Hoang, Jamin, 2022]

$$\Delta C_0 = C_{GG}(\alpha_s, Q/\mu)\,\frac{\mu_f^4}{Q^4}\,\text{PV}\sum_n \alpha_s^{n+1}(\mu_f) K\,(a\beta_0)^n n!\,n^b\left(1 + \frac{s_1}{n} + \ldots\right)$$

Requires an estimate of the non-perturbative $u = 2$ renormalon normalization and leaves open what is the gluon condensate.

- This work: non-perturbatively defined, unambiguous, gauge-invariant "cut-off gluon condensate" whose OPE has the same IR renormalon as $\Pi(Q)$. Subtraction is automatic, do not need to know anything about the explicit large-order behaviour.

Note similarlity of pole mass renormalon subtraction [see review 2108.04861]

Gradient flow [Lüscher, 1006.4518; Lüscher, Weisz 1101.0963]

Here: continuum version ("Wilson flow" on lattice)
Define "flowed" gluon field $B_\mu(t,x)$ by

$$\partial_t B_\mu = \tilde{D}_\nu \tilde{G}_{\nu\mu} + \xi_0 \tilde{D}_\mu \partial_\nu B_\nu\,, \qquad B_\mu|_{t=0} = A_\mu$$

$t =$ flow "time", $\tilde{G}_{\mu\nu}$, $\tilde{D}_\mu$ usual definitions but with B_μ.

Interpretation: Smeared gluon field over distance $\sqrt{8t}$. LO solution

$$B_\mu(t,x) = \int d^d y\, K(t, x-y)\, A_\mu(x) \qquad K(t,z) = \frac{e^{-z^2/(4t)}}{(4\pi t)^{d/2}}$$

Action density

$$E(t) = \frac{g^2}{4} G^A_{\mu\nu}(t) G^{A\mu\nu}(t)$$

Its expectation value, $\langle E(t)\rangle$, can be regarded as a gauge-invariant non-pertubative definition of the gluon condensate with cut-off $\Lambda_{\rm UV} \propto 1/\sqrt{t}$, whch can replace the ill-defined $\langle \frac{\alpha_s}{\pi} G^2\rangle(\mu)$ in the $\overline{\rm MS}$-OPE.

OPE of the action density / subtracted Adler function

For $t \ll 1/\Lambda_{\rm QCD}^2$ [Lüscher, 1006.4518; Harlander, Neumann 1606.03756 (NNLO)]

$$\frac{1}{\pi^2}\langle E(t)\rangle = \frac{\tilde{C}_0(t)}{t^2} + \tilde{C}_{GG}(t)\langle\frac{\alpha_s}{\pi}GG\rangle + \mathcal{O}(t \times \text{dim-6})$$

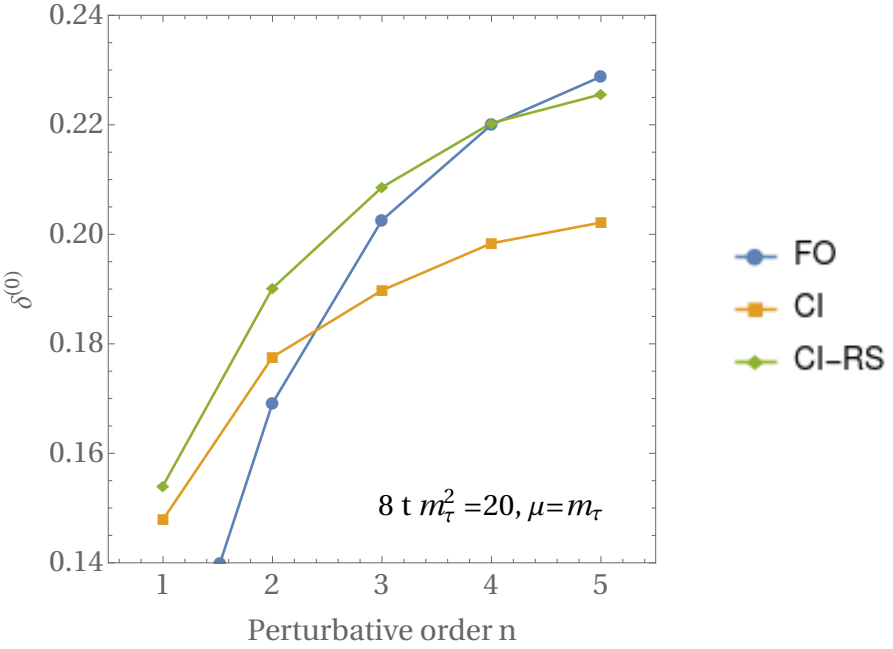
$\tilde{C}_1$ known to NLO [Lüscher, 1006.4518] and NNLO [Harlander, Neumann 1606.03756], $\tilde{C}_{GG}(t)$ to NNLO [Harlander, Kluth, Lange, 1808.09837]

Eliminate $\langle\frac{\alpha_s}{\pi}G^2\rangle(\mu)$ in the standard OPE for the Adler function

$$D(Q) = \underbrace{\left[C_0(Q) - \frac{1}{t^2Q^4}\frac{C_{GG}(Q)}{\tilde{C}_{GG}(t)} \times \tilde{C}_0(t)\right]}_{u\,=\,2\text{ renormalon cancels}} + \frac{1}{Q^4}\underbrace{\frac{C_{GG}(Q)}{\tilde{C}_{GG}(t)}\frac{1}{\pi^2}\langle E(t)\rangle}_{\text{non-perturbatively defined}} + O(1/Q^6)$$

Only need fixed-order calculations to obtain a better-behaved expansion.

Hadronic τ decay with gradient-flow subtraction



$$8tm_\tau^2 = 20$$

FO and subtracted FO identical for NLO $C_{GG}(Q)/\tilde{C}_{GG}(t)$.

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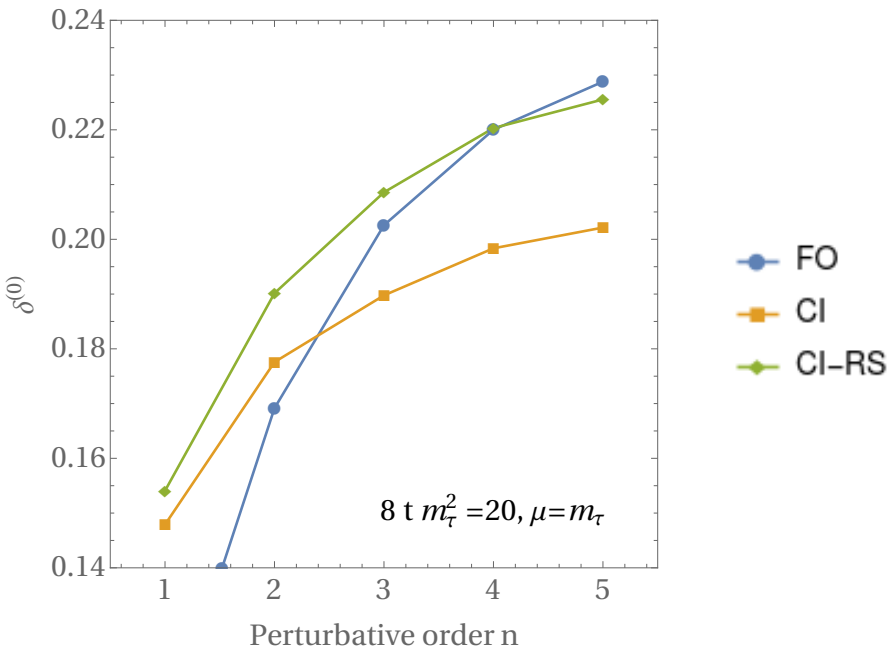

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FO and subtracted FO identical for NLO $C_{GG}(Q)/\tilde{C}_{GG}(t)$.

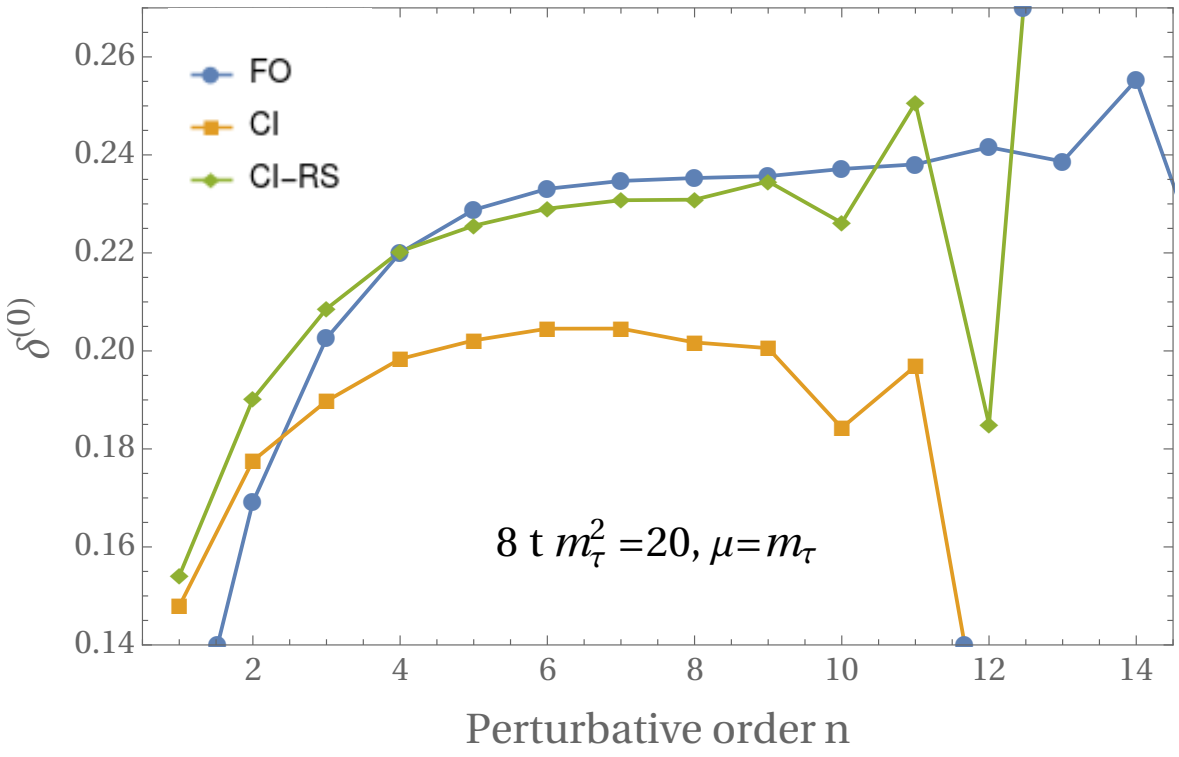
Renormalon model

The gradient flow action density has no UV renormalons. Ansatz (in practice for the entire subtraction term

$$B[E](u) = B[E_2^{\mathrm{IR}}](u) + B[E_3^{\mathrm{IR}}](u) + e_0^{\mathrm{PO}} + e_1^{\mathrm{PO}} u$$

Three unknowns (one fixed by $u = 2$ cancellation). Matches the available three exactly known coefficients of $\tilde{C}_0$

Hadronic τ decay with gradient-flow subtraction



- CI and FO approach now similar values.
  (How well, depends on choice of t.)
- CI may converges more quickly than FO at low orders, now to the correct value.

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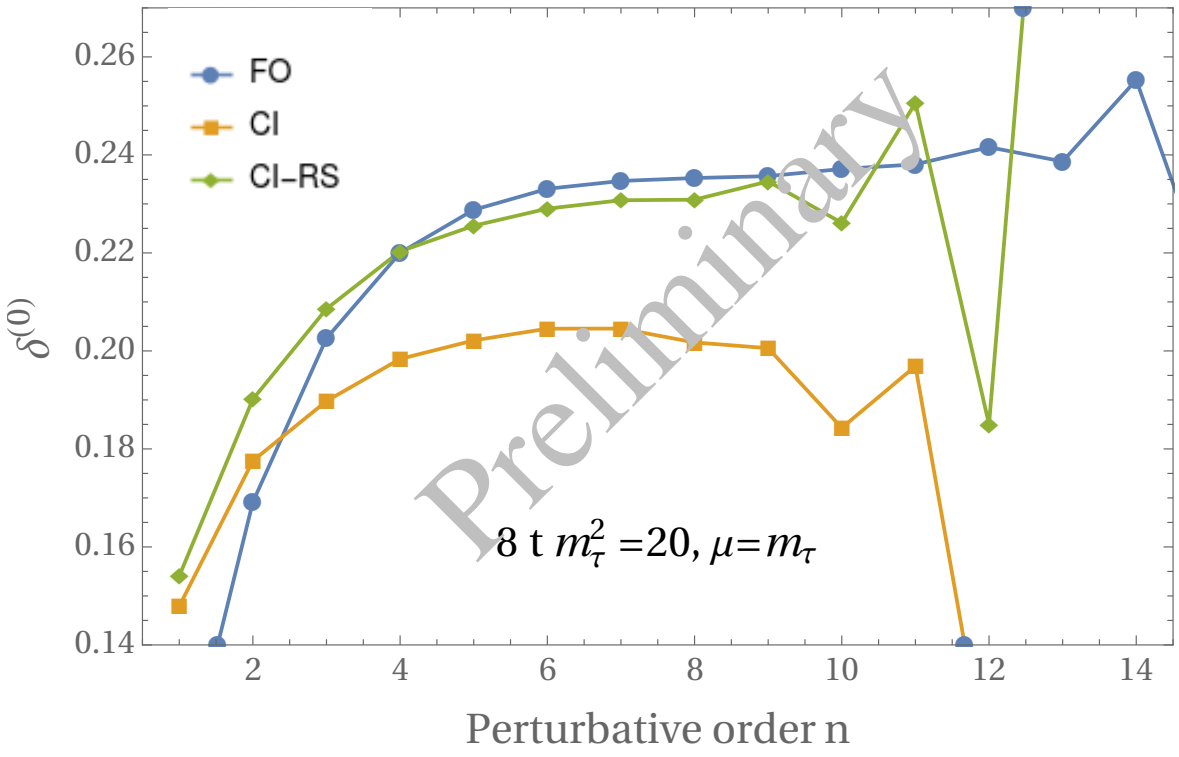

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Conclusion

- FOPT/CIPT resolved by gluon-condensate renormalon subtraction as suggested by [Benitez-Rathgeb, Boito, Hoang, Jamin, 2022]
- The gradient flow subtraction scheme works in low orders without explicit knowledge of asymptotic behaviour at a low subtraction scale.
- It is possible to consistently add the leading power correction from the gluon condensate. The flowed action density at large t can be computed easily on the lattice.
- The gradient flow separates the continuum limit $a \to 0$ on the lattice for the cut-off scale $1/\sqrt{t}$ defining the renormalon subtraction.
- In α_s compilations the value from τ decay should taken to be the smaller value from FOPT (when no renormalon subtraction-method is used) (as advocated in [MB, Jamin, 2008])