UNIVERSITÀ DI TORINO

INFN Istituto Nazionale di Fisica Nucleare

Subtracting NNLO singularities With a fully general analytic algorithm In massless QCD

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In collaboration with L. Magnea, G. Pelliccioli, A. Ratti, C. Signorile-Signorile, P. Torrielli, S. Uccirati

[arXiv:2212.11190]

Motivation

- Vast increase in collider data accuracy as well as in the complexity of observables being probed, as the LHC moves into the high-luminosity phase

- Accurate theoretical predictions of scattering events require handling of **IR singularities beyond NLO**, which proves to be a highly non-trivial challenge

- Several established approaches, from both slicing and subtraction side, are rapidly setting **NNLO in the strong coupling** as the **standard**; in particular, state-of-the-art predictions for $2 \to 3$ collider processes (with al least 2 QCD final-state particles at the Born level)

  $pp \to jjj$ *[M. Czakon, A. Mitov, R. Poncelet 2021]* *Talk by R. Poncelet*

  $pp \to Wb\bar{b}$ (massless) *[H. B. Hartanto, R. Poncelet, A. Popescu, S. Zoia 2022]*

  $pp \to t\bar{t}H$ *[S. Catani, S. Devoto, M. Grazzini, S. Kallweit, J. Mazzitelli, C. Savoini 2022]* *Talk by C. Savoini*

  $pp \to Wb\bar{b}$ (massive) *[L. Buonocore, S. Devoto, S. Kallweit, J. Mazzitelli, L. Rottoli, C. Savoini 2022]* *Talk by L. Buonocore*

  $pp \to \gamma jj$ *[S. Badger, M. Czakon, H. B. Hartanto, R. Moodie, T. Peraro, R. Poncelet, S. Zoia 2023]* *Talks by M. Czakon and S. Zoia*

- On-going effort to generalise/improve schemes, while looking at the extension to higher orders
  Talks by M. Marcoli, C. Signorile-Signorile, O. Braun-White

- Room for improvement in universality and efficiency, to overcome high computational complexity

Ambitious goal:
automation of NNLO QCD predictions

Exploring the framework...

Local Analytic Sector Subtraction

$$\frac{d\sigma}{dX} = \frac{d\sigma_{\rm LO}}{dX} + \boxed{\frac{d\sigma_{\rm NLO}}{dX}} + \frac{d\sigma_{\rm NNLO}}{dX} + \ldots$$

σ = partonic cross section
X = generic IRC-safe observable

Introduction to subtraction strategy at NLO

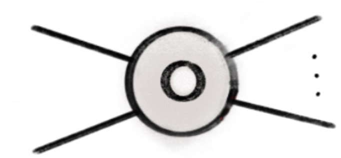
Massless QCD
final-state radiation

Generalities at NLO

- X_i = **IRC-safe** observable computed with i-body kinematics, $\delta_{X_i} \equiv \delta(X - X_i)$

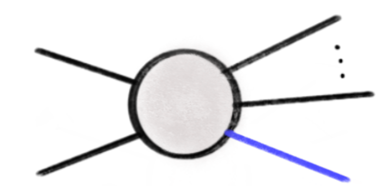
$$\frac{d\sigma_{NLO}}{dX} = \int d\Phi_n \, V \, \delta_{X_n}$$

Explicit ϵ poles



$$+ \int d\Phi_{n+1} \, R \, \delta_{X_{n+1}}$$

Singular in PS

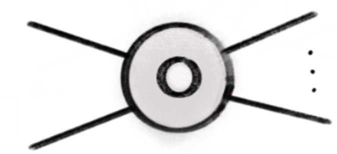


Generalities at NLO

- X_i = **IRC-safe** observable computed with i-body kinematics, $\delta_{X_i} \equiv \delta(X - X_i)$

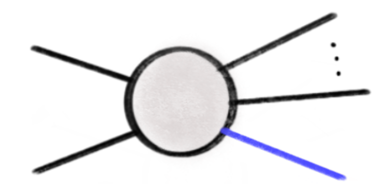
$$\frac{d\sigma_{NLO}}{dX} = \int d\Phi_n \left(V + I\right) \delta_{X_n}$$

Finite in ϵ



$$+ \int d\Phi_{n+1} \left(R\, \delta_{X_{n+1}} - K\, \delta_{X_n}\right)$$

Integrable in PS



- ***Subtraction algorithm:*** introduce **local counterterm** K and **phase-space factorisation**

$$\int d\Phi_{n+1}\, K\, \delta_{X_n} = \int d\Phi_n\, I\, \delta_{X_n} \qquad d\Phi_{n+1} = d\Phi_n\, d\Phi_{\text{rad}}$$

- ***Result*:** subtracted NLO cross section **numerically integrable** in $d = 4$ dimensions

Strategy of the algorithm

- ***Unitary partition*** of radiative phase-space with **sector functions** $\mathcal{W}_{ij}$ *[Frixione, Kunszt, Signer 9512328]*

$$R = \sum_{i,j\neq i} R\mathcal{W}_{ij} \qquad \text{with} \qquad \sum_{i,j\neq i} \mathcal{W}_{ij} = 1$$

Minimal approach to disentangle overlapping singularities

* Single-unresolved configurations

$\mathbf{S}_i$ **soft** parton i $\qquad \mathcal{E}_i = \dfrac{s_{qi}}{s} \to 0$

$\mathbf{C}_{ij}$ **collinear** pair ij $\qquad w_{ij} = \dfrac{s\, s_{ij}}{s_{qi} s_{qj}} \to 0$

* ***Sum rules:*** limits of sector functions still form a unitary partition

$$\mathbf{S}_i \sum_{l\neq i} \mathcal{W}_{il} = 1$$

$$\mathbf{C}_{ij}(\mathcal{W}_{ij} + \mathcal{W}_{ji}) = 1$$

Key for integration

* Example of NLO *sector functions* $\quad (s_{qi} = 2\, q_{\rm cm} \cdot k_i \,, s_{ij} = 2\, k_i \cdot k_j \,, s = q_{\rm cm}^2)$

$$\mathcal{W}_{ij} = \frac{\sigma_{ij}}{\sum_{a,b\neq a} \sigma_{ab}} \qquad \sigma_{ij} = \frac{1}{\mathcal{E}_i\, \omega_{ij}}$$

Strategy of the algorithm

- ***Unitary partition*** of radiative phase-space with **sector functions** $\mathcal{W}_{ij}$ *[Frixione, Kunszt, Signer 9512328]*

- Collect the relevant **IRC limits** for a given sector

$$R\mathcal{W}_{ij} - \Big[\mathbf{S}_i + \mathbf{C}_{ij} - \mathbf{S}_i\mathbf{C}_{ij}\Big] R\mathcal{W}_{ij} \rightarrow \text{integrable}$$

Soft + Collinear - Overlap

Notation

$$\mathbf{L}\, R\,\mathcal{W}_{ij} = (\mathbf{L}\, R)\,(\mathbf{L}\,\mathcal{W}_{ij})$$

for $\mathbf{L} = \mathbf{S}_i, \mathbf{C}_{ij}, \ldots$

Strategy of the algorithm

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Soft + Collinear - Overlap

Notation

$$\mathbf{L}\, R\,\mathcal{W}_{ij} = (\mathbf{L}\, R)(\mathbf{L}\,\mathcal{W}_{ij})$$

for $\mathbf{L} = \mathbf{S}_i, \mathbf{C}_{ij}, \ldots$

* Products of known **splitting kernels** x **Born-level MEs**

Not yet parametrised

$$\mathbf{S}_i\, R = \mathcal{N}_1\, \delta_{f_i g} \sum_{k,l} \frac{s_{kl}}{s_{ik}s_{il}} B_{kl}(\{k\}_{\not i})$$

$$\mathbf{C}_{ij}\, R = \mathcal{N}_1 \frac{P^{\mu\nu}_{ij}}{s_{ij}} B_{\mu\nu}(\{k\}_{\not i \not j}, k_i + k_j)$$

missing proper
n-body on-shell kinematics!

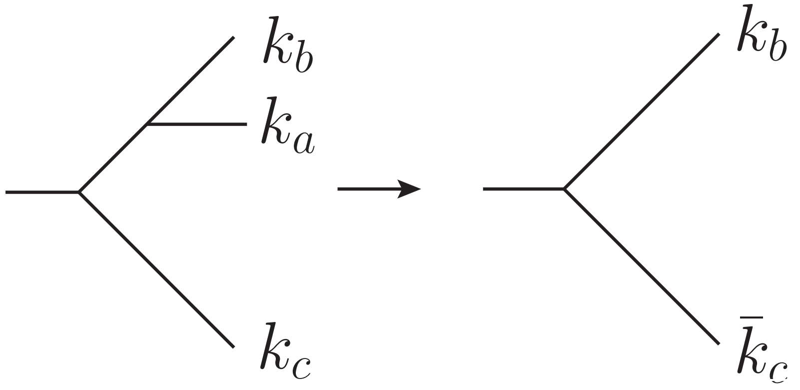
Strategy of the algorithm

- ***Unitary partition*** of radiative phase-space with **sector functions** $\mathcal{W}_{ij}$ *[Frixione, Kunszt, Signer 9512328]*

- Collect the relevant **IRC limits** for a given sector

- ***Catani-Seymour*** final-state **dipole mapping** *[Catani, Seymour 9605323]*

$$\{k_1, ..., k_{n+1}\} \to \{\bar{k}_1, ..., \bar{k}_n\}^{(abc)}$$



$$\bar{k}_b^{(abc)} = k_a + k_b - \frac{y}{1-y} k_c$$
$$\bar{k}_c^{(abc)} = \frac{1}{1-y} k_c$$

$$y = \frac{s_{ab}}{s_{ab} + s_{ac} + s_{bc}}, \quad z = \frac{s_{ac}}{s_{ab} + s_{bc}}$$

✱ Phase-space factorisation and parametrisation

$$d\Phi_{n+1} = d\Phi_n^{(abc)} \, d\Phi_{\text{rad}}^{(abc)} = d\Phi_n(\{\bar{k}\}^{(abc)}) \, d\Phi_{\text{rad}}(\bar{s}_{bc}^{(abc)}; y, z, \phi)$$

$$\int d\Phi_{\text{rad}}^{(abc)} \propto (\bar{s}_{bc}^{(abc)})^{1-\epsilon} \int_0^\pi d\phi \sin^{-2\epsilon}\phi \int_0^1 dy \int_0^1 dz \Big[y(1-y^2)z(1-z)\Big]^{-\epsilon} (1-y)$$

Strategy of the algorithm

- ***Unitary partition*** of radiative phase-space with **sector functions** $\mathcal{W}_{ij}$ *[Frixione, Kunszt, Signer 9512328]*

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- ***Catani-Seymour*** final-state **dipole mapping** *[Catani, Seymour 9605323]*

- Promotion to **counterterms**: ***Improved limits***

Adapt momenta mapping to each kernel,
while tuning action on sector functions when necessary

$$K = \sum_{i,j\neq i} \left[\overline{\mathbf{S}}_i + \overline{\mathbf{C}}_{ij} - \overline{\mathbf{S}}_i\overline{\mathbf{C}}_{ij}\right] R\mathcal{W}_{ij}$$

Notation

$$\overline{\mathbf{L}}\, R\,\mathcal{W}_{ij} = (\overline{\mathbf{L}}\, R)(\overline{\mathbf{L}}\,\mathcal{W}_{ij})$$

$$\overline{\mathbf{S}}_i\, R = \mathcal{N}_1\, \delta_{f_i g} \sum_{k,l} \frac{s_{kl}}{s_{ik}s_{il}} \bar{B}_{kl}^{(ikl)}$$

$$\overline{\mathbf{C}}_{ij}\, R = \mathcal{N}_1 \frac{P_{ij}^{\mu\nu}}{s_{ij}} \bar{B}_{\mu\nu}^{(ijr)}$$

$$\overline{\mathbf{S}}_i\, \mathcal{W}_{ij} \equiv \mathbf{S}_i\, \mathcal{W}_{ij} = \frac{\frac{1}{w_{ij}}}{\sum_{l\neq i} \frac{1}{w_{il}}}$$

$$\overline{\mathbf{C}}_{ij}\, \mathcal{W}_{ij} \equiv \frac{e_j w_{ir}}{e_i w_{ir} + e_j w_{jr}}$$

Strategy of the algorithm

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- Collect the relevant **IRC limits** for a given sector

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- Promotion to **counterterms**: ***Improved limits***

- **Locality** of the cancellation ensured by ***consistency relations***

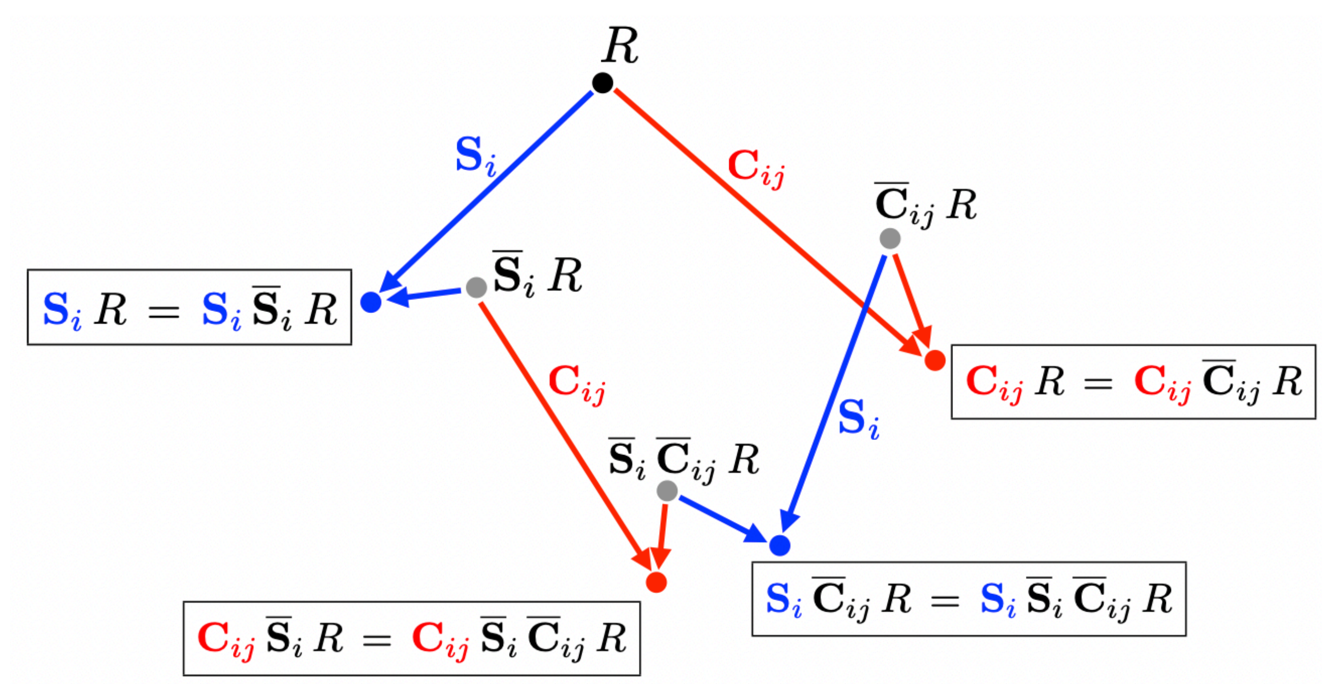
$$\mathbf{S}_i\, R = \mathbf{S}_i \left(\overline{\mathbf{S}}_i + \overline{\mathbf{C}}_{ij} - \overline{\mathbf{S}}_i \overline{\mathbf{C}}_{ij} \right) R$$

$$\mathbf{C}_{ij}\, R = \mathbf{C}_{ij} \left(\overline{\mathbf{S}}_i + \overline{\mathbf{C}}_{ij} - \overline{\mathbf{S}}_i \overline{\mathbf{C}}_{ij} \right) R$$

as well as

$$\mathbf{S}_i\, \mathcal{W}_{ij} = \mathbf{S}_i\, \overline{\mathbf{S}}_i\, \mathcal{W}_{ij}$$

$$\mathbf{C}_{ij}\, \mathcal{W}_{ij} = \mathbf{C}_{ij}\, \overline{\mathbf{C}}_{ij}\, \mathcal{W}_{ij}$$



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$$\frac{d\sigma_{NLO}}{dX} = \int d\Phi_n \Big(V + I\Big)\delta_{X_n} + \int d\Phi_{n+1}\Big(R\,\delta_{X_{n+1}} - K\,\delta_{X_n}\Big)$$

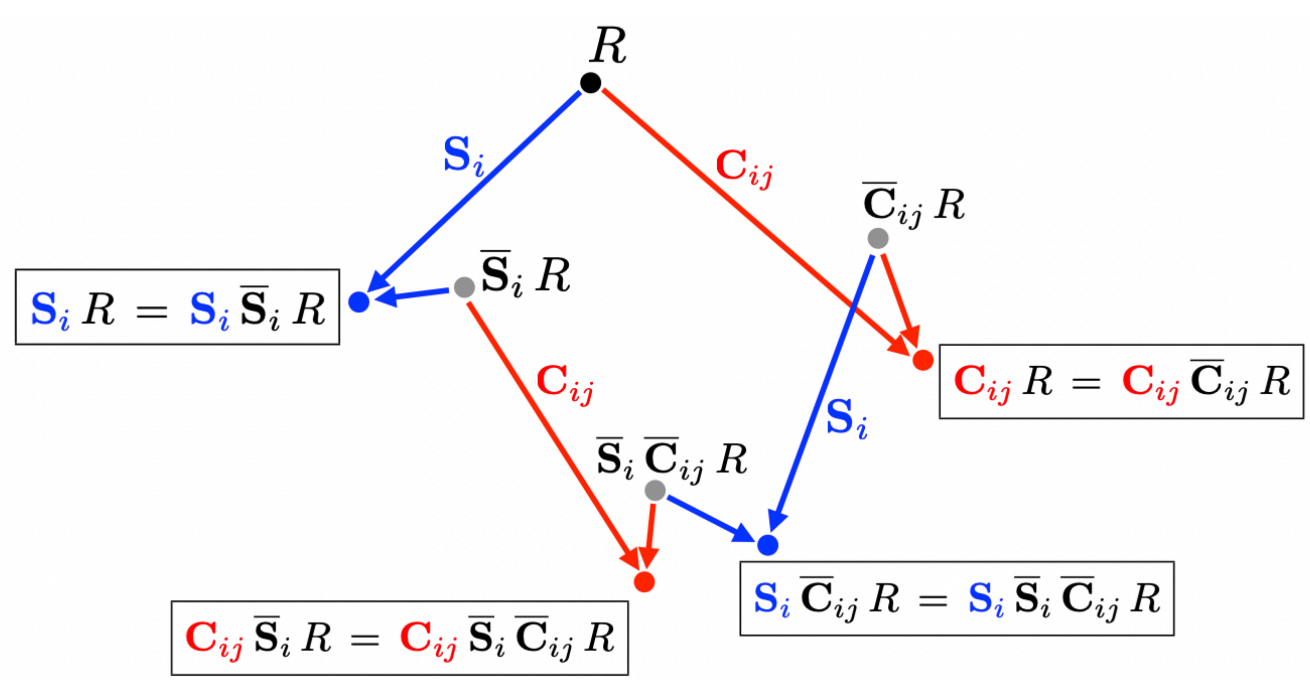
☑ **Finite** in phase space (integrable in $d = 4$)

$$\mathbf{S}_i\, R = \mathbf{S}_i\Big(\overline{\mathbf{S}}_i + \overline{\mathbf{C}}_{ij} - \overline{\mathbf{S}}_i\overline{\mathbf{C}}_{ij}\Big)R$$

$$\mathbf{C}_{ij}\, R = \mathbf{C}_{ij}\Big(\overline{\mathbf{S}}_i + \overline{\mathbf{C}}_{ij} - \overline{\mathbf{S}}_i\overline{\mathbf{C}}_{ij}\Big)R$$

as well as

$$\mathbf{S}_i\,\mathcal{W}_{ij} = \mathbf{S}_i\,\overline{\mathbf{S}}_i\,\mathcal{W}_{ij}$$
$$\mathbf{C}_{ij}\,\mathcal{W}_{ij} = \mathbf{C}_{ij}\,\overline{\mathbf{C}}_{ij}\,\mathcal{W}_{ij}$$



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☑ **Finite** in phase space (integrable in $d = 4$)

- $\mathcal{W}_{ij}$ sum rules + mapping adaptation = **simple analytic** counterterm integration

$$K = \sum_{i,j\neq i}\Big[\overline{\mathbf{S}}_i + \overline{\mathbf{C}}_{ij} - \overline{\mathbf{S}}_i\overline{\mathbf{C}}_{ij}\Big] R\mathcal{W}_{ij} = \sum_i \Big[\overline{\mathbf{S}}_i\, R + \sum_{j>i}\overline{\mathbf{C}}_{ij}(1 - \overline{\mathbf{S}}_i - \overline{\mathbf{S}}_j)\, R\Big]$$

Strategy of the algorithm

- ***Unitary partition*** of radiative phase-space with **sector functions** $\mathcal{W}_{ij}$ *[Frixione, Kunszt, Signer 9512328]*
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$$\frac{d\sigma_{NLO}}{dX} = \int d\Phi_n \Big(V + I\Big)\delta_{X_n} \checkmark$$
$$+ \int d\Phi_{n+1}\Big(R\,\delta_{X_{n+1}} - K\,\delta_{X_n}\Big) \checkmark$$

☑ **Finite** in phase space (integrable in $d = 4$)

☑ **Free** from ϵ poles

- $\mathcal{W}_{ij}$ sum rules + mapping adaptation = **simple analytic** counterterm integration

$$K = \sum_{i,j\neq i}\Big[\overline{\mathbf{S}}_i + \overline{\mathbf{C}}_{ij} - \overline{\mathbf{S}}_i\overline{\mathbf{C}}_{ij}\Big] R\mathcal{W}_{ij} = \sum_i \overline{\mathbf{S}}_i\, R + \sum_{j>i}\overline{\mathbf{C}}_{ij}(1 - \overline{\mathbf{S}}_i - \overline{\mathbf{S}}_j)\, R\Big]$$

$$I^S \propto \sum_{k,l}\bar{B}_{kl}^{(ikl)}\frac{1}{\bar{s}_{kl}^{(ikl)}}\int d\Phi_{\rm rad}(\bar{s}_{kl}^{(ikl)}; y, z, \phi)\frac{1-z}{yz}$$
$$= \sum_{k,l}\bar{B}_{kl}^{(ikl)}\frac{(4\pi)^{\epsilon-2}}{(\bar{s}_{kl}^{(ikl)})^{\epsilon}}\frac{\Gamma(1-\epsilon)\Gamma(2-\epsilon)}{\epsilon^2\Gamma(2-3\epsilon)}$$

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$$\frac{d\sigma_{NLO}}{dX} = \int d\Phi_n \Big(V + I\Big)\delta_{X_n} + \int d\Phi_{n+1}\Big(R\,\delta_{X_{n+1}} - K\,\delta_{X_n}\Big)$$

☑ **Finite** in phase space (integrable in $d = 4$)

☑ **Free** from ϵ poles

$$K = \sum_{i,j\neq i}\Big[\overline{\mathbf{S}}_i + \overline{\mathbf{C}}_{ij} - \overline{\mathbf{S}}_i\overline{\mathbf{C}}_{ij}\Big]R\mathcal{W}_{ij} = \sum_i\Big[\overline{\mathbf{S}}_i\,R + \sum_{j>i}\overline{\mathbf{C}}_{ij}(1-\overline{\mathbf{S}}_i-\overline{\mathbf{S}}_j)\,R\Big]$$

$$I^S \propto \sum_{k,l}\bar{B}^{(ikl)}_{kl}\frac{1}{\bar{s}^{(ikl)}_{kl}}\int d\Phi_{\rm rad}(\bar{s}^{(ikl)}_{kl};y,z,\phi)\frac{1-z}{yz}$$

$$= \sum_{k,l}\bar{B}^{(ikl)}_{kl}\frac{(4\pi)^{\epsilon-2}}{(\bar{s}^{(ikl)}_{kl})^{\epsilon}}\frac{\Gamma(1-\epsilon)\Gamma(2-\epsilon)}{\epsilon^2\Gamma(2-3\epsilon)}$$

Exploring the framework...

Local Analytic Sector Subtraction

$$\frac{d\sigma}{dX} = \frac{d\sigma_{\mathrm{LO}}}{dX} + \frac{d\sigma_{\mathrm{NLO}}}{dX} + \boxed{\frac{d\sigma_{\mathrm{NNLO}}}{dX}} + \dots$$

σ = partonic cross section
X = generic IRC-safe observable

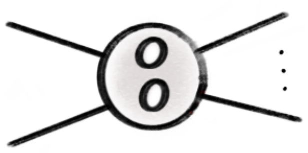
Subtraction formula at NNLO

Massless QCD
final-state radiation

Generalities at NNLO

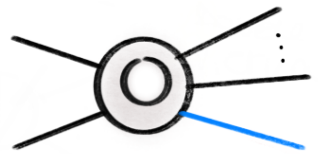
- X_i = **IRC-safe** observable computed with i-body kinematics, $\delta_{X_i} \equiv \delta(X - X_i)$

$$\frac{d\sigma_{NNLO}}{dX} = \int d\Phi_n \, VV \, \delta_{X_n}$$



Up to $1/\epsilon^4$ poles

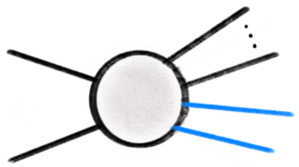
$$+ \int d\Phi_{n+1} \, RV \, \delta_{X_{n+1}}$$



Up to $1/\epsilon^2$ poles

Singular in PS

$$+ \int d\Phi_{n+2} \, RR \, \delta_{X_{n+2}}$$



Singular in PS

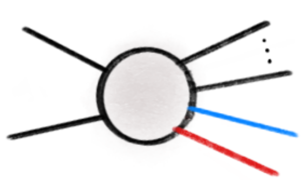
Generalities at NNLO

- X_i = **IRC-safe** observable computed with i-body kinematics, $\delta_{X_i} \equiv \delta(X - X_i)$

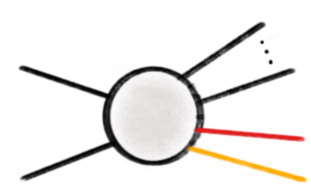
$$
\begin{aligned}
\frac{d\sigma_{NNLO}}{dX} = & \int d\Phi_n \Big(VV \Big) \delta_{X_n} \\
& + \int d\Phi_{n+1} \Big[\Big(RV \Big) \delta_{X_{n+1}} - \Big(K^{(\mathrm{RV})} \Big) \delta_{X_n} \Big] \\
& + \int d\Phi_{n+2} \Big[RR\, \delta_{X_{n+2}} - K^{(1)} \delta_{X_{n+1}} - \Big(K^{(2)} - K^{(12)} \Big) \delta_{X_n} \Big]
\end{aligned}
$$

- Introduce **local counterterms** and proper **phase-space factorisations**

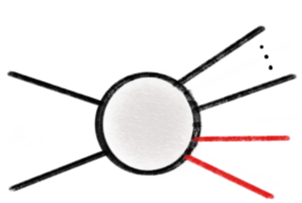
single-unresolved

$K^{(1)}$ 

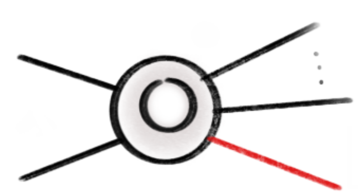
strongly-ordered double-unresolved

$K^{(12)}$ 

double-unresolved

$K^{(2)}$ 

single-unresolved

$K^{(\mathrm{RV})}$ 

Generalities at NNLO

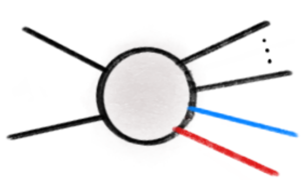
- X_i = **IRC-safe** observable computed with i-body kinematics, $\delta_{X_i} \equiv \delta(X - X_i)$

$$
\begin{aligned}
\frac{d\sigma_{NNLO}}{dX} = & \int d\Phi_n \left(VV + I^{(2)} + I^{(RV)}\right)\delta_{X_n} \\
& + \int d\Phi_{n+1}\left[\left(RV + I^{(1)}\right)\delta_{X_{n+1}} - \left(K^{(\mathrm{RV})} + I^{(12)}\right)\delta_{X_n}\right] \\
& + \int d\Phi_{n+2}\left[RR\,\delta_{X_{n+2}} - K^{(1)}\delta_{X_{n+1}} - \left(K^{(2)} - K^{(12)}\right)\delta_{X_n}\right]
\end{aligned}
$$

- Introduce **local counterterms** and proper **phase-space factorisations**

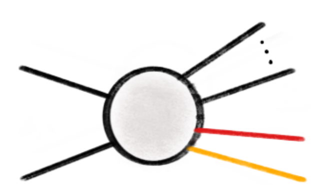
single-unresolved

$K^{(1)}$



strongly-ordered double-unresolved

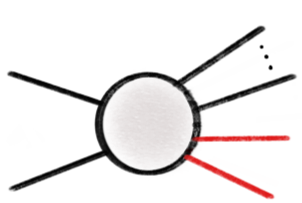
$K^{(12)}$



$$d\Phi_{n+2} = d\Phi_{n+1}\, d\Phi_{\mathrm{rad},1}$$

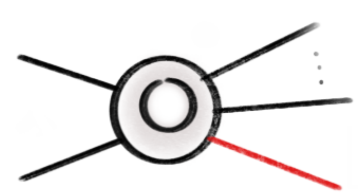
double-unresolved

$K^{(2)}$



single-unresolved

$K^{(\mathrm{RV})}$



$$d\Phi_{n+2} = d\Phi_n\, d\Phi_{\mathrm{rad},2}$$

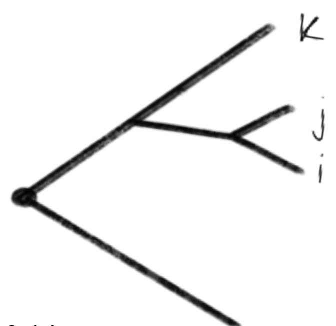
$$d\Phi_{n+1} = d\Phi_n\, d\Phi_{\mathrm{rad},1}$$

Subtracting RR singularities

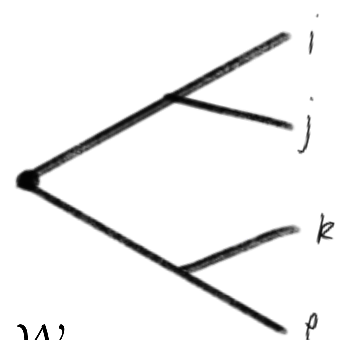
- Smooth **unitary partition** of double-unresolved phase space Φ_{n+2} into ***sectors*** $\mathcal{W}_{ijkl}$

$$RR = \sum_{i,j\neq i} \sum_{\substack{k\neq i \\ l\neq i,k}} RR\,\mathcal{W}_{ijkl} \qquad \text{with} \qquad \sum_{i,j\neq i} \sum_{\substack{k\neq i \\ l\neq i,k}} \mathcal{W}_{ijkl} = 1$$

  - 3 ***topologies*** collecting all types of singularities



$\mathcal{W}_{ijjk}, \mathcal{W}_{ijkj}$



$\mathcal{W}_{ijkl}$

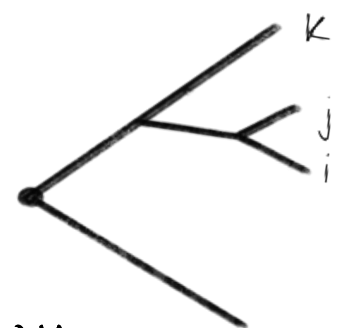
$$\begin{array}{lllllll}
\mathcal{W}_{ijjk} & : & \mathbf{S}_i & \mathbf{C}_{ij} & \mathbf{S}_{ij} & \mathbf{C}_{ijk} & \mathbf{SC}_{ijk} \\
\mathcal{W}_{ijkj} & : & \mathbf{S}_i & \mathbf{C}_{ij} & \mathbf{S}_{ik} & \mathbf{C}_{ijk} & \mathbf{SC}_{ijk} & \mathbf{SC}_{kij} \\
\mathcal{W}_{ijkl} & : & \mathbf{S}_i & \mathbf{C}_{ij} & \mathbf{S}_{ik} & \mathbf{C}_{ijkl} & \mathbf{SC}_{ikl} & \mathbf{SC}_{kij}
\end{array}$$

Subtracting RR singularities

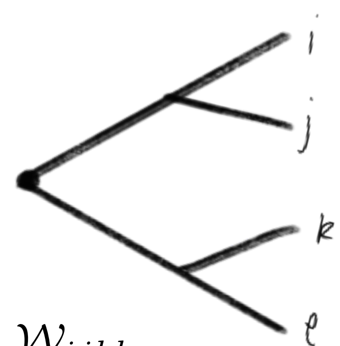
- Smooth **unitary partition** of double-unresolved phase space Φ_{n+2} into ***sectors*** $\mathcal{W}_{ijkl}$

$$RR = \sum_{i,j\neq i} \sum_{\substack{k\neq i \\ l\neq i,k}} RR\, \mathcal{W}_{ijkl} \qquad \text{with} \qquad \sum_{i,j\neq i} \sum_{\substack{k\neq i \\ l\neq i,k}} \mathcal{W}_{ijkl} = 1$$

* 3 ***topologies*** collecting all types of singularities



$\mathcal{W}_{ijjk}, \mathcal{W}_{ijkj}$



$\mathcal{W}_{ijkl}$

$\mathcal{W}_{ijjk}$	:	$\mathbf{S}_{i}$	$\mathbf{C}_{ij}$	$\mathbf{S}_{ij}$	$\mathbf{C}_{ijk}$	$\mathbf{SC}_{ijk}$	
$\mathcal{W}_{ijkj}$	:	$\mathbf{S}_{i}$	$\mathbf{C}_{ij}$	$\mathbf{S}_{ik}$	$\mathbf{C}_{ijk}$	$\mathbf{SC}_{ijk}$	$\mathbf{SC}_{kij}$
$\mathcal{W}_{ijkl}$	:	$\mathbf{S}_{i}$	$\mathbf{C}_{ij}$	$\mathbf{S}_{ik}$	$\mathbf{C}_{ijkl}$	$\mathbf{SC}_{ikl}$	$\mathbf{SC}_{kij}$

single-unresolved limits

double-unresolved limits

$\mathbf{S}_{ij}$ **double-soft** partons *i* and *j*

$\mathbf{C}_{ijk}$ **triple-collinear** partons *(i,j,k)*

$\mathbf{C}_{ijkl}$ **double-collinear** partons *(i,j)* and *(k,l)*

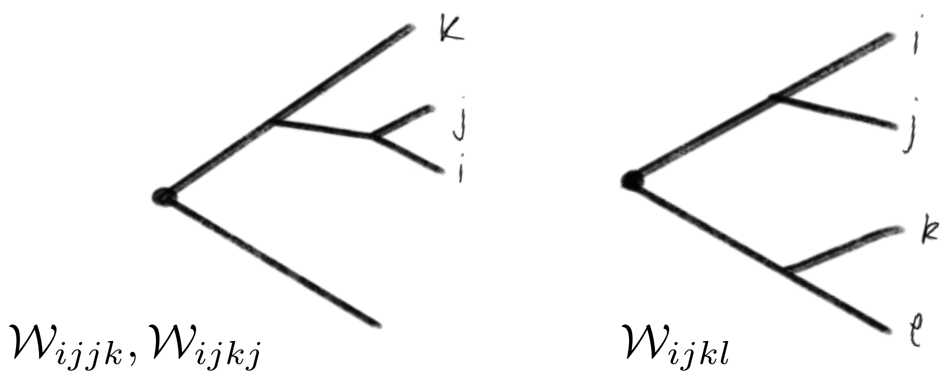
$\mathbf{SC}_{ijk}$ **soft** parton *i* and **collinear** partons *(j,k)*

Subtracting RR singularities

- Smooth **unitary partition** of double-unresolved phase space Φ_{n+2} into ***sectors*** $\mathcal{W}_{ijkl}$

$$RR = \sum_{i,j\neq i} \sum_{\substack{k\neq i \\ l\neq i,k}} RR\, \mathcal{W}_{ijkl} \qquad \text{with} \qquad \sum_{i,j\neq i} \sum_{\substack{k\neq i \\ l\neq i,k}} \mathcal{W}_{ijkl} = 1$$

* 3 ***topologies*** collecting all types of singularities



		single-unresolved		double-unresolved			
$\mathcal{W}_{ijjk}$	:	$\mathbf{S}_i$	$\mathbf{C}_{ij}$	$\mathbf{S}_{ij}$	$\mathbf{C}_{ijk}$	$\mathbf{SC}_{ijk}$	
$\mathcal{W}_{ijkj}$	:	$\mathbf{S}_i$	$\mathbf{C}_{ij}$	$\mathbf{S}_{ik}$	$\mathbf{C}_{ijk}$	$\mathbf{SC}_{ijk}$	$\mathbf{SC}_{kij}$
$\mathcal{W}_{ijkl}$	:	$\mathbf{S}_i$	$\mathbf{C}_{ij}$	$\mathbf{S}_{ik}$	$\mathbf{C}_{ijkl}$	$\mathbf{SC}_{ikl}$	$\mathbf{SC}_{kij}$

single-unresolved limits

double-unresolved limits

* ***Sum rules:*** limits of sector functions still form a unitary partition

Key for integration

$$\mathbf{S}_{ik}\Bigg(\sum_{b\neq i}\sum_{d\neq i,k} \mathcal{W}_{ibkd} + \sum_{b\neq k}\sum_{d\neq k,i} \mathcal{W}_{kbid}\Bigg) = 1$$

$$\mathbf{C}_{ijk} \sum_{abc\,\in\,\pi(ijk)} \Big(\mathcal{W}_{abbc} + \mathcal{W}_{abcb}\Big) = 1$$

$$\ldots$$

$\mathbf{S}_{ij}$	**double-soft** partons *i* and *j*
$\mathbf{C}_{ijk}$	**triple-collinear** partons *(i,j,k)*
$\mathbf{C}_{ijkl}$	**double-collinear** partons *(i,j)* and *(k,l)*
$\mathbf{SC}_{ijk}$	**soft** parton *i* and **collinear** partons *(j,k)*

Subtracting RR singularities

- Smooth **unitary partition** of double-unresolved phase space Φ_{n+2} into ***sectors*** $\mathcal{W}_{ijkl}$

- Collect the limited ***relevant*** **IRC limits** reproducing all singularities

$$RR\,\mathcal{W}_\tau - \left[\mathbf{L}^{(1)}_{ij} + \mathbf{L}^{(2)}_{\tau} - \mathbf{L}^{(1)}_{ij}\mathbf{L}^{(2)}_{\tau}\right] RR\,\mathcal{W}_\tau \;\to\; \text{integrable}$$

single-unresolved

$$\mathbf{L}^{(1)}_{ij} \;=\; \mathbf{S}_i + \mathbf{C}_{ij}(1-\mathbf{S}_i)$$

double-unresolved ($\tau = ijjk, ijkj, ijkl$)

$$\mathbf{L}^{(2)}_{ijjk} \;=\; \mathbf{S}_{ij} + \mathbf{C}_{ijk}(1-\mathbf{S}_{ij}) + \mathbf{SC}_{ijk}(1-\mathbf{S}_{ij})(1-\mathbf{C}_{ijk})$$

$$\mathbf{L}^{(2)}_{ijkj} \;=\; \mathbf{S}_{ik} + \mathbf{C}_{ijk}(1-\mathbf{S}_{ik}) + (\mathbf{SC}_{ijk} + \mathbf{SC}_{kij})(1-\mathbf{S}_{ik})(1-\mathbf{C}_{ijk})$$

$$\mathbf{L}^{(2)}_{ijkl} \;=\; \mathbf{S}_{ik} + \mathbf{C}_{ijkl}(1-\mathbf{S}_{ik}) + (\mathbf{SC}_{ikl} + \mathbf{SC}_{kij})(1-\mathbf{S}_{ik})(1-\mathbf{C}_{ijkl})$$

Notation

$$\mathbf{L}\,RR\,\mathcal{W}_{ijkl} \;=\; (\mathbf{L}\,RR)\,(\mathbf{L}\,\mathcal{W}_{ijkl})$$

Subtracting RR singularities

- Smooth **unitary partition** of double-unresolved phase space Φ_{n+2} into ***sectors*** $\mathcal{W}_{ijkl}$

- Collect the limited ***relevant*** **IRC limits** reproducing all singularities

$$RR\,\mathcal{W}_\tau - \left[\mathbf{L}^{(1)}_{ij} + \mathbf{L}^{(2)}_{\tau} - \mathbf{L}^{(1)}_{ij}\mathbf{L}^{(2)}_{\tau}\right] RR\,\mathcal{W}_\tau \;\to\; \text{integrable}$$

single-unresolved

$$\mathbf{L}^{(1)}_{ij} \;=\; \mathbf{S}_i + \mathbf{C}_{ij}(1-\mathbf{S}_i)$$

double-unresolved $(\tau = ijjk, ijkj, ijkl)$

$$\mathbf{L}^{(2)}_{ijjk} \;=\; \mathbf{S}_{ij} + \mathbf{C}_{ijk}(1-\mathbf{S}_{ij}) + \mathbf{SC}_{ijk}(1-\mathbf{S}_{ij})(1-\mathbf{C}_{ijk})$$
$$\mathbf{L}^{(2)}_{ijkj} \;=\; \mathbf{S}_{ik} + \mathbf{C}_{ijk}(1-\mathbf{S}_{ik}) + (\mathbf{SC}_{ijk} + \mathbf{SC}_{kij})(1-\mathbf{S}_{ik})(1-\mathbf{C}_{ijk})$$
$$\mathbf{L}^{(2)}_{ijkl} \;=\; \mathbf{S}_{ik} + \mathbf{C}_{ijkl}(1-\mathbf{S}_{ik}) + (\mathbf{SC}_{ikl} + \mathbf{SC}_{kij})(1-\mathbf{S}_{ik})(1-\mathbf{C}_{ijkl})$$

Notation

$$\mathbf{L}\,RR\,\mathcal{W}_{ijkl} \;=\; (\mathbf{L}\,RR)\,(\mathbf{L}\,\mathcal{W}_{ijkl})$$

✱ Products of known **splitting kernels** x **lower-multiplicities MEs**

[Catani, Grazzini 9810389, 9908523]

$$\mathbf{S}_{ik}\,RR \;\supset\; \frac{\mathcal{N}_1^2}{2}\sum_{\substack{c\neq i,k\\ d\neq i,k,c}}\left\{\frac{s_{cd}}{s_{ic}s_{id}}\left[\sum_{\substack{e\neq i,k,c,d\\ f\neq i,k,c,d,e}}\frac{s_{ef}}{s_{ke}s_{kf}}\,B_{cdef}(\{k\}_{\not i\not k}) + 2\,\frac{s_{cd}}{s_{kc}s_{kd}}\,B_{cdcd}(\{k\}_{\not i\not k})\right]\right\}$$

Missing momentum conservation out of the double singular region!

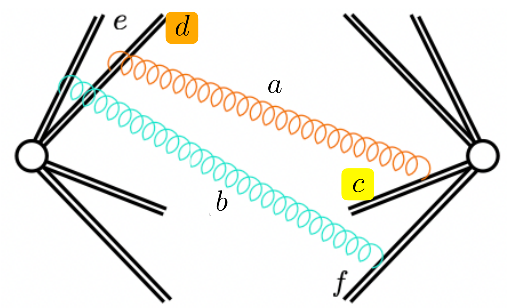
Subtracting RR singularities

- Smooth **unitary partition** of double-unresolved phase space Φ_{n+2} into ***sectors*** $\mathcal{W}_{ijkl}$
- Collect the limited ***relevant*** **IRC limits** reproducing all singularities
- **Nested *Catani-Seymour* mappings** from $(n+2) \to n$ kinematics *[Catani, Seymour 9605323]*

  - Minimal set of involved momenta
  - Still clear factorisation of radiative d.o.f.
  - Smart and adaptive parametrisation simplifies kernel expressions

$$\{k\} \to \{\bar{k}\}^{(abc,def)}$$

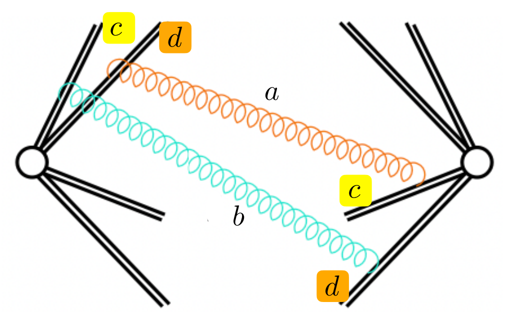
$$d\Phi_{n+2} = d\Phi_n^{(abcd)}\, d\Phi_{\rm rad}^{(abc)}\, d\Phi_{\rm rad}^{(def)}$$



$$\overline{\mathbf{S}}_{ik}\, RR \supset \frac{\mathcal{N}_1^2}{2} \sum_{\substack{c\neq i,k\\ d\neq i,k,c}} \left\{ \frac{s_{cd}}{s_{ic}s_{id}} \left[\sum_{\substack{e\neq i,k,c,d\\ f\neq i,k,c,d,e}} \frac{s_{ef}}{s_{ke}s_{kf}} \bar{B}_{cdef}^{(icd,kef)} + 2\, \frac{s_{cd}}{s_{kc}s_{kd}} \bar{B}_{cdcd}^{(icd,kcd)} \right] \right\}$$

$$\{k\} \to \{\bar{k}\}^{(abcd)}$$

$$d\Phi_{n+2} = d\Phi_n^{(abcd)}\, d\Phi_{\rm rad,2}^{(abcd)}$$



Subtracting RR singularities

- Smooth **unitary partition** of double-unresolved phase space Φ_{n+2} into ***sectors*** $\mathcal{W}_{ijkl}$

- Collect the limited ***relevant*** **IRC limits** reproducing all singularities

- **Nested *Catani-Seymour* mappings** from $(n+2) \to n$ kinematics *[Catani, Seymour 9605323]*

- Promotion to **counterterms**: ***Improved limits*** adapting momenta mapping to each kernel, while tuning action on sector functions when necessary

$$K^{(\mathbf{1})} = \sum_{i,j\neq i} \sum_{\substack{k\neq i \\ l\neq i,k}} \overline{\mathbf{L}}^{(\mathbf{1})}_{ij} \, RR \, \mathcal{W}_{ijkl}$$

single-unresolved limits

$$K^{(\mathbf{2})} = \sum_{i,j\neq i} \sum_{\substack{k\neq i \\ l\neq i,k}} \overline{\mathbf{L}}^{(\mathbf{2})}_{ijkl} \, RR \, \mathcal{W}_{ijkl}$$

uniform double-unresolved limits

$$K^{(\mathbf{12})} = \sum_{i,j\neq i} \sum_{\substack{k\neq i \\ l\neq ,k}} \overline{\mathrm{L}}^{(\mathbf{1})}_{ij} \, \overline{\mathrm{L}}^{(\mathbf{2})}_{ijkl} \, RR \, \mathcal{W}_{ijkl}$$

strongly-ordered double-unresolved limits

Subtracting RR singularities

- Smooth **unitary partition** of double-unresolved phase space Φ_{n+2} into ***sectors*** $\mathcal{W}_{ijkl}$
- Collect the limited ***relevant*** **IRC limits** reproducing all singularities
- **Nested *Catani-Seymour* mappings** from $(n+2) \to n$ kinematics *[Catani, Seymour 9605323]*
- Promotion to **counterterms**: ***Improved limits*** adapting momenta mapping to each kernel, while tuning action on sector functions when necessary

$$K^{(1)} = \sum_{i,j\neq i} \sum_{\substack{k\neq i \\ l\neq i,k}} \overline{\mathbf{L}}^{(1)}_{ij} \, RR \, \mathcal{W}_{ijkl}$$

single-unresolved limits

$$K^{(2)} = \sum_{i,j\neq i} \sum_{\substack{k\neq i \\ l\neq i,k}} \overline{\mathbf{L}}^{(2)}_{ijkl} \, RR \, \mathcal{W}_{ijkl}$$

uniform double-unresolved limits

$$= \Bigg\{ \sum_{i,k>i} \overline{\mathbf{S}}_{ik} + \sum_{i,j>i} \sum_{k>j} \overline{\mathbf{C}}_{ijk} \Big(1 - \overline{\mathbf{S}}_{ij} - \overline{\mathbf{S}}_{ik} - \overline{\mathbf{S}}_{jk}\Big)$$

$$+ \sum_{i,j>i} \sum_{\substack{k\neq j \\ k>i}} \sum_{\substack{l\neq j \\ l>k}} \overline{\mathbf{C}}_{ijkl} \Big[1 - \overline{\mathbf{S}}_{ik} - \overline{\mathbf{S}}_{il} - \overline{\mathbf{S}}_{jk} - \overline{\mathbf{S}}_{jl}$$

$$- \overline{\mathbf{SC}}_{ikl}(1 - \overline{\mathbf{S}}_{ik} - \overline{\mathbf{S}}_{il}) - \overline{\mathbf{SC}}_{jkl}(1 - \overline{\mathbf{S}}_{jk} - \overline{\mathbf{S}}_{jl})$$

$$- \overline{\mathbf{SC}}_{kij}(1 - \overline{\mathbf{S}}_{ik} - \overline{\mathbf{S}}_{jk}) - \overline{\mathbf{SC}}_{lij}(1 - \overline{\mathbf{S}}_{il} - \overline{\mathbf{S}}_{jl})\Big]$$

$$+ \sum_{i,j>i} \sum_{\substack{k\neq i \\ k>j}} \overline{\mathbf{SC}}_{ijk}(1 - \overline{\mathbf{S}}_{ij} - \overline{\mathbf{S}}_{ik})(1 - \overline{\mathbf{C}}_{ijk}) \Bigg\} RR$$

Collection of universal kernels!

Subtracting RR singularities

- Smooth **unitary partition** of double-unresolved phase space Φ_{n+2} into ***sectors*** $\mathcal{W}_{ijkl}$

- Collect the limited ***relevant*** **IRC limits** reproducing all singularities

- **Nested *Catani-Seymour* mappings** from $(n+2) \rightarrow n$ kinematics *[Catani, Seymour 9605323]*

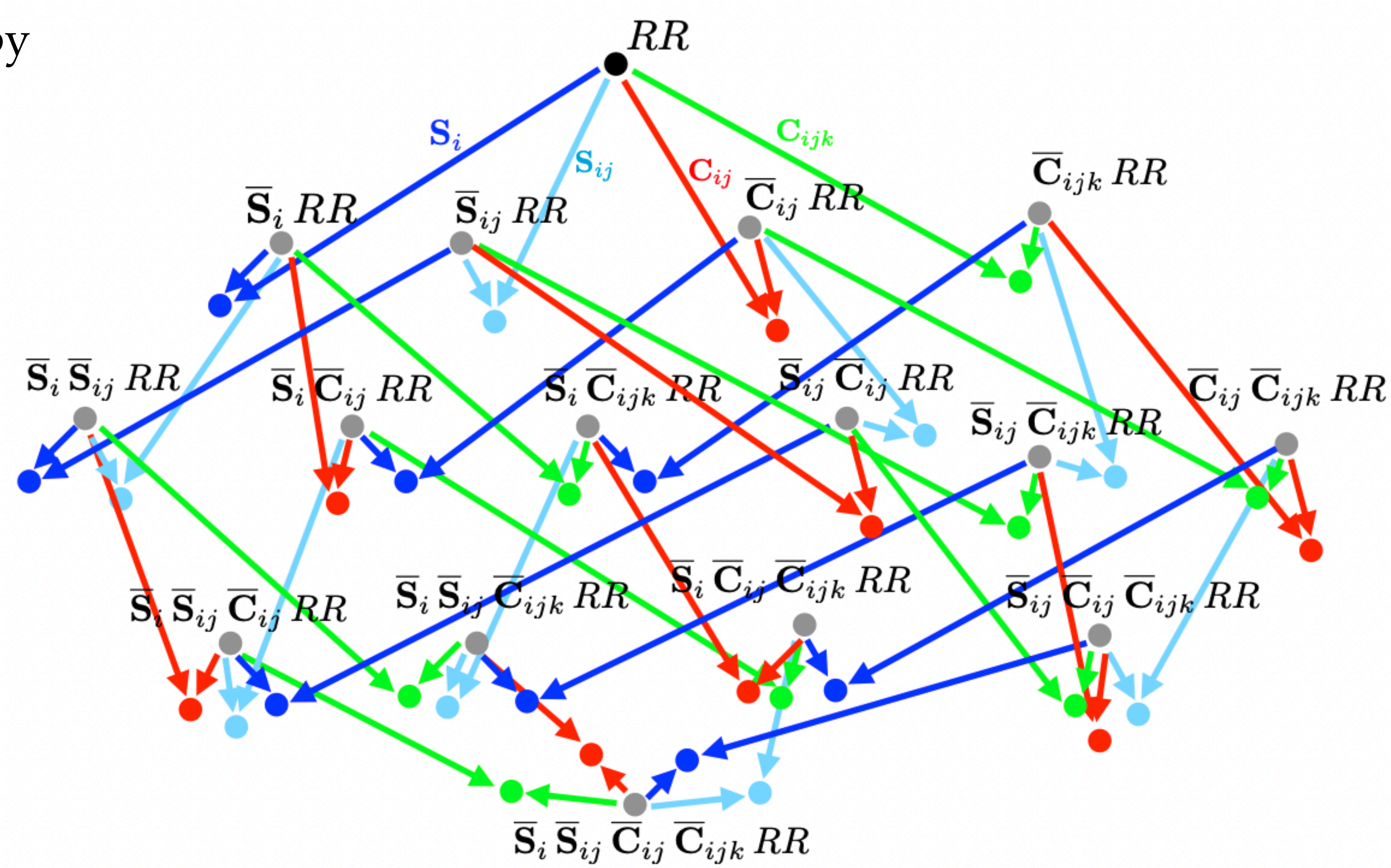
- Promotion to **counterterms**: ***Improved limits*** adapting momenta mapping to each kernel, while tuning action on sector functions when necessary

- **Locality** of the cancellation ensured by ***consistency relations***

verified sector by sector

Selection of displayed limits

$\mathbf{S}_i, \ \mathbf{C}_{ij}, \ \mathbf{S}_{ij}, \ \mathbf{C}_{ijk}$



Subtracting RR singularities

$$\frac{d\sigma_{NNLO}}{dX} = \int d\Phi_n\, VV\, \delta_{X_n} + \int d\Phi_{n+1}\, RV\, \delta_{X_{n+1}} + \int d\Phi_{n+2} \Big[RR\, \delta_{X_{n+2}} - K^{(1)} \delta_{X_{n+1}} - \Big(K^{(2)} - K^{(12)} \Big) \delta_{X_n} \Big]$$

☑ **Finiteness** of double-real correction (integrable in $d = 4$)

Subtracting RR singularities

$$\frac{d\sigma_{NNLO}}{dX} = \int d\Phi_n \Big(VV + I^{(2)} \qquad\qquad \Big) \delta_{X_n}$$
$$+ \int d\Phi_{n+1} \Big[\Big(RV + I^{(1)} \Big) \delta_{X_{n+1}} - \Big(K^{(\mathrm{RV})} \quad + I^{(12)} \Big) \delta_{X_n} \Big]$$
$$+ \int d\Phi_{n+2} \Big[RR\, \delta_{X_{n+2}} - K^{(1)} \delta_{X_{n+1}} - \Big(K^{(2)} - K^{(12)} \Big) \delta_{X_n} \Big] \checkmark$$

☑ **Finiteness** of double-real correction (integrable in $d = 4$)

▷ $\mathcal{W}_{ijkl}$ sum rules + mapping adaptation → **analytic integrations** by means of ***standard techniques***

[Magnea, et al 2010.14493]

$$I^{(\mathbf{1})} = \int d\Phi_{\mathrm{rad}}\, K^{(\mathbf{1})} \qquad I^{(\mathbf{12})} = \int d\Phi_{\mathrm{rad}}\, K^{(\mathbf{12})} \qquad I^{(\mathbf{2})} = \int d\Phi_{\mathrm{rad},2}\, K^{(\mathbf{2})}$$

NNLO complexity

* Logarithmic (trivial) dependence on Mandelstam invariants
* **Note**: no approximations in local terms!

Subtracting RV singularities

$$\frac{d\sigma_{NNLO}}{dX} = \int d\Phi_n \Big(VV + I^{(2)} \Big)\delta_{X_n}$$

$$+ \int d\Phi_{n+1}\Big[\Big(RV + I^{(1)}\Big)\delta_{X_{n+1}} - \Big(K^{(RV)} + I^{(12)}\Big)\delta_{X_n}\Big]$$

$$+ \int d\Phi_{n+2}\Big[RR\,\delta_{X_{n+2}} - K^{(1)}\delta_{X_{n+1}} - \Big(K^{(2)} - K^{(12)}\Big)\delta_{X_n}\Big]$$

- More intricate cancellation pattern involving both ***poles*** and ***phase-space singularities***

$RV + I^{(1)} \to$ finite in ϵ	Still singular in PS
$I^{(1)} - I^{(12)} \to$ integrable	Contains poles in ϵ

Subtracting RV singularities

$$\frac{d\sigma_{NNLO}}{dX} = \int d\Phi_n \left(VV + I^{(2)} \right) \delta_{X_n}$$
$$+ \int d\Phi_{n+1} \left[\left(RV + I^{(1)} \right) \delta_{X_{n+1}} - \left(K^{(\mathrm{RV})} + I^{(12)} \right) \delta_{X_n} \right]$$
$$+ \int d\Phi_{n+2} \left[RR\, \delta_{X_{n+2}} - K^{(1)} \delta_{X_{n+1}} - \left(K^{(2)} - K^{(12)} \right) \delta_{X_n} \right]$$

- More intricate cancellation pattern involving both ***poles*** and ***phase-space singularities***

$RV + I^{(1)} \to$ finite in ϵ

$I^{(1)} - I^{(12)} \to$ integrable

Analytically checked finiteness of RV contribution!

$RV - K^{(\mathrm{RV})} \to$ integrable

$K^{(\mathrm{RV})} + I^{(12)} \to$ finite in ϵ

- Apply NLO strategy to define the **real-virtual local term** *[Bern, at al 9903516]*

$$K^{(\mathrm{RV})} \supset \sum_{i,j\neq i} \left[\overline{\mathbf{S}}_i + \overline{\mathbf{C}}_{ij} (1 - \overline{\mathbf{S}}_i) \right] RV\, \mathcal{W}_{ij}$$

Subtracting RV singularities

$$\frac{d\sigma_{NNLO}}{dX} = \int d\Phi_n \Big(VV + I^{(2)} + I^{(\mathrm{RV})} \Big) \delta_{X_n}$$
$$+ \int d\Phi_{n+1} \Big[\Big(RV + I^{(1)} \Big) \delta_{X_{n+1}} - \Big(K^{(\mathrm{RV})} + I^{(12)} \Big) \delta_{X_n} \Big]$$
$$+ \int d\Phi_{n+2} \Big[RR \, \delta_{X_{n+2}} - K^{(1)} \delta_{X_{n+1}} - \Big(K^{(2)} - K^{(12)} \Big) \delta_{X_n} \Big]$$

- More intricate cancellation pattern involving both ***poles*** and ***phase-space singularities***

$RV + I^{(1)} \to$ finite in ϵ

$I^{(1)} - I^{(12)} \to$ integrable

Analytically checked finiteness of RV contribution!

$RV - K^{(\mathrm{RV})} \to$ integrable

$K^{(\mathrm{RV})} + I^{(12)} \to$ finite in ϵ

- Apply NLO strategy to define the **real-virtual local term** *[Bern, at al 9903516]*

$$K^{(\mathrm{RV})} \supset \sum_{i, j \neq i} \Big[\overline{\mathbf{S}}_i + \overline{\mathbf{C}}_{ij} (1 - \overline{\mathbf{S}}_i) \Big] RV \, \mathcal{W}_{ij}$$

$$I^{(\mathrm{RV})} = \int d\Phi_{\mathrm{rad}} \, K^{(\mathrm{RV})}$$

Subtracting RV singularities

$$\frac{d\sigma_{NNLO}}{dX} = \int d\Phi_n \left(VV + I^{(2)} + I^{(\mathrm{RV})}\right)\delta_{X_n}$$
$$+ \int d\Phi_{n+1}\Big[\left(RV + I^{(1)}\right)\delta_{X_{n+1}} - \left(K^{(\mathrm{RV})} + I^{(12)}\right)\delta_{X_n}\Big]$$
$$+ \int d\Phi_{n+2}\Big[RR\,\delta_{X_{n+2}} - K^{(1)}\delta_{X_{n+1}} - \left(K^{(2)} - K^{(12)}\right)\delta_{X_n}\Big]$$

Removing VV poles

- Extract the ϵ poles of the ***double-virtual*** correction and sum counterterm integrations

☑ **Analytically** verified for an **arbitrary** number of final-state partons

$$VV + I^{(2)} + I^{(\mathrm{RV})} \to \text{free from } \epsilon \text{ poles}$$

NNLO subtraction formula

Massless QCD final-state radiation

$$\frac{d\sigma_{NNLO}}{dX} = \int d\Phi_n \left(VV + I^{(2)} + I^{(\mathrm{RV})}\right)\delta_{X_n}$$
$$+ \int d\Phi_{n+1}\left[\left(RV + I^{(1)}\right)\delta_{X_{n+1}} - \left(K^{(\mathrm{RV})} + I^{(12)}\right)\delta_{X_n}\right]$$
$$+ \int d\Phi_{n+2}\left[RR\,\delta_{X_{n+2}} - K^{(1)}\delta_{X_{n+1}} - \left(K^{(2)} - K^{(12)}\right)\delta_{X_n}\right]$$

- Verified for an **arbitrary number** of final-state coloured particles (as well as of arbitrary massive/massless colourless ones)
- **No approximations** introduced in local and integrated terms
- **Analytic finite remainder** retaining mostly ***simple logarithmic dependence*** on kinematic invariants
- Ready to be implemented in a numerical framework equipped with the relevant matrix elements

NNLO subtraction formula

Massless QCD final-state radiation

$$\frac{d\sigma_{NNLO}}{dX} = \int d\Phi_n \left(VV + I^{(2)} + I^{(\mathrm{RV})}\right)\delta_{X_n}$$

$$+ \int d\Phi_{n+1}\Big[\left(RV + I^{(1)}\right)\delta_{X_{n+1}} - \left(K^{(\mathrm{RV})} + I^{(12)}\right)\delta_{X_n}\Big]$$

$$\mathrm{L}_{ab} = \log\frac{s_{ab}}{\mu^2}$$

$$\begin{aligned}
VV + I^{(2)} + I^{(\mathrm{RV})} = \left(\frac{\alpha_s}{2\pi}\right)^2 \Bigg\{ & \bigg[I^{(0)} + \sum_j I_j^{(1)}\,\mathbf{L}_{jr} + \sum_j I_j^{(2)}\,\mathbf{L}_{jr}^2 + \frac{1}{2}\sum_{j,l\neq j}\gamma_j^{\mathrm{hc}}\,\gamma_l^{\mathrm{hc}}\,\mathbf{L}_{jr'}\mathbf{L}_{lr'}\bigg]\,\mathbf{B} \\
& + \sum_j \Big[I_{jr}^{(0)} + I_{jr}^{(1)}\,\mathbf{L}_{jr}\Big]\,\mathbf{B}_{jr} - 2\,(1-\zeta_2)\sum_{j,c\neq j,r}\gamma_j^{\mathrm{hc}}\,(2-\mathbf{L}_{cr})\,\mathbf{B}_{cr} \\
& + \sum_{c,d\neq c}\mathbf{L}_{cd}\bigg[I_{cd}^{(0)} + I_{cd}^{(1)}\,\mathbf{L}_{cd} + \frac{\beta_0}{12}\,\mathbf{L}_{cd}^2 + (4-\mathbf{L}_{cd})\sum_j \gamma_j^{\mathrm{hc}}\,\mathbf{L}_{jr}\bigg]\,\mathbf{B}_{cd} \\
& + \sum_{c,d\neq c}\bigg[-2 + \zeta_2 + 2\,\zeta_3 - \frac{5}{4}\,\zeta_4 + 2(1-\zeta_3)\,\mathbf{L}_{cd}\bigg]\,\mathbf{B}_{cdcd} \\
& + (1-\zeta_2)\sum_{\substack{c,d\neq c\\ e\neq d}}\mathbf{L}_{cd}\,\mathbf{L}_{ed}\,\mathbf{B}_{cded} + \sum_{\substack{c,d\neq c\\ e,f\neq e}}\mathbf{L}_{cd}\,\mathbf{L}_{ef}\bigg[1 - \frac{1}{2}\,\mathbf{L}_{cd}\left(1-\frac{1}{8}\mathbf{L}_{ef}\right)\bigg]\,\mathbf{B}_{cdef} \\
& + \pi\sum_{\substack{c,d\neq c\\ e\neq c,d}}\bigg[\ln\frac{s_{ce}}{s_{de}}\,\mathbf{L}_{cd}^2 + \frac{1}{3}\ln^3\frac{s_{ce}}{s_{de}} + 2\,\mathrm{Li}_3\left(-\frac{s_{ce}}{s_{de}}\right)\bigg]\mathbf{B}_{cde}\Bigg\} \\
+ \left(\frac{\alpha_s}{2\pi}\right)\Bigg\{ & \bigg[\Sigma_\phi - \sum_j \gamma_j^{\mathrm{hc}}\,\mathbf{L}_{jr}\bigg]\,\mathbf{V}^{\mathrm{fin}} + \sum_{c,d\neq c}\mathbf{L}_{cd}\left(2-\frac{1}{2}\mathbf{L}_{cd}\right)\mathbf{V}_{cd}^{\mathrm{fin}}\Bigg\} + \mathbf{VV}^{\mathrm{fin}}
\end{aligned}$$

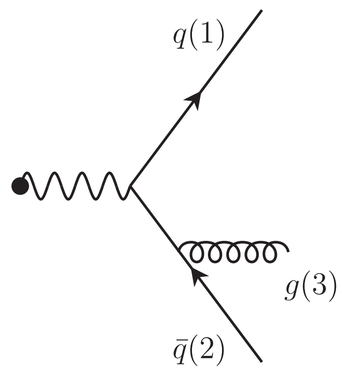
Analytic and compact!

Symmetrised sector functions

$\mathcal{Z}_{ij} \equiv \mathcal{W}_{ij} + \mathcal{W}_{ji}$

$e^+e^- \to jj$ at NLO

Real configuration

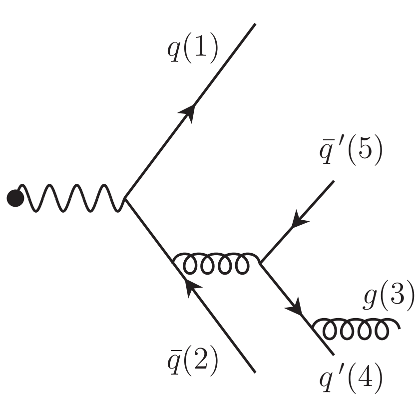


	# limits per sector	# sectors	# limits
$\mathcal{W}_{13}, \mathcal{W}_{23}$	3	4	8
$\mathcal{W}_{31}, \mathcal{W}_{32}$	1		
$\mathcal{Z}_{13}, \mathcal{Z}_{23}$	2	2	4

$e^+e^- \to jjj$ at NNLO

Double-real configuration
for selected channel

$e^+e^- \to q\bar{q}q'\bar{q}'g$



	# limits per sector	# sectors	# limits
$\mathcal{W}_{3445}, \mathcal{W}_{3554}, \mathcal{W}_{3454}, \mathcal{W}_{3545}, \mathcal{W}_{4535}, \mathcal{W}_{5434}$	11	12	88
$\mathcal{W}_{4335}, \mathcal{W}_{4553}, \mathcal{W}_{5334}, \mathcal{W}_{5443}$	3		
$\mathcal{W}_{4353}, \mathcal{W}_{5343}$	5		
$\mathcal{Z}_{345}$	15	1	15

$$\begin{aligned}\mathcal{Z}_{ijk} &= \mathcal{W}_{ijjk} + \mathcal{W}_{ikkj} + \mathcal{W}_{jiik} + \mathcal{W}_{jkki} + \mathcal{W}_{kiij} + \mathcal{W}_{kjji} \\ &+ \mathcal{W}_{ijkj} + \mathcal{W}_{ikjk} + \mathcal{W}_{jiki} + \mathcal{W}_{jkik} + \mathcal{W}_{kiji} + \mathcal{W}_{kjij}\end{aligned}$$

Status & Outlook

☑ General analytic subtraction formula for **massless FSR and ISR at NLO**

*[**GB**, Torrielli, Uccirati, Zaro 2209.09123]*

☑ General analytic **subtraction formula** for **massless FSR at NNLO**

Next goals...

- Implementation and test of the NNLO formula in a numerical framework

On-going work

- Extension to initial-state coloured particles for LHC applications (expected integrals of complexity similar to massless FSR)

- Treatment of the massive case: less singular limits, but more involved integrals

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Thanks for your attention!

Backup slides

NNLO subtraction formula

Massless QCD final-state radiation

$$\frac{d\sigma_{NNLO}}{dX} = \int d\Phi_n \left(VV + I^{(2)} + I^{(\mathrm{RV})}\right)\delta_{X_n}$$

$$+ \int d\Phi_{n+1}\left[\left(RV + I^{(1)}\right)\delta_{X_{n+1}} - \left(K^{(\mathrm{RV})} + I^{(12)}\right)\delta_{X_n}\right]$$

$$\mathrm{L}_{ab} = \log\frac{s_{ab}}{\mu^2}$$

Analytic and compact!

$$VV + I^{(2)} + I^{(\mathrm{RV})} = \left(\frac{\alpha_s}{2\pi}\right)^2 \left\{ \left[I^{(0)} + \sum_j I^{(1)}_j \mathbf{L}_{jr} + \sum_j I^{(2)}_j \mathbf{L}^2_{jr} + \ldots \right.\right.$$

$$+ \sum_j \left[I^{(0)}_{jr} + I^{(1)}_{jr}\mathbf{L}_{jr}\right] \mathbf{B}_{jr} - 2\,(1-\zeta_2 \ldots$$

$$+ \sum_{c,d\neq c} \mathbf{L}_{cd}\left[I^{(0)}_{cd} + I^{(1)}_{cd}\mathbf{L}_{cd} + \frac{\beta_0}{12}\mathbf{L}^2_{cd} + \ldots \right.$$

$$+ \sum_{c,d\neq c} \left[-2 + \zeta_2 + 2\,\zeta_3 - \frac{5}{4}\zeta_4 + 2(1 \ldots \right.$$

$$+ (1-\zeta_2) \sum_{\substack{c,d\neq c\\ e\neq d}} \mathbf{L}_{cd}\,\mathbf{L}_{ed}\,\mathbf{B}_{cded} + \sum_{\substack{c,d\neq c\\ e,f\neq e}} \mathbf{L} \ldots$$

$$+ \pi \sum_{\substack{c,d\neq c\\ e\neq c,d}} \left[\ln\frac{s_{ce}}{s_{de}}\,\mathbf{L}^2_{cd} + \frac{1}{3}\ln^3\frac{s_{ce}}{s_{de}} + 2\,\mathbf{L} \ldots \right.$$

$$+ \left(\frac{\alpha_s}{2\pi}\right) \left\{ \left[\Sigma_\phi - \sum_j \gamma^{\mathrm{hc}}_j \mathbf{L}_{jr} \right] \mathbf{V}^{\mathrm{fin}} + \sum_{c,d\neq c} \mathbf{L}_{cd} \ldots \right.$$

$$I^{(0)} = N_q^2 C_F^2 \left[\frac{101}{8} - \frac{141}{8}\zeta_2 + \frac{245}{16}\zeta_4\right] + N_g N_q C_F \left[C_A\left(\frac{13}{3} - \frac{125}{6}\zeta_2 + \frac{245}{8}\zeta_4\right) + \beta_0\left(\frac{77}{12} - \frac{53}{12}\zeta_2\right)\right]$$
$$+ N_g^2\left[C_A^2\left(\frac{20}{9} - \frac{13}{3}\zeta_2 + \frac{245}{16}\zeta_4\right) + \beta_0^2\left(\frac{73}{72} - \frac{1}{8}\zeta_2\right) + C_A\beta_0\left(-\frac{1}{9} - \frac{11}{3}\zeta_2\right)\right]$$
$$+ N_q C_F\left[C_F\left(\frac{53}{32} - \frac{57}{8}\zeta_2 + \frac{1}{2}\zeta_3 + \frac{21}{4}\zeta_4\right) + C_A\left(\frac{677}{432} + \frac{5}{3}\zeta_2 - \frac{25}{2}\zeta_3 + \frac{47}{8}\zeta_4\right)\right.$$
$$\left. + \beta_0\left(\frac{5669}{864} - \frac{85}{24}\zeta_2 - \frac{11}{12}\zeta_3\right)\right]$$
$$+ N_g\left[C_F C_A\left(-\frac{737}{48} + 11\zeta_3\right) + C_F\beta_0\left(\frac{67}{16} - 3\zeta_3\right) + \beta_0^2\left(\frac{73}{72} - \frac{3}{8}\zeta_2\right)\right.$$
$$\left. + C_A^2\left(-\frac{4289}{216} + \frac{15}{2}\zeta_2 - 14\zeta_3 + \frac{89}{8}\zeta_4\right) + C_A\beta_0\left(\frac{647}{54} - \frac{53}{8}\zeta_2 - \frac{11}{12}\zeta_3\right)\right]$$

$$I^{(1)}_j = \delta_{f_a\{q,\bar q\}} C_F\left[N_q C_F\left(\frac{5}{2} - \frac{7}{4}\zeta_2\right) + N_g C_A\left(\frac{1}{3} - \frac{7}{4}\zeta_2\right) + \frac{2}{3}N_g\beta_0\right.$$
$$\left. + C_F\left(-\frac{3}{8} - 4\zeta_2 + 2\zeta_3\right) + C_A\left(\frac{25}{12} - 3\zeta_2 + 3\zeta_3\right) + \beta_0\left(\frac{1}{24} + \zeta_2\right)\right]$$
$$+ \delta_{f_a g}\left[N_q C_F C_A(10 - 7\zeta_2) - N_q C_F\beta_0\left(\frac{5}{2} - \frac{7}{4}\zeta_2\right) + N_g C_A^2\left(\frac{4}{3} - 7\zeta_2\right) + N_g C_A\beta_0\left(\frac{7}{3} + \frac{7}{4}\zeta_2\right)\right.$$
$$\left. - \frac{2}{3}(N_g + 1)\beta_0^2 + \frac{11}{4}C_F C_A - \frac{3}{4}C_F\beta_0 + C_A^2\left(\frac{28}{3} - \frac{23}{2}\zeta_2 + 5\zeta_3\right) - C_A\beta_0\left(\frac{2}{3} - \frac{5}{2}\zeta_2\right)\right]$$

$$I^{(2)}_j = \frac{1}{8}(15\,C_A - 7\,\beta_0 - 15)C_{f_j} - \frac{1}{4}(5\,C_A - 2\,\beta_0)\gamma_j + 2\,\zeta_2\,C^2_{f_j}$$

$$I^{(0)}_{jr} = (-1 + 3\zeta_2 - 2\zeta_3)C_A - \frac{1}{2}(13 + 10\zeta_2 + 2\zeta_3)C_{f_j} + (5 + 2\zeta_3)\gamma_j$$

$$I^{(1)}_{jr} = (1 - \zeta_2)C_A + \frac{1}{2}(4 + 7\zeta_2)C_{f_j} - (2 + \zeta_2)\gamma_j$$

$$I^{(0)}_{cd} = \left(\frac{20}{9} - 2\zeta_2 - \frac{7}{2}\zeta_3\right)C_A + \frac{31}{9}\beta_0 + 2\,\Sigma_\phi + 8\,(1 - \zeta_2)\,C_{f_d}$$

$$I^{(1)}_{cd} = -\left(\frac{1}{3} - \frac{1}{2}\zeta_2\right)C_A - \frac{11}{12}\beta_0 - \frac{1}{2}\Sigma_\phi$$

Slide by Lorenzo Magnea, HP2 2022

N³LO Subtraction

A systematic generalisation to higher orders is possible. At three loops one finds

$$
\begin{aligned}
\frac{d\sigma_{\rm N3LO}}{dX} &= \int d\Phi_n \left[VVV_n + I_n^{(\mathbf{3})} + I_n^{(\mathbf{RVV})} + I_n^{(\mathbf{RRV},\,\mathbf{2})} \right] \delta_n(X) \\
&+ \int d\Phi_{n+1} \bigg[\Big(RVV_{n+1} + I_{n+1}^{(\mathbf{2})} + I_{n+1}^{(\mathbf{RRV},\,\mathbf{1})} \Big)\, \delta_{n+1}(X) \\
&\qquad - \Big(K_{n+1}^{(\mathbf{RVV})} + I_{n+1}^{(\mathbf{23})} + I_{n+1}^{(\mathbf{RRV},\,\mathbf{12})} \Big)\, \delta_n(X) \bigg] \\
&+ \int d\Phi_{n+2} \bigg\{ \Big(RRV_{n+2} + I_{n+2}^{(\mathbf{1})} \Big)\, \delta_{n+2}(X) - \Big(K_{n+2}^{(\mathbf{RRV},\mathbf{1})} + I_{n+2}^{(\mathbf{12})} \Big)\, \delta_{n+1}(X) \\
&\qquad - \bigg[\Big(K_{n+2}^{(\mathbf{RRV},\,\mathbf{2})} + I_{n+2}^{(\mathbf{13})} \Big) - \Big(K_{n+2}^{(\mathbf{RRV},\mathbf{12})} + I_{n+2}^{(\mathbf{123})} \Big) \bigg]\, \delta_n(X) \bigg\} \\
&+ \int d\Phi_{n+3} \bigg[RRR_{n+3}\, \delta_{n+3}(X) - K_{n+3}^{(\mathbf{1})}\, \delta_{n+2}(X) - \Big(K_{n+3}^{(\mathbf{2})} - K_{n+3}^{(\mathbf{12})} \Big)\, \delta_{n+1}(X) \\
&\qquad - \Big(K_{n+3}^{(\mathbf{3})} - K_{n+3}^{(\mathbf{13})} - K_{n+3}^{(\mathbf{23})} + K_{n+3}^{(\mathbf{123})} \Big)\, \delta_n(X) \bigg] ,
\end{aligned}
$$

A general formula for N^{k}LO subtraction is available, involving $p = 2^{(k+1)} - 2 - k$ counterterms.

Double-unresolved phase space

- Catani-Seymour variables $y, z, y', z', x' \in [0,1]$ for mapping $\{k\} \to \{\bar{k}\}^{(abcd)}$:

$$s_{ab} = y'\, y\, s_{abcd}\,, \qquad s_{cd} = (1-y')\,(1-y)\,(1-z)\; s_{abcd}\,,$$

$$s_{ac} = z'\,(1-y')\;y\, s_{abcd}\,, \qquad s_{bc} = (1-y')\,(1-z')\;y\, s_{abcd}\,,$$

$$s_{ad} = (1-y)\left[y'\,(1-z')\,(1-z) + z'z - 2\,(1-2x')\,\sqrt{y'z'\,(1-z')\;z\,(1-z)}\right] s_{abcd}\,,$$

$$s_{bd} = (1-y)\left[y'z'\,(1-z) + (1-z')\,z + 2\,(1-2x')\,\sqrt{y'z'\,(1-z')\;z\,(1-z)}\right] s_{abcd}\,,$$

- Phase-space factorisation:

$$d\Phi_{n+2} \quad = \quad d\Phi_n^{(abcd)}\, d\Phi_{\mathrm{rad},2}^{(abcd)}\,,$$

$$\int d\Phi_{\mathrm{rad},2}^{(abcd)} \quad = \quad \int d\Phi_{\mathrm{rad},2}\left(s_{abcd};\, y, z, \phi, y', z', x'\right)$$

$$= \quad N^2(\epsilon)\,(s_{abcd})^{2-2\epsilon}\int_0^1 dx' \int_0^1 dy' \int_0^1 dz' \int_0^\pi d\phi\;(\sin\phi)^{-2\epsilon}\int_0^1 dy\int_0^1 dz$$

$$\times\left[4\,x'\,(1-x')\;y'\,(1-y')^2\;z'\,(1-z')\;y^2(1-y)^2\,z\,(1-z)\right]^{-\epsilon}$$

$$\times\left[x'\,(1-x')\right]^{-1/2}(1-y')\;y\,(1-y)\,.$$

Analytic integration of double-unresolved counterterms

- Exploit as much as possible symmetries of $d\Phi^{(abcd)}_{\mathrm{rad},2}$:

$$\mathrm{perm}(k_a, k_b, k_c, k_d)\,, \qquad s_{ab} \leftrightarrow s_{cd}\,, \qquad s_{ac} \leftrightarrow s_{bd}\,, \qquad s_{ad} \leftrightarrow s_{bc}\,.$$

- Possible denominator structures reduce to

$$\begin{aligned}
s_{ab} &= y'\, y\, s_{abcd}\,,\\
s_{ac} &= z'\,(1-y')\; y\, s_{abcd}\,,\\
s_{bc} &= (1-y')\,(1-z')\; y\, s_{abcd}\,,\\
s_{cd} &= (1-y')\,(1-y)\,(1-z)\; s_{abcd}\,,\\
s_{bd} &= (1-y)\left[y'z'\,(1-z) + (1-z')\,z + 2\,(1-2w')\,\sqrt{y'z'\,(1-z')\,z\,(1-z)}\right] s_{abcd}\,,\\
s_{ac} + s_{bc} &= (1-y')\; y\, s_{abcd}\,,\\
s_{ad} + s_{bd} &= (y' + z - y'\,z)\,(1-y)\, s_{abcd}\,,\\
s_{ab} + s_{bc} &= (1 - z' + z'\,y')\; y\, s_{abcd}\,.
\end{aligned}$$

- Integration measure

$$\begin{aligned}
\int d\Phi^{(abcd)}_{\mathrm{rad},2} \;\; = \;\; & N(\epsilon)\,(s_{abcd})^{2-2\epsilon}\int_0^1 dw' \int_0^1 dy' \int_0^1 dz' \int_0^1 dy \int_0^1 dz\,\left[w'\,(1-w')\,\right]^{-1/2-\epsilon}\\
& \times \left[y'\,(1-y')^2\; z'\,(1-z')\; y^2(1-y)^2\, z\,(1-z)\right]^{-\epsilon}\,(1-y')\; y\,(1-y)\;.
\end{aligned}$$

Analytic integration of double-unresolved counterterms

- Integration measure

$$\int d\Phi^{(abcd)}_{\mathrm{rad},2} = N(\epsilon)\,(s_{abcd})^{2-2\epsilon}\int_0^1 dy\int_0^1 dw'\int_0^1 dz\int_0^1 dy'\int_0^1 dz'\,\left[w'\,(1-w')\,\right]^{-1/2-\epsilon}$$
$$\times\left[y'\,(1-y')^2\;z'\,(1-z')\;y^2(1-y)^2\,z\,(1-z)\right]^{-\epsilon}\,(1-y')\;y\,(1-y)\;.$$

- Integrate y: fully factorised dependence $\rightarrow$ Beta functions.

- Integrate w' (azimuth): at worst one gets rational $\times\;{}_2F_1\left[1, 1+\epsilon, 1-\epsilon, \frac{y'z'(1-z)}{z(1-z')}\right]$.

- Integrate z: at worst one gets rational $\times\;{}_2F_1\left[1, n+1-\epsilon, 1-\epsilon, -\frac{y'z'}{1-z'}\right]$.

- ${}_2F_1\;\rightarrow\;$ integral representation in t; integrate in z' and get at worst

$$\int_0^1 dy'\,dt\,t^a\,(1-t)^b\,y'^c\,(1-y')^d\;{}_2F_1\left[n, m-\epsilon, p-2\epsilon, 1-ty'\right], \qquad n, m, p \in \mathbb{N}$$

- Expand in ϵ and integrate in $dt\,dy'$.

- Checked against numerical integration (with no symmetries or relabellings encoded).